Assumption-lean weak limits and tests for two-stage adaptive experiments

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May 15, 2025

Abstract

Adaptive experiments are becoming increasingly popular in real-world applications for effectively maximizing in-sample welfare and efficiency by data-driven sampling. Despite their growing prevalence, however, the statistical foundations for valid inference in such settings remain underdeveloped. Focusing on two-stage adaptive experimental designs, we address this gap by deriving new weak convergence results for mean outcomes and their differences. In particular, our results apply to a broad class of estimators, the *weighted inverse probability weighted* (WIPW) estimators. In contrast to prior works, our results require significantly weaker assumptions and sharply characterize phase transitions in limiting behavior across different signal regimes. Through this common lens, our general results unify previously fragmented results under the two-stage setup. To address the challenge of potential non-normal limits in conducting inference, we propose a computationally efficient and provably valid plug-in bootstrap method for hypothesis testing. Our results and approaches are sufficiently general to accommodate various adaptive experimental designs, including batched bandit and subgroup enrichment experiments. Simulations and semi-synthetic studies demonstrate the practical value of our approach, revealing statistical phenomena unique to adaptive experiments.

1 Introduction

Adaptive experiments are able to achieve substantial efficiency gains compared with traditional non-adaptive experimental designs. They often allocate resources more effectively and require fewer samples or observations to attain the same statistical power or estimation precision. Such designs have been successfully applied in areas such as clinical trials (Sampson et al., 2005; Hu et al., 2006; Magnusson et al., 2013), online learning (Slivkins et al., 2019; Lattimore et al., 2020), mobile health interventions (Klasnja et al., 2019; Liao et al., 2020), and online education platforms (Rafferty et al., 2019; Kizilcec et al., 2020).

However, the adaptive design of these experiments introduces dependencies among observations, violating the independence and identical distribution (i.i.d.) assumptions underlying classical inference methods. As a result, widely used estimators—such as the sample mean and inverse probability weighted estimator—may exhibit bias and non-normal sampling distributions under adaptive data collection (Bowden et al., 2017; Shin et al., 2019; Hadad et al., 2021; Shin et al., 2021). In practice, analyzing adaptive experiments using conventional statistical tools while ignoring the dependencies can lead to severe selection bias (Dwork et al., 2015). The limited theoretical understanding of the statistical behavior of adaptive experiments continues to hinder the development of valid and generalizable inference methods. This represents a major barrier to their reliable use in real-world applications.

In this paper, we study the *two-stage adaptive experiments*, a design framework that has been widely adopted in practice (Sampson et al., 2005; Sladek et al., 2007; Sill et al., 2009; Wu et al., 2010; Gasperini et al., 2019; Lin et al., 2021; Kasy et al., 2021; Che et al., 2023; Schraivogel et al., 2023). The typical data generating process in these experiments can be outlined as follows: there are two stages of data collection, the *pilot stage* and the *follow-up stage*. In the pilot stage, i.i.d. data $\mathcal{D}_P \sim \mathbb{P}_P$ are collected, and informs a selection algorithm $\mathcal{S}(\mathcal{D}_P)$. Then in the follow-up stage, new data $\mathcal{D}_F \sim \mathbb{P}_F(\mathcal{D}_P)$ are gathered according to the output of selection algorithm $\mathcal{S}$, resulting in data that are conditionally i.i.d. given $\mathcal{D}_P$ (see Algorithm 1 for a complete description). The choice of $\mathcal{S}$ depends on the goal of the experiment. Common objectives include welfare maximization (Sampson et al., 2005; Wu et al., 2010; Che et al., 2023) and scientific exploration (Sladek et al., 2007; Gasperini et al., 2019).

Algorithm 1: Two-stage adaptive experiment.

Input: Selection algorithm $\mathcal{S}$.

Pilot stage:

1 Observe i.i.d. data $\mathcal{D}_P$ from the law $\mathbb{P}_P$.

2 Apply the selection criteria $\mathcal{S}(\mathcal{D}_P)$ to modify the law $\mathbb{P}_P$ to $\mathbb{P}_F(\mathcal{D}_P)$.

Follow-up stage:

3 Collect conditionally i.i.d. data $\mathcal{D}_F$ from the modified law $\mathbb{P}_F(\mathcal{D}_P)$.

Output: Pooled outcome $\mathcal{D}_P \cup \mathcal{D}_F$.

The main challenge of conducting valid statistical inference with data $\mathcal{D}_P \cup \mathcal{D}_F$ is to handle the complex dependence structure introduced by the selection algorithm $\mathcal{S}$.

1.1 Relevant literature

Existing work on inference for adaptive experiments largely falls into two main categories, distinguished by the type of inference they offer.

Conditional inference provides valid inference conditional on the output of selection algorithm $\mathcal{S}(\mathcal{D}_P)$. Approaches that achieve this guarantee include data splitting (Cox, 1975), data carving (Fithian et al., 2014; Chen et al., 2023), and randomization-based selective inference (Freidling et al., 2024). This type of guarantee captures the effect of the selection procedure and adjusts for conditional bias. However, when the estimand is a marginal parameter (e.g., outcome mean), conditional inference typically incurs an efficiency loss (Hu et al., 2006; Marschner, 2021). Moreover, due to the potential complex conditioning event, methods developed for conditional inference can be computationally demanding, with the exception of data splitting.

Marginal inference accounts for all sources of randomness, leading to more straightforward interpretation when inferential target is the marginal estimand. Towards this end, different approaches have been proposed, with finite-sample or asymptotic guarantees. Among the former, some seek to achieve exact finite-sample validity (Sampson et al., 2005; Sill et al., 2009; Wu et al., 2010; Neal et al., 2011; Nair et al., 2023). However, these methods are relatively restrictive as they are either computationally intensive and/or are highly sensitive to distributional assumptions. Anytime-valid inference methods (Johari et al., 2015; Howard et al., 2021; Howard et al., 2022; Maharaj et al., 2023; Ramdas et al., 2023; Waudby-Smith et al., 2024) provide finite-sample validity via probabilistic bounds. These bounds, however, are usually conservative and can substantially reduce power (although they generalize well beyond two-stage settings like Algorithm 1). In contrast, asymptotic inferential methods (Zhang et al., 2020; Hadad et al., 2021; Lin et al., 2021; Adusumilli, 2023; Hirano et al., 2023) tend to be statistically more efficient than conditional or anytime-valid approaches. Relying solely on large-sample behavior, asymptotic methods are generally agnostic to outcome distributions.

Our work falls into the category of marginal inference with asymptotic validity. The works of Zhang et al. (2020), Hadad et al. (2021), Adusumilli (2023), and Hirano et al. (2023) are the most related ones. Zhang et al. (2020) and Hadad et al. (2021) establish asymptotic normality results on the outcome means by imposing strong assumptions on the signal strength or the distribution of the potential outcomes.[1] Hirano et al. (2023) provide general representation of the limiting distribution of test statistics in a multi-stage setup, similar to that in Algorithm 1. Later, Adusumilli (2023) uses these representations to study the optimality of the tests under the same setup. Their results are built upon Le Cam's theory on limits of experiments (Le Cam et al., 1972), which are valid only under contiguous alternatives and smooth (semi-)parametric outcome distributions. Despite the elegance of the results, the general representation requires verifying the existence of certain weak limits, which must be addressed on a case-by-case basis and therefore poses a barrier for practitioners. A more detailed comparison with the related literature is provided in Section 3.3.

From the inferential perspective, strong assumptions about signal strength and data generating distributions in this line of work can have significant practical consequences. Misusing the limiting distribution in hypothesis testing may yield Type-I error inflation (see an example in Figure 6 in Appendix E.2). Likewise, the stringent distributional requirements on $\mathbb{P}_F$ and $\mathbb{P}_P$ are often unrealistic when outcomes display complex or non-standard behavior. A vivid illustration arises in exploratory biological studies, such as single-cell CRISPR screens (Dixit et al., 2016; Gasperini et al., 2019). In these experiments, the main outcomes, gene-expression measurements, typically exhibit overdispersion, measurement error, and technical batch effects, all of which may violate the prespecified distributional assumptions. These caveats highlight the need for a robust inferential framework that can accommodate a broader range of signal strengths and remain agnostic to the outcome distribution.

Beyond inference, assumption-lean weak limits offer a compelling tool for experimental design and power analysis. Alternatively, simulation methods have been pro-

[1] The setting of Zhang et al. (2020) and Hadad et al. (2021) can be more general than the two-stage adaptive experiments we consider in Algorithm 1.

posed to design new adaptive experiments (Chapter VII; Food et al., 2019), which can bypass the derivation of limiting distributions. Simulation-based strategies typically rely on stringent parametric assumptions, are largely heuristic, and become computationally intensive when navigating a vast parameter space. In contrast, distribution-agnostic weak limits provide a far more efficient alternative—so long as one can sample from the limiting distribution effectively. Pursuing this direction, Che et al. (2023) prove a joint weak-limit result for batched multi-armed bandits and leverage it to design batch-level allocation rules that minimize Bayes simple regret. Their framework, however, prioritizes regret minimization rather than hypothesis testing, and is not sufficient for inference or for designing experiments that aim to maximize statistical testing power.

1.2 Our contributions

To address these gaps in asymptotic inference of the two-stage experiments, we study the asymptotic distribution of a broad class of *weighted inverse probability weighted* (WIPW) statistics. This class includes several widely used statistics, such as the IPW statistic (Bowden et al., 2017) and the variance-stabilizing IPW statistic (Luedtke et al., 2016; Bibaut et al., 2021; Hadad et al., 2021). We establish weak convergence results under minimal assumptions. Building on this foundation, we propose a valid and computationally efficient bootstrap procedure for hypothesis testing in the presence of non-normal limiting null distributions. Specifically, our main contributions are summarized as follows.

1. **Assumption-lean weak limits:** We derive new weak convergence results for WIPW estimators in two-stage experimental settings (Algorithm 1). These results apply to a wide range of signal strengths under minimal distributional assumptions. Such generality ensures valid inference across the null (zero signal), contiguous (weak signal), and fixed (strong signal) regimes, addressing key demands in hypothesis testing. Our analysis also uncovers a smooth transition of the limiting distribution across signal regimes, offering a unified perspective that connects several existing results in the literature. The proofs of these results require a new set of probabilistic tools to account for the dependence structure induced by adaptive data collection. To this end, we derive new results on conditional normal approximation (Lemma 15) and continuous mapping theorems (Lemma 17 and 18), generalizing the existing unconditional tools (Chatterjee et al., 2008; Meckes, 2009a). These new tools may be of independent interest for analyzing other adaptive experiments.

2. **A fast bootstrap testing procedure:** Building on the general weak convergence results, we define a class of asymptotically valid tests using WIPW test statistics. The critical values in the tests are determined by quantiles of the non-normal limiting distributions under the null. To obtain the analytically intractable critical values, we propose a computationally efficient and provably valid plug-in bootstrap procedure to estimate the critical values. The procedure rests on the key insight that the derived weak limits can be expressed as a randomly weighted sum of dependent Gaussian random variables. By leveraging the

new bootstrap procedure, valid and practical hypothesis tests can be conducted despite the complexity of the limiting distribution. Importantly, the procedure is nonparametric and thus is agnostic to the outcome distribution. Moreover, it has time complexity that is independent of the sample size (conditional on estimated nuisance parameters), making it highly scalable.

We apply our general results to two real-world adaptive experimental designs: *batched bandit experiments* and *subgroup enrichment experiments*. Although these experiments arise in distinct scientific contexts, both can be naturally accommodated within our theoretical framework. We then conduct extensive numerical simulations to evaluate the finite-sample performance of the proposed bootstrap-based testing procedures. Additionally, we perform a semi-synthetic data analysis based on the Systolic Blood Pressure Intervention Trial (SPRINT) (Ambrosius et al., 2014), a large-scale randomized controlled study. The results demonstrate the practical utility of our methods in realistic settings. Notably, our analysis reveals several phenomena that are unique to adaptive designs and would not arise in classical randomized controlled experiments. Moreover, our results can be readily applied to the design of adaptive experiments, as they offer a clear understanding of the limiting behavior of the test statistics. Code to reproduce these analyses is available at `https://github.com/ZiangNiu6/AdaInf-manuscript`.

1.3 Organization of the paper

Section 2 introduces the two-stage adaptive data collection procedure and the WIPW test statistic. In Section 3, we present the formal results on weak convergence and the bootstrap methodology, instantiate our general theory in various adaptive experiments, and establish the connection to existing works. In Section 4, we evaluate the finite-sample performance of the derived tests. We conclude the paper with a discussion in Section 5.

2 Data generating procedure and test statistic

We will discuss the data collection procedure in Section 2.1 and test statistics of interest in Section 2.2.

2.1 Two-stage adaptive data collection

We denote the sample sizes for the pilot and follow-up stages as N_1 and N_2, respectively, and treat them as given. The total sample size is defined as $N \equiv N_1 + N_2$, and the sample size ratio for the two stages is fixed as $q_t \equiv N_t/N \in (0,1)$ for $t \in \{1,2\}$. Throughout this paper, we adopt the triangular array framework, allowing the distribution to vary with N. To emphasize this dependence, we use the subscript N when defining the random variables. Also, we define $[I] \equiv \{1,\ldots,I\}$ for any integer $I \geq 1$.

In our setup, there are two competing treatments indexed by 0 and 1. Let $A \in \{0,1\}$ denote the assigned treatment. Suppose $(A_{uN}^{(t)}, Y_{uN}^{(t)})_{u\in[N_t]}$ denotes the observed data at stage t, where $Y_{uN}^{(t)}$ is the observed outcome corresponding to assigned treatment

$A_{uN}^{(t)}$. Let $\mathcal{H}_t = \sigma\big((A_{uN}^{(t)}, Y_{uN}^{(t)})_{u\in[N_t]}\big)$ be the σ-algebra generated by the observed data at stage t. Additionally, define $\mathcal{H}_0 \equiv \{\varnothing, \Omega\}$. Adopting the potential outcome framework, for N subjects, the potential outcomes are denoted as

$$\{(Y_{uN}^{(t)}(0), Y_{uN}^{(t)}(1)) : t \in [2], u \in [N_t]\}.$$

They are independently and identically distributed as $(Y_{uN}(0), Y_{uN}(1))$ for any fixed N. To identify the distribution of potential outcome variables, we assume the following *consistency* and *unconfoundedness* conditions throughout this paper.

- *Consistency:* $Y_{uN}^{(t)} = A_{uN}^{(t)} Y_{uN}^{(t)}(1) + (1 - A_{uN}^{(t)}) Y_{uN}^{(t)}(0), \quad u \in [N_t],\ t \in [2];$
- *Unconfoundedness:* $(Y_{uN}^{(t)}(0), Y_{uN}^{(t)}(1)) \perp\!\!\!\perp A_{uN}^{(t)} \mid \mathcal{H}_{t-1}, \quad u \in [N_t],\ t \in [2].$

These are two assumptions that are commonly made in the literature of causal inference (Imbens et al., 2015). The consistency assumption states that the observed outcome is equal to the potential outcome under the assigned treatment. The unconfoundedness assumption states that the potential outcomes are independent of the treatment assignment, given the information from previous stages. Now we describe the observed data generating procedure.

1. **Pilot stage:** In the pilot stage, we observe $(A_{uN}^{(1)}, Y_{uN}^{(1)})$ for $u \in [N_1]$, with treatment assignment probabilities $e(s) \equiv \mathbb{P}[A_{uN}^{(1)} = s]$ [2] for $s \in \{0, 1\}$, where $e(0) + e(1) = 1$. A selection algorithm $\mathcal{S}$ (introduced in Algorithm 1) determines treatment assignment for the follow-up stage. For an estimator $S_N^{(1)}(0) - S_N^{(1)}(1)$ for the difference-in-means $\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)]$, the sampling probabilities are then updated based on $\mathcal{S}(S_N^{(1)}(0) - S_N^{(1)}(1))$.

2. **Follow-up stage:** In the follow-up stage, we define the new sampling probabilities as

   $$\mathbb{P}[A_{uN}^{(2)} = 0 \mid \mathcal{H}_1] = \mathcal{S}(S_N^{(1)}(0) - S_N^{(1)}(1)) \quad \text{and} \quad \mathbb{P}[A_{uN}^{(2)} = 1 \mid \mathcal{H}_1] = 1 - \mathbb{P}[A_{uN}^{(2)} = 0 \mid \mathcal{H}_1].$$

   With these updated probabilities, we collect the data $(A_{uN}^{(2)}, Y_{uN}^{(2)})$ for $u \in [N_2]$. These observations are independently and identically distributed, *conditional* on the information from the pilot stage ($\mathcal{H}_1$).

We make one comment on the selection algorithm $\mathcal{S}$.

Remark 1 (Generality of $\mathcal{S}$). *Our main results readily generalize to settings where the selection algorithm depends on more complex functions of the data beyond the simple difference in means. However, for clarity of presentation and broad applicability, we focus on selection algorithms $\mathcal{S}$ based on the estimator of the difference-in-means, $S_N^{(1)}(1) - S_N^{(1)}(0)$. In fact, such algorithm reflects several important practical objectives in adaptive experimentation. A natural sampling strategy is to assign higher probability to the treatment yielding a higher average outcome in the pilot stage. This*

[2] We will assume that there exist $1 > c_u > c_l > 0$ such that $e(s) \in (c_l, c_u)$ for any $s \in \{0, 1\}$ to guarantee enough exploration for two competing treatments in the pilot stage.

strategy aligns well with the goal of revenue maximization, a widely studied objective in e-commerce, online recommendation systems, and inventory control (Bakshy et al., 2018; Che et al., 2024). Similar selection algorithms are also applicable to the identification of optimal policies in political science (Offer-Westort et al., 2021). Moreover, such adaptive assignment rules can support objectives related to welfare and ethics (Burnett et al., 2020) by minimizing the allocation of inferior treatments. From another perspective, the treatment indicators 0 *and* 1 *may represent subgroups within a population. Selecting the subgroup that appears more beneficial based on observed outcomes is a common practice in clinical trials. These so-called* subgroup enrichment designs *have been extensively studied in the literature (Magnusson et al., 2013; Tanniou et al., 2016; Lin et al., 2021).*

The choice of the interim statistic $S_N^{(1)}(0) - S_N^{(1)}(1)$ for estimating the difference in treatment means using pilot data is flexible. Throughout the paper, we consider the (scaled) IPW estimator for $S_N^{(1)}(s)$, defined as:

$$S_N^{(1)}(s) \equiv \frac{1}{N_1^{1/2}} \sum_{u=1}^{N_1} \frac{\mathbb{1}(A_{uN}^{(1)} = s) Y_{uN}^{(1)}}{e(s)} \quad \text{for } s \in \{0, 1\}. \tag{1}$$

This estimator is a popular choice in the literature on batched bandit algorithms (Agarwal et al., 2014; Dimakopoulou et al., 2017) and multi-stage clinical trials (Shen et al., 2014; Bowden et al., 2017). We believe our general theoretical framework can be extended to accommodate more sophisticated estimators, such as the augmented IPW estimator (Dimakopoulou et al., 2021).[3] For simplicity and consistency between the estimators used for selection and inference, we adopt the IPW estimator.

2.2 Weighted IPW test statistic

We are interested in estimating the mean of the potential outcomes $\mathbb{E}[Y_{uN}(s)]$ for $s \in \{0, 1\}$, as well as the difference in means $\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)]$. The WIPW estimator is a natural choice for estimating these quantities. As the name suggests, the WIPW estimator can be viewed as a weighted average of the IPW estimators from the two stages:

$$\text{WIPW}(s) \equiv \sum_{t=1}^{2} \frac{N_t h_N^{(t)}(s)}{\sum_{t=1}^{2} N_t h_N^{(t)}(s)} \hat{\Lambda}_N^{(t)}(s) \quad \text{where} \quad h_N^{(t)}(s) \text{ is a weight function,} \tag{2}$$

and $\hat{\Lambda}_N^{(t)}(s)$ is a IPW statistic using data from stage t, and defined as

$$\hat{\Lambda}_N^{(t)}(s) \equiv \frac{1}{N_t} \sum_{u=1}^{N_t} \hat{\Lambda}_{uN}^{(t)}(s) \quad \text{where} \quad \hat{\Lambda}_{uN}^{(t)}(s) \equiv \frac{\mathbb{1}(A_{uN}^{(t)} = s) Y_{uN}^{(t)}}{\bar{e}_N(s, \mathcal{H}_{t-1})}$$

and $\bar{e}_N(s, \mathcal{H}_{t-1}) \equiv \mathbb{P}[A_{uN}^{(t)} = s | \mathcal{H}_{t-1}]$. The choice of weights $h_N^{(t)}(s)$ plays a critical role in the performance of the WIPW estimator. In what follows, we describe how the weights can be selected in practice.

[3] When there is no covariate, augmented IPW estimator is asymptotically equivalent to sample mean estimator.

A broad class of weighting choices. We consider the class of weights in the form $h_N^{(t)}(s) \equiv \bar{e}_N^m(s, \mathcal{H}_{t-1})/N^{1/2}$, where different choices of m allow for a variety of weighting strategies:

- $m = 0$**:** *Constant weighting,* $h_N^{(t)}(s) \equiv 1/N^{1/2}$;
- $m = 1/2$**:** *Adaptive weighting,* $h_N^{(t)}(s) \equiv \bar{e}_N^{1/2}(s, \mathcal{H}_{t-1})/N^{1/2}$.

The constant weighting corresponds to the usual IPW estimator (Bowden et al., 2017). The adaptive weighting method, is sometimes referred to as the variance-stabilizing weighting (Luedtke et al., 2016; Hadad et al., 2021; Bibaut et al., 2024). It is particularly useful for compensating the high variability in $\hat{\Lambda}_N^{(2)}(s)$, arising from the potential downsampling in the follow-up stage caused by selection algorithm $\mathcal{S}$. Other data-dependent weighting choices under the same class of statistics include $m = 1$. The resulting statistic can be shown to be (asymptotically) equivalent to the sample mean statistic after proper augmentation (Lemma 27). Our results in the main text can be applied to $m = 0$ and $m = 1/2$ and we extend the results to $m = 1$ in Appendix N.1.

Two scaling schemes. Motivated by the normalized statistic proposed in Hadad et al. (2021), we study the following two test statistics based on the WIPW estimator, depending on whether normalization is applied. Define the normalization $\hat{S}_N \equiv (N\hat{V}_N(0) + N\hat{V}_N(1))^{1/2}$, where the variance esitmator $\hat{V}_N(s)$ is defined as

$$\hat{V}_N(s) \equiv \sum_{t=1}^{2} \Big(\frac{N_t h_N^{(t)}(s)}{\sum_{t=1}^{2} N_t h_N^{(t)}(s)} \Big)^2 \frac{1}{N_t^2} \sum_{u=1}^{N_t} \big(\hat{\Lambda}_{uN}^{(t)}(s) - \mathrm{WIPW}(s) \big)^2. \tag{3}$$

Then we can define the corresponding unnormalized and normalized test statistics as

$$T_N \equiv \mathrm{WIPW}(0) - \mathrm{WIPW}(1) \quad \text{and} \quad W_N \equiv T_N/\hat{S}_N.$$

It is commonly understood that tests based on normalized and unnormalized statistics are asymptotically equivalent as the normalization factor converges to a constant. This holds true in many settings with i.i.d. data, as exemplified by the Wald and score tests. With adaptively collected data, however, normalization can impact the performance of the testing procedure. We refer the readers to simulation results in Section 4.1.

A peek at the sampling distribution. Understanding the asymptotic distributions of test statistics $\mathrm{WIPW}(s), T_N$ and W_N is important for downstream inferential tasks. To build intuition about the sampling distributions, we begin with a simulation study. Consider the scaled difference in means: $c_N \equiv \sqrt{N}(\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])$. *Without loss of generality, we assume* $c_N \leq 0$ *in this paper.* We plot the sampling distribution of centered statistic $\sqrt{N}T_N - c_N$, where c_N takes values in $\{0, -5, -10, -15\}$ and repeat the computation 5000 independent times. The detailed simulation setup is provided in Appendix B and the results are shown in Figure 1.

In the figure, we observe the following pattern: when $c_N = 0$ (left-most panel), the sampling distribution of the estimator is highly skewed; as the magnitude of c_N

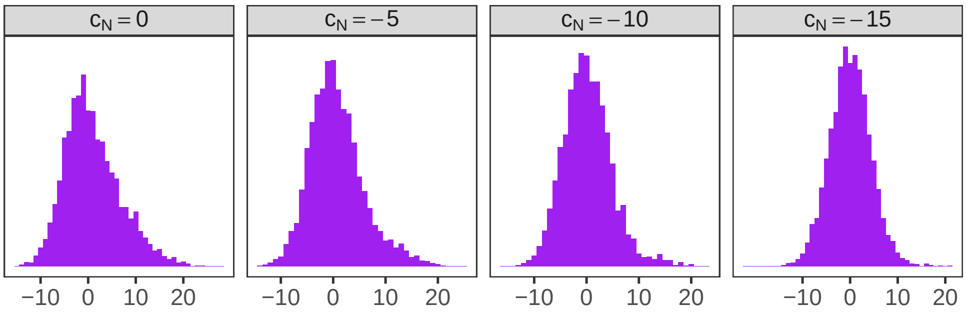

Figure 1: Sampling distribution of $\sqrt{N}T_N - c_N$ with adaptive weighting ($m = 1/2$).

increases, the distribution becomes more symmetric and eventually approaches a near-normal distribution (right-most panel). The shape transition of the sampling distribution from $c_N = 0$ to $c_N = -15$ suggests the limiting distribution of the test statistic depends on the signal strength c_N. The results presented in the next section provide an exact characterization of such dependence.

3 Theoretical results

The organization of this section is as follows. We state our main weak limit results in Section 3.1, followed by several remarks in Section 3.2. Section 3.3 describes the phase transition of the limiting distribution of the test statistic across different signal strengths. We then apply the general results to two specific adaptive experimental designs in Section 3.4. In Section 3.5, we propose a bootstrap procedure for hypothesis testing based on the weak limits.

3.1 General theory: signal-dependent weak limits

We first explicitly define the selection algorithm $\mathcal{S}(S_N^{(1)}(0) - S_N^{(1)}(1))$. We allow the $\mathcal{S}$ to vary across sample size N so we will write $\mathcal{S}_N$ to emphasize such dependence. Suppose $\bar{e}(s, x) \in [0, 1]$ is a sampling function for $s \in \{0, 1\}$ such that $\bar{e}(0, x) + \bar{e}(1, x) = 1$ for any $x \in \mathbb{R}$. Then define the sampling function

$$\mathcal{S}_N(S_N^{(1)}(0) - S_N^{(1)}(1)) \equiv \min\{1 - l_N, \max\{l_N, \bar{e}(0, S_N^{(1)}(0) - S_N^{(1)}(1))\}\}, \quad (4)$$

where l_N is a positive sequence $l_N \in [0, 1/2)$. We note that the minimum and maximum functions are mainly used for ensuring both treatments are assigned with nonzero probability in the follow-up stage when $l_N > 0$. This is a reasonable assumption for many practical applications/algorithms, as it is often desirable to maintain a certain level of exploration in the follow-up stage. Such purpose includes reducing the risk of assigning treatment to the inferior group or stablizing the variance of the downstream test statistic. Such clipping strategy has been widely adopted in the literature of adaptive experiments (Zhang et al., 2020; Hadad et al., 2021). The results in this section assume $l_N \in (0, 1/2)$, i.e. strictly positive, but we will extend the results to $l_N = 0$ in Section N.1. We defer the discussion on the choice of l_N to Remark 2 and 3.

Define the extended $\mathbb{R}$ space as $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty\}$. We consider the following assumptions.

Assumption 1 (Moment conditions). *For any $s \in \{0, 1\}$,*

$$0 < \inf_N \left(\mathbb{E}\left[Y_{uN}^2(s)\right] - \mathbb{E}\left[Y_{uN}(s)\right]^2 \right) \leq \sup_N \left(\mathbb{E}\left[Y_{uN}^4(s)\right] \right)^{1/2} < \infty.$$

For $p \in \{1, 2\}, \lim_{N\to\infty} \mathbb{E}[Y_{uN}^p(s)]$ exist. Recalling $c_N \equiv \sqrt{N}(\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])$, we assume $\lim_{N\to\infty} c_N = c$ for $c \in [-\infty, 0]$.

Assumption 2 (Sampling designs). *The sampling function $\bar{e}(s, x)$ satisfies $\bar{e}(0, x) + \bar{e}(1, x) = 1$ for any $x \in \bar{\mathbb{R}}$. Moreover, one of the following assumptions holds:*

1. ***Lipschitz condition:*** *For any $s \in \{0, 1\}$, $\bar{e}(s, x)$ is a Lipschitz function over $x \in \bar{\mathbb{R}}$, with universal Lipschitz constant $L > 0$ and $\bar{e}(s, -\infty)$ takes values in $\{0, 1\}$;*

2. ***Step-function condition:*** *There exist some $m_1 \in \mathbb{R}, K \in \mathbb{N}$ and continuous function $g : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ with $g(-\infty) = -\infty$, such that for any $s \in \{0, 1\}$,*

$$\bar{e}(s, x) = \sum_{k=1}^{K} c_k \mathbb{1}(g(x) \in C_k) \quad \textit{where} \quad c_k \in [0, 1] \textit{ and } C_k \textit{ are disjoint sets,}$$

   where $C_1 = [-\infty, m_1]$ and C_k are open sets for $k \geq 2$, such that $\cup_{k=2}^K C_k = (m_1, \infty)$.

Assumption 3 (Constant weighting). *Suppose $m = 0$ is used and clipping rate l_N (introduced in (4)) satisfies $0 < c_l < \bar{l} = l_N < c_u < 1/2$ for any $N \in \mathbb{N}$.*

Assumption 4 (Adaptive weighting). *Suppose $m = 1/2$ is used and clipping rate l_N satisfies $l_N \in (0, 1/2)$ for any $N \in \mathbb{N}$. Moreover, $\lim_{N\to\infty} l_N = 0$ and $N l_N \to \infty$.*

We postpone the discussion on the assumptions to Section 3.2. Before stating the theorem, we first describe the limiting distributions of the WIPW estimator and the test statistics T_N, W_N.

Form of limiting distributions. To ease the presentation, we use $(a)_{2\times 2}$ to denote a symmetric matrix with dimension 2, diagonal values to be 1 and off-diagonal values to be a. The limiting distirbutions of both WIPW(s) and the test stiatistics T_N and W_N, can be expressed as weighted sums of two dependent Gaussian vectors, $A^{(t)} \equiv (A^{(t)}(0), A^{(t)}(1))^\top$ for $t \in [2]$. Intuitively, $A^{(1)}$ corresponds to the randomness in the pilot stage and $A^{(2)}$ to that in the follow-up stage and is dependent on $A^{(1)}$. Concretely, the distributions of $A^{(t)}$ can be defined as

$$A^{(1)} \sim N(\mathbf{0}, \mathbf{\Sigma}^{(1)}) \quad \text{and} \quad A^{(2)} | A^{(1)} \sim N(\mathbf{0}, \mathbf{\Sigma}^{(2)}(A^{(1)})),$$

where $\mathbf{\Sigma}^{(1)} \equiv (\mathrm{Cov}^{(1)})_{2\times 2}$ and $\mathbf{\Sigma}^{(2)}(A^{(1)}) \equiv (\mathrm{Cov}^{(2)}(A^{(1)}))_{2\times 2}$. The covariance $\mathrm{Cov}^{(2)}(A^{(1)})$ depends on the realization of $A^{(1)}$, and their explicit definitions can be found in Appendix A.

- **Weak limits of** $\mathrm{WIPW}(s)$**.** Suppose $\mathcal{W}$ stands for weighting scheme and takes values in $\{\mathcal{A}, \mathcal{C}\}$, standing for adaptive and constant weighting, respectively. We use $\bar{\mathbb{W}}_{\mathcal{W}}(s)$ to denote the limiting distributions of $\mathrm{WIPW}(s)$ after proper centering and scaling. We will use $\stackrel{d}{=}$ to denote equality in distribution. Then $\bar{\mathbb{W}}_{\mathcal{W}}(s)$ can be written as

$$\bar{\mathbb{W}}_{\mathcal{W}}(s) \stackrel{d}{=} \sum_{t=1}^{2} A^{(t)}(s)\bar{w}_{\mathcal{W}}^{(t)}(s) \quad \text{for any } s \in \{0,1\}, \tag{5}$$

where $\bar{w}_{\mathcal{W}}^{(t)}(s)$ varies with different weighting schemes and $\bar{w}_{\mathcal{W}}^{(2)}(s)$ may be dependent on $A^{(1)}$. The form of $\bar{w}_{\mathcal{W}}^{(t)}(s)$ can be found in Appendix A.

- **Weak limits of** T_N **and** W_N**.** Suppose $\mathcal{V}$ stands for scaling scheme and takes value in $\{\mathcal{N}, \mathcal{U}\}$, standing for normalized (W_N) and unnormalized statistics (T_N), respectively. We use $\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}$ to denote the limiting distributions for T_N and W_N after proper centering and scaling. We emphasize the dependence of $\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}$ on the limiting signal strength c by writing $\mathbb{W}_{\mathcal{V}}^{\mathcal{W}} = \mathbb{W}_{\mathcal{V}}^{\mathcal{W}}(c)$. Then we can write

$$\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}(c) \stackrel{d}{=} \sum_{t=1}^{2} A^{(t)}(0)w_{\mathcal{W},\mathcal{V}}^{(t)}(0) - \sum_{t=1}^{2} A^{(t)}(1)w_{\mathcal{W},\mathcal{V}}^{(t)}(1) \tag{6}$$

where the weights $w_{\mathcal{W},\mathcal{V}}^{(t)}(s)$ can be written as

$$w_{\mathcal{W},\mathcal{U}}^{(t)}(s) = \bar{w}_{\mathcal{W}}^{(t)}(s) \quad \text{and} \quad w_{\mathcal{W},\mathcal{N}}^{(t)}(s) = \frac{\bar{w}_{\mathcal{W}}^{(t)}(s)}{(\sum_{s=0}^{1}\sum_{t=1}^{2}(\bar{w}_{\mathcal{W}}^{(t)}(s))^2)^{1/2}}.$$

Note both $\bar{w}_{\mathcal{W}}^{(t)}(s)$ and $A^{(t)}(s)$ also depend on the signal strength c but we omit this dependence for simplicity.

We summarize the definition of different limiting distributions in Table 1.

Table 1: Summary of different limiting distributions. $\mathcal{W}$ takes values in $\{\mathcal{A}, \mathcal{C}\}$.

Test statistic	Estimand	Weak limit	Limiting weight
$\mathrm{WIPW}(s)$	$\mathbb{E}[Y_{uN}(s)]$	$\bar{\mathbb{W}}_{\mathcal{W}}$	$\bar{w}_{\mathcal{W}}^{(t)}(s)$
T_N	$\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)]$	$\mathbb{W}_{\mathcal{U}}^{\mathcal{W}}$	$w_{\mathcal{W},\mathcal{U}}^{(t)}(s)$
W_N	$\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)]$	$\mathbb{W}_{\mathcal{N}}^{\mathcal{W}}$	$w_{\mathcal{W},\mathcal{N}}^{(t)}(s)$

Now we are ready to state the main results.

Theorem 1 (Weak convergence)**.** *Suppose Assumptions 1-2 hold. Recall the definition* $c_N \equiv \lim_{N\to\infty}\sqrt{N}(\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])$. *The following statements hold.*

1. *Suppose Assumption 3 holds. Then,* $\sqrt{N}(\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)])$ *converges weakly to* $\bar{\mathbb{W}}_{\mathcal{C}}(s)$ *for any* $s \in \{0,1\}$*. Moreover,* $\sqrt{N}T_N - c_N$ *and* $W_N - c_N/\hat{S}_N$ *converge weakly to* $\mathbb{W}_{\mathcal{U}}^{\mathcal{C}}(c)$ *and* $\mathbb{W}_{\mathcal{N}}^{\mathcal{C}}(c)$*, respectively.*

2. *Suppose Assumption 4 holds. Then,* $\sqrt{N}(\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)])$ *converges weakly to* $\bar{\mathbb{W}}_{\mathcal{A}}(s)$ *for any* $s \in \{0,1\}$*. Moreover,* $\sqrt{N}T_N - c_N$ *and* $W_N - c_N/\hat{S}_N$ *converge weakly to* $\mathbb{W}_{\mathcal{U}}^{\mathcal{A}}(c)$ *and* $\mathbb{W}_{\mathcal{N}}^{\mathcal{A}}(c)$*, respectively.*

Proof of Theorem 1 can be found in Appendix J.

3.2 Remarks on the assumptions and technical challenges

We present several remarks on the assumptions and technical challenges of proving Theorem 1.

Remark 2 (Comments on Assumptions 1-4). *Assumption 1 is a mild regularity assumption. Assumption 2 imposes mild restrictions on the sampling function, providing substantial flexibility for the choice of sampling function* $\bar{e}(s,x)$ *(and hence the selection algorithm* $\mathcal{S}$*) to encode different experimental designs. To demonstrate the generality of these assumptions, we present two classes of experiments in Section 3.4. Now we comment on Assumption 3 and Assumption 4. Constant weighting, informed by Assumption 3, requires the minimum sampling probability to be uniformly bounded away from* 0*. In the causal inference literature, Assumption 3 is known as the positivity assumption (Crump et al., 2009; Imbens et al., 2015). The adaptive weighting enables less stringent requirement on the sampling probability in the second stage encouraging further exploitation. Assumption 4 allows minimum sampling probability in the second stage to go to zero at the rate slower than* $1/N$*. Similar assumption has also been adopted in Zhang et al. (2020) and Hadad et al. (2021).*

Remark 3 (Early-dropping experiments). *Relevant to Remark 2, one kind of adaptive sampling Theorem 1 does not cover is the so-called early-dropping experiments (Sampson et al., 2005; Sill et al., 2009). In these experiments, the inferior treatment will be dropped from the follow-up stage. In this case,* $\mathcal{S}_N(S_N^{(1)}(0) - S_N^{(1)}(1))$ *can be* 0 *or* 1*. We will show in Appendix N.1 that when* $m = 1$*, we can get rid of the clipping* (4) *by allowing* $l_N = 0$*.*

Remark 4 (Technical challenges behind Theorem 1). *Despite the clean and nice results, proving Theorem 1 is technically challenging. Most existing weak convergence results for adaptive experiments rely on variants of the martingale central limit theorem (Zhang et al., 2020; Hadad et al., 2021). However, the limiting distributions derived in Theorem 1 are generally non-normal when* $c \in (-\infty, 0]$*, rendering those results inapplicable in our setting. Instead, we establish weak convergence from first principles using the* test function approach *(Dudley, 2002). While inspired by the framework of Che et al. (2023), our proofs are considerably more technical due to two key challenges. First, unlike the test statistic adopted in Che et al. (2023),*[4] *our test statistics* T_N *and* W_N *have more complex dependencies on the two-stage sampled data. In particular, we must establish joint convergence for a vector of statistics that includes possibly nonlinear functionals, such as* $\mathcal{S}_N(S_N^{(1)}(0) - S_N^{(1)}(1))$*. This poses a challenge for establishing the conditional convergence of the vector of statistics contributed from the second stage, necessitating an extension of classical asymptotic tools. Notably, we introduce two new continuous mapping theorems to handle convergence under conditioning (Lemma 17 and 18). Second, under adaptive weighting (Assumption 4), we allow the clipping rate* l_N *to decay to zero relatively fast. As a result, higher-order moments of the test statistics (e.g.* T_N *and* W_N*) may diverge, depending on the rate at which* l_N *vanishes. This imposes constraints on the choice of test functions used for establishing the weak convergence and we choose to work with bounded Lipschitz test*

[4] We refer the reader to Appendix E.4 for more details on the test statistic analyzed in Che et al. (2023).

functions. As a by-product of our analysis, we establish a new version of the conditional CLT with bounded Lipschitz test functions (Lemma 15). These new theoretical results can be applied to general setting involving dependent data and thus may be of independent interest.

3.3 Phase transition and implication on hypothesis testing

A key strength of our results in Theorem 1 is the weak limits can be expressed explicitly using weighted sums of two dependent Gaussian variables $A^{(t)}$ as shown in (5) and (6). This allows us to understand the limiting distributions of different test statistics in a more intuitive way. We will discuss how the limiting distributions of T_N and W_N change as signal strength c changes. Specifically, the shapes of limiting distributions are determined by the covariance $\mathrm{Cov}^{(2)}(A^{(1)})$ and random weights $w^{(2)}_{\mathcal{W},\mathcal{V}}(s)$, which are both influenced by the signal strength c. Consider the following two regimes:

- **Strong signal regime.** When $c = -\infty$, i.e., the absolute difference between two expected outcomes is much larger than $1/\sqrt{N}$, it can be shown that $\mathrm{Cov}^{(2)}(A^{(1)})$ and $w^{(2)}_{\mathcal{W},\mathcal{V}}(s)$ are deterministic constants. This implies that the flucutation of the first stage estimator does not affect the final limit through the limiting covariance. Therefore the final limit $\mathbb{W}^{\mathcal{W}}_{\mathcal{V}}(-\infty)$ follows a Gaussian distribution.

- **Zero and weak signal regimes.** When $c \in (-\infty, 0]$, the limiting distribution may no longer be a normal distribution. This is because $A^{(1)}$ will appear in the conditional distribution of $A^{(2)}|A^{(1)}$ through the limiting covariance $\mathrm{Cov}^{(2)}(A^{(1)})$. Similarly, $w^{(2)}_{\mathcal{W},\mathcal{V}}(s)$ depends on the realization of $A^{(1)}$. Therefore, when the signal is "weak", the non-normal behavior is the "price" one needs to pay for choosing to use the adaptive sampling scheme.

Additional insights on the non-normal limiting behaviors from double-dipping and data generating process perspectives can be found in Appendix D. To get better intuition, we simulate $\mathbb{W}^{\mathcal{W}}_{\mathcal{V}}(c)$ with $\mathcal{V} = \mathcal{U}, \mathcal{W} = \mathcal{A}$—this is the limiting distribution corresponding to the sampling distribution presented in Figure 1. We vary the limiting signal strength c and show the simulated results in Figure 2, which align closely with those in Figure 1. As c approaches $-\infty$, i.e., as the signal gets stronger, the limiting distribution of $\mathbb{W}^{\mathcal{W}}_{\mathcal{V}}(c)$ approaches a normal distribution. Such phase transition, as informed by Theorem 2, is smooth with respect to signal strength c under *1-Wasserstein distance*, W_1.

Theorem 2 (Smooth transition of limiting distributions)**.** *Suppose the assumptions of Theorem 1 hold. Then for any $\mathcal{V} \in \{\mathcal{U}, \mathcal{N}\}$ and $\mathcal{W} \in \{\mathcal{A}, \mathcal{C}\}$, we have*

$$W_1(\mathbb{W}^{\mathcal{W}}_{\mathcal{V}}(-\infty), \mathbb{W}^{\mathcal{W}}_{\mathcal{V}}(c)) \to 0 \quad \textit{as} \quad c \to -\infty.$$

Proof of Theorem 2 can be found in Appendix L, where the definition of Wasserstein distance can also be found. Gathering these insights, we now discuss the implications of our results on hypothesis testing.

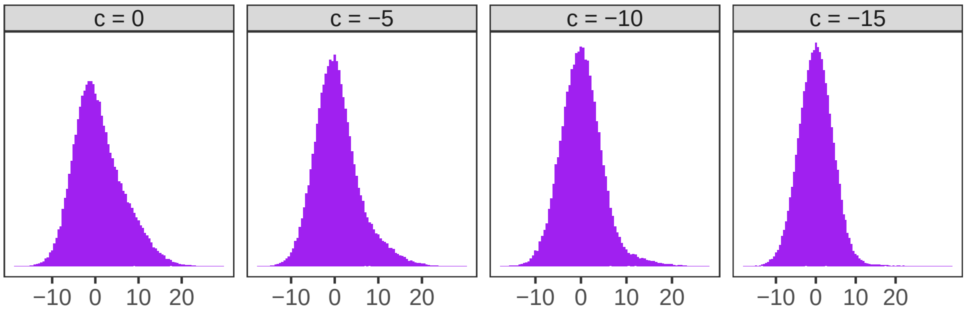

Figure 2: Distribution $\mathbb{W}_{\mathcal{U}}^{\mathcal{A}}(c)$ as a function of limiting signal strength c.

Implication on hypothesis testing. Another strength of our results lies in establishing weak convergence under minimal moment conditions, mild assumptions on the sampling functions, and broad signal strength regimes. These assumption-lean properties are not merely of theoretical interest—they carry practical significance for downstream hypothesis testing. To demonstrate the implication of our results on the hypothesis testing, consider the null hypothesis $H_{0N} : \mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)] = 0$ and two alternatives within the general hypothesis $H_{1N} : \mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)] \neq 0$:

$$H_{2N} : \mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)] = \frac{b_2}{\sqrt{N}} \quad \text{and} \quad H_{3N} : \mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)] = \frac{b_3}{N^{\beta}},$$

where $b_2, b_3 \in (-\infty, 0)$ and $\beta \in [0, 1/2)$. The contiguous alternative H_{2N} corresponds to a weak signal regime, while H_{3N} reflects a strong signal setting. Ideally, a test should control the Type-I error under H_{0N} and achieve non-trivial power under H_{2N}, while attaining power approaching one under H_{3N}. It is also desirable for the test to remain assumption-lean with respect to the potential outcome distributions. Our results accommodate this full range of signal strengths while maintaining minimal assumptions on the sampling functions and underlying distributions.

Comparison to existing literature. To highlight the significance of our results, we compare our results to those in related work. Zhang et al. (2020) establish asymptotic normality via stage-wise normalization under general hypotheses. Their method, however, relies on restrictive outcome distribution assumptions and yields lower power than tests based on sample means with pooled two-stage data (Hirano et al., 2023). Hadad et al. (2021) analyze the WIPW estimator for $m = 1/2$, but only in strong signal regimes (e.g., H_{3N}), excluding H_{0N} and local alternatives. Both Zhang et al. (2020) and Hadad et al. (2021) achieve asymptotic normality under strong signals, potentially at the cost of power. Hirano et al. (2023) and Adusumilli (2023) derive asymptotic representations for general test statistics under batched designs and contiguous alternatives H_{2N}, relying on classical limit-of-experiment theory by Le Cam et al. (1972). However, to apply Le Cam's theory, one must first establish the required weak convergence of certain statistics (see, for example, Theorem 2 in (Hirano et al., 2023)). Their results also assume quadratic mean differentiability (QMD) (or smooth semiparametric models), which may be restrictive in practice. A complementary line of work uses diffusion approximations with increasing batch numbers (Fan et al., 2021;

Kuang et al., 2024), which differs from our two-stage setup with growing per-stage samples. Further discussion appears in Appendix E.

3.4 Case study: application to different adaptive experiments

In this section, we demonstrate the applicability of Theorem 1 to a variety of adaptive experimental designs. Specifically, we focus on two widely used paradigms: *batched bandit experiments* and *subgroup enrichment designs*. These designs have been studied in recent statistical and machine learning literature (Russo, 2016; Lin et al., 2021; Che et al., 2024; Freidling et al., 2024).

Batched bandit experiments. Batched bandit experiments typically employ adaptive algorithms to balance exploration and exploitation. Two commonly studied strategies are *Thompson sampling* and the *ε-greedy algorithm*. Thompson sampling is a Bayesian approach that selects actions according to their posterior probabilities of being optimal. The ε-greedy algorithm chooses the empirically best arm with probability $1-\varepsilon/2$ and explores the inferior arm with probability $\varepsilon/2$ when there are two arms. We will show that Theorem 1 applies to two-batch bandit experiments employing these algorithms.

- **Modified Thompson sampling:** Assuming suitable prior distributions, the posterior for each expected outcome $\mathbb{E}[Y_{uN}(s)]$, conditional on pilot data, is (approximately) $N(S_N^{(1)}(s), 1/2)/\sqrt{N_1}$, where N represents the normal distribution. This yields a sampling function:

  $$\bar{e}(s,x) = (1-\Phi(x))\mathbb{1}(s=1) + \Phi(x)\mathbb{1}(s=0),$$

  which is Lipschitz continuous over $x \in \bar{\mathbb{R}}$ and satisfies the **Lipschitz condition** of Assumption 2. This algorithm has been used in Hadad et al. (2021). Incorporating a clipping rate l_N (see Eq. (4)), the follow-up sampling probability $\mathbb{P}[A_{uN}^{(2)} = 0|\mathcal{H}_1]$ becomes a modified Thompson sampling rule:

  $$\max\{l_N, \min\{1-l_N, \Phi(S_N^{(1)}(0) - S_N^{(1)}(1))\}\}. \tag{7}$$

- **ε-greedy algorithm:** Consider the non-smooth sampling function:

  $$\bar{e}(s,x) = \mathbb{1}(x<0)\mathbb{1}(s=1) + \mathbb{1}(x \geq 0)\mathbb{1}(s=0),$$

  which satisfies **Step-function condition** in Assumption 2. With clipping rate l_N, the follow-up sampling probability $\mathbb{P}[A_{uN}^{(2)} = 0|\mathcal{H}_1]$ corresponds to an ε-greedy algorithm with $\varepsilon = 2l_N$:

  $$(1-l_N)\mathbb{1}(S_N^{(1)}(0) \geq S_N^{(1)}(1)) + l_N\mathbb{1}(S_N^{(1)}(0) < S_N^{(1)}(1)). \tag{8}$$

Subgroup enrichment experiments. The assignment variable A may indicate subgroup membership rather than treatment assignment. In this context, $Y_{uN}(0)$ and $Y_{uN}(1)$ represent outcomes for two distinct subgroups. Adaptive enrichment designs aim to identify and focus on the subgroup that benefits more from the treatment, based on interim results from pilot stage. Common strategies include enrichment based on estimated effect size or interim p-value (Food et al., 2019; Ben-Eltriki et al., 2024).

- **Enrichment based on effect size:** The sampling function in this case is:

$$\bar{e}(s,x) = \mathbb{1}(x < \beta)\mathbb{1}(s=1) + \mathbb{1}(x \geq \beta)\mathbb{1}(s=0),$$

  where β is a pre-specified threshold for the effect size.

- **Enrichment based on interim p-values:** Let $\hat{\sigma}$ denote the estimated standard deviation of $S_N^{(1)}(0) - S_N^{(1)}(1)$ under the null hypothesis H_{0N}. Define left-sided and right-sided p-values as $p_l = \Phi(x/\hat{\sigma})$ and $p_r = 1 - p_l$, respectively. Given a pre-specified significance level α, the sampling function based on interim p-values becomes:

$$\bar{e}(s,x) = \mathbb{1}(p_l < \alpha)\mathbb{1}(s=0) + \mathbb{1}(p_r < \alpha)\mathbb{1}(s=1) + \mathbb{1}(p_l \in [\alpha, 1-\alpha]) \cdot 0.5,$$

  which introduces randomization when the interim result is inconclusive.

The follow-up sampling probabilities $\mathbb{P}[A_{uN}^{(2)} = 0|\mathcal{H}_1]$ and $\mathbb{P}[A_{uN}^{(2)} = 1|\mathcal{H}_1]$ can be similarly obtained based on these sampling functions.

Remark 5 (Generalization of Theorem 1 to accommodate nuisance parameter)**.** *The second example of subgroup enrichment experiments falls outside the scope of Theorem 1 due to the presence of the nuisance parameter $\hat{\sigma}$ in the sampling mechanism. To address this, we extend our theoretical results to accommodate such nuisance parameters. This generalization is presented in Theorem 5 in Appendix N.3.*

3.5 Asymptotically valid tests with plug-in bootstrap

Theorem 1 characterizes the limiting behavior of WIPW test statistics, forming the basis for constructing asymptotically valid tests. In this section, we focus on testing whether or not the difference in means $\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)] = 0$ for demonstration. Similar results can be established for other hypotheses, for example, single outcome hypothesis $\mathbb{E}[Y_{uN}(s)] = 0$ for some $s \in \{0,1\}$. Based on Theorem 1, we can define the following asymptotically valid tests: for any $\mathcal{W} \in \{\mathcal{A}, \mathcal{C}\}$,

$$\phi_{\mathcal{U}}^{\mathcal{W}} \equiv \mathbb{1}\big(\sqrt{N}T_N \geq \mathbb{Q}_{1-\alpha}(\mathbb{W}_{\mathcal{U}}^{\mathcal{W}}(0))\big) \quad \text{and} \quad \phi_{\mathcal{N}}^{\mathcal{W}} \equiv \mathbb{1}\big(W_N \geq \mathbb{Q}_{1-\alpha}(\mathbb{W}_{\mathcal{N}}^{\mathcal{W}}(0))\big). \tag{9}$$

The critical values $\mathbb{Q}_{1-\alpha}(\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}(0))$ are the $(1-\alpha)$-th quantiles of the limiting distribution $\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}(0)$ defined in Theorem 1. However, tests in (9) cannot be implemented in practice for two reasons. First, the asymptotic distributions $\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}(0)$ are generally non-normal (see also the left-most subplot in Figure 2). Second, the limiting distribution involves unknown nuisance parameters, which have to be estimated using observed data. In this section, we propose a bootstrap procedure to address these challenges and construct valid tests, which can be implemented in practice.

A fast bootstrap procedure. Note that the expression of limiting distribution in (6) is a weighted sum of two dependent Gaussian variables. Motivated by such observation, we propose a plug-in bootstrap procedure to obtain the quantile information $\mathbb{Q}_{1-\alpha}(\mathbb{W}_{\mathcal{V}}^{\mathcal{W}}(0))$. For the ease of presentation, we will omit the definition of nuisance estimators and present a simplified algorithm in Algorithm 2. The complete bootstrap procedure with the estimators of nuisance parameters can be found in Appendix C.

Algorithm 2: Simplified two-stage sampling and weighting procedure

---

Input: Outcome mean estimators $\hat{\mathbb{E}}[Y_{uN}(s)]$ and $\hat{\mathbb{E}}[Y_{uN}^2(s)]$; estimators $\hat{\text{Cov}}^{(1)}$ and $\hat{\text{Cov}}^{(2)}(\cdot)$; weighting and scaling schemes $\mathcal{W} \in \{\mathcal{A}, \mathcal{C}\}$ and $\mathcal{V} \in \{\mathcal{U}, \mathcal{N}\}$; sampling funciton $\bar{e}(s, \cdot)$; clipping rate l_N.

1 **First stage sampling:** Compute $\tilde{A}^{(1,b)} = (\hat{\mathbf{\Sigma}}^{(1)})^{1/2} S_1^{(b)}$, $S_1^{(b)} \sim N(0, \boldsymbol{I}_2)$, with $\hat{\mathbf{\Sigma}}^{(1)} = (\hat{\text{Cov}}^{(1)})_{2\times 2}$.

2 **Second stage sampling:** Compute $\tilde{A}^{(2,b)} = (\hat{\mathbf{\Sigma}}^{(2,b)})^{1/2} S_2^{(b)}$, $S_2^{(b)} \sim N(0, \boldsymbol{I}_2)$, with $\hat{\mathbf{\Sigma}}^{(2,b)} = (\hat{\text{Cov}}^{(2)}(\tilde{A}^{(1,b)}))_{2\times 2}$.

3 **Weighting procedure:** Compute estimates $\hat{w}_{\mathcal{W},\mathcal{V}}^{(t,b)}(s)$ for $w_{\mathcal{W},\mathcal{V}}^{(t,b)}(s)$ with $\tilde{A}^{(t,b)}$, $\hat{\mathbb{E}}[Y_{uN}(s)], \hat{\mathbb{E}}[Y_{uN}^2(s)], \bar{e}(s, \cdot)$ and l_N. Defining $(\tilde{A}^{(t,b)}(0), \tilde{A}^{(t,b)}(1))^\top \equiv \tilde{A}^{(t,b)}$, generate bootstrap sample

$$\mathcal{D}_{\mathcal{W},\mathcal{V}}^{(b)} = \sum_{t=1}^{2} \hat{w}_{\mathcal{W},\mathcal{V}}^{(t,b)}(0)\tilde{A}^{(t,b)}(0) - \sum_{t=1}^{2} \hat{w}_{\mathcal{W},\mathcal{V}}^{(t,b)}(1)\tilde{A}^{(t,b)}(1).$$

4 **Repeated sampling:** Repeat steps 1-3 to get B bootstrap resamples.

Output: Bootstrap sample $\{\mathcal{D}_{\mathcal{W},\mathcal{V}}^{(b)} : b \in [B]\}$.

We make the following remarks regarding the proposed bootstrap procedure.

Remark 6 (Computational efficiency)**.** *The computational cost of generating a boostrap sample in Algorithm 2 is $O(1)$, leading to a total cost of $O(B)$ for the whole procedure. This is much more efficient than the general nonparametric bootstrap even with linear computation cost for computing each test statistic, which takes $O(BN)$ to generate B samples.*

Remark 7 (A nonparametric simulation-based approach)**.** *Notably, Algorithm 2 mimics a two-stage asymptotic experiment to approximate the limiting distribution of the test statistic in (6). From this perspective, our bootstrap approach is conceptually aligned with the simulation-based methods that leverage asymptotic representation results, as proposed in Hirano et al. (2023). Our method is nonparametric and free of distributional assumptions, thereby enabling more robust and widely applicable inference.*

Asymptotic valid tests with plug-in bootstrap. Based on Algorithm 2, we construct the plug-in bootstrap tests using the bootstrap samples $\mathcal{D}_{\mathcal{W},\mathcal{V}}^{(b)}$. Denote the σ-algebra generated by the observed data as $\mathcal{G}_N \equiv \sigma(\mathcal{H}_1 \cup \mathcal{H}_2)$. Suppose $\mathcal{D}_{\mathcal{W},\mathcal{V}} \overset{d}{=} \mathcal{D}_{\mathcal{W},\mathcal{V}}^{(b)}$, conditional on $\mathcal{G}_N$. Then for $\mathcal{W} \in \{\mathcal{A}, \mathcal{C}\}$, define

$$\hat{\phi}_{\mathcal{U}}^{\mathcal{W}} \equiv \mathbb{1}\big(\sqrt{N}T_N > \mathbb{Q}_{1-\alpha}(\mathcal{D}_{\mathcal{W},\mathcal{U}} \mid \mathcal{G}_N)\big) \quad \text{and} \quad \hat{\phi}_{\mathcal{N}}^{\mathcal{W}} \equiv \mathbb{1}\big(W_N > \mathbb{Q}_{1-\alpha}(\mathcal{D}_{\mathcal{W},\mathcal{N}} \mid \mathcal{G}_N)\big).$$

The following theorem shows the validity of the plug-in bootstrap procedure and corresponding bootstrap tests.

Theorem 3 (Validity of plug-in bootstrap and tests $\hat{\phi}_{\mathcal{V}}^{\mathcal{W}}$)**.** *Suppose Assumption 1-2 hold. Then, the following statements hold.*

1. *Suppose Assumption 3 holds, we have for $\mathcal{V} \in \{\mathcal{U}, \mathcal{N}\}$,*

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}\left[\mathcal{D}_{\mathcal{C},\mathcal{V}} \le x | \mathcal{G}_N\right] - \mathbb{P}[\mathbb{W}_{\mathcal{V}}^{\mathcal{C}}(0) \le x]\right| \xrightarrow{p} 0 \quad \textit{and} \quad \lim_{N\to\infty} \mathbb{E}_{H_{0N}}[\hat{\phi}_{\mathcal{V}}^{\mathcal{C}}] = \alpha;$$

2. *Suppose Assumption 4 holds, we have for $\mathcal{V} \in \{\mathcal{U}, \mathcal{N}\}$,*

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}\left[\mathcal{D}_{\mathcal{A},\mathcal{V}} \le x | \mathcal{G}_N\right] - \mathbb{P}[\mathbb{W}_{\mathcal{V}}^{\mathcal{A}}(0) \le x]\right| \xrightarrow{p} 0 \quad \textit{and} \quad \lim_{N\to\infty} \mathbb{E}_{H_{0N}}[\hat{\phi}_{\mathcal{V}}^{\mathcal{A}}] = \alpha.$$

The proof of Theorem 3 can be found in Appendix M. There is an implicit assumption we make behind Theorem 3, which is the knowledge of the sampling function $\bar{e}(s, \cdot)$ and l_N. In practice, this assumption is reasonable since the selection algorithm or the experimental protocol is usually pre-specified before the data is collected and thus is at the hand of the experiment designer. In fact, this is the case in many empirical studies including Collins et al. (2007), Li et al. (2010), Offer-Westort et al. (2021), and Jin et al. (2023).

4 Finite-sample evaluation

In this section, we conduct extensive numerical simulations and a semi-synthetic data analysis to investigate the finite-sample performance of the tests studied in the previous section. In particular, we include all four bootstrap tests with two scaling and two weighting schemes proposed in Section 3.5. As a benchmark, we also include the IPW test statistic based on sample splitting, which only uses the data from follow-up stage for inference; the comparison demonstrates the benefit of using data from both stages. The significance level is taken to be 0.05 throughout this section.

4.1 Numerical simulation

Data generation procedure. We consider two potential outcome distributions:

1. **Binary outcome:** $Y_{uN}(0) \sim \text{Bern}(\theta + 0.5)$, $Y_{uN}(1) \sim \text{Bern}(0.5)$;
2. **Continuous outcome:** $Y_{uN}(0) \sim N(\theta, 1)$, $Y_{uN}(1) \sim N(0, 0.25)$.

In the pilot stage, we employ the equal sampling in the pilot stage $e(1) = 0.5$, which mimics the common practice in real world when there is no prior information which treatment is better. Insipired by batched bandit setup, we consider two selection algorithms: the modified version of Thompson sampling (7) and ε-greedy algorithm (8), both with clipping $l_N = \varepsilon/2$. In the second stage, we sample two treatments based on the results of these selection algorithm. The task is to test if the treatment effect is significantly different from 0, i.e. $\theta = \mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)] = 0$, or not.

Parameter setup. In both selection algorithms, we set $\varepsilon \in \{0.1, 0.2, 0.4\}$. The signal strength θ ranges from -0.2 to 0.2 with equal sapce 0.05, with $\theta = 0$ corresponding to the null hypothesis. The number of total samples $N = 1000$ and each batch has the same sample size $N_1 = N_2 = 500$.

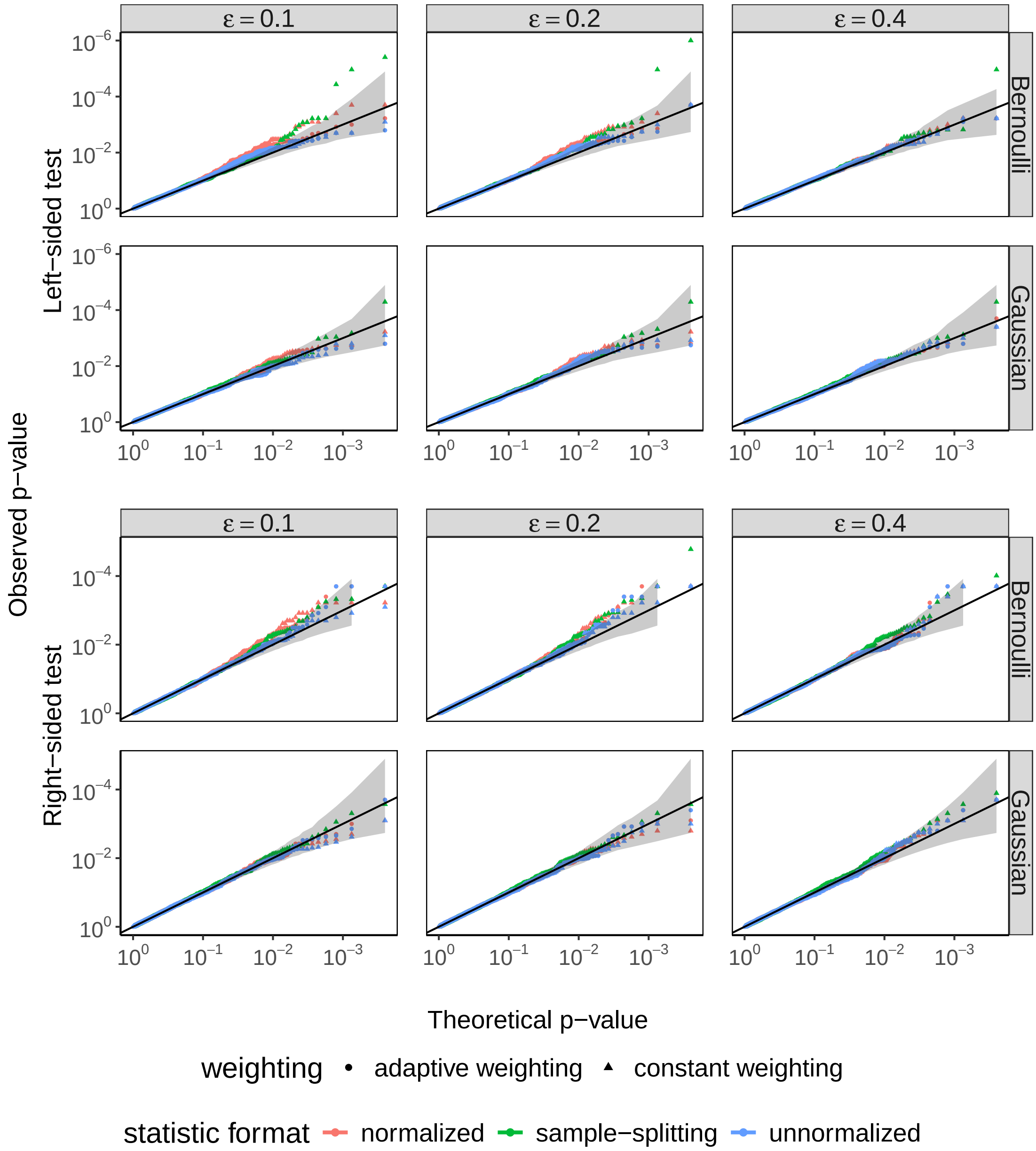

Figure 3: QQ plots for the 5 tests under different signal strengths. The simulation is repeated for 2000 times. The number of bootstrap used in each test is 5000.

Results analysis and interpretation. For the ease of presentation, we select the representative results with Thompson sampling being the selection algorithm. For additional simulation results with ε-greedy algorithm, we refer the readers to Appendix O. The calibration results are summarized as QQ plots in Figure 3 and the power

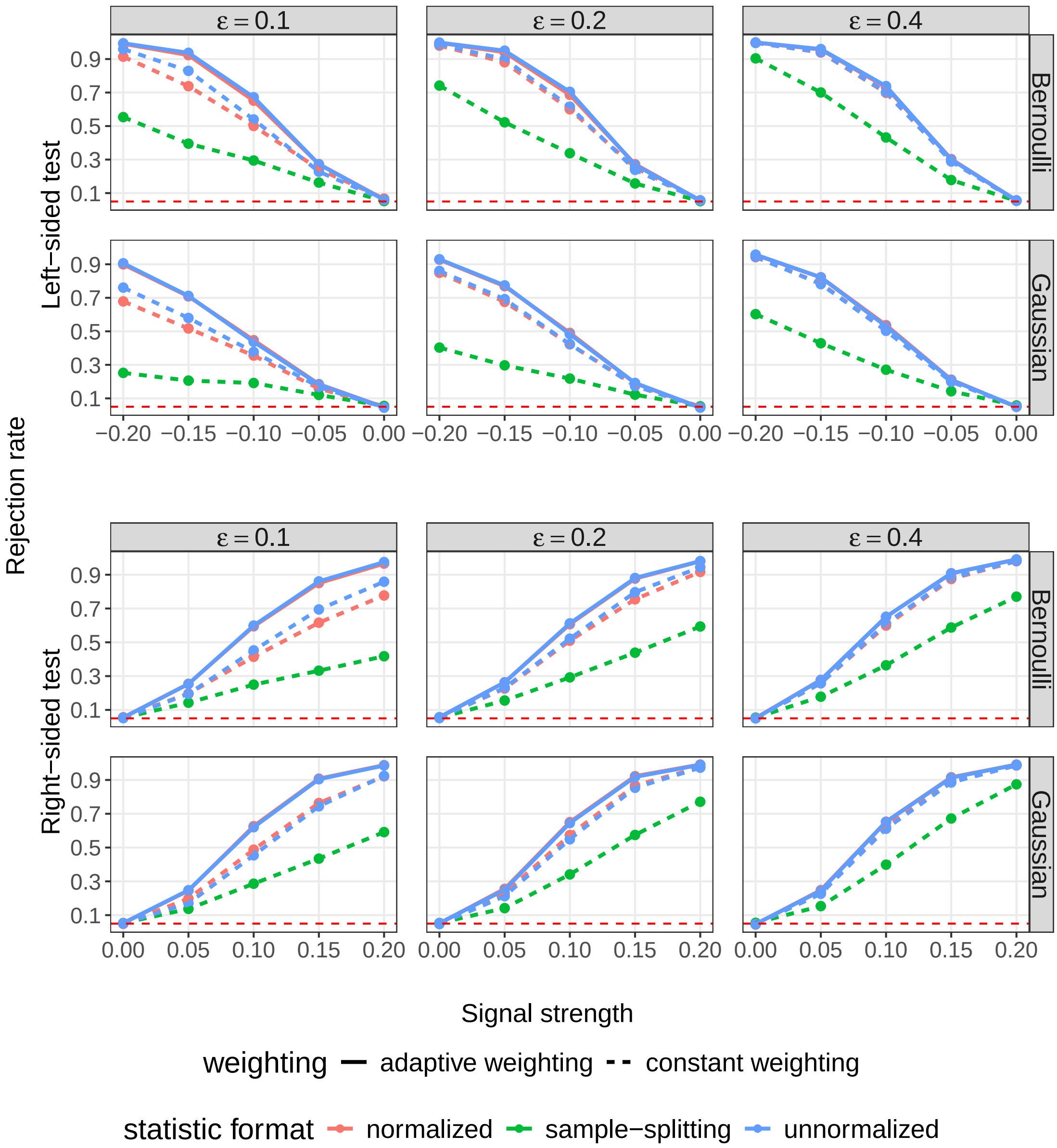

Figure 4: Rejection rate for the 5 tests under different signal strength. The simulation is repeated for 2000 times. The number of bootstrap used in each test is 5000.

results are summarized in Figure 4. From Figure 3, we observe that all the tests can produce relatively well-calibrated p-values. This validates the bootstrap procedure proposed in Algorithm 2. For power comparison, it is unsurprising to see that our approach using pooled two-stage data can improve power over sample splitting. Moreover, we make the following intriguing obsevations:

1. **Tests based on adaptive weighting are more powerful.** Tests with adaptive weighting (solid lines) show substantial power improvement compared to tests with constant weighting, no matter which scaling is used. The improvement is especially pronounced when ε is small. This is mainly because when constant weighting is used, WIPW estimator boils down to the usual IPW estimator. It

is well-known that IPW estimator is highly variable when the downsampling on one arm in the second stage is substantial due to small ε.

2. **Tests based on adaptive weighting are robust to ε.** Relevant to the previous point, tests based on constant weighting (including sample splitting) are more sensitive to the ε in the selection algorithm. The adaptive weighting scheme, on the other hand, is more robust to the choice of ε. This is because the adaptive weighting scheme can adjust the weights based on the observed data, making it less sensitive to ε. In experimental practice, employing adaptive weighting scheme for inference enables more aggressive exploitation strategies in sampling.

3. **Normalization influences each test in distinct ways.** Unlike the randomized controlled experiments, the normalization can make a difference in the power of the tests, even asymptotically (see Theorem 1 on results with T_N and W_N). We can observe a power gain when using unnormalized test with constant weighting (see plots in first column of Figure 4). However, from a theoretical standpoint, it remains unclear whether the unnormalized test exhibits greater power under more general data generating processes.

4. **Power curve depends on the sideness of the test.** We observe that the power performance differs between left-sided and right-sided tests. Under Bernoulli sampling, the left-sided test tends to reject more often across all methods, even when the absolute signal magnitudes are the same. In contrast, under Gaussian sampling, the right-sided test exhibits greater power. These observations highlight the potential need for side-dependent experimental design strategies.

4.2 Semi-synthetic data analysis

To further investigate the performance of different tests on real data, we conduct a semi-synthetic data analysis designed to better mimic real-world settings. The data is derived from the Systolic Blood Pressure Intervention Trial (SPRINT) (Ambrosius et al., 2014). This is a randomized controlled trial and evaluates whether a new treatment program for lowering systolic blood pressure reduces the risk of cardiovascular disease (CVD). The population is divided into treatment (new treatment) and control (placebo) group and primary clinical outcome is the occurrence of a major CVD event. We generate the semi-synthetic data, apply the tests, and evaluate their performance through the following procedures.

1. **Permute data to break the dependence.** We first permute the outcomes within the whole population, generating $B = 500$ permuted samples. This permutation effectively removes any treatment effect, ensuring that the treatment and control groups have the same expected outcome level.

2. **Add signal back to the data.** For these 500 permuted samples, we manually introduce a treatment effect by increasing the mean outcome (i.e. the major CVD event occurrence) in the control group, since the new treatment is intended

to reduce the risk of CVD. Let N_c^0 denote the total number of control-group participants who did not experience a CVD event. We set n_0 of these zero outcomes to 1, where $n_0 \sim \text{Bin}(N_c^0, \eta)$. The added signal η varies within the set $\{0, 0.015, 0.03, 0.045, 0.06\}$.

3. **Adaptively sample the data to maximize welfare.** For each permuted sample, we simulate adaptive sampling. We first draw $N_1 = 1000$ random samples. Because the new treatment could be beneficial for the patients, we apply the ε-greedy algorithm (8) to collect additional $N_2 = 1000$ samples in the second stage, encouraging assignment of new treatment. We vary $\varepsilon \in \{0.1, 0.2, 0.4\}$.

4. **Evaluate Type-I error control and power.** We apply the five tests introduced in Section 4.1 to the synthetically generated data. We consider the right-sided test to see if the CVD event rate in the control group ($\mathbb{E}[Y_{uN}(0)]$) is higher than that in the treatment group ($\mathbb{E}[Y_{uN}(1)]$). We evaluate Type-I error control before introducing signal and statistical power after introducing the signal.

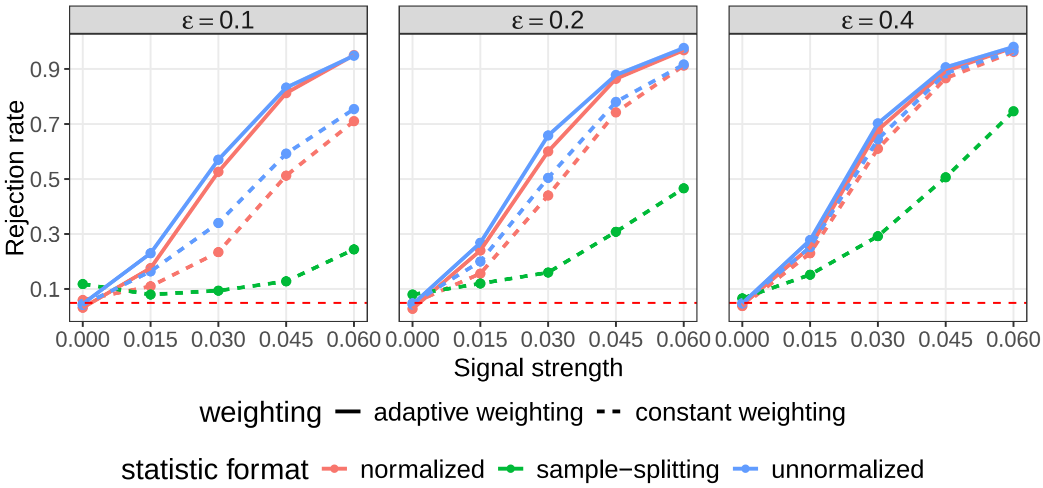

Figure 5: Type-I error and power for the five tests under semi-synthetic data.

The results are presented in Figure 5. Perhaps a bit surprisingly, we observe that the test based on sample splitting suffers from Type-I error inflation in the absence of signal. This issue is primarily due to the sparsity of the outcome: the average rate of CVD occurrence is less than 0.1. Such sparsity may prevent the central limit theorem from taking effect, posing a particular challenge for sample splitting, which uses only half of the data. We also find that the choice of ε influences calibration performance: in the ε-greedy algorithm, a smaller ε results in a smaller effective sample size in the second stage. Consequently, Type-I error inflation in the sample-splitting method is mitigated as ε increases. In contrast, our methods control Type-I error well and ε has a much smaller effect on these tests, which combine data from both stages and thus have higher effective sample size. A further investigation of calibration performance is provided in Appendix P. Regarding power, the benefit of adaptive sampling remains evident compared to the sample-splitting approach. Among all 4 tests proposed in this paper, the unnormalized test exhibits slightly higher power than the normalized test, although the difference is modest.

5 Conclusion and discussion

In this paper, we establish a set of general and assumption-lean weak convergence results for the WIPW estimator under a two-stage adaptive sampling scheme. These results are largely agnostic to any specific outcome distribution, allowing for broad applicability across a wide range of potential outcomes. Moreover, they accommodate a broad spectrum of signal strengths, making them especially useful for downstream hypothesis testing. To facilitate asymptotically valid tests based on these weak convergence results, we propose a plug-in bootstrap procedure that is highly scalable. Finally, we validate our theoretical claims through extensive numerical simulations and a semi-synthetic data analysis, illustrating strong finite-sample performance of the proposed tests.

There are two directions that can be directly pursued with the results and techniques developed in this paper.

- **Experimental design:** In practice, designing the adaptive experiments often requires balancing statistical goals (e.g., power) with non-statistical considerations (e.g., regret, welfare) under budget constraints. Investigating the optimal design of adaptive experiments that trade off these competing objectives is a compelling direction for future study. See, for instance, recent work on adaptive experimental design in Che et al. (2023), Liang et al. (2023), Simchi-Levi et al. (2023), and Li et al. (2024). Our results on limiting distributions and proposed bootstrap procedure simplify power calculations, which in turn can inform the design of adaptive experiments.

- **Covariate adjustment:** In randomized controlled experiments, it is well established that appropriately adjusting for predictive covariates can improve the efficiency of statistical inference and increase the power of hypothesis testing (Lin, 2013). It would be valuable to explore how such covariate adjustments can be incorporated into the analysis of data collected from adaptive experiments, and how they may enhance the efficiency of the proposed tests. These investigations require the study of asymptotic efficiency of different tests and the techniques developed in this paper may provide a useful starting point for exploring this line of research. In Appendix N.2, we present preliminary results on augmenting the WIPW test statistics.

There are several limitations in our current work that point to promising directions for future research. We summarize them below.

- **Beyond two-stage experiments:** In this paper, we focus on a two-stage adaptive sampling scheme. However, other adaptive designs exist that fall outside this framework, such as fully adaptive sampling schemes (Lai et al., 1985), early-stopping experiments (Sampson et al., 2005; Sill et al., 2009) and experiments with adaptive stopping rules (Bauer et al., 1994). We have sketched the extension of our results to the latter two classes of adaptive sampling strategies in Section N.1 and N.4, respectively. For the fully adaptive sampling schemes, we refer the readers to Khamaru et al. (2024), Han et al. (2024), and Ren et al. (2024) for recent works on this topic.

- **Statistical optimality:** The statistical optimality of the proposed tests remains an open question. We investigate this question by providing preliminary comparisons of power between the $m = 1/2$ and $m = 1$ weightings in Appendix O.2. It would be interesting to study semiparametrically efficient test statistic for testing the hypothesis $H_{0N} : \mathbb{E}[Y_{uN}(0)] = \mathbb{E}[Y_{uN}(1)]$ under suitable sub-classes of the general nonparametric data generating process. Although Hirano et al. (2023) and Adusumilli (2023) derive power functions via the Neyman-Pearson lemma, devising practical tests that attain the stated optimal power remains an open problem. Furthermore, for more complex scenarios—such as composite alternatives—the corresponding optimality theory remains undeveloped.

Acknowledgments

We thank Keisuke Hirano and Stefan Wager for helpful discussions and suggestions. Z.R. is supported by the National Science Foundation (NSF) under grant DMS-2413135 and Wharton Analytics.

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Appendix

Notation. Throughout the appendix, we will use (a, b) to denote a column vector for $a \in \mathbb{R}^k, b \in \mathbb{R}^d$ when there is no ambiguity. In other words, we use (a, b) to represent $(a^\top, b^\top)^\top$ and we omit the transpose operator for ease of presentation. Thus when we write $f(x, y)$ for x, y as vectors, we will use $f(x, y)$ to denote the function $f((x^\top, y^\top)^\top)$. We use $\mathcal{C}^2(\mathcal{X})$ as the class of functions that are twice continuously differentiable on the domain $\mathcal{X}$. We use $\mathbf{0}_k$ to denote a k-dimension vector with each dimension being 0 and $\boldsymbol{I}_k$ to denote the identify matrix with dimension k. We will drop the subscript k if there is no ambiguity. We use $\mathbb{N}_+$ to denote the positive natural number. We denote ∂C_k as the boundary set of C_k. We denote ∇g as the gradient of a differentiable function g. We denote $a_N \lesssim b_N$ if there exists $c > 0$ such that $|a_N/b_N| \leq c$ for large enough N. *Without loss of generality, we will only prove the results in all the main text when $q_t = 1/2$ for $t \in [2]$, i.e., two batches have the same sample size.*

A Explicit form of the asymptotic distributions

We define the limiting probabilities

$$H^{(1)}(s) \equiv \lim_{N\to\infty} \bar{e}_N(s, \mathcal{H}_0) = e(s), \ \ H^{(2)}(s) \equiv \max\Big\{ \lim_{N\to\infty} l_N, \bar{e}(s, S((A^{(1)}, V^{(1)}), c))\Big\}. \quad (10)$$

The function $S(x, y) : \mathbb{R}^4 \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is defined as

$$S(x, y) \equiv x_1 \cdot x_3^{1/2}/e^{1/2}(0) - x_2 \cdot x_4^{1/2}/e^{1/2}(1) + y/\sqrt{2}, \quad (11)$$

where $x = (x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4$ and $y \in \bar{\mathbb{R}}$. Also define the scaled asymptotic variance as

$$V^{(t)}(s) \equiv \lim_{N\to\infty} \mathbb{E}[Y_{uN}^2(s)] - H^{(t)}(s)(\lim_{N\to\infty} \mathbb{E}[Y_{uN}(s)])^2, \tag{12}$$

and denote $R^{(t)}(s) \equiv (H^{(t)}(s)/V^{(t)}(s))^{1/2}$. Now we define the covariances for $A^{(t)}$ as follows.

Distribution of $A^{(1)}$. The covariance $\mathrm{Cov}^{(1)}$ can be defined as

$$\mathrm{Cov}^{(1)} \equiv -\left(\frac{H^{(1)}(0)H^{(1)}(1)}{V^{(1)}(0)V^{(1)}(1)}\right)^{1/2} \lim_{N\to\infty} \left(\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]\right). \tag{13}$$

Distribution of $A^{(2)}$. The asymptotic covariance structure $\mathrm{Cov}^{(2)}(A^{(1)})$ can be written as

$$\mathrm{Cov}^{(2)}(A^{(1)}) \equiv -\left(\frac{H^{(2)}(0)H^{(2)}(1)}{V^{(2)}(0)V^{(2)}(1)}\right)^{1/2} \lim_{N\to\infty} \left(\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]\right). \tag{14}$$

Now we define the weights $\bar{w}_{\mathcal{W}}^{(t)}(s)$. To this end, we need the following auxiliary random variables,

$$M_{\mathcal{A}}^{(t)}(s) \equiv q_t\Big(\frac{(H^{(t)}(s))^{1/2}}{\sum_{t=1}^2 q_t(H^{(t)}(s))^{1/2}}\Big)^2 \quad \text{and} \quad M_{\mathcal{C}}^{(t)}(s) \equiv q_t.$$

Then we can write the weights as

$$\bar{w}_{\mathcal{W}}^{(t)}(s) = \left(M_{\mathcal{W}}^{(t)}(s)/(R^{(t)}(s))^2\right)^{1/2} \quad \text{for any } s \in \{0,1\} \quad \text{and} \quad \mathcal{W} \in \{\mathcal{A}, \mathcal{C}\}.$$

B Simulation detail for Figures 1-2

We present the simulation details for Figure 1 and Figure 2 respectively.

Figure 1. To generate Figure 1, we consider the following potential outcome model:

$$Y_{uN}(0) \sim N(0,1),\ Y_{uN}(1) \sim N(-c_N/\sqrt{N}, 9),\ c_N \in \{0, -5, -10, -15\}. \tag{15}$$

We set the sample size $N = 1000$ and the batch size $N_1 = N_2 = 500$. The initial sampling $e(0) = e(1) = 0.5$ and ε-greedy algorithm is used with $\varepsilon = 0.05$. The simulations are repeated 5000 times.

Figure 2. We use the covariance structure (13) and (14), to simulate the limiting distribution (6). We set $q_t = 1/2$ and assume the knowledge of the first and second moments $\mathbb{E}[Y_{uN}(s)], \mathbb{E}[Y_{uN}^2(s)]$ from model (15). Set the limiting signal strength $c \in \{0, -5, -10, -15\}$. The simulations are repeated 100,000 times.

C Details of bootstrap algorithm in Section 3.5

We begin by estimating the necessary nuisance parameters for the bootstrap in Appendix C.1 and then provide the detailed bootstrap algorithm in Appendix C.2.

C.1 Nuisance parameter estimation

Analogous to the estimator $\hat{\mathbb{E}}[Y_{uN}(s)]$ in (2), we estimate $\mathbb{E}[Y_{uN}^2(s)]$ using:

$$\mathrm{WIPWS}(s) \equiv \sum_{t=1}^{2} \frac{N_t h_N^{(t)}(s)}{\sum_{t=1}^{2} N_t h_N^{(t)}(s)} \cdot \frac{1}{N_t} \sum_{u=1}^{N_t} \tilde{\Lambda}_{uN}^{(t)}(s) \quad \text{and} \quad \tilde{\Lambda}_{uN}^{(t)}(s) \equiv \frac{\mathbb{1}(A_{uN}^{(t)} = s)(Y_{uN}^{(t)})^2}{\mathbb{P}[A_{uN}^{(t)} = s \mid \mathcal{H}_{t-1}]}. \tag{16}$$

Using these, we estimate the first-stage variances and covariances as:

$$\hat{V}^{(1)}(s) = \hat{\mathbb{E}}[Y_{uN}^2(s)] - H^{(1)}(s)(\hat{\mathbb{E}}[Y_{uN}(s)])^2 \quad \text{and} \quad \hat{\mathrm{Cov}}^{(1)} = -\frac{(\bar{H}^{(1)})^{1/2}\hat{\mathbb{E}}[Y_{uN}(0)]\hat{\mathbb{E}}[Y_{uN}(1)]}{(\hat{V}^{(1)}(0)\hat{V}^{(1)}(1))^{1/2}},$$

where $\bar{H}^{(1)} \equiv H^{(1)}(0)H^{(1)}(1)$. To define $\hat{\mathrm{Cov}}^{(2)}(\cdot)$, first define $\hat{V}^{(1)} \equiv (\hat{V}^{(1)}(0), \hat{V}^{(1)}(1))$ and consider the function $\hat{H}^{(2)}(x) \equiv \hat{H}^{(2)}(x,0) \cdot \hat{H}^{(2)}(x,1)$, where

$$\hat{H}^{(2)}(x,s) \equiv \begin{cases} \max\{\bar{l}, \bar{e}(s, S((x, \hat{V}^{(1)}), 0))\} & \text{if Assumption 3 holds;} \\ \bar{e}(s, S((x, \hat{V}^{(1)}), 0)) & \text{if Assumption 4 holds.} \end{cases}$$

Furthermore, we define $\hat{V}^{(2)}(x) \equiv \hat{V}^{(2)}(x,0) \cdot \hat{V}^{(2)}(x,1)$, where $\hat{V}^{(2)}(x,s) \equiv \hat{\mathbb{E}}[Y_{uN}^2(s)] - \hat{H}^{(2)}(x,s)(\hat{\mathbb{E}}[Y_{uN}(s)])^2$. Finally, we can define the second-stage covariance function as

$$\hat{\mathrm{Cov}}^{(2)}(x) \equiv -(\hat{H}^{(2)}(x)/\hat{V}^{(2)}(x))^{1/2}\hat{\mathbb{E}}[Y_{uN}(0)]\hat{\mathbb{E}}[Y_{uN}(1)].$$

C.2 Bootstrap algorithm

Now we consider the bootstrap procedure.

1. **First stage sampling:** Sample $S_1^{(b)} \sim N(0, \boldsymbol{I}_2)$ and let $\tilde{A}^{(1,b)} = (\hat{\boldsymbol{\Sigma}}^{(1)})^{1/2} S_1^{(b)}$, where $\hat{\boldsymbol{\Sigma}}^{(1)} = (\hat{\mathrm{Cov}}^{(1)})_{2\times 2}$.

2. **Second stage sampling:** Sample $S_2^{(b)} \sim N(0, \boldsymbol{I}_2)$ and let $\tilde{A}^{(2,b)} = (\hat{\boldsymbol{\Sigma}}^{(2,b)})^{1/2} S_2^{(b)}$, where $\hat{\boldsymbol{\Sigma}}^{(2,b)} = (\hat{\mathrm{Cov}}^{(2)}(\tilde{A}^{(1,b)}))_{2\times 2}$.

3. **Weighting procedure:** Compute weights $\hat{w}_{\mathcal{W},\mathcal{V}}^{(t,b)}(s)$ by replacing $H^{(1)}(s), H^{(2)}(s)$ and $V^{(1)}(s), V^{(2)}(s)$ in (6) by $H^{(1)}(s), \hat{H}^{(2)}(\tilde{A}^{(1,b)}, s)$, and $\hat{V}^{(1)}(s), \hat{V}^{(2)}(\tilde{A}^{(1,b)}, s)$, respectively. Then obtain the bootstrap sample:

$$\mathcal{D}_{\mathcal{W},\mathcal{V}}^{(b)} = \sum_{t=1}^{2} \hat{w}_{\mathcal{W},\mathcal{V}}^{(t,b)}(0)\tilde{A}^{(t,b)}(0) - \sum_{t=1}^{2} \hat{w}_{\mathcal{W},\mathcal{V}}^{(t,b)}(1)\tilde{A}^{(t,b)}(1),$$

   where $\tilde{A}^{(t,b)}(s)$ is the $(s+1)$-th coordinate of $\tilde{A}^{(t,b)}$.

4. **Repeat sampling:** Repeat steps 1-3 for B iterations to obtain B bootstrap samples.

D Intuition from data collection and double-dipping

In this section, we discuss the intuition behind the different phases of the limiting distribution, drawing on the data collection procedure outlined in Section 2.1. We argue that the key driver of the various limiting behaviors is the signal strength. To build intuition, consider the ε-greedy selection algorithm as an illustrative example, and suppose we use a fixed clipping rate $l_N = \varepsilon > 0$. Then the updated treatment assignment probability for treatment s after the pilot stage is given by

$$\bar{e}_N(s, \mathcal{H}_1) = \frac{\varepsilon}{2} \cdot \mathbb{1}(D_N(s) < 0) + \left(1 - \frac{\varepsilon}{2}\right) \cdot \mathbb{1}(D_N(s) \geq 0), \; D_N(s) \equiv S_N^{(1)}(s) - S_N^{(1)}(1 - s).$$

When the signal strength c_N converges to a finite constant, the pilot-stage data does not provide sufficiently strong evidence for the ε-greedy algorithm to confidently prefer one arm over the other based on the summary statistic $D_N(s)$. As a result, the assignment remains uncertain, and this added randomness in the selection procedure leads to a non-normal limiting distribution. In contrast, when the signal strength is strong (i.e., $c = -\infty$), the algorithm can confidently conclude that treatment 1 is superior to treatment 0 in terms of the expected potential outcome, with ignorable randomness in the selection. In this case, the limiting distribution approaches a normal distribution.

An alternative perspective comes from the concept of "double-dipping." It is widely believed that sample splitting is necessary to avoid selection bias (Fithian et al., 2014). When the signal is weak (i.e., c is finite), the selection procedure exhibits non-negligible randomness, and reusing data without accounting for this selection randomness can be problematic. However, when the signal is strong and the selection becomes deterministic, the impact of "double-dipping" becomes negligible. In this regime (i.e., $c = -\infty$), the two-stage data collection process can be viewed as a non-adaptive procedure: sample treatment 0 (1) with probability $e(0)$ ($e(1)$) in the first stage and with probability $\varepsilon/2$ ($1 - \varepsilon/2$) in the second stage. Standard asymptotic inference applies, ensuring asymptotic validity.

E A closer look at literature

In this section, we provide a detailed comparison between our results and the existing literature.

E.1 Investigating Zhang et al. (2020)

They show that the non-normal limiting distribution can happen for classical sample mean statistic under a batched bandit setup (see Figure 1 in their paper). To address this issue, the same paper proposes a batch-wise Hájek estimator. In particular, for each batch $t \in [2]$, the test statistic can be computed as

$$\sqrt{\frac{(\sum_{u=1}^{N_t} A_{uN}^{(t)})(\sum_{u=1}^{N_t}(1 - A_{uN}^{(t)}))}{N}} \left(\frac{\sum_{u=1}^{N_t}(1 - A_{uN}^{(t)})Y_{uN}^{(t)}}{\sum_{u=1}^{N_t}(1 - A_{uN}^{(t)})} - \frac{\sum_{u=1}^{N_t} A_{uN}^{(t)} Y_{uN}^{(t)}}{\sum_{u=1}^{N_t} A_{uN}^{(t)}} - \Delta_n \right) \tag{17}$$

where $\Delta_n \equiv \mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)]$. A crucial assumption they make to recover the conditional asymptotic normality is that the conditional variance of the observed outcome is constant, i.e., $\mathrm{Var}[Y_u^{(t)}|\mathcal{H}_{t-1}] = \text{Cons.} \in (0,\infty)$. This assumption is very stringent and essentially rules out the possible heterogeneity in the distribution of potential outcomes $Y_u(0), Y_u(1)$. Let us consider a concrete data generating model to illustrate the failure of this assumption. Consider the potential outcome distribution $Y_u(0) \sim N(0,1)$ and $Y_u(1) \sim N(0,4)$. Then we can compute, assuming the ε-greedy algorithm between stages, using the consistency assumption

$$
\begin{aligned}
\mathrm{Var}[Y_u^{(t)}|\mathcal{H}_{t-1}] &= \mathbb{E}[(Y_u^{(t)})^2|\mathcal{H}_{t-1}] - (\mathbb{E}[Y_u^{(t)}|\mathcal{H}_{t-1}])^2 \\
&= \bar{e}(0,\mathcal{H}_{t-1})\mathbb{E}[Y_u(0)^2] + \bar{e}(1,\mathcal{H}_{t-1})\mathbb{E}[Y_u(1)^2] \\
&= \bar{e}(0,\mathcal{H}_{t-1}) + 4\bar{e}(1,\mathcal{H}_{t-1}) \\
&= 1 + 3\bar{e}(1,\mathcal{H}_{t-1}) \\
&= 1 + 3\left(1 - \frac{\varepsilon}{2}\right)\mathbb{1}(S_N^{(1)}(0) < S_N^{(1)}(1)) + \frac{3\varepsilon}{2}\mathbb{1}(S_N^{(1)}(0) \geq S_N^{(1)}(1)).
\end{aligned}
$$

Moreover, test based on the pooled batch-wise estimator (17) is later on shown to be less powerful than the test statistic based on data pooled from all stages (see Hirano et al. (2023)). In fact, the sample mean test statistic considered in Section 5 of Hirano et al. (2023) is a special case of our proposed WIPW test statistic with $m = 1$. We refer readers to Appendix N.1 for more details.

E.2 Investigating Hadad et al. (2021)

Hadad et al. (2021) consider a class of WIPW estimators. Their paper observes that when estimating the expected outcome $\mathbb{E}[Y_{uN}(s)]$, even the classical inverse probability weighted estimator can exhibit non-normal behavior (see Figure 1 in Hadad et al.). To address this, Hadad et al. (2021) shows asymptotic normality can be recovered through the use of adaptive weighting—similar in spirit to the method proposed in our work (see Theorem 4 in their paper). However, their theoretical guarantees rely on a key assumption: that the ratio of variance estimators converges to a constant. This assumption, however, fails to hold under the null hypothesis H_{0N}.

Indeed, as we will demonstrate shortly, the ratio of variance estimators converges to a non-degenerate, positive random variable whenever $c \in (-\infty,\infty)$—that is, under both the null hypothesis H_{0N} and the weak signal regime H_{2N}. Moreover, our simulations reveal that applying a normal approximation in the absence of this convergence can lead to inflated Type-I error rates when using the WIPW estimator. Consequently, the theoretical results in Hadad et al. (2021) are not directly applicable for hypothesis testing, due to the unknown limiting distribution under the null.

To be specific, Theorem 4 in Hadad et al. (2021) hinges on the assumption that $\hat{V}_N(0)/\hat{V}_N(1)$ converges weakly to a constant, where $\hat{V}_N(s)$ is defined as in Equation (3). However, under the null H_{0N} and local alternative H_{2N}, this condition generally fails. As shown in **step 2** and **step 3** in section J.1: with adaptive weighting ($m = 1/2$) and $q_t = 1/2$,

$$
\hat{V}_N(0)/\hat{V}_N(1) \xrightarrow{d} \frac{\sum_{t=1}^2 V^{(t)}(0)/(\sum_{t=1}^2 (H^{(t)}(0))^{1/2})^2}{\sum_{t=1}^2 V^{(t)}(1)/(\sum_{t=1}^2 (H^{(t)}(1))^{1/2})^2}. \tag{18}
$$

where $V^{(t)}(s)$ and $H^{(t)}(s)$ are defined as in (12) and (10).

To further illustrate this issue, we present empirical evidence showing that using the normal distribution to calibrate the test statistic leads to inflated Type-I error. Consider a two-batch setup under the following model:

$$Y_u(0) \sim N(2,1), \;\; Y_u(1) \sim N(2,9).$$

with a total sample size of $N = 500$ and equal batch sizes $N_1 = N_2 = 250$. We apply the ε-greedy algorithm with $\varepsilon = 0.05$, and compute the weighted augmented IPW estimator (WAIPW), in line with the original setup in Hadad et al. (2021):

$$\mathrm{WAIPW}(s) \equiv \sum_{t=1}^{2} \frac{h_N^{(t)}(s)}{\sum_{t=1}^{2} h_N^{(t)}(s)} \Gamma_N^{(t)}(s),$$

where

$$\Gamma_N^{(t)}(s) \equiv \frac{\sum_{u=1}^{N_t} \Gamma_u^{(t)}(s)}{N_t} + \hat{\mathbb{E}}[Y_u(s)] \quad \text{and} \quad \Gamma_u^{(t)}(s) \equiv \frac{\mathbb{1}(A_u^{(t)} = s)(Y_u^{(t)} - \hat{\mathbb{E}}[Y_u(s)])}{\mathbb{P}[A_u^{(t)} = s | \mathcal{H}_{t-1}]}.$$

According to Theorem 4 in Hadad et al., 2021, we know

$$\frac{\mathrm{WAIPW}(0) - \mathrm{WAIPW}(1)}{(\tilde{V}_N(0) + \tilde{V}_N(1))^{1/2}} \xrightarrow{d} N(0,1), \;\; \tilde{V}_N(s) \equiv \frac{\sum_{t=1}^{2} (h_N^{(t)}(s))^2 \sum_{u=1}^{N_t} \left(\Gamma_u^{(t)}(s)\right)^2}{(\sum_{t=1}^{2} h_N^{(t)}(s) N_t)^2}.$$

Setting $\hat{\mathbb{E}}[Y_u(0)] = 2$ and $\hat{\mathbb{E}}[Y_u(1)] = 6$, we can satisfy the required condition that for **at least one** $s \in \{0,1\}, \hat{\mathbb{E}}[Y_u(s)] \to \mathbb{E}[Y_u(s)]$. Then we use $N(0,1)$ to calibrate the test statistic and obtain the Type-I error control results as shown in Figure 6. The results show substantial Type-I error inflation when the normal approximation is used.

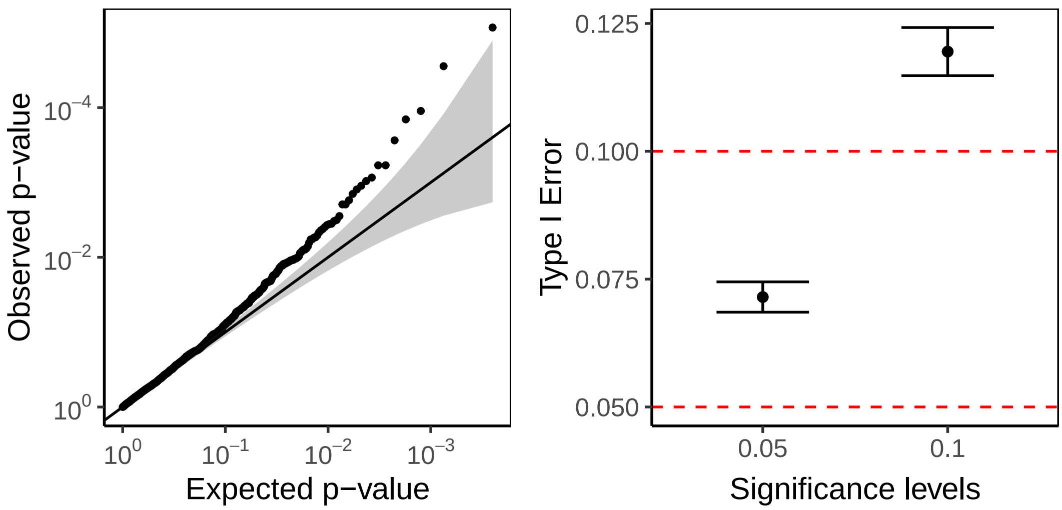

Figure 6: Type-I error inflation using normal approximation in Hadad et al. (2021). The simulation is repeated for 2000 times.

E.3 Investigating Hirano et al. (2023) and Adusumilli (2023)

To address non-normality directly, Hirano et al. (2023) develop asymptotic representations for a broad class of test statistics under batched designs, which are subsequently used to derive power functions and optimal tests in Adusumilli (2023). The validity of this theory requires establishing weak convergence of a vector of statistics, as well as assuming that the potential outcome distributions are differentiable in quadratic mean (QMD). In contrast, our Theorem 1 focuses on the widely used WIPW class of statistics and provides explicit weak convergence results that go beyond the QMD framework, based on transparent and interpretable assumptions.

E.4 Investigating Che et al. (2023)

While Che et al. (2023) broaden the framework of Hirano et al. (2023) to encompass settings beyond QMD, their emphasis lies in experimental design within batched bandit designs rather than in inferential validation. Specifically, they analyze the statistic

$$\frac{1}{\sqrt{N_t}} \sum_{u=1}^{N_t} A_{uN}^{(t)} Y_{uN}^{(t)},$$

deriving its asymptotic distribution under conditions similar to ours. Their results, however, are not sufficient for the purpose of inference. For example, one can show that—after an additional scaling by $\sqrt{N_t}$—this statistic generally fails to converge to the target estimand $\mathbb{E}[Y_{uN}(s)]$, except in the degenerate case $\mathbb{E}[Y_{uN}(s)] = 0$. Moreover, the asymptotic distribution therein contains unknown parameters, making it infeasible to construct valid hypothesis tests or confidence intervals based on this statistic.

By contrast, our work prioritizes the inferential integrity of adaptive experiments. Our results incorporate consistent and pivotal estimators, which guarantees that hypothesis tests are valid and that confidence intervals faithfully reflect uncertainty around the desired parameter.

F Probabilistic preliminaries

In Section F.1, we discuss technique of proving weak convergence of random variables with test functions. In Section F.2, we discuss the regular conditional distribution (RCD) and its properties.

F.1 Bounded Lipschitz test function class

We will use the following definition of convergence in distribution throughout the appendix.

Definition 1 (Convergence in distribution). *Suppose $W_N \in \mathbb{R}^d$ is a sequence of random variables and $W \in \mathbb{R}^d$ is a random variable. We say W_N converge in distribution to W if for any bounded and continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we have*

$$\mathbb{E}[f(W_N)] \to \mathbb{E}[f(W)].$$

Moreover, we use the notation $W_N \xrightarrow{d} W$ to denote the convergence in distribution. If W_N and W are univariate, we will interchangeably use the following equivalent deifnition

$$\mathbb{P}[W_N \le t] \to \mathbb{P}[W \le t], \text{ for each } t \in \mathbb{R} \text{ at which } t \mapsto \mathbb{P}[W \le t] \text{ is continuous.}$$

Beyond the bounded continuous functions, there are other classes of functions that are useful in the context of weak convergence. In particular, we will use the bounded Lipschitz function class to prove our main results. Suppose f is a function from $\mathbb{R}^d$ to $\mathbb{R}$. We call f is a m-Lipschitz function as long as $\|f\|_{\mathrm{L}} \le m$, where $\|f\|_{\mathrm{L}} \equiv \sup_{x \neq y \in \mathbb{R}^d} |f(x) - f(y)|/\|x - y\|_2$ and $\|\cdot\|_2$ is the Euclidean norm. Then we define

$$\|f\|_{\mathrm{BL}} \equiv \|f\|_{\mathrm{L}} + \|f\|_\infty \quad \text{where} \quad \|f\|_\infty \equiv \sup_{x \in \mathbb{R}^d} |f(x)|.$$

The following lemma shows that the BL function class is rich enough to characterize the weak convergence of random variables.

Lemma 1 (Theorem 11.3.3 in Dudley (2002))**.** *Suppose the sequence of random variable $W_N \in \mathbb{R}$ and $W \in \mathbb{R}$ have law $\mathbb{P}_N$ and $\mathbb{P}$, respectively. Then the following two statements are equivalent:*

1. $W_N \xrightarrow{d} W$;
2. $\left|\int f(x)\mathrm{d}\mathbb{P}_N(x) - \int f(x)\mathrm{d}\mathbb{P}(x)\right| \to 0$ *for any f such that* $\|f\|_{\mathrm{BL}} < \infty$.

Despite the fruitful results in the literature on normal approximation on independent observations (Chatterjee et al., 2008) and weakly dependent observation Chen et al. (2004), these existing results do not apply directly to our case since the adaptive sampling scheme introduces a *strong* dependence structure. We will develop new tools for proving our results, based on these existing results. Thus we review the relevant results in the following section. First comes the finite-sample bound proved in Chatterjee et al. (2008) and an interpolation result Meckes (2009b).

Lemma 2 (Chatterjee et al. (2008), Theorem 3.1)**.** *Let $W_{1N}, \ldots, W_{NN}$ be a sequence of independent, identically distributed random vectors in $\mathbb{R}^k$. Suppose*

$$\mathbb{E}[W_{uN}] = 0, \ \mathbb{E}[W_{uN} W_{uN}^\top] = \boldsymbol{I}_k.$$

Let $W_N \equiv \sum_{u=1}^N W_{uN}/\sqrt{N}$ and $Z \sim N(\mathbf{0}, \boldsymbol{I}_k)$. Then for any $g \in \mathcal{C}^2(\mathbb{R}^k)$,

$$|\mathbb{E}[g(W_N)] - \mathbb{E}[g(Z)]| \le \frac{\|g\|_{\mathrm{L}}}{2\sqrt{N}}(\mathbb{E}[\|W_{uN}\|_2^4])^{1/2} + \frac{\sqrt{2\pi}}{3\sqrt{N}}\|\nabla g\|_{\mathrm{L}}\mathbb{E}[\|W_{uN}\|_2^3].$$

Lemma 2 shows that when the test function is chosen to be $\mathcal{C}^2$, the weak convergence for W_N can be derived as long as the third moment of W_{uN} diverging slower than $\sqrt{N}$ and fourth moment diverging slower than N.

Lemma 3 (Meckes (2009b), Corollary 3.5). *Consider the density function of a multivariate Gaussian random variable* $\phi_\delta(x) \equiv \frac{1}{(2\pi\delta^2)^{k/2}} \exp\left(-\frac{1}{2\delta^2}\|x\|_2^2\right)$. *For any 1-Lipschitz function* f, *consider the Gaussian convolution* $(f * \phi_\delta) \equiv \mathbb{E}[f(x + \delta Z)]$ *where* $Z \sim N(\mathbf{0}, \boldsymbol{I}_k)$ *and* $\delta > 0$. *Then we have*

$$\|\nabla(f * \phi_\delta)\|_{\mathrm{L}} \leq \|f\|_{\mathrm{L}} \times \sup_{\theta:\|\theta\|_2=1} \left\|\nabla \phi_\delta^\top \theta\right\|_1 \leq \frac{\sqrt{2}}{\pi^{1/2}} \frac{1}{\delta}$$

where $\|f\|_1$ *denotes the* L_1 *norm of function* f. *Moreover, for any random variable* X

$$\mathbb{E}[(f * \phi_\delta)(X) - f(X)] \leq \mathbb{E}[\delta\|Z\|_2] \leq \delta\sqrt{k}.$$

Applying a smoothing argument to the Lipschitz function class with the help of Lemma 3, we can prove the weak convergence of W_N under Lipschitz class with the same requirement on the third and fourth moment as in Lemma 2. We formalize such result in the following lemma.

Lemma 4 (Upper bound with bounded Lipschitz function). *Suppose* $W_{uN} \in \mathbb{R}^k$ *are i.i.d. random variables for any fixed* $N \in \mathbb{N}_+$. *Then if we have* $\mathbb{E}[W_{uN}] = \mathbf{0}, \mathbb{E}[W_{uN}W_{uN}^\top] = \boldsymbol{I}_k$ *and*

$$\frac{\mathbb{E}\left[\|W_{uN}\|_2^4\right]}{N} \to 0, \ \frac{\mathbb{E}[\|W_{uN}\|_2^3]}{N^{1/2}} \to 0,$$

then for any sequence of Lipschitz function r_N *such that* $\|r_N\|_{\mathrm{BL}} \leq 1$, *we have*

$$\left|\mathbb{E}\left[r_N\left(\frac{1}{\sqrt{N}}\sum_{u=1}^N W_{uN}\right)\right] - \mathbb{E}\left[r_N(Z)\right]\right| \to 0, \ Z \sim N(\mathbf{0}, \boldsymbol{I}_k).$$

F.2 Preliminaries on regular conditional distribution

To better understand the argument involving conditional distribution, we briefly discuss the basic definition of regular conditional distribution. Let $\mathcal{B}(\mathbb{R}^N)$ be the Borel σ-algebra on $\mathbb{R}^N$ and $\Omega, \mathcal{F}_N$ be the sample space and a sequence of σ-algebras. For any $N \in \mathbb{N}_+, \kappa_N : \Omega \times \mathcal{B}(\mathbb{R}^N)$ is a regular conditional distribution of $W_N \equiv (W_{1N}, \ldots, W_{NN})$ given $\mathcal{F}_N$ if

$$\begin{aligned} \omega \mapsto \kappa_N(\cdot, B) &\text{ is measurable with respect to } \mathcal{F}_N \text{ for any fixed } B \in \mathcal{B}(\mathbb{R}^N); \\ B \mapsto \kappa_N(\omega, B) &\text{ is a probability measure on } (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N)) \text{ for any } \omega \in \Omega; \\ \kappa_N(\omega, B) = \mathbb{P}[(W_{1N}, \ldots, W_{NN}) \in B | \mathcal{F}_N](\omega) &\text{ for almost all } \omega \in \Omega \text{ and all } B \in \mathcal{B}(\mathbb{R}^N). \end{aligned}$$

The following lemma from Klenke (2017, Theorem 8.37) ensures the general existence of regular conditional distribution.

Lemma 5 (Theorem 8.37 in Klenke (2017)). *Suppose* $(\Omega, \mathcal{G}, \mathbb{P})$ *is the Probability triple. Let* $\mathcal{F} \subset \mathcal{G}$ *be a sub-*σ*-algebra. Let* Y *be a random variable with values in a Borel space* $(E, \mathcal{E})$ *(for example,* E *is Polish,* $E = \mathbb{R}^k$*). Then there exists a regular conditional distribution* $\kappa_{Y,\mathcal{F}}$ *of* Y *given* $\mathcal{F}$.

Result from Klenke (2017, Theorem 8.38) guarantees that the conditional expectation and the integral of measurable function with respect to regular conditional distribution are almost surely same.

Lemma 6 (Modified version of Theorem 8.38 in Klenke (2017))**.** *Let Y be a random variable $(\Omega, \mathcal{G}, \mathbb{P})$ taking values in a Borel space $(E, \mathcal{E})$. Let $\mathcal{F} \subset \mathcal{G}$ be a σ-algebra and let $\kappa_{Y,\mathcal{F}}$ be a regular conditional distribution of Y given $\mathcal{F}$. Further, let $f : E \to \mathbb{R}$ be measurable and $\mathbb{E}[|f(Y)|] < \infty$. Then we can define a version of the conditional expectation using regular conditional distribution, i.e.,*

$$\mathbb{E}[f(Y)|\mathcal{F}](\omega) = \int f(y)\mathrm{d}\kappa_{Y,\mathcal{F}}(\omega, y), \ \forall \omega \in \Omega.$$

Throughout this paper we will fix a version of the conditional expectation and the version is defined by applying Lemma 6. The following lemma is useful as well.

Lemma 7 (Conditional expectation with variable measurable with respect to $\mathcal{F}$)**.** *Let Y be a random variable $(\Omega, \mathcal{G}, \mathbb{P})$ with values in a Borel space $(E, \mathcal{E})$. Let $\mathcal{F} \subset \mathcal{G}$ be a σ-algebra and let $\kappa_{Y,\mathcal{F}}$ be a regular conditional distribution of Y given $\mathcal{F}$. Suppose $X \in \mathcal{F}$ is a random variable in another Borel space $(B, \mathcal{B})$. Further, let $f : E \times B \to \mathbb{R}$ be measurable and $\mathbb{E}[|f(X, Y)|] < \infty$. Then we can define a version of the following conditional expectation using regular conditional distribution:*

$$\mathbb{E}[f(X, Y)|\mathcal{F}](\omega) = \int f(X(\omega), y)\mathrm{d}\kappa_{Y,\mathcal{F}}(\omega, y), \ \forall \omega \in \Omega.$$

F.3 Definition of conditional convergence

Since we are under the setup where the data is generated adaptively, we need to extensively work with the conditional convergence where the conditioning is on the data collected in the first stage. We need to first define the notions of conditional convergence, with the definition of regular conditional distribution. In particular, we adopt the definition of conditional convergence in distribution and probability from Niu et al. (2024).

Definition 2. *For each N, let W_N be a random variable and let $\mathcal{F}_N$ be a σ-algebra. Then, we say W_N converges in distribution to a random variable W conditionally on $\mathcal{F}_N$ if*

$$\mathbb{P}[W_N \leq t \mid \mathcal{F}_N] \xrightarrow{p} \mathbb{P}[W \leq t] \text{ for each } t \in \mathbb{R} \text{ at which } t \mapsto \mathbb{P}[W \leq t] \text{ is continuous.} \tag{19}$$

We denote this relation via $W_N \mid \mathcal{F}_N \xrightarrow{d,p} W$.

Definition 3. *For each N, let W_N be a random variable and let $\mathcal{F}_N$ be a σ-algebra. Then, we say W_N converges in probability to a constant c conditionally on $\mathcal{F}_N$ if W_N converges in distribution to the delta mass at c conditionally on $\mathcal{F}_N$ (recall Definition 2). We denote this convergence by $W_N \mid \mathcal{F}_N \xrightarrow{p,p} c$. In symbols,*

$$W_N \mid \mathcal{F}_N \xrightarrow{p,p} c \quad \text{if} \quad W_N \mid \mathcal{F}_N \xrightarrow{d,p} \delta_c. \tag{20}$$

F.4 Proof of Lemma 4

Proof of Lemma 4. Define the Gaussian convolution of $r_N(x)$ as $F_N(x,\delta) \equiv \mathbb{E}[r_N(x+\delta Y)]$ where $Y \sim N(\mathbf{0}, \boldsymbol{I}_k)$ and the expectation is taken with respect to Y. We first decompose the desired difference to three parts:

$$
\begin{aligned}
A_N &\equiv \mathbb{E}\left[r_N\left(\frac{1}{\sqrt{N}}\sum_{u=1}^{N} W_{uN}\right)\right] - \mathbb{E}\left[r_N\left(Z\right)\right] \\
&= \mathbb{E}\left[r_N\left(\frac{1}{\sqrt{N}}\sum_{u=1}^{N} W_{uN}\right)\right] - \mathbb{E}\left[F_N\left(\frac{1}{\sqrt{N}}\sum_{u=1}^{N} W_{uN}, \delta\right)\right] \\
&\quad + \mathbb{E}\left[F_N\left(\frac{1}{\sqrt{N}}\sum_{u=1}^{N} W_{uN}, \delta\right)\right] - \mathbb{E}\left[F_N\left(Z,\delta\right)\right] \\
&\quad + \mathbb{E}\left[F_N\left(Z,\delta\right)\right] - \mathbb{E}\left[r_N\left(Z\right)\right] \\
&\equiv A_N^{(1)} + A_N^{(2)} + A_N^{(3)}.
\end{aligned}
$$

Now we decompose the proof into two steps.

Control of $A_N^{(2)}$: We want to apply Lemma 2 and 3 to the variables W_{uN}. First we notice for any fixed N and $\delta, F_N(x,\delta) \in \mathcal{C}^2(\mathbb{R}^k)$. Indeed, we can write, by change of variable,

$$
F_N(x,\delta) = \mathbb{E}[r_N(x+\delta A)] = \int r_N(x+\delta a)\phi(a)da = \int r_N(t)\frac{1}{\delta^d}\phi\left(\frac{t-x}{\delta}\right)\mathrm{d}t.
$$

By dominated convergence theorem, we can interchange the derivative and integral so that we can verify for any fixed $N, \delta, F_N(x,\delta)$ is 2-times continuously differentiable. Then applying Lemma 2 with W_{uN}, we have

$$
|A_N^{(2)}| \leq \frac{\|F_N(\cdot,\delta)\|_{\mathrm{L}} \left(\mathbb{E}[\|W_{uN}\|_2^4]\right)^{1/2}}{2N^{1/2}} + \frac{\sqrt{2\pi}\|\nabla F_N(\cdot,\delta)\|_{\mathrm{L}}\mathbb{E}[\|W_{uN}\|_2^3]}{3N^{1/2}}.
$$

By the definition of r_N that $\|r_N\|_{\mathrm{BL}} \leq 1$ so that $\|F_N(\cdot,\delta)\|_{\mathrm{L}} \leq 1$. Then applying Lemma 3, we obtain $\|\nabla F_N(\cdot,\delta)\|_{\mathrm{L}} \leq \sqrt{2}/(\pi^{1/2}\delta)$. Therefore we have

$$
|A_N^{(2)}| \leq \frac{1}{2}\frac{\left(\mathbb{E}[\|W_{uN}\|_2^4]\right)^{1/2}}{N^{1/2}} + \frac{2}{3\delta}\frac{\mathbb{E}[\|W_{uN}\|_2^3]}{N^{1/2}}.
$$

Control of $A_N^{(1)}, A_N^{(3)}$: Applying Lemma 3, we know $|A_N^{(1)}| \leq \delta\sqrt{k}$, $|A_N^{(3)}| \leq \delta\sqrt{k}$.

Conclusion: Collecting all the results above, we have

$$
|A_N| \leq \frac{1}{2}\frac{\left(\mathbb{E}[\|W_{uN}\|_2^4]\right)^{1/2}}{N^{1/2}} + \frac{2}{3\delta}\frac{\mathbb{E}[\|W_{uN}\|_2^3]}{N^{1/2}} + 2\delta\sqrt{k}.
$$

Since

$$
\frac{\mathbb{E}[\|W_{uN}\|_2^4]}{N} \to 0, \ \frac{\mathbb{E}[\|W_{uN}\|_2^3]}{N^{1/2}} \to 0, \ \text{almost surely}
$$

as assumed, we can optimize δ such that $|A_N| \to 0$ to complete the proof. $\square$

F.5 Proof of Lemma 7

Proof of Lemma 7. By Lemma 6, we know that

$$\mathbb{E}[f(X,Y)|\mathcal{F}](\omega) = \int f(x,y)\mathrm{d}\kappa_{(X,Y),\mathcal{F}}(\omega,(x,y)).$$

Now we prove that for almost every $\omega \in \Omega$,

$$\kappa_{(X,Y),\mathcal{F}}(\omega, S) = \kappa_{Y,\mathcal{F}}(\omega, S_1) \cdot \mathbb{1}(X(\omega) \in S_2),\ \forall S = S_1 \times S_2 \subset E \times B.$$

In other words, $\kappa_{(X,Y),\mathcal{F}}(\omega,\cdot)$ is product measure of another measure $\kappa_{Y,\mathcal{F}}(\omega,\cdot)$ and counting measure supported on the value $X(\omega)$. This can be proved by using the definition of regular conditional distribution. For any $S = S_1 \times S_2 \subset E \times B$, we have

$$\begin{aligned}\kappa_{(X,Y),\mathcal{F}}(\omega, S) = \mathbb{P}[(X,Y) \in S|\mathcal{F}](\omega) &= \mathbb{1}(X(\omega) \in S_2)\mathbb{P}[Y \in S_1|\mathcal{F}](\omega)\\ &= \kappa_{Y,\mathcal{F}}(\omega, S_1) \cdot \mathbb{1}(X(\omega) \in S_2)\end{aligned}$$

for almost every $\omega \in \Omega$. Thus by Fubini's theorem, we conclude

$$\int f(x,y)\mathrm{d}\kappa_{(X,Y),\mathcal{F}}(\omega,(x,y)) = \int f(X(\omega),y)\mathrm{d}\kappa_{Y,\mathcal{F}}(\omega,y).$$

□

G Useful lemmas and the proofs

G.1 Lemma statements

Lemma 8 (Conditional Polya's theorem, Theorem 5 in Niu et al. (2024)). *Let W_N be a sequence of random variables. If $W_N \mid \mathcal{F}_N \xrightarrow{d,p} W$ for some random variable W with continuous CDF, then*

$$\sup_{t\in\mathbb{R}} |\mathbb{P}[W_N \leq t \mid \mathcal{F}_N] - \mathbb{P}[W \leq t]| \xrightarrow{p} 0. \tag{21}$$

Lemma 9 (Slutsky's theorem, Theorem 13.18 in Dudley (2002)). *Let $X_1, X_2 \ldots$ and $Y_1, Y_2, \ldots,$ be random variables with values in $\mathbb{R}^k$. Suppose $X_N \xrightarrow{d} X$ and $\|X_N - Y_N\|_2 \xrightarrow{p} 0$. Then $Y_N \xrightarrow{d} X$.*

Lemma 10 (Conditional weak law of large numbers, Theorem 7 in Niu et al. (2024)). *Let W_{uN} be a triangular array of random variables, such that W_{uN} are independent conditionally on $\mathcal{F}_N$ for each N. If for some $\delta > 0$ we have*

$$\frac{1}{N^{1+\delta}} \sum_{u=1}^{N} \mathbb{E}[|W_{uN}|^{1+\delta} \mid \mathcal{F}_N] \xrightarrow{p} 0, \tag{22}$$

then

$$\frac{1}{N}\sum_{u=1}^{N}(W_{uN} - \mathbb{E}[W_{uN}|\mathcal{F}_N]) \mid \mathcal{F}_N \xrightarrow{p,p} 0. \tag{23}$$

Applying the dominated convergence theorem, we know

$$\frac{1}{N}\sum_{u=1}^{N}(W_{uN}-\mathbb{E}[W_{uN}|\mathcal{F}_N]) \xrightarrow{p} 0.$$

The condition (22) *is satisfied when*

$$\sup_{1\le u\le N}\mathbb{E}[|W_{uN}|^{1+\delta}\mid \mathcal{F}_N]=o_p(N^{\delta}). \tag{24}$$

Lemma 11 (Durrett (2019), Theorem 2.3.2)**.** *A sequence of random variables W_N converges to a limit W in probability if and only if every subsequence of W_N has a further subsequence that converges to W almost surely.*

Lemma 12 (Skorohod's representation theorem)**.** *Let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of probability measures on a metric space S such that μ_N converges weakly to some probability measure μ_∞ on S as $N\to\infty$. Suppose also that the support of μ_∞ is separable. Then there exist S-valued random variables W_N defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that the law of W_N is μ_N for all N (including $N=\infty$) and such that $(W_N)_{N\in\mathbb{N}}$ converges to W_∞, $\mathbb{P}$-almost surely.*

Lemma 13 (Lipschitz continuity of square root matrix under F-norm)**.** *Suppose $A,B\in\mathbb{R}^{k\times k}$ are two positive definite matrices, then we have*

$$\|A^{1/2}-B^{1/2}\|_{\mathrm{F}}\le\sqrt{k}\|A-B\|_{\mathrm{F}}^{1/2}.$$

Lemma 14 (Boundedness on the covariance)**.** *Suppose X_N,Y_N are two sequences of random variables, with finite first moment, satisfying*

1. $\mathbb{E}[X_N^p]$ *and* $\mathbb{E}[Y_N^p]$ *converge to finite constants for* $p=1,2$;
2. $\liminf_{N\to\infty}\mathrm{Var}[X_N]>0,\ \liminf_{N\to\infty}\mathrm{Var}[Y_N]>0$.

Suppose a random sequence $a_N\in(0,1)$ almost surely. Then we have

$$\limsup_{N\to\infty}\left|\frac{a_N^{1/2}\mathbb{E}[X_N]}{(\mathbb{E}[X_N^2]-a_N\mathbb{E}[X_N]^2)^{1/2}}\times\frac{(1-a_N)^{1/2}\mathbb{E}[Y_N]}{(\mathbb{E}[Y_N^2]-(1-a_N)\mathbb{E}[Y_N]^2)^{1/2}}\right|<C<1 \tag{25}$$

almost surely and the constant C only depends on the limit of the moments $\lim_{N\to\infty}\mathbb{E}[X_N^p]$ and $\lim_{N\to\infty}\mathbb{E}[Y_N^p]$ for $p\in\{1,2\}$.

G.2 Proof of Lemma 13

Proof of Lemma 13. It suffices to prove

$$\|A^{1/2}-B^{1/2}\|_2\le\|A-B\|_2^{1/2}, \tag{26}$$

where $\|A\|_2$ is the operator norm A. This is because if the statement (26) holds, then

$$\|A^{1/2}-B^{1/2}\|_{\mathrm{F}}\le\sqrt{k}\|A^{1/2}-B^{1/2}\|_2\le\sqrt{k}\|A-B\|_2^{1/2}\le\sqrt{k}\|A-B\|_{\mathrm{F}}^{1/2}.$$

Now we prove (26). If vector x with $\|x\| = 1$ is an eigenvector of $\sqrt{A} - \sqrt{B}$ with eigenvalue μ then

$$x^\top (A - B)x = x^\top (\sqrt{A} - \sqrt{B})\sqrt{A}x + x^\top \sqrt{B}(\sqrt{A} - \sqrt{B})x = \mu x^\top (\sqrt{A} + \sqrt{B})x.$$

Now if we choose $\mu = \pm \|\sqrt{A} - \sqrt{B}\|_2$ (depending on the sign of the eigenvalue which has the largest magnitude), we have

$$\begin{aligned}\|\sqrt{A} - \sqrt{B}\|_2^2 = (x^\top (\sqrt{A} - \sqrt{B})x)^2 &\le |x^\top (\sqrt{A} - \sqrt{B})x| x^\top (\sqrt{A} + \sqrt{B})x \\ &= |x^\top (A - B)x| \le \|A - B\|_2.\end{aligned}$$

This completes the proof. □

G.3 Proof of Lemma 14

Proof of Lemma 14. We now divide the proof into two cases.

1. **When** $\lim_{N\to\infty} \mathbb{E}[X_N] = 0$ **or** $\lim_{N\to\infty} \mathbb{E}[Y_N] = 0$**.** Since

$$\liminf_{N\to\infty} (\mathbb{E}[X_N^2] - a_N \mathbb{E}[X_N]^2) \ge \liminf_{N\to\infty} (\mathbb{E}[X_N^2] - \mathbb{E}[X_N]^2) > 0$$

   and

$$\liminf_{N\to\infty} (\mathbb{E}[Y_N^2] - (1 - a_N)\mathbb{E}[Y_N]^2) \ge \liminf_{N\to\infty} (\mathbb{E}[Y_N^2] - \mathbb{E}[Y_N]^2) > 0,$$

   we know the claim is true with $C = 0$ almost surely.

2. **When both** $\lim_{N\to\infty} \mathbb{E}[X_N] \ne 0$ **and** $\lim_{N\to\infty} \mathbb{E}[Y_N] \ne 0$**:** Define the sequence $E_N \equiv a_N \mathbb{E}[Y_N^2](\mathbb{E}[X_N^2] - \mathbb{E}[X_N^2]) + (1 - a_N)(\mathbb{E}[Y_N^2] - \mathbb{E}[Y_N]^2)\mathbb{E}[X_N^2]$. We know

$$\begin{aligned} D_N &\equiv (\mathbb{E}[X_N^2] - a_N \mathbb{E}[X_N]^2)(\mathbb{E}[Y_N^2] - (1 - a_N)\mathbb{E}[Y_N]^2) \\ &= a_N(1 - a_N)(\mathbb{E}[X_N]\mathbb{E}[Y_N])^2 + E_N \\ &\equiv C_N + E_N, \end{aligned}$$

   Note that conclusion (25) is equivalent to proving $\limsup_{N\to\infty} |C_N^{1/2}/D_N^{1/2}| < 1$ almost surely. To this end, we observe that

$$\begin{aligned} \frac{D_N}{C_N} &= 1 + \frac{E_N}{C_N} \\ &= 1 + \frac{a_N \mathbb{E}[Y_N^2](\mathbb{E}[X_N^2] - \mathbb{E}[X_N]^2)}{C_N} + \frac{(1 - a_N)(\mathbb{E}[Y_N^2] - \mathbb{E}[Y_N]^2)\mathbb{E}[X_N^2]}{C_N} \\ &= 1 + \frac{\mathbb{E}[Y_N^2](\mathbb{E}[X_N^2] - \mathbb{E}[X_N]^2)}{(1 - a_N)(\mathbb{E}[X_N]\mathbb{E}[Y_N])^2} + \frac{(\mathbb{E}[Y_N^2] - \mathbb{E}[Y_N]^2)\mathbb{E}[X_N^2]}{a_N(\mathbb{E}[X_N]\mathbb{E}[Y_N])^2} \\ &\equiv 1 + R_{1N} + R_{2N}. \end{aligned}$$

   We can bound

$$\liminf_{N\to\infty} R_{1N} \ge \liminf_{N\to\infty} \frac{\mathbb{E}[Y_N^2](\mathbb{E}[X_N^2] - \mathbb{E}[X_N]^2)}{(\mathbb{E}[X_N]\mathbb{E}[Y_N])^2}$$

$$= \lim_{N\to\infty} \frac{\mathbb{E}[Y_N^2]}{(\mathbb{E}[Y_N])^2} \lim_{N\to\infty} \frac{(\mathbb{E}[X_N^2] - \mathbb{E}[X_N]^2)}{(\mathbb{E}[X_N])^2} > 0$$

almost surely, since $\lim_{N\to\infty}(\mathbb{E}[X_N^2] - \mathbb{E}[X_N]^2) = \lim_{N\to\infty} \mathrm{Var}[X_N] > 0$. Similarly, we can prove $\liminf_{N\to\infty} R_{2N} > 0$ almost surely. Thus we have

$$\liminf_{N\to\infty} \frac{D_N}{C_N} \geq 1 + \liminf_{N\to\infty} R_{1N} + \liminf_{N\to\infty} R_{2N} > 1$$

almost surely. Thus the desired bound can be obtained by checking

$$\limsup_{N\to\infty} \left| \frac{a_N^{1/2}\mathbb{E}[X_N]}{(\mathbb{E}[X_N^2] - a_N\mathbb{E}[X_N]^2)^{1/2}} \times \frac{(1-a_N)^{1/2}\mathbb{E}[Y_N]}{(\mathbb{E}[Y_N^2] - (1-a_N)\mathbb{E}[Y_N]^2)^{1/2}} \right|$$
$$= \limsup_{N\to\infty} \left| \frac{C_N^{1/2}}{D_N^{1/2}} \right| = \frac{1}{\liminf_{N\to\infty} D_N^{1/2}/C_N^{1/2}} < 1$$

almost surely.

□

H New results on conditional CLT and CMT

In this section, we present a new conditional central limit theorem (CLT) and a new continuous mapping theorem (CMT) under the conditional convergence framework.

H.1 A new conditional CLT

Lemma 15 (Conditional CLT under Lipschitz class). *Consider σ-alegbras $\mathcal{F}_N$. Suppose $W_{1N}, \ldots, W_{NN} \in \mathbb{R}^k$ are i.i.d. random variables conditional on $\mathcal{F}_N$ for any fixed $N \in \mathbb{N}$. Define $W_N \equiv \sum_{u=1}^N W_{uN}/\sqrt{N}$. Suppose*

$$\mathbb{E}[W_{uN}|\mathcal{F}_N] = \mathbf{0},\ \mathbb{E}[W_{uN}W_{uN}^\top|\mathcal{F}_N] = \boldsymbol{I}_k$$

and

$$\frac{\mathbb{E}[\|W_{uN}\|_2^4|\mathcal{F}_N]}{N} = o_p(1),\ \frac{\mathbb{E}[\|W_{uN}\|_2^3|\mathcal{F}_N]}{\sqrt{N}} = o_p(1).$$

Furthermore, consider random variables $X_N \in \mathbb{R}^k$ and X_N is measurable with respect to $\mathcal{F}_N$. Then for any measurable function $g_N : \mathbb{R}^k \mapsto \mathbb{R}^d$ and any Lipschitz function $f : \mathbb{R}^{d+k} \to \mathbb{R}$ such that $\|f\|_{\mathrm{BL}} \leq 1$, we have

$$\mathbb{E}[f(g_N(X_N), W_N)|\mathcal{F}_N] - \mathbb{E}[f(g_N(X_N), Z)|\mathcal{F}_N] = o_p(1) \quad \textit{where} \quad Z \sim N(0, \boldsymbol{I}_k).$$

Lemma 15 can be viewed a conditional version of Lemma 4.

H.2 New conditional CMT results

In this section, we extend the classical continuous mapping theorem to conditional convergence. The following lemma is the classical result.

Lemma 16 (Classical CMT). *Let W_N, W be random variables on a metric space S. Suppose a function $g : S \mapsto S'$, where S' is another metric space, has the set of discontinuity points D_g such that $\mathbb{P}[W \in D_g] = 0$. Then if $W_N \xrightarrow{d} W$, the following is true: $g(W_N) \xrightarrow{d} g(W)$.*

To account for the varying sequence and conditioning, we present the following lemmas. These wo results provide new conditional convergence in probability and distribution, respectively.

Lemma 17 (CMT: convergence in conditional probability). *Suppose $X_N, Y_N \in \mathcal{X} \subset \mathbb{R}^k$ and $\|X_N - Y_N\|_2 = o_p(1)$. Furthermore, suppose function $f : \mathcal{X} \to \mathbb{R}$ is uniformly continuous in the support $\mathcal{X}$ and uniformly bounded: $\sup_{x\in\mathcal{X}} |f(x)| < \infty$. Then we have $\mathbb{E}[|f(X_N) - f(Y_N)|] \to 0$. Moreover, we have $\mathbb{E}[|f(X_N) - f(Y_N)|\,|\mathcal{F}_N] \xrightarrow{p} 0$.*

Lemma 18 (CMT: convergence in conditional distribution). *Consider the σ-algebra $\mathcal{F}_N$ and a measurable function g with discontinuity set D_g. Suppose random variable $W \sim \mathbb{P}_W$ satisfies $\mathbb{P}[W \in D_g] = 0$ and $g(W)$ is a continuous random variable. If the sequence of random variable W_N satisfies*

$$|\mathbb{E}[f(W_N)|\mathcal{F}_N] - \mathbb{E}[f(W)]| \xrightarrow{p} 0, \text{ for any } f \text{ such that } \|f\|_{\mathrm{BL}} < \infty,$$

then we have $\sup_{t\in\mathbb{R}} |\mathbb{P}[g(W_N) \le t|\mathcal{F}_N] - \mathbb{P}[g(W) \le t]| \xrightarrow{p} 0$.

H.3 Proof of Lemma 15

Proof of Lemma 15. We prove the result by regular conditional distribution argument. Define the regular conditional distribution $\kappa_{\tilde{W}_N,\mathcal{F}_N}(\omega,\cdot)$ for $\tilde{W}_N = (W_{1N}, \ldots, W_{NN})$ given $\mathcal{F}_N$ such that for almost every $\omega \in \Omega$,

$$\kappa_{\tilde{W}_N,\mathcal{F}_N}(\omega, B) = \mathbb{P}[\tilde{W}_N \in B|\mathcal{F}_N](\omega), \forall B \in \mathcal{B}(\mathbb{R}^{Nk}).$$

Now suppose $(\tilde{W}_{1N}(\omega), \ldots, \tilde{W}_{NN}(\omega))$ is a draw from $\kappa_{\tilde{W}_N,\mathcal{F}_N}(\omega,\cdot)$. Then by Lemma 7 and 11, it suffices to prove for any subsequence N_k of N, there exists a further subsequence N_{k_j} such that for almost every $\omega \in \Omega$

$$A_{N_{k_j}} \equiv \int f\left(g_{N_{k_j}}(X_{N_{k_j}}(\omega)), x\right) \mathrm{d}\kappa_{\tilde{W}_N,\mathcal{F}_N}(\omega, x) - \int f\left(g_{N_{k_j}}(X_{N_{k_j}}(\omega)), Z\right) \mathrm{d}\mathbb{P}_Z \to 0,$$

where $\mathbb{P}_Z$ is the law of Z. The way we choose the subsequence N_{k_j} is to search N_{k_j} such that

$$\frac{\mathbb{E}[\|W_{uN_{k_j}}\|_2^4|\mathcal{F}_{N_{k_j}}](\omega)}{N_{k_j}} \to 0, \ \frac{\mathbb{E}[\|W_{uN_{k_j}}\|_2^3|\mathcal{F}_{N_{k_j}}](\omega)}{N_{k_j}^{1/2}} \to 0, \text{ for almost every } \omega \in \Omega.$$

This is doable guaranteed by the assumption in the lemma. We apply Lemma 4 with $r_{N_{k_j}}(\cdot) = f(g_{N_{k_j}}(X_{N_{k_j}}(\omega)), \cdot)$ and $U_{uN_{k_j}} = \tilde{W}_{uN_{k_j}}(\omega)$ so that we have $A_{N_{k_j}} \to 0$ almost surely. This completes the proof. $\square$

H.4 Proof of Lemma 17

Proof of Lemma 17. Fix any $\delta > 0$, we can choose $\varepsilon(\delta) > 0$ such that whenever $\|X_N - Y_N\|_2 < \varepsilon(\delta)$ we can guarantee $|f(X_N) - f(Y_N)| \le \delta$ since f is uniformly continuous in $\mathcal{X}$. Then for any $\delta > 0$, we have

$$\begin{aligned}\mathbb{E}[|f(X_N) - f(Y_N)|] &= \mathbb{E}\left[|f(X_N) - f(Y_N)|\mathbf{1}(\|X_N - Y_N\|_2 \le \varepsilon(\delta))\right] \\ &\quad + \mathbb{E}\left[|f(X_N) - f(Y_N)|\mathbf{1}(\|X_N - Y_N\|_2 > \varepsilon(\delta))\right] \\ &\le \delta + 2\sup_{x\in\mathcal{X}}|f(x)|\mathbb{P}[\|X_N - Y_N\|_2 > \varepsilon(\delta)].\end{aligned}$$

Letting $N \to \infty$, we know $\mathbb{P}[\|X_N - Y_N\|_2 > \varepsilon(\delta)] \to 0$ so that we have

$$\lim_{N\to\infty} \mathbb{E}[|f(X_N) - f(Y_N)|] \le \delta.$$

Since δ is chosen arbitrarily, we have $\lim_{N\to\infty}\mathbb{E}[|f(X_N) - f(Y_N)|] = 0$ so that we complete the first part proof. By the uniform integrability of $\mathbb{E}[|f(X_N) - f(Y_N)|\mathcal{F}_N]$, guaranteed by the uniform boundedness of f, we conclude the second part of the proof. $\square$

H.5 Proof of Lemma 18

Proof of Lemma 18. Using Lemma 6 and definition of regular conditional distribution, we know

$$\left|\int f(x)\mathrm{d}\kappa_{W_N,\mathcal{F}_N}(\omega, x) - \int f(x)\mathrm{d}\mathbb{P}_W(x)\right| \to 0, \text{ for any } f \text{ such that } \|f\|_{\mathrm{BL}} < \infty$$

for any $\omega \in \mathcal{E} \subset \Omega$, where $\mathbb{P}[\mathcal{E}] \to 1$. Applying Lemma 1 to $\kappa_{W_N,\mathcal{F}_N}(\omega,\cdot), \mathbb{P}_W(\cdot)$, we know on the event $\mathcal{E}$, random variable $\tilde{W}_N$ from measure $\kappa_{W_N,\mathcal{F}_N}(\omega,\cdot)$ converges to measure $\mathbb{P}_W(\cdot)$ in distribution. In other words, we have for any bounded and continuous function h,

$$\mathbb{E}[h(\tilde{W}_N(\omega))] = \int h(x)\mathrm{d}\kappa_{W_N,\mathcal{F}_N}(\omega, x) \to \int h(x)\mathrm{d}\mathbb{P}_W(x) = \mathbb{E}[h(W)],\ \forall\omega \in \mathcal{E}.$$

Then by continuous mapping theorem, we have for any $t \in \mathbb{R}$,

$$\mathbb{P}[g(\tilde{W}_N(\omega)) \le t] \to \mathbb{P}[g(W) \le t],\ \forall\omega \in \mathcal{E} \cap \{W \notin D_g\}.$$

Again applying Lemma 6, we know $\forall\omega \in \mathcal{E} \cap \{W \notin D_g\}$,

$$|\mathbb{P}[g(W_N) \le t|\mathcal{F}_N](\omega) - \mathbb{P}[g(W) \le t]| \to 0.$$

Since $\mathbb{P}[\mathcal{E} \cap \{W \notin D_g\}] = \mathbb{P}[\mathcal{E}] \to 1$ by the assumption that $\mathbb{P}[W \in D_g] = 0$, we have

$$|\mathbb{P}[g(W_N) \le t|\mathcal{F}_N] - \mathbb{P}[g(W) \le t]| \xrightarrow{p} 0.$$

Finally, applying Lemma 8, we have

$$\sup_{t\in\mathbb{R}} |\mathbb{P}[g(W_N) \le t|\mathcal{F}_N] - \mathbb{P}[g(W) \le t]| \xrightarrow{p} 0.$$

$\square$

I Preparation for the proof of Theorem 1

I.1 Auxiliary definitions for the proof of Theorem 1

Recall the definition of c_N and c in Assumption 1. Then consider the function $h(\cdot, c) : \mathbb{R}^{10} \to \bar{\mathbb{R}}$ such that for $x = (x_1, \ldots, x_{10}) \in \mathbb{R}^{10}$,

$$h(x, c) = (\sqrt{x_3/x_5}, -\sqrt{x_4/x_6})A(x_1, x_2)^\top + \frac{1}{\sqrt{2}}c \quad \text{where} \quad A = \begin{pmatrix} x_7 & x_8 \\ x_9 & x_{10} \end{pmatrix}, \tag{27}$$

and $h_N(x) \equiv h(x, c_N)$. Recall the sampling function $\bar{e}(s, x)$, defined as in Assumption 2. Define for any $x \in \mathbb{R}^{10}$,

$$H_N(s, x) \equiv \min\{1 - l_N, \max\{l_N, \bar{e}(s, h_N(x))\}\}$$

and

$$V_N(s, x) \equiv \mathbb{E}[Y_{uN}^2(s)] - H_N(s, x)\mathbb{E}[Y_{uN}(s)]^2.$$

In particular, we define the weight and variance functions:

$$H_N^{(2)}(x) \equiv (H_N(0, x), H_N(1, x)) \quad \text{and} \quad V_N^{(2)}(x) \equiv (V_N(0, x), V_N(1, x))$$

and the covariance function

$$\mathrm{Cov}_N^{(2)}(x) \equiv \frac{-(H_N(0, x)H_N(1, x))^{1/2}}{(V_N(0, x))^{1/2}(V_N(1, x))^{1/2}}\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]. \tag{28}$$

Also, define the matrix function $\mathbf{\Sigma}_N^{(2)}(x) \equiv (\mathrm{Cov}_N^{(2)}(x))_{2\times 2}$.

I.2 Generic lemmas on the random functions

Lemma 19 (Asymptotic lower and upper bound of $V_N^{(t)}(s)$ and $V_N(s, x)$)**.** *Suppose the Assumption 1-2 hold and either Assumption 3 or Assumption 4 holds. Then for any $t = 1, 2$ and $s = 0, 1$, we have*

$$0 < \liminf_{N\to\infty} V_N^{(t)}(s) \le \limsup_{N\to\infty} V_N^{(t)}(s) < \infty,$$

and

$$0 < \liminf_{N\to\infty} \inf_{x\in\mathbb{R}^{10}} V_N(s, x) \le \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^{10}} V_N(s, x) < \infty.$$

Similarly, $V^{(t)}(s)$ is uniformly lower and upper bounded for $s = 0, 1$ and $t = 1, 2$.

Lemma 20 (Lipschitz property of weight, variance and covariance matrix function)**.** *Suppose the Assumption 1-2 hold and either Assumption 3 or Assumption 4 holds. Then there exist universal constants C_1, C_2, C_3, C_4 such that for any $x_1, x_2 \in \mathbb{R}^{10}$*

$$\begin{aligned}
|H_N(s, x_1) - H_N(s, x_2)| &\le C_1|\bar{e}(s, h_N(x_1)) - \bar{e}(s, h_N(x_2))| \\
|H_N^{1/2}(s, x_1) - H_N^{1/2}(s, x_2)| &\le C_2|\bar{e}^{1/2}(s, h_N(x_1)) - \bar{e}^{1/2}(s, h_N(x_2))| \\
|V_N(s, x_1) - V_N(s, x_2)| &\le C_3|\bar{e}(s, h_N(x_1)) - \bar{e}(s, h_N(x_2))| \\
\|(\mathbf{\Sigma}_N^{(2)}(x_1))^{1/2} - (\mathbf{\Sigma}_N^{(2)}(x_2))^{1/2}\|_{\mathrm{F}} &\le C_4 \sum_{s=0,1} |\bar{e}^{1/2}(s, h_N(x_1)) - \bar{e}^{1/2}(s, h_N(x_2))|^{1/2}.
\end{aligned}$$

I.3 Proof of Lemma 19

Proof of Lemma 19. We first prove the result for $V_N^{(t)}(s)$. We divide the proof into two parts.

Proof of $\liminf_{N\to\infty} V_N^{(t)}(s) > 0$: For any s, t, guaranteed by Assumption 1,

$$\begin{aligned}\liminf_{N\to\infty} V_N^{(t)}(s) &= \liminf_{N\to\infty} \left(\mathbb{E}[Y_{uN}^2(s)] - \mathbb{E}[Y_{uN}(s)]^2 + (1 - \bar{e}_N(s, \mathcal{H}_{t-1}))\mathbb{E}[Y_{uN}(s)]^2\right) \\ &\geq \liminf_{N\to\infty} \left(\mathbb{E}[Y_{uN}^2(s)] - \mathbb{E}[Y_{uN}(s)]^2\right) > 0.\end{aligned}$$

Thus we have $\liminf_{N\to\infty} V_N^{(t)}(s) > 0$ for any s, t.

Proof of $\limsup_{N\to\infty} V_N^{(t)}(s) < \infty$: We bound

$$\limsup_{N\to\infty} V_N^{(t)}(s) \leq \limsup_{N\to\infty} \mathbb{E}[Y_{uN}^2(s)] \leq \limsup_{N\to\infty} (\mathbb{E}[Y_{uN}^4(s)])^{1/2} < \infty$$

where the last inequality is due to Assumption 1. The proof for $V_N(s, x)$ is similar so that we omit it. □

I.4 Proof of Lemma 20

Proof of Lemma 20. We prove the claim subsequently.

Proof for $H_N(s, x)$: Write $H_N(s, x) = \min\{1 - l_N, \max\{l_N, \bar{e}(s, h_N(x))\}\}$. We first notice that $\min\{1-l_N, \max\{l_N, x\}\}$ is a Lipschitz function of x with Lipschitz constant 1 so that we have

$$|H_N(s, x_1) - H_N(s, x_2)| \leq |\bar{e}(s, h_N(x_1)) - \bar{e}(s, h_N(x_2))|.$$

Proof for $H_N^{1/2}(s, x)$: Since

$$\sqrt{\min\{1 - l_N, \max\{l_N, x\}\}} = \min\{\sqrt{1 - l_N}, \max\{\sqrt{l_N}, \sqrt{x}\}\}$$

is a Lipschitz function of $\sqrt{x}$, we can conclude

$$|H_N^{1/2}(s, x_1) - H_N^{1/2}(s, x_2)| \leq |\bar{e}^{1/2}(s, h_N(x_1)) - \bar{e}^{1/2}(s, h_N(x_2))|.$$

Proof for $V_N(s, x)$: Recall the definition $V_N(s, x_1) = \mathbb{E}[Y_{uN}^2(s)] - H_N(s, x_1)\mathbb{E}[Y_{uN}(s)]^2$ so that we can bound

$$\begin{aligned}|V_N(s, x_1) - V_N(s, x_2)| &\leq \mathbb{E}[Y_{uN}(s)]^2 |H_N(s, x_1) - H_N(s, x_2)| \\ &\leq \mathbb{E}[Y_{uN}(s)]^2 |\bar{e}(s, h_N(x_1)) - \bar{e}(s, h_N(x_2))|\end{aligned}$$

where the last inequality is true due to the result already proved for $H_N(s, x)$. Since $\mathbb{E}[Y_{uN}(s)]^2$ is uniformly bounded by Assumption 1, we complete the proof for $V_N(s, x)$.

Proof for $(\mathbf{\Sigma}_N^{(2)}(x))^{1/2}$: By Lemma 13, we have

$$\|(\mathbf{\Sigma}_N^{(2)}(x_1))^{1/2} - (\mathbf{\Sigma}_N^{(2)}(x_2))^{1/2}\|_{\mathrm{F}} \leq \sqrt{2}\|\mathbf{\Sigma}_N^{(2)}(x_1) - \mathbf{\Sigma}_N^{(2)}(x_2)\|_{\mathrm{F}}^{1/2}.$$

Define $V_N(x) \equiv V_N(0,x)V_N(1,x)$. Then recalling the definition of $\mathbf{\Sigma}_N^{(2)}$, we notice

$$\begin{aligned}
&\|\mathbf{\Sigma}_N^{(2)}(x_1) - \mathbf{\Sigma}_N^{(2)}(x_2)\|_{\mathrm{F}} \\
&= \sqrt{2}|\mathrm{Cov}_N^{(2)}(x_1) - \mathrm{Cov}_N^{(2)}(x_2)| \\
&\leq \sqrt{2}(H_N(0,x_1)H_N(1,x_1))^{1/2}|\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]| \frac{\left|V_N^{1/2}(x_1) - V_N^{1/2}(x_2)\right|}{V_N^{1/2}(x_1)V_N^{1/2}(x_2)} \\
&\quad + \sqrt{2}\left|(H_N(0,x_1)H_N(1,x_1))^{1/2} - (H_N(0,x_2)H_N(1,x_2))^{1/2}\right| \frac{|\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]|}{V_N^{1/2}(x_2)} \\
&\equiv C_{N,1} + C_{N,2}.
\end{aligned}$$

We bound $C_{N,1}$ and $C_{N,2}$ separately.

1. **Treatment for $C_{N,1}$.** Notice $V_N(0,x_1)$ is uniformly lower and upper bounded, proved as in Lemma 19. Then we denote the uniform lower and upper bounds respectively as c_v, C_v, i.e.,

   $$c_v \leq \liminf_{N\to\infty} \inf_x V_N(s,x) \leq \limsup_{N\to\infty} \sup_x V_N(s,x) \leq C_v \qquad \text{for any } s = 0,1.$$

   Then we have

   $$\begin{aligned}
   \frac{\left|V_N^{1/2}(x_1) - V_N^{1/2}(x_2)\right|}{V_N^{1/2}(x_1)V_N^{1/2}(x_2)} &\leq \frac{|V_N(x_1) - V_N(x_2)|}{c_v^2(V_N^{1/2}(x_1) + V_N^{1/2}(x_2))} \\
   &\leq \frac{|V_N(x_1) - V_N(x_2)|}{2c_v^3} \\
   &\leq \frac{C_v}{2c_v^3}(|V_N(0,x_1) - V_N(0,x_2)| + |V_N(1,x_1) - V_N(1,x_2)|) \\
   &\leq \frac{C_v}{2c_v^3}(\mathbb{E}[Y_{uN}(s)])^2 \sum_{s=0,1} |\bar{e}(s,h_N(x_1)) - \bar{e}(s,h_N(x_2))|
   \end{aligned}$$

   where the last inequality is due to the result already proved for $V_N(s,x)$. Therefore, by the bound $H_N(s,x_1) \leq 1$, we have

   $$\begin{aligned}
   |C_{N,1}| &\lesssim \sum_{s=0,1} |\bar{e}(s,h_N(x_1)) - \bar{e}(s,h_N(x_2))| \\
   &= \sum_{s=0,1} |\bar{e}^{1/2}(s,h_N(x_1)) - \bar{e}^{1/2}(s,h_N(x_2))||\bar{e}^{1/2}(s,h_N(x_1)) + \bar{e}^{1/2}(s,h_N(x_2))| \\
   &\leq 2\sum_{s=0,1} |\bar{e}^{1/2}(s,h_N(x_1)) - \bar{e}^{1/2}(s,h_N(x_2))|.
   \end{aligned}$$

2. **Treatment for $C_{N,2}$.** Since $\mathbb{E}[Y_{uN}(s)]$ and $V_N(x)$ are uniformly lower and upper bounded, we have

   $$C_{N,2} \lesssim \left|(H_N(0,x_1)H_N(1,x_1))^{1/2} - (H_N(0,x_2)H_N(1,x_2))^{1/2}\right|$$

$$\leq \sum_{s=0,1} |H_N^{1/2}(s, x_1) - H_N^{1/2}(s, x_2)|$$
$$\leq \sum_{s=0,1} |\bar{e}^{1/2}(s, h_N(x_1)) - \bar{e}^{1/2}(s, h_N(x_2))|$$

We conclude the proof. □

J Proof of Theorem 1

The organization of this section is as follows. We first present the general proof roadmap in Appendix J.1. Then we work out the first two steps involved in the general proof roadmap under different weighting schemes in Appendix J.2 and Appendix J.3, respectively.

J.1 General proof roadmap for weak convergence result

Before presenting the general proof roadmap, we first define the following notations.

Random variables in the limiting distributions. We first recall the following definitions.

$$V^{(t)}(s) = \lim_{N\to\infty} \mathbb{E}[Y_{uN}^2(s)] - H^{(t)}(s) \lim_{N\to\infty} \mathbb{E}[Y_{uN}(s)]^2 \tag{29}$$

and

$$H^{(1)}(s) = e(s) \quad \text{and} \quad H^{(2)}(s) = \begin{cases} \max\{\bar{l}, \bar{e}(s, S((A^{(1)}, V^{(1)}), c))\} & \text{under Assumption 3} \\ \bar{e}(s, S((A^{(1)}, V^{(1)}), c)) & \text{under Assumption 4} \end{cases}. \tag{30}$$

Define $A^{(t)} \equiv (A^{(t)}(0), A^{(t)}(1)), V^{(t)} \equiv (V^{(t)}(0), V^{(t)}(1))$ and $H^{(t)} \equiv (H^{(t)}(0), H^{(t)}(1))$. Furthermore, recall the asymptotic covariance matrix $\mathbf{\Sigma}^{(t)} \equiv (\mathrm{Cov}^{(t)})_{2\times 2}$, where the covariance is defined as in Appendix A. We will just denote $\mathrm{Cov}^{(2)} = \mathrm{Cov}^{(2)}(A_1)$ for simplicity.

Random variables related to observed data. For the ease of presentation, we rewrite the weight vector as

$$H_N^{(t)} = (H_N^{(t)}(0), H_N^{(t)}(1)),\ H_N^{(1)}(s) = e(s),\ H_N^{(2)}(s) \equiv H_N(s, E_N^{(1)}) = \bar{e}_N(s, \mathcal{H}_{t-1})$$

and the variance vector as

$$V_N^{(t)} \equiv (V_N^{(t)}(0), V_N^{(t)}(1)) \quad \text{where} \quad V_N^{(t)}(s) \equiv \mathbb{E}[Y_{uN}^2(s)] - H_N^{(t)}(s)\mathbb{E}[Y_{uN}(s)]^2.$$

Also, to slightly abuse the notation, we define

$$\Lambda_N^{(t)} \equiv (\Lambda_N^{(t)}(0), \Lambda_N^{(t)}(1)) \quad \text{where} \quad \Lambda_N^{(t)}(s) \equiv \frac{(H_N^{(t)}(s))^{1/2}}{(V_N^{(t)}(s))^{1/2}} \frac{1}{N_t^{1/2}} \sum_{u=1}^{N_t} (\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)]).$$

Define the matrix $\boldsymbol{\Sigma}_N^{(t)} \equiv (\mathrm{Cov}_N^{(t)})_{2\times 2}$, where the covariance is defined as

$$\mathrm{Cov}_N^{(t)} \equiv \frac{-(H_N^{(t)}(0)H_N^{(t)}(1))^{1/2}}{(V_N^{(t)}(0))^{1/2}(V_N^{(t)}(1))^{1/2}}\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]. \tag{31}$$

Random vector joining two stages. Recalling the definitions of $H^{(t)}(s)$ and $V^{(t)}(s)$ in (30) and (29), we define the limiting vectors

$$\mathbb{W} \equiv (W_1, W_2) \quad \text{where} \quad W_t \equiv (Z_t, V^{(t)}, H^{(t)}, \mathrm{Vec}((\boldsymbol{\Sigma}^{(t)})^{1/2})). \tag{32}$$

where Z_1, Z_2 are independently generated from $N(\mathbf{0}, \boldsymbol{I}_2)$. We use $\mathrm{Vec}(\boldsymbol{\Sigma})$ to denote the vectorization of matrix $\boldsymbol{\Sigma}$ and the order is by column. Now we define the empirical counterparts for W_t:

$$E_N^{(t)} \equiv \left((\boldsymbol{\Sigma}_N^{(t)})^{-1/2}\Lambda_N^{(t)}, V_N^{(t)}, H_N^{(t)}, \mathrm{Vec}((\boldsymbol{\Sigma}_N^{(t)})^{1/2})\right), \ t = 1, 2. \tag{33}$$

It is easy to see that $S_N^{(1)}(0) - S_N^{(1)}(1) = h_N(E_N^{(1)})$, recalling the definition of h_N as in (27). In fact,

$$S_N^{(1)}(0) - S_N^{(1)}(1) = \Lambda_N^{(1)}(0) \cdot \frac{(V_N^{(1)}(0))^{1/2}}{(H_N^{(1)}(0))^{1/2}} - \Lambda_N^{(1)}(1) \cdot \frac{(V_N^{(1)}(1))^{1/2}}{(H_N^{(1)}(1))^{1/2}} + \frac{1}{\sqrt{2}}c_N = h_N(E_N^{(1)}).$$

Next, we define an intermediate **auxiliary** random vector joining the limiting random vector M_2 and empirical random vector $E_N^{(2)}$, with Z_2 considered in (32):

$$E_N^{\mathbf{a}}(x) \equiv (Z_2, V_N^{(2)}(x), H_N^{(2)}(x), \mathrm{Vec}((\boldsymbol{\Sigma}_N^{(2)}(x))^{1/2})), \ \forall x \in \mathbb{R}^{10}. \tag{34}$$

Recall $V_N^{(2)}(x), H_N^{(2)}(x)$ and $\boldsymbol{\Sigma}_N^{(2)}(x)$ are defined as in Section I.1.

Proofs for both weighting choices in Theorem 1 follow from the same proof roadmap. Now we present this general proof roadmap.

Proof roadmap. First, the following lemma shows the consistency of $\mathrm{WIPW}(s)$ and $\mathrm{WIPWS}(s)$.

Lemma 21 (Consistency of WIPW and WIPWS)**.** *Under Assumption 1-2 and either Assumption 3 or Assumption 4, we have*

$$\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)] \xrightarrow{p} 0 \quad \textit{and} \quad \mathrm{WIPWS}(s) - \mathbb{E}[Y_{uN}^2(s)] \xrightarrow{p} 0,$$

where we recall the estimators $\mathrm{WIPW}(s)$ *and* $\mathrm{WIPWS}(s)$ *as in* (2) *and* (16)*.*

The proof of Lemma 21 can be found in Appendix K.1.1. Now we prove the weak convergence. We summarize the roadmap as follows:

Step 1: We first prove that for $t = 1, 2$

$$R_V^{(t)}(s) \equiv \frac{\sum_{u=1}^{N_t}(\hat{\Lambda}_{uN}^{(t)}(s) - \mathrm{WIPW}(s))^2}{\sum_{u=1}^{N_t}(\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)])^2} \xrightarrow{p} 1$$

and

$$S_V^{(t)}(s) \equiv \frac{H_N^{(t)}(s)}{N_t V_N^{(t)}(s)} \sum_{u=1}^{N_t}(\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)])^2 \xrightarrow{p} 1.$$

Step 2: Prove $(E_N^{(1)}, E_N^{(2)}) \xrightarrow{d} \mathbb{W}$ where $E_N^{(t)}$ is defined as in (33) and $\mathbb{W}$ is defined as in (32). This step involves the analysis on the different choice of $h_N^{(t)}(s)$ for Assumption 3 and 4;

Step 3: We define

$$\hat{I}_N \equiv W_N - \frac{\sqrt{N}(\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])}{(N\hat{V}_N(0) + N\hat{V}_N(1))^{1/2}} \quad \text{and} \quad \hat{I}_U \equiv \sqrt{N}(T_N - (\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])).$$

In order to find the asymptotic distribution of $\hat{I}_N, \hat{I}_U$, we can rewrite $\hat{I}_N, \hat{I}_U$ with different weighting methods as a function of the weak limit of $(E_N^{(1)}, E_N^{(2)})$ and apply Slutsky's Lemma and the continuous mapping theorem. We consider the following four cases:

1. W_N **with constant weighting:** we can write $\hat{I}_N$ with constant weighting as
$$\frac{\sum_{t=1}^2 \Lambda_N^{(t)}(0)(V_N^{(t)}(0)/H_N^{(t)}(0))^{1/2} - \sum_{t=1}^2 \Lambda_N^{(t)}(1)(V_N^{(t)}(1)/H_N^{(t)}(1))^{1/2}}{(\sum_{t=1}^2 V_N^{(t)}(0)S_V^{(t)}(0)R_V^{(t)}(0)/H_N^{(t)}(0) + \sum_{t=1}^2 V_N^{(t)}(1)S_V^{(t)}(1)R_V^{(t)}(1)/H_N^{(t)}(1))^{1/2}};$$
2. W_N **with adaptive weighting:** we can write $\hat{I}_N$ with adaptive weighting as
$$\frac{R_N(0)\sum_{t=1}^2 \Lambda_N^{(t)}(0)(V_N^{(t)}(0))^{1/2} - R_N(1)\sum_{t=1}^2 \Lambda_N^{(t)}(1)(V_N^{(t)}(1))^{1/2}}{(R_N^2(0)\sum_{t=1}^2 V_N^{(t)}(0)S_V^{(t)}(0)R_V^{(t)}(0) + R_N^2(1)\sum_{t=1}^2 V_N^{(t)}(1)S_V^{(t)}(1)R_V^{(t)}(1))^{1/2}},$$
where $R_N^{-1}(s) \equiv \sum_{t=1}^2 (H_N^{(t)}(s))^{1/2}$;
3. T_N **with constant weighting:** we can write $\hat{I}_U$ with constant weighting as
$$\frac{1}{\sqrt{2}}\left(\sum_{t=1}^2 \Lambda_N^{(t)}(0)(V_N^{(t)}(0)/H_N^{(t)}(0))^{1/2} - \sum_{t=1}^2 \Lambda_N^{(t)}(1)(V_N^{(t)}(1)/H_N^{(t)}(1))^{1/2}\right);$$
4. T_N **with adaptive weighting:** we can write $\hat{I}_U$ with adaptive weighting as
$$\sqrt{2}\left(R_N(0)\sum_{t=1}^2 \Lambda_N^{(t)}(0)(V_N^{(t)}(0))^{1/2} - R_N(1)\sum_{t=1}^2 \Lambda_N^{(t)}(1)(V_N^{(t)}(1))^{1/2}\right).$$

Then we use the results $R_V^{(t)}(s), S_V^{(t)}(s) \xrightarrow{p} 1, t = 1, 2$ as well as the weak convergence of $(E_N^{(1)}, E_N^{(2)})$ to derive the weak convergence with the help of Slutsky's Lemma and continuous mapping theorem.

To see this, we will only work out the case for W_N with constant weighting since the proof for the other cases are similar and even simpler. We define the random vector $(\bar{E}_N^{(1)}, \bar{E}_N^{(2)})$ and $(\check{E}_N^{(1)}, \check{E}_N^{(2)})$, where $\bar{E}_N^{(t)} \equiv (E_N^{(t)}, S_V^{(t)}, R_V^{(t)})$, $\check{E}_N^{(t)} \equiv (E_N^{(t)}, \mathbf{1}_2, \mathbf{1}_2)$ and

$$S_V^{(t)} \equiv (S_V^{(t)}(0), S_V^{(t)}(1)),\ R_V^{(t)} \equiv (R_V^{(t)}(0), R_V^{(t)}(1)),\ \mathbf{1}_2 \equiv (1, 1).$$

Then we apply Lemma 9 with $X_N = (\check{E}_N^{(1)}, \check{E}_N^{(2)})$ and $Y_N = (\bar{E}_N^{(1)}, \bar{E}_N^{(2)})$ by noticing that X_N converge weakly to $(W_1, \mathbf{1}_2, \mathbf{1}_2, W_2, \mathbf{1}_2, \mathbf{1}_2)$ and $R_V^{(t)}, S_V^{(t)} \xrightarrow{p} \mathbf{1}_2$ for $t = 1, 2$. Therefore we obtain $Y_N \xrightarrow{d} (W_1, \mathbf{1}_2, \mathbf{1}_2, W_2, \mathbf{1}_2, \mathbf{1}_2)$. Then we can use continuous mapping lemma (Lemma 16) to derive the weak convergence of $\hat{I}_N$ by writing $\hat{I}_N$ as a continuous function of Y_N.

J.2 Proof of Step 1 in Appendix J.1

We will show the convergence of $R_V^{(t)}(s)$ and $S_V^{(t)}(s)$ for $s = 0, 1$ with different choice of $h_N^{(t)}(s)$.

J.2.1 Proof of convergence of $R_V^{(t)}(s)$

To first show the convergence of $R_V^{(t)}(s)$, we give the following lemma.

Lemma 22 (Convergence of $R_V^{(t)}(s)$)**.** *Suppose the Assumption 1-2 hold and either Assumption 4 or Assumption 3 holds. If the following statements are true:*

1. $\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)] = o_p(1)$ *for any* $s \in \{0, 1\}$*;*
2. $W_N^{(t)}(s) \equiv \sum_{u=1}^{N_t} \bar{e}_N(s, \mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)} - \mathbb{E}[Y_{uN}(s)])^2/N_t$ *is asymptotically lower bounded;*

then we have for any s, t, $R_V^{(t)}(s) \xrightarrow{p} 1$.

Since the consistency has been proved in Lemma 21, it suffices to prove $W_N^{(t)}(s)$ is stochastically lower bounded. We first present a useful lemma.

Lemma 23 (Asymptotic representation of $W_N^{(t)}(s)$)**.** *Suppose the Assumption 1-2 hold and either Assumption 4 or Assumption 3 holds. Then we have*

$$W_N^{(t)}(s) = \mathbb{E}[Y_{uN}^2(s)] - \bar{e}_N(s, \mathcal{H}_{t-1})(\mathbb{E}[Y_{uN}(s)])^2 + o_p(1).$$

By Lemma 23, we know $W_N^{(t)}(s) = \mathbb{E}[Y_{uN}^2(s)] - \bar{e}_N(s, \mathcal{H}_{t-1})(\mathbb{E}[Y_{uN}(s)])^2 + o_p(1)$. Since Assumption 1 guarantees that

$$\liminf_{N\to\infty}(\mathbb{E}[Y_{uN}^2(s)] - \bar{e}_N(s, \mathcal{H}_{t-1})(\mathbb{E}[Y_{uN}(s)])^2) \geq \liminf_{N\to\infty}(\mathbb{E}[Y_{uN}^2(s)] - (\mathbb{E}[Y_{uN}(s)])^2) > 0,$$

we know $W_N^{(t)}(s)$ is asymptotically lower bounded.

J.2.2 Proof of convergence of $S_V^{(t)}(s)$

We write

$$\begin{aligned}
S_V^{(t)}(s) &= \frac{H_N^{(t)}(s)}{N_t V_N^{(t)}(s)} \sum_{u=1}^{N_t} (\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)])^2 \\
&= \frac{1}{N_t}\bar{e}_N(s, \mathcal{H}_{t-1}) \sum_{u=1}^{N_t} (\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)])^2 \cdot \frac{1}{V_N^{(t)}(s)} \\
&= \frac{W_N^{(t)}(s)}{V_N^{(t)}(s)}.
\end{aligned}$$

Then we know from Lemma 23 that

$$W_N^{(t)}(s) - \left(\mathbb{E}[Y_{uN}^2(s)] - \bar{e}_N(s, \mathcal{H}_{t-1})\mathbb{E}[Y_{uN}(s)]^2\right) = W_N^{(t)}(s) - V_N^{(t)}(s) \xrightarrow{p} 0.$$

By Lemma 19, we know $\liminf_{N\to\infty} V_N^{(t)}(s) > 0$. Therefore we have

$$S_V^{(t)}(s) = \frac{W_N^{(t)}(s)}{V_N^{(t)}(s)} \xrightarrow{p} 1.$$

J.3 Proof of Step 2 in Appendix J.1

To proceed with the proof of Step 2, we will show that $(E_N^{(1)}, E_N^{(2)}) \xrightarrow{d} \mathbb{W}$. It suffices to show that for any bounded Lipschitz function f,

$$\mathbb{E}\left[f(E_N^{(1)}, E_N^{(2)})\right] \to \mathbb{E}\left[f(\mathbb{W})\right] \tag{35}$$

Without loss of generality, we assume

$$\|f\|_\infty \leq \frac{1}{2}, \sup_{x,y\in\mathbb{R}^{20}} |f(x) - f(y)| \leq 1, \sup_{x,y\in\mathbb{R}^{20}} \frac{|f(x) - f(y)|}{\|x - y\|} \leq \frac{1}{2}. \tag{36}$$

We divide the proofs into the following steps

$$\begin{aligned}
\mathbb{E}\left[f(E_N^{(1)}, E_N^{(2)})\right] - \mathbb{E}\left[f(\mathbb{W})\right] &= \mathbb{E}[f(W_1, E_N^{\mathbf{a}}(W_1))] - \mathbb{E}[f(W_1, W_2)] \\
&\quad + \mathbb{E}[f(E_N^{(1)}, E_N^{(2)})] - \mathbb{E}[f(W_1, E_N^{\mathbf{a}}(W_1))] \\
&\equiv F_1 + F_2.
\end{aligned}$$

It suffices to $F_1 = o(1)$ and $F_2 = o(1)$. We will prove these two claims subsequently. Before doing so, we present a useful lemma, characterizing the asymptotic behavior of sampling function in the second stage.

Lemma 24 (Convergence of sampling function)**.** *Suppose the Assumption 1-2 hold and either Assumption 3 or Assumption 4 holds. Then if a sequence of random variable M_N satisfying*

$$M_N \to W_1 \text{ almost surely}$$

where W_1 is defined as in (32), then we have

$$\bar{e}(s, h_N(M_N)) \to \bar{e}(s, h(W_1, c)) \quad \text{and} \quad \bar{e}^{1/2}(s, h_N(M_N)) \to \bar{e}^{1/2}(s, h(W_1, c))$$

almost surely. Consequently, we know

$$\bar{e}(s, h_N(M_N)) - \bar{e}(s, h_N(W_1)) \quad \text{and} \quad \bar{e}^{1/2}(s, h_N(M_N)) - \bar{e}^{1/2}(s, h_N(W_1))$$

coverge to 0 almost surely.

J.3.1 Proof of $F_1 = o(1)$

To prove $F_1 = o(1)$, by the boundedness and Lipschitz property of f and Lemma 17, it suffices to show the following quantities are $o(1)$:

$$A_N \equiv \|H_N^{(2)}(W_1) - H^{(2)}\|_2, \;\; B_N \equiv \|V_N^{(2)}(W_1) - V^{(2)}\|_2$$

and

$$C_N \equiv \|\mathrm{Vec}((\boldsymbol{\Sigma}_N^{(2)}(W_1))^{1/2}) - \mathrm{Vec}((\boldsymbol{\Sigma}^{(2)})^{1/2})\|_2.$$

We prove these claims subsequently.

Proof of $A_N = o(1)$: Recall the definition of $H_N^{(2)}(W_1)$

$$H_N^{(2)}(W_1) = (H_N(0, W_1), H_N(1, W_1))$$

where $H_N(s, W_1) = \min\{1 - l_N, \max\{l_N, \bar{e}(s, h_N(W_1))\}\}$. It suffices to prove for any $s \in \{0, 1\}$

$$|H_N(s, W_1) - H^{(2)}(s)| \to 0 \text{ almost surely.}$$

We will prove the result depending on the choice of l_N under either Assumption 3 or 4 .

1. **Under Assumption 3:** In this case, $0 < c_l < \bar{l} = l_N < c_u < 1/2$. By the Lipschitz property of $\min\{1 - \bar{l}, \max\{\bar{l}, x\}\}$ in x, we have

   $$|H_N(s, W_1) - H^{(2)}(s)| \leq |\bar{e}(s, h_N(W_1)) - \bar{e}(s, h(W_1, c))|.$$

   Convergence in RHS has been proved in Lemma 24.

2. **Under Assumption 4:** In this case $\lim_{N\to\infty} l_N = 0$. We can bound by Lemma 24 that

   $$\begin{aligned}
   |H_N(s, W_1) - H^{(2)}(s)| &= |\min\{1 - l_N, \max\{l_N, \bar{e}(s, h_N(W_1))\}\} - \bar{e}(s, h(W_1, c))| \\
   &\leq |\bar{e}(s, h_N(W_1)) - \bar{e}(s, h(W_1, c))| \\
   &\quad\quad + |l_N - \bar{e}(s, h(W_1, c))| \mathbb{1}(\bar{e}(s, h_N(W_1)) < l_N) \\
   &\quad\quad + |1 - l_N - \bar{e}(s, h(W_1, c))| \mathbb{1}(\bar{e}(s, h_N(W_1)) > 1 - l_N) \\
   &\leq 3|\bar{e}(s, h_N(W_1)) - \bar{e}(s, h(W_1, c))| + 2l_N \to 0
   \end{aligned}$$

   almost surely. Thus we have proved $A_N = o(1)$.

Proof of $B_N = o(1)$: For B_N, recall the expression

$$V_N^{(2)}(W_1) = (V_N(0, W_1), V_N(1, W_1)) \quad \text{and} \quad V_N(s, W_1) = \mathbb{E}[Y_{uN}^2(s)] - H_N(s, W_1)\mathbb{E}[Y_{uN}(s)]^2.$$

Notice it suffices to prove $H_N(s, W_1)$ converges to $H^{(2)}(s)$ and this can be implied by the convergence of $H_N(s, W_1)$ as shown in the proof of $A_N = o(1)$. Thus we have proved $B_N = o(1)$.

Proof of $C_N = o(1)$: We notice that by Lemma 13,

$$C_N \lesssim \|\mathbf{\Sigma}_N^{(2)}(W_1) - \mathbf{\Sigma}^{(2)}\|_{\mathrm{F}}^{1/2} = \sqrt{2}|\mathrm{Cov}_N^{(2)}(W_1) - \mathrm{Cov}^{(2)}|^{1/4}.$$

Recall the definition

$$\mathrm{Cov}_N^{(2)}(W_1) = \frac{-(H_N(0, W_1)H_N(1, W_1))^{1/2}}{V_N^{1/2}(0, W_1)V_N^{1/2}(1, W_1)} \mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]$$

and

$$\mathrm{Cov}^{(2)} = -\frac{(H^{(t)}(0)H^{(t)}(1))^{1/2}}{(V^{(t)}(0)V^{(t)}(1))^{1/2}} \lim_{N\to\infty} (\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]).$$

It suffices to prove the following claims:

1. $|H_N^{1/2}(s, W_1) - (H^{(2)}(s))^{1/2}| = o(1)$;

2. $V_N(s, W_1)$ is uniformly lower bounded, and $|V_N(s, W_1) - V^{(2)}(s)| \overset{a.s.}{\to} 0$.

First we show $|H_N^{1/2}(s, W_1) - (H^{(2)}(s))^{1/2}| = o(1)$ for any $s \in \{0, 1\}$.

1. **Under Assumption 3:** We can easily obtain

$$|H_N^{1/2}(s, W_1) - (H^{(2)}(s))^{1/2}| \leq \frac{1}{2\bar{l}^{1/2}} |\bar{e}(s, h_N(W_1)) - \bar{e}(s, h(W_1, c))| \to 0$$

   by Lemma 24;

2. **Under Assumption 4:** we decompose

$$\begin{aligned}
&|H_N^{1/2}(s, W_1) - (H^{(2)}(s))^{1/2}| \\
&= |\min\{(1 - l_N)^{1/2}, \max\{l_N^{1/2}, \bar{e}^{1/2}(s, h_N(W_1))\}\} - \bar{e}^{1/2}(s, h(W_1, c))| \\
&\leq |\min\{(1 - l_N)^{1/2}, \max\{l_N^{1/2}, \bar{e}^{1/2}(s, h_N(W_1))\}\} - \bar{e}^{1/2}(s, h_N(W_1))| \\
&\qquad + |\bar{e}^{1/2}(s, h_N(W_1)) - \bar{e}^{1/2}(s, h(W_1, c))| \\
&\leq l_N^{1/2} + l_N + |\bar{e}^{1/2}(s, h_N(W_1)) - \bar{e}^{1/2}(s, h(W_1, c))| \to 0
\end{aligned}$$

   almost surely by Lemma 24.

Then, we only need to prove that $V_N(s, W_1)$ is uniformly lower bounded, and

$$|V_N(s, W_1) - V^{(2)}(s)| \overset{a.s.}{\to} 0. \tag{37}$$

For the lower bound, by Lemma 19, we know $\liminf_{N\to\infty} V_N(s, W_1) > 0$. For (37), this can be implied by the convergence $B_N = o(1)$. Thus we have proved $C_N = o(1)$ and thus we have $F_1 = o(1)$.

J.3.2 Proof of $F_2 = o(1)$

We further divide the proof into two steps. Define

$$\begin{aligned}
A_N(f) &\equiv \mathbb{E}[f(E_N^{(1)}, E_N^{(2)})] - \mathbb{E}[f(E_N^{(1)}, E_N^{\mathbf{a}}(E_N^{(1)}))], \\
B_N(f) &\equiv \mathbb{E}[f(E_N^{(1)}, E_N^{\mathbf{a}}(E_N^{(1)}))] - \mathbb{E}[f(A_1, E_N^{\mathbf{a}}(A_1))].
\end{aligned}$$

Noticing $F_2 = A_N(f) + B_N(f)$, it suffices to prove $A_N(f) = o(1)$, $B_N(f) = o(1)$.

Proof of $\lim_{N\to\infty} |A_N(f)| = 0$ We first present a useful lemma.

Lemma 25. *Suppose the Assumption 1-2 hold and either Assumption 4 or Assumption 3 holds. For any bounded Lipschitz function f satisfying (36), we have*

$$\mathbb{E}\left[f(E_N^{(1)}, E_N^{(2)}) - f(E_N^{(1)}, E_N^{\boldsymbol{a}}(E_N^{(1)})) \mid \mathcal{H}_N^{(1)}\right] = o_p(1)$$

where $\mathcal{H}_N^{(1)}$ is the σ-algebra generated by the data in first batch $(A_{1N}^{(1)}, Y_{1N}^{(1)}), \ldots, (A_{N_1N}^{(1)}, Y_{N_1N}^{(1)})$.

Then by dominated convergence theorem, we know $\lim_{N\to\infty} |A_N(f)| = 0$.

Proof of $\lim_{N\to\infty}|B_N(f)| = 0$: We first show that $E_N^{(1)} \xrightarrow{d} W_1$ in the following lemma.

Lemma 26 (Weak convergence of $E_N^{(1)}$)**.** *Suppose the Assumption 1-2 hold and either Assumption 4 or Assumption 3 holds. Then we have $E_N^{(1)} \xrightarrow{d} W_1$.*

By Skorohod's representation theorem (Lemma 12) and Lemma 26, there exists a sequence of random variables $\tilde{E}_N^{(1)}$ such that $\tilde{E}_N^{(1)} \stackrel{d}{=} W_1$ and $\lim_{N\to\infty}\tilde{E}_N^{(1)} = W_1$. Then we write

$$(\tilde{E}_N^{(1)}, E_N^{\mathbf{a}}(\tilde{E}_N^{(1)})) = (\tilde{E}_N^{(1)}, Z_2, V_N^{(2)}(\tilde{E}_N^{(1)}), H_N^{(2)}(\tilde{E}_N^{(1)}), \mathrm{Vec}((\mathbf{\Sigma}_N^{(2)}(\tilde{E}_N^{(1)}))^{1/2}))$$

and

$$(W_1, E_N^{\mathbf{a}}(W_1)) = (W_1, Z_2, V_N^{(2)}(W_1), H_N^{(2)}(W_1), \mathrm{Vec}((\mathbf{\Sigma}_N^{(2)}(W_1))^{1/2})).$$

We intend to apply Lemma 17 and in order to do so, we need to show, in addition to $\tilde{E}_N^{(1)} \to W_1$, that $\|V_N^{(2)}(\tilde{E}_N^{(1)}) - V_N^{(2)}(W_1)\|_2$, $\|H_N^{(2)}(\tilde{E}_N^{(1)}) - H_N^{(2)}(W_1)\|_2$ and $\|(\mathbf{\Sigma}_N^{(2)}(\tilde{E}_N^{(1)}))^{1/2} - (\mathbf{\Sigma}_N^{(2)}(W_1))^{1/2}\|_{\mathrm{F}}$ converge to 0 almost surely. In fact, by Lemma 20, it suffices to prove

$$\bar{e}(s, h_N(\tilde{E}_N^{(1)})) - \bar{e}(s, h_N(W_1)) = o(1) \quad \text{and} \quad \bar{e}^{1/2}(s, h_N(\tilde{E}_N^{(1)})) - \bar{e}^{1/2}(s, h_N(W_1)) = o(1).$$

Applying Leamm 24 by noticing $\tilde{E}_N^{(1)} \to W_1$ almost surely, we complete the proof for $\lim_{N\to\infty}|B_N(f)| = 0$.

K Proof of lemmas in Appendix J

The organization of this section is as follows. We will prove the lemmas appearing in Appendix J.1 in Appendix K.1. We will prove the lemmas appearing in Appendix J.2 in Appendix K.2. Finally, we will prove the lemmas appearing in Appendix J.3 in Appendix K.3. We will inherit all the notations in Appendix J.

K.1 Proof of lemmas in Appendix J.1

K.1.1 Proof of Lemma 21

Proof of Lemma 21. We will only prove the consistency for $\mathbb{E}[Y_{uN}(s)]$. The proof for the second part is similar. We divide the proof into two cases depending if Assumption 4 or Assumption 3 holds.

1. **Under Assumption 3:** Compute

$$\begin{aligned}
\mathrm{Var}[\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)]] &= \mathbb{E}\left[\frac{1}{N^2}\sum_{t=1}^{2}\sum_{u=1}^{N_t}\mathbb{E}\left\{\left(\frac{\mathbb{1}(A_{uN}^{(t)} = s)}{\bar{e}_N(s, \mathcal{H}_{t-1})}Y_{uN}^{(t)} - \mathbb{E}[Y_{uN}(s)]\right)^2 |\mathcal{H}_1\right\}\right] \\
&= \frac{1}{N^2}\sum_{t=1}^{2}\sum_{u=1}^{N_t}\mathbb{E}\left[\left(\frac{\mathbb{E}[Y_{uN}^2(s)]}{\bar{e}_N(s, \mathcal{H}_{t-1})} - \mathbb{E}[Y_{uN}(s)]^2\right)\right]
\end{aligned}$$

$$\leq \mathbb{E}[Y_{uN}^2(s)]\mathbb{E}\left[\frac{1}{2N\bar{e}_N(s,\mathcal{H}_0)} + \frac{1}{2N\bar{e}_N(s,\mathcal{H}_1)}\right].$$

Thus we know by Assumption 3 that $N\bar{e}_N(s,\mathcal{H}_t) \geq N\min\{e(s),\bar{l}\}$ for any $t = 0,1$. This implies $\mathrm{Var}[\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)]] \to 0$. This implies $\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)] \xrightarrow{p} 0$ since $\mathbb{E}[\mathrm{WIPW}(s)] = \mathbb{E}[Y_{uN}(s)]$.

2. **Under Assumption 4:** We first can show that

$$\begin{aligned}
W_N(s) \equiv \sum_{t=1}^{2}\sum_{u=1}^{N_t} h_N^{(t)}(s) &= \sum_{t=1}^{2} N_t h_N^{(t)}(s) \\
&= \frac{1}{2}(N^{1/2}\bar{e}_N^{1/2}(s,\mathcal{H}_0) + N^{1/2}\bar{e}_N^{1/2}(s,\mathcal{H}_1)) \\
&\geq \frac{1}{2}\left(N^{1/2}l_N^{1/2} + N^{1/2}e^{1/2}(s)\right) \\
&\geq \frac{1}{2}N^{1/2}e^{1/2}(s). \qquad (38)
\end{aligned}$$

Compute

$$\begin{aligned}
&\mathrm{Var}[\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)]] \\
&= \mathbb{E}\left[\frac{\left(\sum_{t=1}^{2}\sum_{u=1}^{N_t} h_N^{(t)}(s)\left(\frac{\mathbb{1}(A_{uN}^{(t)}=s)}{\bar{e}_N(s,\mathcal{H}_{t-1})}Y_{uN}^{(t)} - \mathbb{E}[Y_{uN}(s)]\right)\right)^2}{W_N^2(s)}\right] \\
&\leq \frac{4}{Ne(s)}\mathbb{E}\left[\left(\sum_{t=1}^{2}\sum_{u=1}^{N_t} h_N^{(t)}(s)\left(\frac{\mathbb{1}(A_{uN}^{(t)}=s)}{\bar{e}_N(s,\mathcal{H}_{t-1})}Y_{uN}^{(t)} - \mathbb{E}[Y_{uN}(s)]\right)\right)^2\right] \\
&= \frac{4}{N^2e(s)}\sum_{t=1}^{2}\sum_{u=1}^{N_t}\left(\mathbb{E}[Y_{uN}^2(s)] - \bar{e}_N(s,\mathcal{H}_{t-1})\mathbb{E}[Y_{uN}(s)]^2\right) \\
&\leq \frac{4\mathbb{E}[Y_{uN}^2(s)]}{Ne(s)}.
\end{aligned}$$

Then it suffices to show

$$|\mathbb{E}[\mathrm{WIPW}(s)] - \mathbb{E}[Y_{uN}(s)]| \to 0.$$

In fact, we can compute

$$\begin{aligned}
|\mathbb{E}[\mathrm{WIPW}(s)] - \mathbb{E}[Y_{uN}(s)]| &= \left|\mathbb{E}\left[\frac{h_N^{(1)}(s)\sum_{u=1}^{N_1}(\hat{\Lambda}_{uN}^{(1)}(s) - \mathbb{E}[Y_{uN}(s)])}{W_N(s)}\right]\right| \\
&\leq \mathbb{E}\left[\frac{h_N^{(1)}(s)|\sum_{u=1}^{N_1}(\hat{\Lambda}_{uN}^{(1)}(s) - \mathbb{E}[Y_{uN}(s)])|}{W_N(s)}\right] \\
&\leq \mathbb{E}\left[\frac{|\sum_{u=1}^{N_1}(\hat{\Lambda}_{uN}^{(1)}(s) - \mathbb{E}[Y_{uN}(s)])|}{N_1}\right]
\end{aligned}$$

$$\leq \sqrt{\mathbb{E}\left[\left(\frac{\sum_{u=1}^{N_1}(\hat{\Lambda}_{uN}^{(1)}(s) - \mathbb{E}[Y_{uN}(s)])}{N_1}\right)^2\right]}$$

$$= \sqrt{\frac{\mathbb{E}[Y_{uN}^2(s)] - e(s)(\mathbb{E}[Y_{uN}(s)])^2}{N_1}} \to 0,$$

where the second inequality is due to the lower bound (38) and the third inequality is due to Jensen's inequality. Then we know $\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)] \xrightarrow{p} 0$.

□

K.2 Proof of lemmas in Appendix J.2

K.2.1 Proof of Lemma 22

Proof of Lemma 22. The idea is to decompose $R_V^{(t)}(s)$ into different pieces and show each piece converges to zero. Recall the definition of $W_N^{(t)}(s)$ as in Appendix J.2.1. Then consider

$$\begin{aligned}
R_V^{(t)}(s) &= \frac{\sum_{u=1}^{N_t}(\hat{\Lambda}_{uN}^{(t)}(s) - \mathrm{WIPW}(s))^2}{\sum_{u=1}^{N_t}(\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)])^2} \\
&= \frac{\bar{e}_N(s, \mathcal{H}_{t-1})}{N_t W_N^{(t)}(s)} \sum_{u=1}^{N_t}\left(\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)] + \mathbb{E}[Y_{uN}(s)] - \mathrm{WIPW}(s)\right)^2 \\
&= 1 + \frac{2\left(\mathbb{E}[Y_{uN}(s)] - \mathrm{WIPW}(s)\right)}{W_N^{(t)}(s)} \frac{\bar{e}_N(s, \mathcal{H}_{t-1})}{N_t} \sum_{u=1}^{N_t}\left(\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)]\right) \\
&\quad + \frac{\bar{e}_N(s, \mathcal{H}_{t-1})}{W_N^{(t)}(s)}(\mathbb{E}[Y_{uN}(s)] - \mathrm{WIPW}(s))^2 \\
&\equiv 1 + R_{1N}^{(t)}(s) + R_{2N}^{(t)}(s).
\end{aligned}$$

In order to show $R_{1N}^{(t)}(s) = o_p(1)$ and $R_{2N}^{(t)}(s) = o_p(1)$, it suffices to show

1. $$\frac{\bar{e}_N(s, \mathcal{H}_{t-1})}{N_t} \sum_{u=1}^{N_t}(\hat{\Lambda}_{uN}^{(t)}(s) - \mathbb{E}[Y_{uN}(s)]) = o_p(1); \tag{39}$$

2. $W_N^{(t)}(s)$ is asymptotically lower bounded;

3. $\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)] = o_p(1)$ for any $s \in \{0, 1\}$.

The second and the last claims follow by the assumption so we only need to show the first claim. The intuition behind the validity of (39) is the summand is mean zero and thus we only need to show the variance of the summand converges to 0. To this end, we can compute

$$\mathrm{Var}\left[\bar{e}_N(s, \mathcal{H}_{t-1}) \frac{1}{N_t} \sum_{u=1}^{N_t}(\hat{\Lambda}_{uN,s}^{(t)} - \mathbb{E}[Y_{uN}(s)])\right]$$

$$= \mathbb{E}\left[\mathrm{Var}\left[\frac{1}{N_t}\bar{e}_N(s,\mathcal{H}_{t-1})\sum_{u=1}^{N_t}\left(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)]\right)|\mathcal{H}_{t-1}\right]\right]$$

$$= \mathbb{E}\left[\frac{\bar{e}_N^2(s,\mathcal{H}_{t-1})}{N_t^2}\sum_{u=1}^{N_t}\mathbb{E}\left[\left(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)]\right)^2|\mathcal{H}_{t-1}\right]\right]$$

$$= \frac{1}{N_t}\mathbb{E}\left[\bar{e}_N(s,\mathcal{H}_{t-1})\left(\mathbb{E}\left[Y_{uN}^2(s)\right]-\bar{e}_N(s,\mathcal{H}_{t-1})\mathbb{E}[Y_{uN}(s)]^2\right)\right] \le \frac{1}{N_t}\mathbb{E}[Y_{uN}^2(s)] = o(1).$$

Thus we proved (39). □

K.2.2 Proof of Lemma 23

Proof of Lemma 23. Since the summand in $W_N^{(t)}(s)$ is conditionally independent and identically distributed so we want to apply Lemma 10 to prove the claim. We will verify (24) with $\delta = 1$ and $W_{uN} = \bar{e}_N(s,\mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)])^2$. In other words, we need to verify

$$\mathbb{E}\left[\bar{e}_N^2(s,\mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)])^4|\mathcal{H}_{t-1}\right] = o_p(N).$$

To see this, we can compute

$$\mathbb{E}\left[\bar{e}_N^2(s,\mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)])^4|\mathcal{H}_{t-1}\right] \le \frac{8\mathbb{E}[Y_{uN}^4(s)]}{\bar{e}_N(s,\mathcal{H}_{t-1})} + 8\bar{e}_N^2(s,\mathcal{H}_{t-1})(\mathbb{E}[Y_{uN}(s)])^4.$$

Since $N\bar{e}_N(s,\mathcal{H}_{t-1}) \ge N\min\{e(s),\bar{l}_N\} \to \infty$ in probability by Assumption 4 or 3 and Assumption 1 guarantees that $\mathbb{E}[Y_{uN}(s)], \mathbb{E}[Y_{uN}^4(s)]$ are uniformly bounded, then we know

$$\frac{\mathbb{E}[Y_{uN}^4(s)]}{N\bar{e}_N(s,\mathcal{H}_{t-1})} = o_p(1), \quad \frac{(\mathbb{E}[Y_{uN}(s)])^4}{N} = o(1).$$

Thus we can apply Lemma 10 with $\delta = 1$ such that

$$W_N^{(t)}(s) - \mathbb{E}[\bar{e}_N(s,\mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)])^2|\mathcal{H}_{t-1}] = o_p(1).$$

Thus it suffices to prove $\mathbb{E}[\bar{e}_N(s,\mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)])^2|\mathcal{H}_{t-1}]$ is stochastically lower bounded. We compute

$$\mathbb{E}[\bar{e}_N(s,\mathcal{H}_{t-1})(\hat{\Lambda}_{uN}^{(t)}(s)-\mathbb{E}[Y_{uN}(s)])^2|\mathcal{H}_{t-1}] = \mathbb{E}[Y_{uN}^2(s)] - \bar{e}_N(s,\mathcal{H}_{t-1})(\mathbb{E}[Y_{uN}(s)])^2.$$

The above quantity is lower bounded by $\mathbb{E}[Y_{uN}^2(s)] - (\mathbb{E}[Y_{uN}(s)])^2$, which is asymptotically lower bounded by Assumption 1. Thus we have proved the claim. □

K.3 Proof of lemmas in Appendix J.3

K.3.1 Proof of Lemma 24

Proof of Lemma 24. We notice by the definition of $h(\cdot,c)$ in (27), and the definition of W_1 in definition (32), we know $h_N(M_N) \to h(W_1,c)$, almost surely, using continuous mapping theorem, for any $c \in [-\infty, 0]$. Based on the Assumption 2, we divide the proof into two cases.

1. **When $\bar{e}(s,x)$ is Lipschitz continuous on x.** We note by Lemma 17 that
$$|\bar{e}(s,h_N(M_N)) - \bar{e}(s,h(W_1,c))| \overset{a.s.}{\to} 0$$
is true. Moreover, if a nonnegative function f is Lipschitz continuous and the range is in $[0,1]$, then $f^{1/2}$ is uniformly continuous. This is because $\sqrt{x}$ is a uniformly continuous function in the compact support $[0,1]$. Thus we apply Lemma 17 again with $f = \bar{e}(s,x)$ to get
$$|\bar{e}^{1/2}(s,h_N(M_N)) - \bar{e}^{1/2}(s,h(W_1,c))| \overset{a.s.}{\to} 0.$$

2. **When $\bar{e}(s,x)$ takes the form $\sum_{k=1}^K c_k \mathbb{1}(g(x) \in C_k)$.** For both functions $\bar{e}^{1/2}(s,x)$ and $\bar{e}(s,x)$, we only need to prove that
$$\mathbb{1}(g(h_N(M_N)) \in C_k) - \mathbb{1}(g(h(W_1,c)) \in C_k) \overset{a.s.}{\to} 0, \ \forall k \in [K]$$
is true. Notice when $c = -\infty$, we know by Assumption 2 that $g(-\infty) = -\infty \in C_1$. Then we know
$$\mathbb{1}(g(h_N(M_N)) \in C_1) - \mathbb{1}(g(h(W_1,-\infty)) \in C_1) = \mathbb{1}(g(h_N(M_N)) \in C_1) - 1 \overset{a.s.}{\to} 0.$$
When $c \in (-\infty, 0]$, we know $g(h(W_1,c))$ is a continuous random variable. Indeed, $h(W_1,c)$ is a continuous random variable and g is a continuous function. This means $\mathbb{P}[g(h(W_1,c)) \in \partial C_k] = 0$ since ∂C_k is of Lebesgue measure zero by the definition of C_k in Assumption 2. Then by Lemma 16, we know
$$\mathbb{1}(g(h_N(M_N)) \in C_k) - \mathbb{1}(g(h(W_1,c)) \in C_k) \overset{a.s.}{\to} 0.$$

This completes the proof. □

K.3.2 Proof of Lemma 25

Proof of Lemma 25. Recall the definition of $E_N^{(2)}$ as in definition (33)
$$E_N^{(2)} = \left((\boldsymbol{\Sigma}_N^{(2)})^{-1/2}\Lambda_N^{(2)}, V_N^{(2)}, H_N^{(2)}, \mathrm{Vec}((\boldsymbol{\Sigma}_N^{(2)})^{1/2}) \right).$$
Define
$$\Lambda_{uN}^{(2)}(s) \equiv \frac{(H_N^{(2)}(s))^{1/2}(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])}{(V_N^{(2)}(s))^{1/2}}.$$
Recall $\Lambda_N^{(2)} = \frac{1}{N_2^{1/2}} \sum_{u=1}^{N_2} (\Lambda_{uN}^{(2)}(0), \Lambda_{uN}^{(2)}(1)), V_N^{(2)} = (V_N(0,E_N^{(1)}), V_N(1,E_N^{(1)})), \boldsymbol{\Sigma}_N^{(2)} = \boldsymbol{\Sigma}_N^{(2)}(E_N^{(1)})$ and $H_N^{(2)} = (H_N(0,E_N^{(1)}), H_N(1,E_N^{(1)}))$. We will use Lemma 15 to prove the result. Define $W_{uN_2} \equiv (\boldsymbol{\Sigma}_N^{(2)})^{-1/2}(\Lambda_{uN}^{(2)}(0), \Lambda_{uN}^{(2)}(1))$. In order to apply Lemma 15 with $N = N_2$ and $W_{uN} = W_{uN_2}$, it suffices to verify the following conditions hold:
$$\frac{1}{N_2}\mathbb{E}\left[\|W_{uN_2}\|_2^4 | \mathcal{H}_N^{(1)}\right] \overset{p}{\to} 0 \quad \text{and} \quad \frac{\mathbb{E}\left[\|W_{uN_2}\|_2^3 | \mathcal{H}_N^{(1)}\right]}{N_2^{1/2}} \overset{p}{\to} 0. \tag{40}$$
Notice we can bound
$$\|W_{uN_2}\|_2 \le \|(\boldsymbol{\Sigma}_N^{(2)})^{-1/2}\|_2 \sum_{s=0,1} (H_N^{(2)}(s))^{1/2} \left| \frac{\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)]}{(V_N^{(2)}(s))^{1/2}} \right|. \tag{41}$$
Now we prove the claims in (40).

Proof of the first claim in (40) By the bound (41),

$$\frac{\mathbb{E}\left[\|W_{uN_2}\|_2^4|\mathcal{H}_N^{(1)}\right]}{N_2} \leq \|(\boldsymbol{\Sigma}_N^{(2)})^{-1/2}\|_2^4 \sum_{s=0,1} \frac{8(H_N^{(2)}(s))^2\mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])^4|\mathcal{H}_N^{(1)}\right]}{N_2(V_N^{(2)}(s))^2}.$$

It suffices to show

$$\|(\boldsymbol{\Sigma}_N^{(2)})^{-1/2}\|_2 = O_p(1) \tag{42}$$

and

$$\frac{1}{N_2(V_N^{(2)}(s))^2}(H_N^{(2)}(s))^2\mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])^4|\mathcal{H}_N^{(1)}\right] \xrightarrow{p} 0, \ \ s = 0, 1. \tag{43}$$

We separate the proof further into two parts.

1. **Proof of (42):** It suffices to prove that the eigenvalues of $\boldsymbol{\Sigma}_N^{(2)} = \boldsymbol{\Sigma}_N^{(2)}(E_N^{(1)})$ are uniformly bounded from 0 and ∞. Recall the expression of $\boldsymbol{\Sigma}_N^{(2)}$ as in (31) and (28) $\boldsymbol{\Sigma}_N^{(2)} = (\mathrm{Cov}_N^{(2)}(E_N^{(1)}))_{2\times 2}$. To show this, we only need to show that there exists universal constant C such that, adopting the abbreviation $\bar{e}_N(s) \equiv H_N(s, E_N^{(1)})$, $\limsup_{N\to\infty} |\mathrm{Cov}_N^{(2)}(E_N^{(1)})| < C < 1$. We will apply Lemma 14 to prove the result. Recall

   $$\mathrm{Cov}_N^{(2)}(E_N^{(1)}) = \frac{(\bar{e}_N(0)(1-\bar{e}_N(0)))^{1/2}\mathbb{E}[Y_{uN}(0)]\mathbb{E}[Y_{uN}(1)]}{(\mathbb{E}[Y_{uN}^2(0)] - \bar{e}_N(0)\mathbb{E}[Y_{uN}(0)]^2)^{1/2}(\mathbb{E}[Y_{uN}^2(1)] - (1-\bar{e}_N(0))\mathbb{E}[Y_{uN}(1)]^2)^{1/2}}.$$

   Note $\bar{e}_N(s) \in (0,1)$ almost surely by the clip assumption, $\lim_{N\to\infty} \mathbb{E}[Y_{uN}^p(s)]$ converges for $p = 1, 2$ and $s = 0, 1$ by Assumption 1, and $\liminf_{N\to\infty} \mathrm{Var}[Y_{uN}(s)] > 0$ for $s = 0, 1$ by Assumption 1. Then we can apply Lemma 14 with $a_N = \bar{e}_N(0), X_N = Y_{uN}(0)$ and $Y_N = Y_{uN}(1)$, we know $\limsup_{N\to\infty} |\mathrm{Cov}_N^{(2)}(E_N^{(1)})| < C < 1$ almost surely where C is a universal constant.

2. **Proof of (43):** First consider

   $$\begin{aligned}\mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])^4|\mathcal{H}_N^{(1)}\right] &\leq 8\left(\mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s))^4|\mathcal{H}_N^{(1)}\right] + \mathbb{E}[Y_{uN}(s)]^4\right) \\ &= 8\frac{\mathbb{E}[Y_{uN}^4(s)]}{(H_N^{(2)}(s))^2 H_N^{(2)}(s)} + 8\mathbb{E}[Y_{uN}(s)]^4.\end{aligned}$$

   Then we have

   $$(H_N^{(2)}(s))^2\mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])^4|\mathcal{H}_N^{(1)}\right] \leq 8\frac{\mathbb{E}[Y_{uN}^4(s)]}{H_N^{(2)}(s)} + 8(H_N^{(2)}(s))^2\mathbb{E}[Y_{uN}(s)]^4.$$

   Since $H_N^{(t)}(s) \leq 1$ and by Assumption 1 that $\mathbb{E}[Y_{uN}^4(s)]$ is uniformly bounded, we know

   $$\limsup_{N\to\infty}(H_N^{(2)}(s))^2\mathbb{E}[Y_{uN}(s)]^4 < \infty \Rightarrow \limsup_{N\to\infty} \frac{(H_N^{(2)}(s))^2\mathbb{E}[Y_{uN}(s)]^4}{N_2} = 0.$$

Now since $N\bar{e}_N(s, \mathcal{H}_{t-1}) \geq Nl_N \to \infty$ by Assumption 3 or 4, we know

$$\frac{1}{N_2 H_N^{(2)}(s)} \mathbb{E}[Y_{uN}^4(s)] = \frac{1}{N_2 \bar{e}_N(s, \mathcal{H}_1)} \mathbb{E}[Y_{uN}^4(s)] \xrightarrow{p} 0.$$

Collecting these results, we have

$$\frac{1}{N_2} (H_N^{(2)}(s))^2 \mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])^4 | \mathcal{H}_N^{(1)} \right] \xrightarrow{p} 0.$$

Since we have proved in Lemma 19 that $\liminf_{N\to\infty} V_N^{(2)}(s) > 0$, we have

$$\frac{1}{N_2 (V_N^{(2)}(s))^2} 8 (H_N^{(2)}(s))^2 \mathbb{E}\left[(\hat{\Lambda}_{uN}^{(2)}(s) - \mathbb{E}[Y_{uN}(s)])^4 | \mathcal{H}_N^{(1)} \right] \xrightarrow{p} 0.$$

Proof of the second claim in (40). The proof is similar to the proof of the first claim so we omit it. □

K.3.3 Proof of Lemma 26

Proof of Lemma 26. Recall the expression of $E_N^{(1)}$. We have

$$E_N^{(1)} \equiv \left((\mathbf{\Sigma}_N^{(1)})^{-1/2} \Lambda_N^{(1)}, V_N^{(1)}, H_N^{(1)}, \mathrm{Vec}((\mathbf{\Sigma}_N^{(1)})^{1/2}) \right).$$

The proof can be decomposed to two steps.

1. We first prove that

$$\|H_N^{(1)} - H^{(1)}\|_2 = o(1), \|V_N^{(1)} - V^{(1)}\|_2 = o(1), \|(\mathbf{\Sigma}_N^{(1)})^{1/2} - (\mathbf{\Sigma}^{(1)})^{1/2}\|_{\mathrm{F}} = o(1). \tag{44}$$

2. Then we prove

$$(\mathbf{\Sigma}_N^{(1)})^{-1/2} \Lambda_N^{(1)} \xrightarrow{d} Z, \ Z \sim N(\mathbf{0}, \boldsymbol{I}_2). \tag{45}$$

Proof of (44): The convergence of $H_N^{(1)}$ and $V_N^{(1)}$ are obvious. For $\mathbf{\Sigma}_N^{(1)}$, we use Lemma 13 so that it suffices to prove

$$\|\mathbf{\Sigma}_N^{(1)} - \mathbf{\Sigma}^{(1)}\|_{\mathrm{F}} = \sqrt{2} |\mathrm{Cov}_N^{(1)} - \mathrm{Cov}^{(1)}| = o(1).$$

To this end, recall the definition of $\mathrm{Cov}_N^{(1)}$ as in (31),

$$\mathrm{Cov}_N^{(1)} = \frac{-(H_N^{(1)}(0) H_N^{(1)}(1))^{1/2}}{(V_N^{(1)}(0))^{1/2} (V_N^{(1)}(1))^{1/2}} \mathbb{E}[Y_{uN}(0)] \mathbb{E}[Y_{uN}(1)].$$

Since $\|V_N^{(1)} - V^{(1)}\|_2, \|H_N^{(1)} - H^{(1)}\|_2 = o(1)$, and

$$0 < \liminf_{N\to\infty} V_N^{(1)}(s) \leq \limsup_{N\to\infty} V_N^{(1)}(s) < \infty$$

as proved in Lemma 19, we know $|\mathrm{Cov}_N^{(1)} - \mathrm{Cov}^{(1)}| = o(1)$. The completes the proof for (44).

Proof of (45)**:** This can be proved easily by applying Lemma 15. We omit the proof. □

L Proof of Theorem 2

Given two random variables X and Y, the 1-Wasserstein distance is defined as

$$W_1(X,Y) \equiv \sup_{\|f\|_{\mathrm{L}}\le 1} \big|\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]\big|.$$

L.1 Proof preparation for Theorem 2

We will only prove the result for $\mathbb{W}_{\mathcal{U}}^{\mathcal{C}}(c)$ since the other proofs are very similar. We will drop the subscript in $w_{\mathcal{C},\mathcal{U}}^{(t)}(s)$ and just write $w^{(t)}(s)$. Similarly, we use $\mathbb{W}(c)$ to denote $\mathbb{W}_{\mathcal{U}}^{\mathcal{C}}(c)$. Recall the definition of $\mathbb{W}(c)$ as

$$\mathbb{W}(c) = \sum_{t=1}^{2} A^{(t)}(0)w^{(t)}(0) - \sum_{t=1}^{2} A^{(t)}(1)w^{(t)}(1),\ w^{(t)}(s) = \frac{(2V^{(t)}(s))^{1/2}}{2\sqrt{H^{(t)}(s)}}.$$

To further ease the burden of notation, we define $V_p(s) \equiv \lim_{N\to\infty}\mathbb{E}[Y_{uN}^p(s)]$. Then we can rewrite $V^{(2)}(s) \equiv V_2(s) - H^{(2)}(s)V_1^2(s)$, where

$$H^{(2)}(s) = \min\{1-\bar{l}, \max\{\bar{l}, \bar{e}(s, S((A^{(1)}, V^{(1)}), c))\}\}.$$

We will use $V^{(2)}(s,c), H^{(2)}(s,c)$ to denote $V^{(2)}(s)$ and $H^{(2)}(s)$ stress the dependence on c. In particular, we can write

$$H^{(2)}(s,-\infty) \equiv \min\{1-\bar{l}, \max\{\bar{l}, \bar{e}(s,-\infty)\}\} \quad \text{and} \quad V^{(2)}(s,-\infty) \equiv V_2(s) - H^{(2)}(s,-\infty)V_1^2(s).$$

Similarly, we can define

$$\mathrm{Cov}^{(2)}(-\infty) = -(H^{(2)}(0,-\infty)H^{(2)}(1,-\infty))^{1/2}/(V^{(2)}(0,-\infty)V^{(2)}(1,-\infty))^{1/2}V_1(0)V_1(1).$$

Then with $V^{(2)}(s,-\infty), H^{(2)}(s,-\infty)$, we can define the corresponding weight $w_{-\infty}^{(2)}(s) \equiv (V^{(2)}(s,-\infty))^{1/2}/(2H^{(2)}(s,-\infty))^{1/2}$. Moreover, we define

$$(A_{-\infty}^{(2)}(0), A_{-\infty}^{(2)}(1))^\top \sim N(\mathbf{0}, \mathbf{\Sigma}_{-\infty}^{(2)}) \quad \text{where} \quad \mathbf{\Sigma}_{-\infty}^{(2)} \equiv (\mathrm{Cov}^{(2)}(-\infty))_{2\times 2}.$$

Finally, define $S_{2,s} \equiv A_{-\infty}^{(2)}(s)w_{-\infty}^{(2)}(s)$.

L.2 Proof of Theorem 2

Proof of Theorem 2. We rewrite the random variable $\mathbb{W}(-\infty)$ as

$$\mathbb{W}(-\infty) = A^{(1)}(0)w^{(1)}(0) - A^{(1)}(1)w^{(1)}(1) + S_{2,0} - S_{2,1}.$$

Notice that $\mathbb{W}(-\infty)$ is a Gaussian random variable with mean zero since $S_{2,0} - S_{2,1}$ is independent with $A^{(1)}(0)w^{(1)}(0) - A^{(1)}(1)w^{(1)}(1)$. The proof will be divded into three steps:

- We first decompose the desired $W_1(\mathbb{W}(-\infty), \mathbb{W}(c))$ distance into different pieces and bound different pieces by W_1 distances;
- We bound the W_1 distance by $|w^{(2)}(s) - w^{(2)}_{-\infty}(s)|$ and further obtain a bound $|w^{(2)}(s) - w^{(2)}_{-\infty}(s)|$, which just involves $|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s, -\infty)|$;
- We collect all the results to prove the claim.

Decomposition of $W_1(\mathbb{W}(-\infty), \mathbb{W}(c))$**.** Now we first decompose the KS distance into two parts, using the triangle inequality:

$$W_1(\mathbb{W}(-\infty), \mathbb{W}(c)) \leq W_1(A^{(2)}(0)w^{(2)}(0), S_{2,0}) + W_1(A^{(2)}(1)w^{(2)}(1), S_{2,1}) \equiv K_0 + K_1.$$

By triangle inequality, we have for $s \in \{0, 1\}$,

$$K_s \leq W_1(A^{(2)}(s)w^{(2)}(s), A^{(2)}(s)w^{(2)}_{-\infty}(s)) + W_1(A^{(2)}(s)w^{(2)}_{-\infty}(s), S_{2,s}).$$

In fact, $W_1(A^{(2)}(s)w^{(2)}_{-\infty}(s), S_{2,s}) = 0$ since $w^{(2)}_{-\infty}(s)$ is a constant and $A^{(2)}(s), A^{(2)}_{-\infty}(s)$ have the same distribution. It suffices to study $W_1(A^{(2)}(s)w^{(2)}(s), A^{(2)}(s)w^{(2)}_{-\infty}(s))$.

Bounding $W_1(A^{(2)}(s)w^{(2)}(s), A^{(2)}(s)w^{(2)}_{-\infty}(s))$**.** We compute

$$\begin{aligned}
& W_1(A^{(2)}(s)w^{(2)}(s), A^{(2)}(s)w^{(2)}_{-\infty}(s)) \\
&= \sup_{\|f\|_{\mathrm{L}} \leq 1} \left| \mathbb{E}\left[f(A^{(2)}(s)w^{(2)}(s))\right] - \mathbb{E}\left[f(A^{(2)}(s)w^{(2)}_{-\infty}(s))\right] \right| \\
&\leq \mathbb{E}\left[|A^{(2)}(s)w^{(2)}(s) - A^{(2)}(s)w^{(2)}_{-\infty}(s)|\right] \\
&\leq \sqrt{\mathbb{E}[|A^{(2)}(s)|^2]\mathbb{E}[|w^{(2)}(s) - w^{(2)}_{-\infty}(s)|^2]} = \sqrt{\mathbb{E}[|w^{(2)}(s) - w^{(2)}_{-\infty}(s)|^2]}.
\end{aligned}$$

Define

$$D(c, -\infty) \equiv (2V^{(2)}(s, c)H^{(2)}(s, -\infty))^{1/2} + (2V^{(2)}(s, -\infty)H^{(2)}(s, c))^{1/2}.$$

Notice that when N is large,

$$D(c, -\infty) \geq (2V^{(2)}(s, c)H^{(2)}(s, -\infty))^{1/2} \geq (2 \liminf_{N \to \infty} \mathrm{Var}[Y_{uN}(s)]\bar{l})^{1/2}.$$

Then we can bound

$$\begin{aligned}
\left|w^{(2)}(s) - w^{(2)}_{-\infty}(s)\right| &= \left| \frac{(2V^{(2)}(s, c)H^{(2)}(s, -\infty))^{1/2} - (2V^{(2)}(s, -\infty)H^{(2)}(s, c))^{1/2}}{2(H^{(2)}(s, c)H^{(2)}(s, -\infty))^{1/2}} \right| \\
&= \left| \frac{V^{(2)}(s, c)H^{(2)}(s, -\infty) - V^{(2)}(s, -\infty)H^{(2)}(s, c)}{(H^{(2)}(s, c)H^{(2)}(s, -\infty))^{1/2}} \right| \frac{1}{D(c, -\infty)} \\
&\lesssim \left| \frac{V^{(2)}(s, c)H^{(2)}(s, -\infty) - V^{(2)}(s, -\infty)H^{(2)}(s, c)}{(H^{(2)}(s, c)H^{(2)}(s, -\infty))^{1/2}} \right|.
\end{aligned} \tag{46}$$

Next, notice that $\min\{H^{(2)}(s,-\infty), H^{(2)}(s,c)\} \geq \bar{l}$, so that we can futher bound

$$\begin{aligned}\left|w^{(2)}(s) - w^{(2)}_{-\infty}(s)\right| &\lesssim \left|V^{(2)}(s,c)H^{(2)}(s,-\infty) - V^{(2)}(s,-\infty)H^{(2)}(s,c)\right| \\ &\lesssim V^{(2)}(s,c)|H^{(2)}(s,-\infty) - H^{(2)}(s,c)| + |V^{(2)}(s,c) - V^{(2)}(s,-\infty)|H^{(2)}(s,c).\end{aligned}$$

We now further bound the RHS by the quantity invovling just $|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|$. Then we show respectively that

1. $|H^{(2)}(s,c) - H^{(2)}(s,-\infty)| \leq |\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|$;
2. $|V^{(2)}(s,c) - V^{(2)}(s,-\infty)| \leq V_1^2(s)|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|$.

For $H^{(2)}(s,c) - H^{(2)}(s,-\infty)$, the claim is true by the Lipschitz property of $\min\{1 - l_N, \max\{l_N, x\}\}$. For $V^{(2)}(s,c) - V^{(2)}(s,-\infty)$, we can compute

$$\begin{aligned}|V^{(2)}(s,c) - V^{(2)}(s,-\infty)| &= V_1^2(s)|H^{(2)}(s,c) - H^{(2)}(s,-\infty)| \\ &\leq V_1^2(s)|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|.\end{aligned}$$

By Lemma 19, we know $V_2(s) - V_1^2(s) \lesssim V^{(2)}(s,c) \lesssim V_2(s)$. By the definition of $H^{(2)}(s,c)$ we know $H^{(2)}(s,c) \in (0,1)$. Then combining bound (46), we have

$$\left|w^{(2)}(s) - w^{(2)}_{-\infty}(s)\right| \lesssim (V_2(s) + V_1^2(s))|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|.$$

Therefore, we obtain the bound

$$\begin{aligned}W_1(A^{(2)}(s)w^{(2)}(s), A^{(2)}(s)w^{(2)}_{-\infty}(s)) &\leq \sqrt{\mathbb{E}[|w^{(2)}(s) - w^{(2)}_{-\infty}(s)|^2]} \\ &\lesssim \left(\mathbb{E}[|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|^2]\right)^{1/2}.\end{aligned}$$

Concluding the proof: Therefore we can bound

$$K_s \leq W_1(A^{(2)}(s)w^{(2)}(s), A^{(2)}(s)w^{(2)}_{-\infty}(s)) \lesssim \left(\mathbb{E}[|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|^2]\right)^{1/2}.$$

Thus we conclude

$$W_1(\mathbb{W}(-\infty), \mathbb{W}(c)) \leq C \sum_{s=0,1} \left(\mathbb{E}[|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|^2]\right)^{1/2}.$$

Now we prove the convergence of $\mathbb{E}[|\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty)|^2]$ as $c \to -\infty$. This is true by dominated convergence theorem since $\bar{e}(s, S((A^{(1)}, V^{(1)}), c)) - \bar{e}(s,-\infty) \to 0$ by Assumption 2. □

M Proof of Theorem 3

M.1 Necessary definitions

Random vectors. We define the random vector

$$W^{(1,b)} \equiv \left(S_1^{(b)}, \hat{V}^{(1)}, H^{(1)}, \mathrm{Vec}((\hat{\mathbf{\Sigma}}^{(1)})^{1/2})\right)$$

where $H^{(1)} = (H^{(1)}(0), H^{(1)}(1))$ is defined as in (30), and

$$W^{(2,b)} \equiv \left(S_2^{(b)}, \hat{V}^{(2,b)}, \hat{H}^{(2,b)}, \mathrm{Vec}((\hat{\mathbf{\Sigma}}^{(2,b)})^{1/2})\right)$$

where $\hat{H}^{(2,b)} = (\hat{H}^{(2,b)}(0), \hat{H}^{(2,b)}(1))$ and $\hat{V}^{(1)} \equiv (\hat{V}^{(1)}(0), \hat{V}^{(1)}(1))$.

Random functions. For $x \in \mathbb{R}^{10}$, recalling the definiton of function h in (27), the weight function is defined as

$$\hat{H}^{(2,b)}(s,x) \equiv \begin{cases} \bar{e}(s, h(x,0)) & \text{under Assumption 4,} \\ \min\{1-\bar{l}, \max\{\bar{l}, \bar{e}(s, h(x,0))\}\} & \text{under Assumption 3,} \end{cases}$$

and the variance function is defined as $\hat{V}^{(2,b)}(s,x) \equiv \hat{\mathbb{E}}[Y_{uN}^2(s)] - \hat{H}^{(2,b)}(s,x)(\hat{\mathbb{E}}[Y_{uN}(s)])^2$. By slightly abusing the notation, we define $\hat{H}^{(2,b)}(x) \equiv (\hat{H}^{(2,b)}(0,x), \hat{H}^{(2,b)}(1,x))$, the random variance vector function $\hat{V}^{(2,b)}(x) \equiv (\hat{V}^{(2,b)}(0,x), \hat{V}^{(2,b)}(1,x))$ and the covariance function

$$\hat{\mathrm{Cov}}^{(2,b)}(x) \equiv -\frac{(\hat{H}^{(2,b)}(0,x)\hat{H}^{(2,b)}(1,x))^{1/2}}{(\hat{V}^{(2,b)}(0,x)\hat{V}^{(2,b)}(1,x))^{1/2}} \hat{\mathbb{E}}[Y_{uN}(0)]\hat{\mathbb{E}}[Y_{uN}(1)]$$

and the covariance matrix function $\hat{\boldsymbol{\Sigma}}^{(2,b)}(x) \equiv (\hat{\mathrm{Cov}}^{(2,b)}(x))_{2\times 2}$. Last, define the function

$$\begin{aligned} W^{(2,b)}(x) &\equiv \left(S_2^{(b)}, \hat{V}^{(2,b)}(x), \hat{H}^{(2,b)}(x), \mathrm{Vec}((\hat{\boldsymbol{\Sigma}}^{(2,b)}(x))^{1/2})\right) \\ W^{(a,b)}(x) &\equiv \left(Z_2, \hat{V}^{(2,b)}(x), \hat{H}^{(2,b)}(x), \mathrm{Vec}((\hat{\boldsymbol{\Sigma}}^{(2,b)}(x))^{1/2})\right), \end{aligned}$$

where Z_2 is defined as in (32).

M.2 Proof of Theorem 3

Proof of Theorem 3. Recall $\mathbb{W}$ as defined in (32). We use Lemma 18 to prove the result, with $W_N = (W^{(1,b)}, W^{(2,b)})$ and $W = \mathbb{W}$. The continuous function g is chosen to be as in **Step 3** in Section J.1. Thus it suffices to prove the following statement is true:

$$\mathbb{E}[f(W^{(1,b)}, W^{(2,b)})|\mathcal{G}_N] - \mathbb{E}[f(\mathbb{W})] \xrightarrow{p} 0, \text{ for any } f \text{ such that } \|f\|_{\mathrm{BL}} < \infty.$$

Consider the following decomposition:

$$\begin{aligned} &\mathbb{E}[f(W^{(1,b)}, W^{(2,b)})|\mathcal{G}_N] - \mathbb{E}[f(\mathbb{W})] \\ &= \mathbb{E}[f(W^{(1,b)}, W^{(2,b)}(W^{(1,b)}))|\mathcal{G}_N] - \mathbb{E}[f(W_1, W_2)] \\ &= \mathbb{E}[f(W_1, W^{(a,b)}(W_1))|\mathcal{G}_N] - \mathbb{E}[f(W_1, W_2)] \\ &\quad + \mathbb{E}[f(W^{(1,b)}, W^{(a,b)}(W^{(1,b)}))|\mathcal{G}_N] - \mathbb{E}[f(W_1, W^{(a,b)}(W_1))|\mathcal{G}_N] \\ &\quad + \mathbb{E}[f(W^{(1,b)}, W^{(2,b)}(W^{(1,b)}))|\mathcal{G}_N] - \mathbb{E}[f(W^{(1,b)}, W^{(a,b)}(W^{(1,b)}))|\mathcal{G}_N] \\ &\equiv M_1 + M_2 + M_3. \end{aligned}$$

We prove $M_1, M_2, M_3 = o_p(1)$ respectively.

1. **Proof of $M_1 = o_p(1)$.** Notice $(W_1, W_2) \perp\!\!\!\perp \mathcal{G}_N$. Then it suffices to prove

$$\left|\mathbb{E}[f(W_1, W^{(a,b)}(W_1))|\mathcal{G}_N] - \mathbb{E}[f(W_1, W_2)|\mathcal{G}_N]\right| = o_p(1).$$

Then by the Lipschitz property and boundedness of f and Lemma 17, it suffices to prove

$$\left\|W^{(a,b)}(W_1) - W_2\right\|_2 \xrightarrow{p} 0,\ \mathcal{F}_N = \mathcal{G}_N.$$

In other words, we need to show, by Lemma 13,

$$\|\hat{V}^{(2,b)}(W_1) - V^{(2)}\|_2 \xrightarrow{p} 0,\ \|\hat{H}^{(2,b)}(W_1) - H^{(2)}\|_2 \xrightarrow{p} 0,\ \|\hat{\boldsymbol{\Sigma}}^{(2,b)} - \boldsymbol{\Sigma}^{(2)}\|_{\mathrm{F}} \xrightarrow{p} 0.$$

We can compute

$$\|\hat{\boldsymbol{\Sigma}}^{(2,b)} - \boldsymbol{\Sigma}^{(2)}\|_{\mathrm{F}} = \sqrt{2}|\hat{\mathrm{Cov}}^{(2,b)} - \mathrm{Cov}^{(2)}|.$$

We observe that if we can prove $\|\hat{H}^{(2,b)}(W_1) - H^{(2)}\|_2 \xrightarrow{p} 0$, the other claims are true by the consistency of $\hat{\mathbb{E}}[Y_{uN}(s)]$ and $\hat{\mathbb{E}}[Y_{uN}^2(s)]$. However, by observing that $\hat{H}^{(2,b)}(W_1) = H^{(2)}$, we know the claim is true.

2. **Proof of $M_2 = o_p(1)$.** Define $W^{(c,b)} \equiv (S_1^{(b)}, V^{(1)}, H^{(1)}, \mathrm{Vec}((\boldsymbol{\Sigma}^{(1)})^{1/2}))$. Since

$$(W^{(c,b)}, W^{(a,b)}(W^{(c,b)}))|\mathcal{G}_N \overset{d}{=} (W_1, W^{(a,b)}(W_1))|\mathcal{G}_N,$$

it suffices to show

$$\left|\mathbb{E}[f(W^{(1,b)}, W^{(a,b)}(W^{(1,b)}))|\mathcal{G}_N] - \mathbb{E}[f(W^{(c,b)}, W^{(a,b)}(W^{(c,b)}))|\mathcal{G}_N]\right| = o_p(1).$$

By the Lipschitz property and boundedness of f and Lemma 17, it suffices to show $\|W^{(1,b)} - W^{(c,b)}\|_2 = o_p(1)$ and

$$\|\hat{V}^{(2,b)}(W^{(1,b)}) - \hat{V}^{(2,b)}(W^{(c,b)})\|_2 \xrightarrow{p} 0,\ \|\hat{H}^{(2,b)}(W^{(1,b)}) - \hat{H}^{(2,b)}(W^{(c,b)})\|_2 \xrightarrow{p} 0$$

and, by Lemma 13

$$\|\hat{\boldsymbol{\Sigma}}^{(2,b)}(W^{(1,b)}) - \hat{\boldsymbol{\Sigma}}^{(2,b)}(W^{(c,b)})\|_{\mathrm{F}} = \sqrt{2}|\hat{\mathrm{Cov}}^{(2,b)}(W^{(1,b)}) - \hat{\mathrm{Cov}}^{(2,b)}(W^{(c,b)})| \xrightarrow{p} 0.$$

By the consistency of $\hat{\mathbb{E}}[Y_{uN}(s)]$ and $\hat{\mathbb{E}}[Y_{uN}^2(s)]$ proved in Lemma 21, it suffices to show

$$\|W^{(1,b)} - W^{(c,b)}\|_2 \xrightarrow{p} 0,\ \|\hat{H}^{(2,b)}(W^{(1,b)}) - \hat{H}^{(2,b)}(W^{(c,b)})\|_2 \xrightarrow{p} 0$$

We prove these two claims subsequently.

Proof of $\|W^{(1,b)} - W^{(c,b)}\|_2 = o_p(1)$: It suffices and it is easy to show

$$\|\hat{V}^{(1)} - V^{(1)}\|_2 = o_p(1),\ \|\hat{\boldsymbol{\Sigma}}^{(1)} - \boldsymbol{\Sigma}^{(1)}\|_{\mathrm{F}} = o_p(1). \tag{47}$$

Proof of $\|\hat{H}^{(2,b)}(W^{(1,b)}) - \hat{H}^{(2,b)}(W^{(c,b)})\|_2 = o_p(1)$ By the Lipschitz property of $\min\{1-l_N, \max\{l_N, x\}\}$, it suffices to show

$$\begin{aligned}
&|\bar{e}(s, h(W^{(1,b)}, 0)) - \bar{e}(s, S(((\boldsymbol{\Sigma}^{(1)})^{1/2} S_1^{(b)}, V^{(1)}), 0))| \\
&= |\bar{e}(s, h(W^{(1,b)}, 0)) - \bar{e}(s, h(W^{(c,b)}, 0))| \xrightarrow{p} 0, \;\; s = 0, 1.
\end{aligned}$$

Depending on the smoothness of $\bar{e}(s, x)$, we divide the proof into two cases based on Assumption 2.

- **When $\bar{e}(s, x)$ is Lipschitz continuous in x:** It suffices to show

$$|h(W^{(1,b)}, 0) - h(W^{(c,b)}, 0)| \xrightarrow{p} 0. \tag{48}$$

  By the definition of h in (27), it suffices to prove, by the continuous mapping theorem, that

$$\|(\hat{\boldsymbol{\Sigma}}^{(1)})^{1/2} S_1^{(b)} - (\boldsymbol{\Sigma}^{(1)})^{1/2} S_1^{(b)}\|_2 \xrightarrow{p} 0, \;\; \|\hat{V}^{(1)} - V^{(1)}\|_2 \xrightarrow{p} 0.$$

  This is obvious by result (47) and

$$\begin{aligned}
\|(\hat{\boldsymbol{\Sigma}}^{(1)})^{1/2} S_1^{(b)} - (\boldsymbol{\Sigma}^{(1)})^{1/2} S_1^{(b)}\|_2 &\le \|S_1^{(b)}\|_2 \|(\hat{\boldsymbol{\Sigma}}^{(1)})^{1/2} - (\boldsymbol{\Sigma}^{(1)})^{1/2}\|_2 \\
&\le \|S_1^{(b)}\|_2 \|(\hat{\boldsymbol{\Sigma}}^{(1)})^{1/2} - (\boldsymbol{\Sigma}^{(1)})^{1/2}\|_{\mathrm{F}} \\
&\le \|S_1^{(b)}\|_2 \sqrt{2} \|\hat{\boldsymbol{\Sigma}}^{(1)} - \boldsymbol{\Sigma}^{(1)}\|_{\mathrm{F}}.
\end{aligned}$$

- **When $\bar{e}(s, x) = \sum_{k=1}^{K} c_k \mathbb{1}(g(x) \in C_k)$:** It suffices to prove for any $k \in [K]$,

$$|\mathbb{1}(g(h(W^{(1,b)}, 0)) \in C_k) - \mathbb{1}(g(h(W^{(c,b)}, 0)) \in C_k)| \xrightarrow{p} 0.$$

  Then with result (48), applying Lemma 16, we know the claim is true since $g(h(W^{(c,b)}, 0))$ is a continuous random variable and $\mathbb{P}[g(h(W^{(c,b)}, 0)) \in \partial C_k] = 0$.

3. **Proof of $M_3 = o_p(1)$.** It is easy to see that $W^{(2,b)} | W^{(1,b)}, \mathcal{G}_N \stackrel{d}{=} W^{(a,b)} | W^{(1,b)}, \mathcal{G}_N$ so we know the following statement is true almost surely:

$$\mathbb{E}[f(W^{(1,b)}, W^{(2,b)}(W^{(1,b)})) | \mathcal{G}_N] - \mathbb{E}[f(W^{(1,b)}, W^{(a,b)}(W^{(1,b)})) | \mathcal{G}_N] = 0.$$

□

N Extenstion

N.1 Extension to $m = 1$

In some adaptive experimental designs, unpromising treatments are dropped in the second stage—a strategy known as the "drop-the-loser" approach (Sampson et al., 2005; Sill et al., 2009). In such cases, the sampling probability for one of the treatment arms is set to exactly zero in the second stage. Recall Assumptions 3 and 4; neither allows the sampling probability of any treatment to be exactly zero in the follow-up stage. In this section, we further extend the results in Theorem 1 to incorporate the weighting $m = 1$ in $h_N^{(t)}(s) = \bar{e}_N^m(s, \mathcal{H}_{t-1})/N^{1/2}$ considered in the general estimator (2). In fact, we can write the resulting test statistic as

$$\mathrm{WIPW}(s) = \frac{\sum_{t=1}^{2} N_t \mathbb{1}(A_{uN}^{(t)} = s) Y_{uN}^{(t)}}{\sum_{t=1}^{2} N_t \bar{e}_N(s, \mathcal{H}_t)}. \tag{49}$$

Consider the following assumption and theorem.

Assumption 5 (Adaptive weighting with $m = 1$)**.** *Suppose adpative weighting ($m = 1$) is used and clipping rate $l_N = 0$ as in Assumption 2.*

Theorem 4 (Adaptive weighting with $m = 1$)**.** *Suppose Assumption 1-2 and Assumption 5 hold. Then, for any $s \in \{0, 1\}$, we have $\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)] = o_p(1)$. Furthermore, define*

$$M^{(t)}(s) \equiv q_t \left(\frac{H^{(t)}(s)}{\sum_{t=1}^{2} q_t H^{(t)}(s)} \right)^2 \quad \text{and} \quad \bar{w}^{(t)}(s) = \left(M^{(t)}(s)/(R^{(t)}(s))^2 \right)^{1/2}.$$

Then considering the test statistic (49), we have

$$\sqrt{N}(\mathrm{WIPW}(s) - \mathbb{E}[Y_{uN}(s)]) \xrightarrow{d} \sum_{t=1}^{2} A^{(t)}(0) \bar{w}^{(t)}(0)$$

$$\sqrt{N}\{T_N - (\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])\} \xrightarrow{d} \sum_{t=1}^{2} A^{(t)}(0) \bar{w}^{(t)}(0) - \sum_{t=1}^{2} A^{(t)}(1) \bar{w}^{(t)}(1).$$

Defining $w^{(t)}(s) = \bar{w}^{(t)}(s)/(\sum_{s=0}^{1} \sum_{t=1}^{2} (\bar{w}^{(t)}(s))^2)^{1/2}$, then we have

$$W_N - \frac{\sqrt{N}(\mathbb{E}[Y_{uN}(0)] - \mathbb{E}[Y_{uN}(1)])}{(N\hat{V}_N(0) + N\hat{V}_N(1))^{1/2}} \xrightarrow{d} \sum_{t=1}^{2} A^{(t)}(0) w^{(t)}(0) - \sum_{t=1}^{2} A^{(t)}(1) w^{(t)}(1).$$

Theorem 4 demonstrates that $m = 1$ weighting can accommodate the early-dropping experiments. We now comment on the proof of Theorem 4.

Remark 8 (Comment on the proof of Theorem 4)**.** *The proof of Theorem 4 is similar to the proof of Theorem 1, following the general proof roadmap sketched in Appendix J.1. The difference happens in the* ***Step 2****, where we need to show the weak convergence of the random vector $(E_N^{(1)}, E_N^{(2)})$, where*

$$E_N^{(t)} = \Big(\frac{1}{N_t^{1/2}} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s)(Y_{uN}^{(t)} - H_N^{(t)} \mathbb{E}[Y_{uN}^{(t)}]), H_N^{(t)} \Big) \quad \text{and} \quad H_N^{(t)} = \bar{e}_N(s, \mathcal{H}_{t-1}).$$

The other proofs are similar.

N.2 Extension to test statistics with augmentation

When data is generated independently as in non-adaptive experiments, efficiency of IPW estimator may be improved by augmenting the statistic with a consistent sample mean estimator. In particular, we consider the weighted augmented inverse probability weighted (WAIPW) estimator, WAIPW(s), defined as

$$\sum_{t=1}^{2} \frac{N_t h_N^{(t)}(s)}{\sum_{t=1}^{2} N_t h_N^{(t)}(s)} \left(\frac{1}{N_t} \sum_{u=1}^{N_t} \frac{\mathbb{1}(A_{uN}^{(t)} = s)(Y_{uN}^{(t)} - \hat{\mathbb{E}}[Y_{uN}(s)])}{\bar{e}_N(s, \mathcal{H}_{t-1})} + \hat{\mathbb{E}}[Y_{uN}(s)] \right). \tag{50}$$

Furthermore, define WIPW$_a(s)$ as

$$\sum_{t=1}^{2} \frac{N_t h_N^{(t)}(s)}{\sum_{t=1}^{2} N_t h_N^{(t)}(s)} \left(\frac{1}{N_t} \sum_{u=1}^{N_t} \frac{\mathbb{1}(A_{uN}^{(t)} = s)(Y_{uN}^{(t)} - \mathbb{E}[Y_{uN}(s)])}{\bar{e}_N(s, \mathcal{H}_{t-1})} \right) + \mathbb{E}[Y_{uN}(s)].$$

It is not hard to show that $\sqrt{N}\,(\mathrm{WAIPW}(s) - \mathrm{WIPW}_a(s))$ converges to 0 in probability as long as $\hat{\mathbb{E}}[Y_{uN}(s)] - \mathbb{E}[Y_{uN}(s)] \xrightarrow{p} 0$. Then we can apply Theorem 1 and Theorem 4 with $Y_{uN} = Y_{uN} - \mathbb{E}[Y_{uN}(s)]$ to prove that $\sqrt{N}(\mathrm{WIPW}_a(s) - \mathbb{E}[Y_{uN}(s)])$ converges weakly. Similarly, we can show the weak limits of T_N and W_N.

Asymptotic equivalence between WAIPW(s) **and sample mean.** We can show that when $m = 1$, the WAIPW(s) is asymptotically equivalent to the sample mean estimator, SM(s), defined as

$$\mathrm{SM}(s) \equiv \frac{\sum_{t=1}^{2} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s) Y_{uN}^{(t)}}{\sum_{t=1}^{2} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s)}.$$

By convention, we define $0/0 = 1$. We formalize the equivalence claim in the following lemma.

Lemma 27 (Asymptotic equivalence between sample mean and WAIPW(s))**.** *Suppose $m = 1$ and $\hat{\mathbb{E}}[Y_{uN}(s)] - \mathbb{E}[Y_{uN}(s)] \xrightarrow{p} 0$. Furthermore, suppose Assumption 1 holds. Then we have $\sqrt{N}\,(\mathrm{SM}(s) - \mathrm{WAIPW}(s)) \xrightarrow{p} 0$.*

The proof of Lemma 27 can be found in Appendix N.5.

N.3 Extension to selection with nuisance parameter

In practice, the selection algorithm can depend on statistic beyond $S_N^{(1)}(0) - S_N^{(1)}(1)$. The example includes the case when the selection depends on the interim p-value, which involves the standard deviation estimate, going beyond the difference of the two statistics. The following theorem shows that our results can be extended to such case.

Theorem 5 (Extension to selection algorithm with nuisance parameter)**.** *Suppose the follow-up stage sampling probability is given by*

$$\mathcal{S}_N(S_N^{(1)}(0) - S_N^{(1)}(1)) = \min\{1 - l_N, \max\{l_N, \bar{e}(0, (S_N^{(1)}(0) - S_N^{(1)}(1))/\hat{\sigma})\}\}, \tag{51}$$

where $\hat{\sigma} \in (0, \infty)$. Then if there exists $\sigma > 0$ such that $\hat{\sigma} \xrightarrow{p} \sigma \in (0, \infty)$, then the conclusion of Theorem 1 still holds.

Proof sketch for Theorem 5. The proof is similar to the proof of Theorem 1. In particular, it follows the general proof roadmap sketched in Appendix J.1. We need to choose the appropriate joint vector $E_N^{(t)}$ to accommodate the nuisance parameter $\hat{\sigma}$. In particular, keeping $E_N^{(1)}$ as defined in (33), we can define the new $E_N^{(2)}$ as

$$E_N^{(2)} = \Big((\mathbf{\Sigma}_N^{(2)})^{-1/2} \Lambda_N^{(2)}, V_N^{(2)}, H_N^{(2)}, \mathrm{Vec}((\mathbf{\Sigma}_N^{(2)})^{1/2}) \Big)$$

where $H_N^{(2)}$ is defined as in (51). The other proofs are similar.

N.4 Extension to adaptive experiments with stopping time

We outline how our main results can be extended to adaptive experiments involving a stopping time. Specifically, we consider a stopping time τ that depends on the quantity $S_N^{(1)}(0) - S_N^{(1)}(1)$. To describe the stopping criterion, we define the event $\mathcal{E} \equiv \{D(S_N^{(1)}(0) - S_N^{(1)}(1)) \in [0, \beta] \subset \mathbb{R}\}$, where D is a decision rule and $\beta > 0$ lies on the decision boundary. The stopping time is then defined as $\tau \equiv \mathbb{1}(\mathcal{E})$, where $\tau = 1$ indicates continuation of the experiment and $\tau = 0$ indicates early termination. This setup captures scenarios in which strong preliminary evidence warrants stopping the experiment at the pilot stage. For example, if $D(x) = x$, then $\tau = 0$ when the evidence in favor of treatment 0 over treatment 1 is sufficiently strong, if β is large. Then the test statistic can be written as

$$\mathrm{WIPW}(s) = \frac{N_1 h_N^{(1)}(s)}{N_1 h_N^{(1)}(s) + N_2 h_N^{(2)}(s)\mathbb{1}(\mathcal{E})} \hat{\Lambda}_N^{(1)}(s) + \frac{N_2 h_N^{(2)}(s)\mathbb{1}(\mathcal{E})}{N_1 h_N^{(1)}(s) + N_2 h_N^{(2)}(s)\mathbb{1}(\mathcal{E})} \hat{\Lambda}_N^{(2)}(s). \quad (52)$$

Sketched derivation of weak limit. Weak limit $\mathrm{WIPW}(s)$ defined in (52) can be derived following the similar proof roadmap sketched in Appendix J.1. In particular, we will need to derive the joint weak convergence of the random vector including $\hat{\Lambda}_N^{(1)}(s), \hat{\Lambda}_N^{(2)}(s), h_N^{(1)}(s), h_N^{(2)}(s)\mathbb{1}(\mathcal{E})$.

N.5 Proof of Lemma 27

Proof of Lemma 27. When $m = 1$, we can write the $\mathrm{WAIPW}(s)$ as

$$\mathrm{WAIPW}(s) = \frac{\sum_{t=1}^{2} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s)(Y_{uN}^{(t)} - \hat{\mathbb{E}}[Y_{uN}(s)])}{\sum_{t=1}^{2} N_t \bar{e}_N(s, \mathcal{H}_{t-1})} + \hat{\mathbb{E}}[Y_{uN}(s)].$$

For the ease of notation, define

$$\mathcal{I}_N(s) \equiv \sum_{t=1}^{2} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s) \quad \text{and} \quad \mathcal{Y}_N(s) \equiv \sum_{t=1}^{2} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s) Y_{uN}^{(t)}.$$

Also, define

$$\mathcal{R}_N(s) = \mathcal{Y}_N(s) - \sum_{t=1}^{2} \sum_{u=1}^{N_t} \mathbb{1}(A_{uN}^{(t)} = s) \mathbb{E}[Y_{uN}^{(t)}].$$

Then we can write the difference as

$$\begin{aligned}\mathrm{SM}(s)-\mathrm{WAIPW}(s) &= \left(\frac{\mathcal{Y}_N(s)}{\mathcal{I}_N(s)}-\hat{\mathbb{E}}[Y_{uN}(s)]\right)\left(1-\frac{\mathcal{I}_N(s)}{\sum_{t=1}^{2}N_t\bar{e}_N(s,\mathcal{H}_{t-1})}\right)\\ &\equiv A_N(s)\times B_N(s).\end{aligned}$$

It suffices to prove $A_N(s)=o_p(1)$ and $B_N(s)=O_p(1/\sqrt{N})$.

Proof of $A_N(s)=o_p(1)$. By the definition of $\mathcal{I}_N(s)$, we can write

$$A_N(s)=\frac{\mathcal{R}_N(s)}{\mathcal{I}_N(s)}+\mathbb{E}[Y_{uN}(s)]-\hat{\mathbb{E}}[Y_{uN}(s)]\equiv A_{1N}(s)+\mathbb{E}[Y_{uN}(s)]-\hat{\mathbb{E}}[Y_{uN}(s)].$$

Since by the assumption, $\mathbb{E}[Y_{uN}(s)]-\hat{\mathbb{E}}[Y_{uN}(s)]=o_p(1)$, it suffices to prove that $\mathrm{Var}[A_{1N}(s)]=o(1)$. In fact, we can show something stronger: $\mathrm{Var}[\sqrt{N}A_{1N}(s)]=O(1)$. Since the denominator $\mathcal{I}_N=\sum_{t=1}^{2}\sum_{u=1}^{N_t}\mathbb{1}(A_{uN}^{(t)}=s)$ is likely to be 0, we divide the proof into three cases. Define $\mathcal{I}_N^{(t)}(s)\equiv\sum_{u=1}^{N_t}\mathbb{1}(A_{uN}^{(t)}=s)$.

1. **When $\mathcal{I}_N(s)=0$.** In this case, we have $A_{1N}(s)=1$. However, this is event is exponentially unlikely since $\mathcal{I}_N(s)\geq\mathcal{I}_N^{(1)}(s)=\sum_{u=1}^{N_1}\mathbb{1}(A_{uN}^{(1)}=s)$ and $\mathbb{E}[\mathcal{I}_N^{(1)}(s)]=N_1e(s)$. Therefore, $\mathrm{Var}[\sqrt{N}A_{1N}(s)\mathbb{1}(\mathcal{I}_N(s)=0)]\to 0$.

2. **When $\mathcal{I}_N(s)>0$ but $\mathcal{I}_N^{(1)}(s)=0$.** In this case, we have $|A_{1N}(s)|\leq|Y_{uN}^{(2)}(s)-\mathbb{E}[Y_{uN}(s)]|$. Then we have

$$\begin{aligned}\mathrm{Var}[\sqrt{N}A_{1N}(s)\mathbb{1}(\mathcal{I}_N^{(1)}(s)=0)] &\leq \mathbb{E}[NA_{1N}^2(s)\mathbb{1}(\mathcal{I}_N^{(1)}(s)=0)]\\ &= N\mathrm{Var}[Y_{uN}^{(2)}(s)]\mathbb{P}[\mathcal{I}_N^{(1)}(s)=0].\end{aligned}$$

   Since $\mathbb{P}[\mathcal{I}_N^{(1)}(s)=0]\to 0$ exponentially, we have $\mathrm{Var}[\sqrt{N}A_{1N}(s)\mathbb{1}(\mathcal{I}_N^{(1)}(s)=0)]\to 0$.

3. **When $\mathcal{I}_N^{(1)}(s)>0$.** We compute

$$\mathrm{Var}[A_{1N}(s)\mathbb{1}(\mathcal{I}_N^{(1)}(s)>0)]\leq\mathbb{E}\left[\frac{(\mathcal{R}_N(s))^2}{(\mathcal{I}_N(s))^2}\mathbb{1}(\mathcal{I}_N^{(1)}(s)>0)\right].$$

   Since $\mathcal{I}_N(s)\geq\mathcal{I}_N^{(1)}(s)=\sum_{u=1}^{N_1}\mathbb{1}(A_{uN}^{(1)}=s)$, we know

$$\begin{aligned}\mathbb{E}\left[\frac{(\mathcal{R}_N(s))^2}{(\mathcal{I}_N(s))^2}\mathbb{1}(\mathcal{I}_N^{(1)}(s)>0)\right] &\leq \mathbb{E}\left[\frac{(\mathcal{R}_N(s))^2}{(\mathcal{I}_N^{(1)}(s))^2}\mathbb{1}(\mathcal{I}_N^{(1)}(s)>0)\right]\\ &= \mathbb{E}\left[\frac{\mathbb{E}\left[(\mathcal{R}_N(s))^2\,|\mathcal{H}_1\right]}{(\sum_{u=1}^{N_1}\mathbb{1}(A_{uN}^{(1)}=s))^2}\mathbb{1}(\mathcal{I}_N^{(1)}(s)>0)\right].\end{aligned}$$

   Further, we can decompose

$$\mathbb{E}\left[(\mathcal{R}_N(s))^2\,|\mathcal{H}_1\right]=\sum_{u=1}^{N_1}\mathbb{1}(A_{uN}^{(1)}=s)(Y_{uN}^{(1)}-\mathbb{E}[Y_{uN}(s)])^2+N_2\bar{e}_N(s,\mathcal{H}_1)\mathrm{Var}[Y_{uN}(s)]$$

$$\leq \sum_{u=1}^{N_1} (Y_{uN}^{(1)} - \mathbb{E}[Y_{uN}(s)])^2 + N_2 \bar{e}_N(s, \mathcal{H}_1) \mathrm{Var}[Y_{uN}(s)].$$

Then define

$$A_{2N}(s) \equiv \mathbb{E}\left[\frac{\sum_{u=1}^{N_1} (Y_{uN}^{(1)} - \mathbb{E}[Y_{uN}(s)])^2}{(\sum_{u=1}^{N_1} \mathbb{1}(A_{uN}^{(1)} = s))^2} \mathbb{1}(\mathcal{I}_N^{(1)}(s) > 0) \right]$$

and

$$A_{3N}(s) \equiv \mathbb{E}\left[\frac{N_2 \mathbb{1}(\mathcal{I}_N^{(1)}(s) > 0)}{(\sum_{u=1}^{N_1} \mathbb{1}(A_{uN}^{(1)} = s))^2} \right] \mathrm{Var}[Y_{uN}(s)].$$

It suffices to prove that $A_{2N}(s) = O(1/N)$ and $A_{3N}(s) = O(1/N)$. We first prove the claim for $A_{2N}(s)$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(A_{2N}(s))^2 &\leq \mathbb{E}\left[\left(\frac{\sum_{u=1}^{N_1} (Y_{uN}^{(1)} - \mathbb{E}[Y_{uN}(s)])^2}{N_1} \right)^2 \right] \mathbb{E}\left[\left(\frac{N_1 \mathbb{1}(\mathcal{I}_N^{(1)}(s) > 0)}{(\sum_{u=1}^{N_1} \mathbb{1}(A_{uN}^{(1)} = s))^2} \right)^2 \right] \\
&\leq \mathbb{E}[(Y_{uN}^{(1)} - \mathbb{E}[Y_{uN}(s)])^4] \mathbb{E}\left[\left(\frac{N_1 \mathbb{1}(\mathcal{I}_N^{(1)}(s) > 0)}{(\sum_{u=1}^{N_1} \mathbb{1}(A_{uN}^{(1)} = s))^2} \right)^2 \right] \\
&\leq \mathbb{E}[(Y_{uN}^{(1)} - \mathbb{E}[Y_{uN}(s)])^4] \mathbb{E}\left[\left(\frac{4N_1}{(1 + \sum_{u=1}^{N_1} \mathbb{1}(A_{uN}^{(1)} = s))^2} \right)^2 \right].
\end{aligned}$$

For the first term, by Assumption 1, we have $\mathbb{E}[(Y_{uN}^{(1)} - \mathbb{E}[Y_{uN}(s)])^4] = O(1)$. For the second term, we use the following lemma to conclude the proof.

Lemma 28 (Cribari-Neto et al. (2000))**.** *Suppose $X_1, \ldots, X_N$ are i.i.d. Bernoulli random variables with $\mathbb{E}[X_i] = p$. Then we have*

$$\mathbb{E}\left[\frac{N^k}{(1 + \sum_{i=1}^N X_i)^k} \right] = O\left(1/p^k\right).$$

Now we can apply Lemma 28 with $k = 4$ so that we have $A_{2N}(s) = O(1/N_1) = O(1/N)$. Similarly, $A_{3N}(s) = O(1/N)$. Thus $\mathrm{Var}[\sqrt{N} A_{1N}(s) \mathbb{1}(\mathcal{I}_N^{(1)}(s) > 0)] \to 0$.

This concludes the proof of $A_N(s) = o_p(1)$.

Proof of $B_N(s) = O_p(1/\sqrt{N})$. We will prove that $\mathrm{Var}[\sqrt{N} B_N(s)] = O(1)$. The proof follows the similar argument as the proof of $\mathrm{Var}[\sqrt{N} A_{1N}(s)] = O(1)$. We omit it. □

O Additional simulation results

We provide additional simulation results in this section. In Section O.1, we show the simulation results with ε-greedy selection algorithm applied. In Section O.2, we compare the power of the adaptive weighting with $m = 1/2$ and $m = 1$.

O.1 Additional simulation results with ε-greedy algorithm

We show the additional results for the simulation in Section 4.1 with ε-greedy selection algorithm applied. The ε is chosen within $0.1, 0.2, 0.4$. Results are shown in Figure 7 and 8. We do not employ clipping in this case.

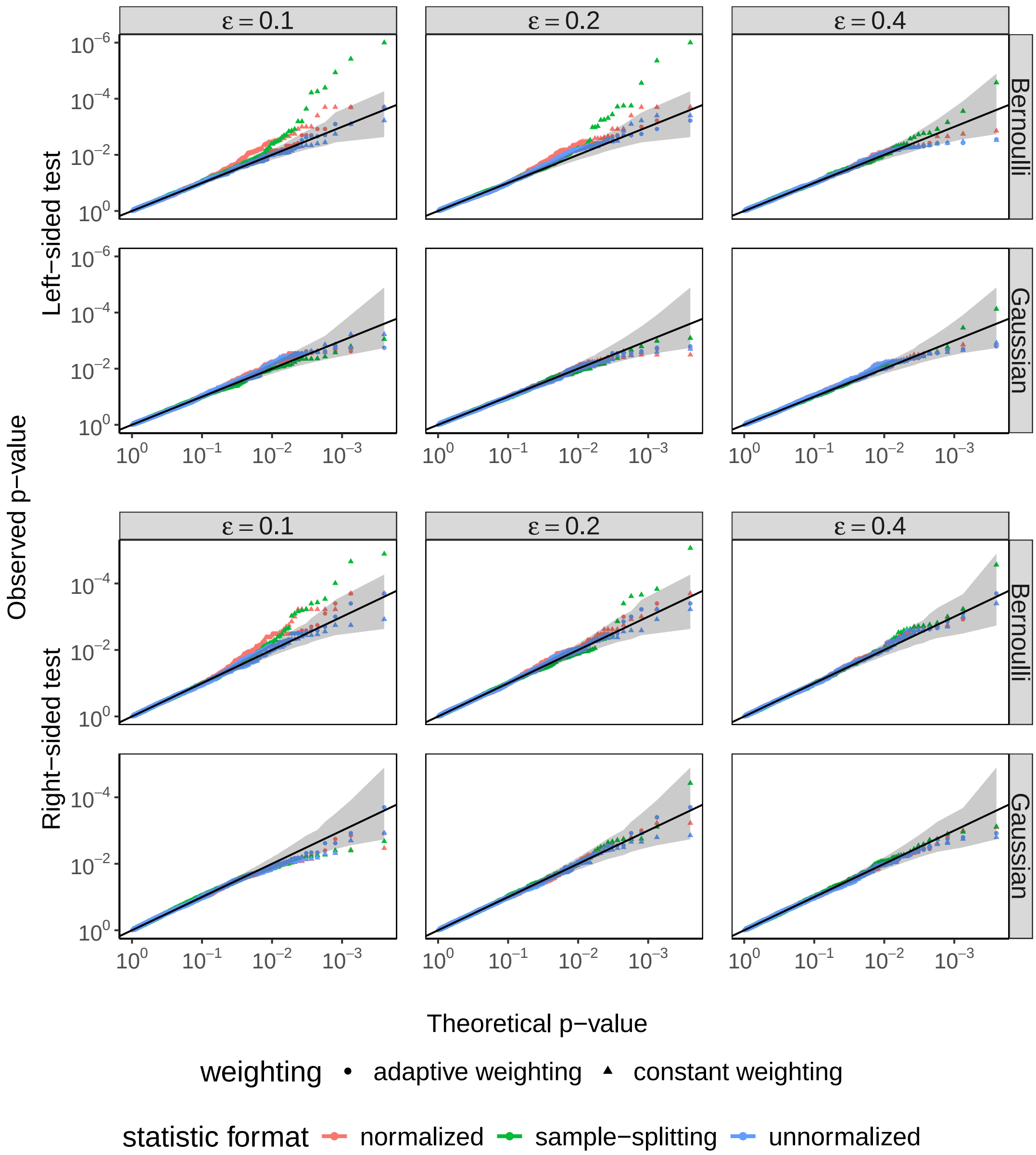

Figure 7: QQ-plot for the 5 tests under different signal strength. The simulation is repeated for 2000 times. The number of bootstrap used in each test is 5000.

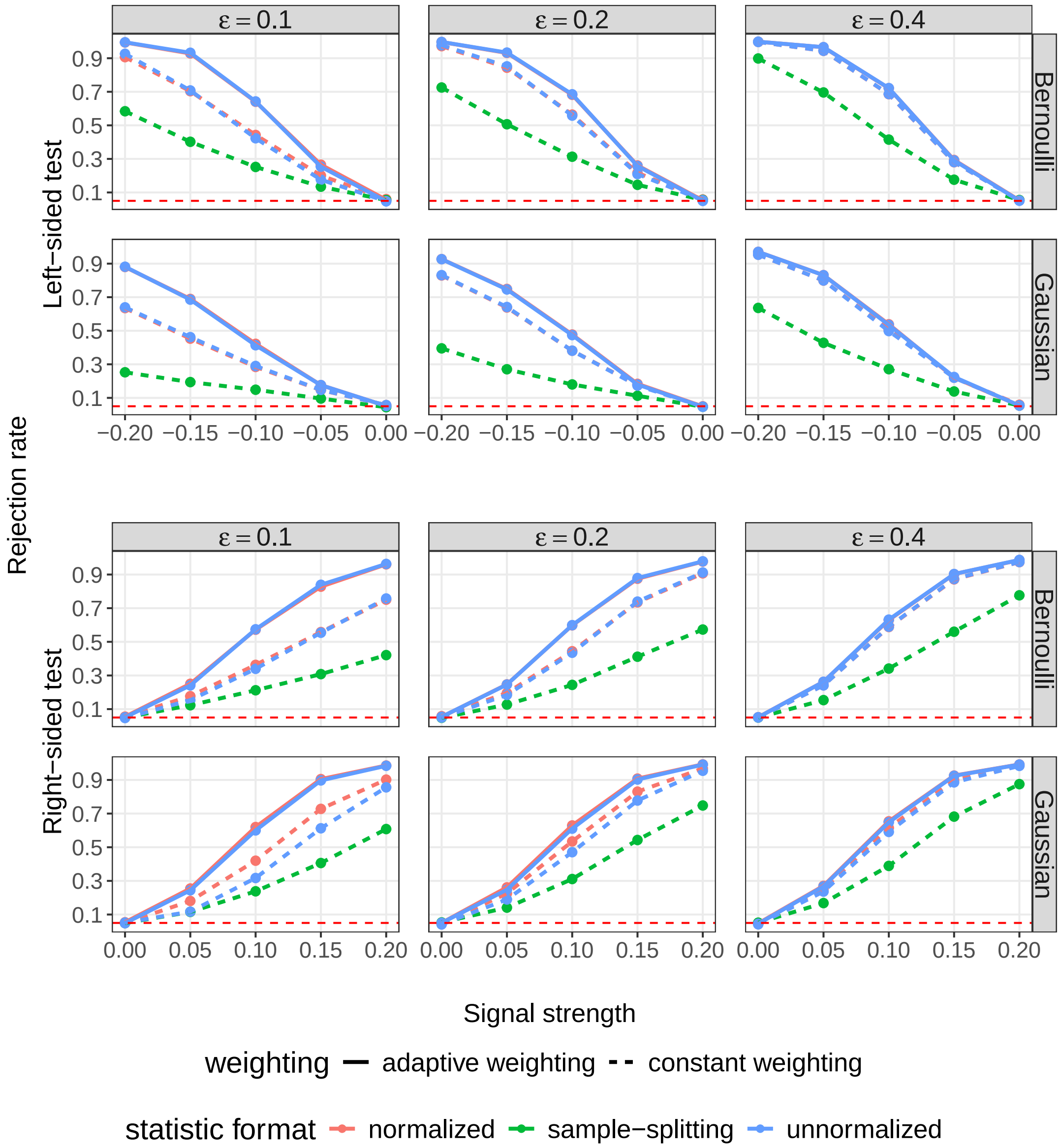

Figure 8: Rejection rate for the 5 tests under different signal strength. The simulation is repeated for 2000 times. The number of bootstrap used in each test is 5000.

O.2 Power comparison: $m = 1/2$ versus $m = 1$

We have observed from Figure 4 that the adaptive weighting with $m = 1/2$ is more powerful than the constant weighting ($m = 0$). The natural question is then to ask which adaptive weighting scheme, $m = 1$ or $m = 1/2$, is better in terms of power. To investigate this question, we conduct a set of simulation with both unnormalized and normalized tests and the simulation invovles 4 common distributions: Gaussian, Bernoulli, Poisson and Student distributions. We consider the Thompson sampling (7) with $l_N = 0.2$ and consider the sampel size $N = 20,000$ and $N_1 = N_2 = 10,000$. We set the significance level to be 0.05. The distribution information can be summarized as below:

- **Gaussian:** $Y_u(0) \sim N(\theta, 1), Y_u(1) \sim N(0, 0.25)$.
- **Bernoulli:** $Y_u(0) \sim \text{Bern}(0.5 + \theta), Y_{uN}(1) \sim \text{Bern}(0.5)$.
- **Poisson:** $Y_u(0) \sim \text{Pois}(1 + \theta), Y_u(1) \sim \text{Pois}(1)$.
- **Student:** $Y_u(0) \sim \theta + \text{t}(4), Y_u(1) \sim \text{t}(10)$ where 4 and 10 are degrees of freedom in the student distributions.

In particular, we choose $\theta \in \{0, 0.01, 0.02, 0.03, 0.04\}$. The results are presented in Figure 9.

Implementation details. We center the outcome $Y_{uN}(s)$ by generating $\tilde{Y}_{uN}(s) = Y_{uN}(s) - \mathbb{E}[Y_{uN}(1)]$ and compare the test power for the average treatment effect $\mathbb{E}[\tilde{Y}_{uN}(0)] - \mathbb{E}[\tilde{Y}_{uN}(1)]$. The motivation for this operation is we do not want the test to be affected by the absolute signal strength. Asymptotically, such operation is equivalent to use $\text{WAIPW}(s)$, defind as in (50), when testing with the original data $(Y_{uN}(0), Y_{uN}(1))$. This is because the magnitude of θ is very small. Also, we want to point out that when $\mathbb{E}[\tilde{Y}_{uN}(s)] \sim 1/\sqrt{N}$ and $\mathbb{E}[\tilde{Y}_{uN}(1)] = 0$ statistic $\text{WIPW}(s)$ with $m = 1$ is asymptotically equivalent to the sample mean, as proved in Lemma 27. In other words, the asymptotic power function for the sample mean test statistic is the same as the power function for the $m = 1$ weighting when testing is performed on the transformed data $(\tilde{Y}_{uN}(0), \tilde{Y}_{uN}(1))$.

Interpretation of the results. For all the other setups, it seems both $m = 1$ and $m = 1/2$ have very similar power performance. This means we would expect the sample mean test statistic should also have very comparable power performance with the weighting $m = 1/2$. It is generally unclear if the test statistics considered in this paper are optimal or not under these distributions. As an exception, the sample mean test stiatistic has been shown to be near-optimal in some Gaussian setup, shown in Section 4.4 of Hirano et al. (2023).

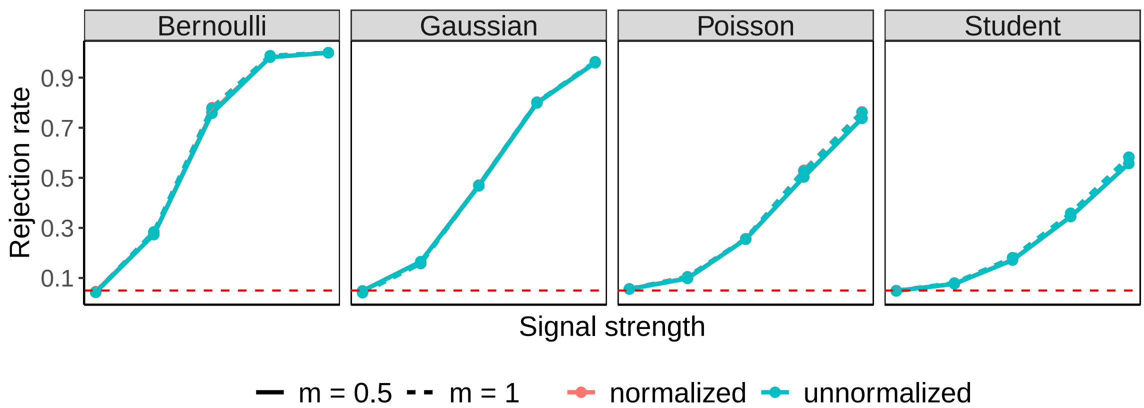

Figure 9: Rejection plots for adaptive weighting with $m = 1$ and $m = 1/2$ on the centered data $(\tilde{Y}_{uN}(0), \tilde{Y}_{uN}(1))$.

P Additional semi-synthetic data analysis results

We present additional results for the semi-synthetic data analysis in Section 4.2. Following the same procedure outlined in Section 4.2, we use 5000 permuted sample to compute the p-values for the 5 tests. The QQ-plot is shown in Figure 10. We can see the message is similar to the one in Figure 7 in Section 4.2 when there is no signal.

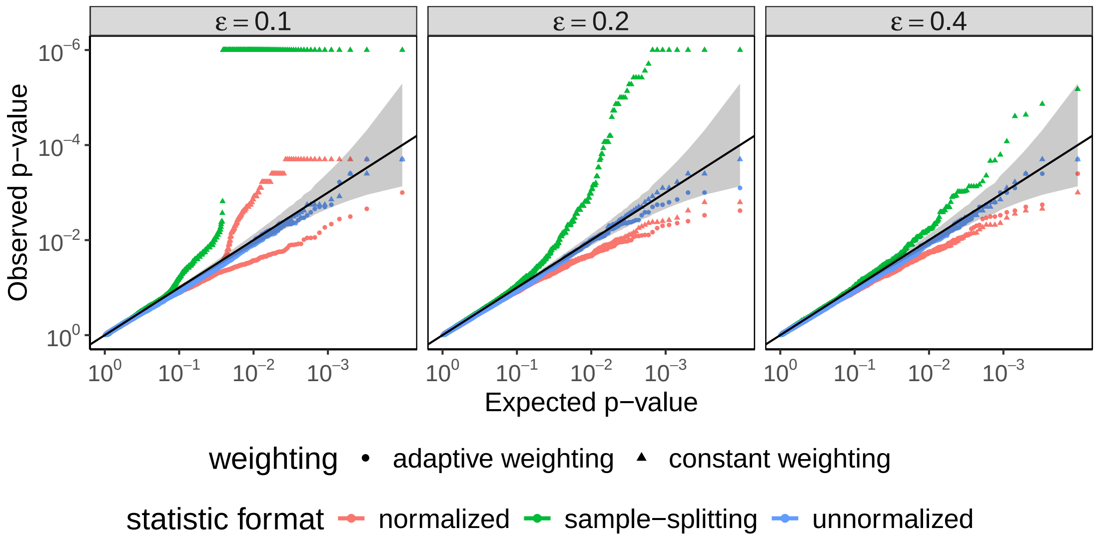

Figure 10: QQ-plot for the semi-synthetic data analysis.