Detect model miscalibration via your nearest neighbor

Bernoulli-*ims*
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Ziang Niu

Collaborators

Anirban Chatterjee

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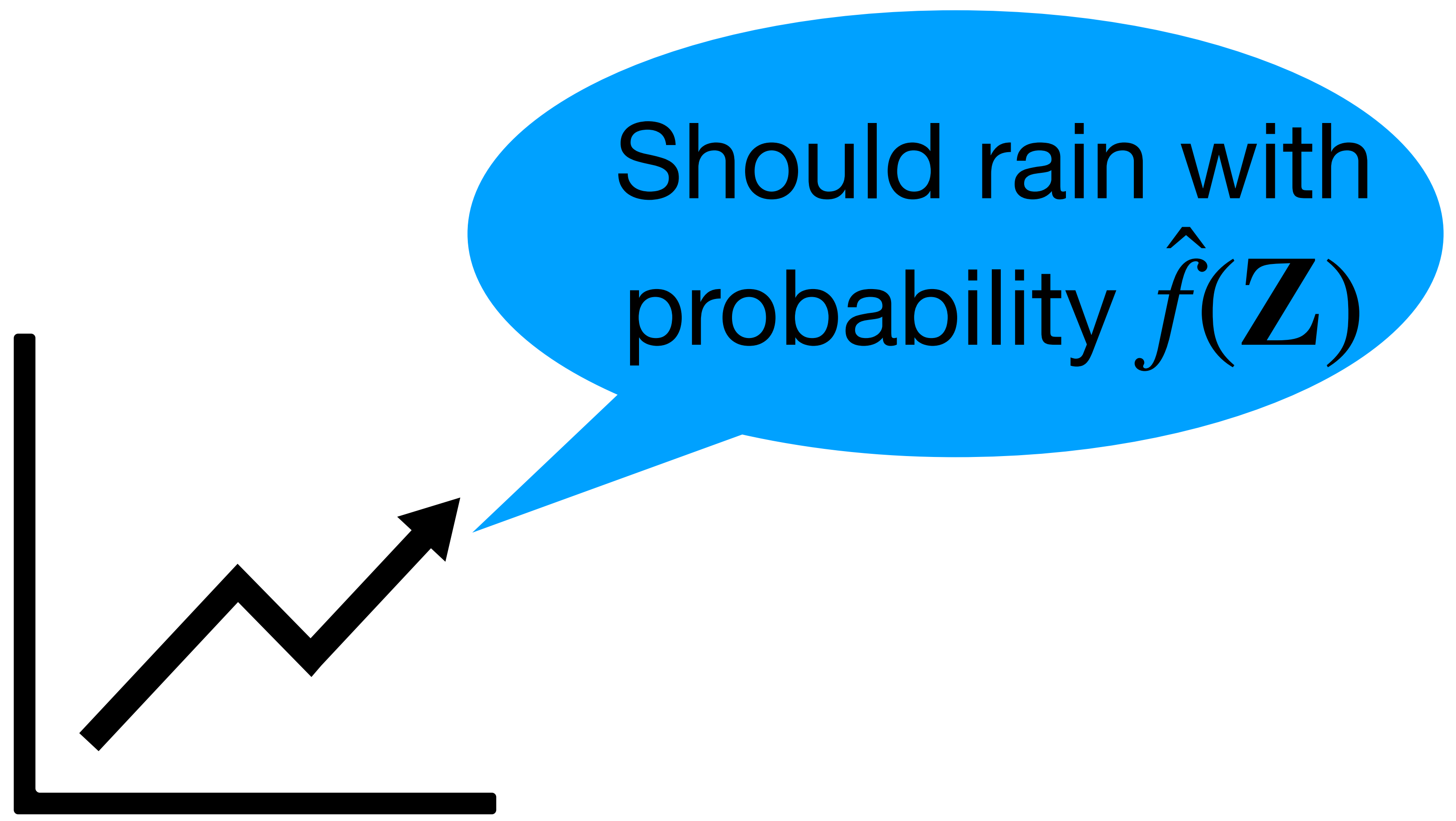

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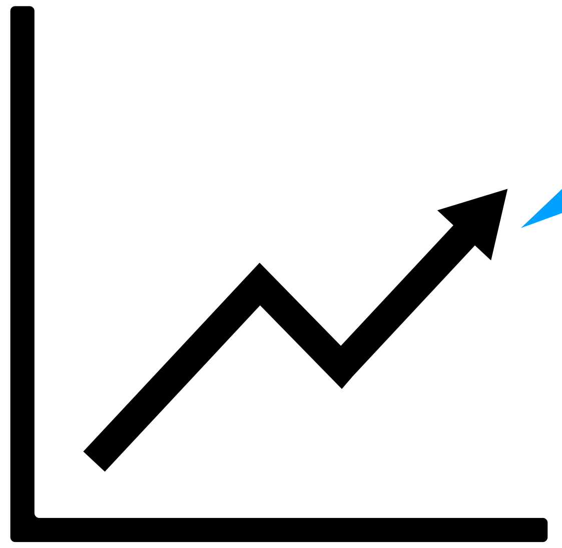

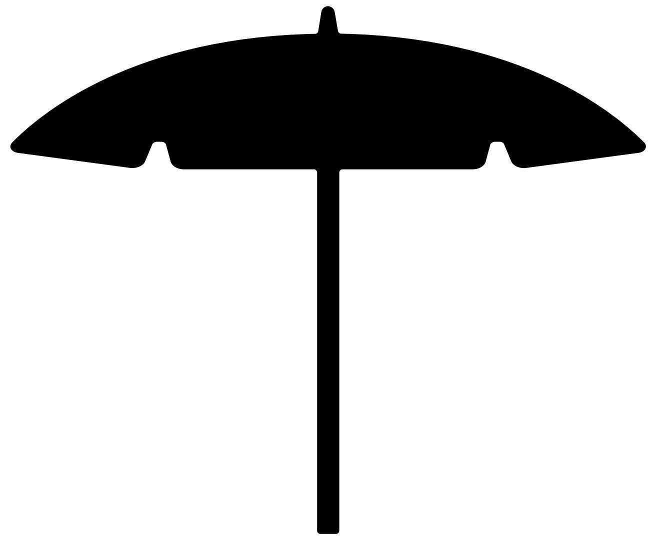

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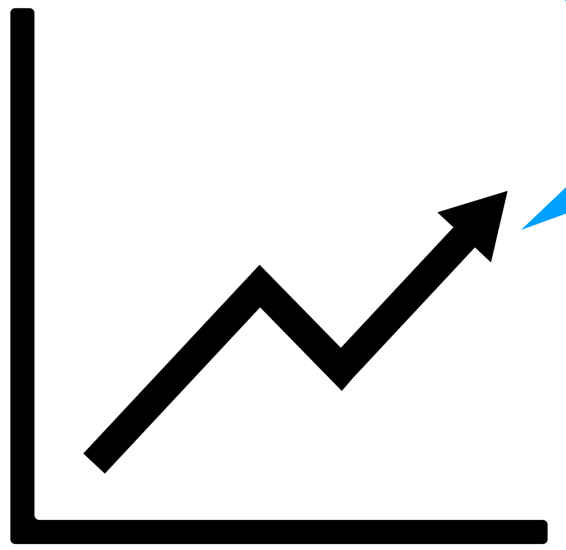

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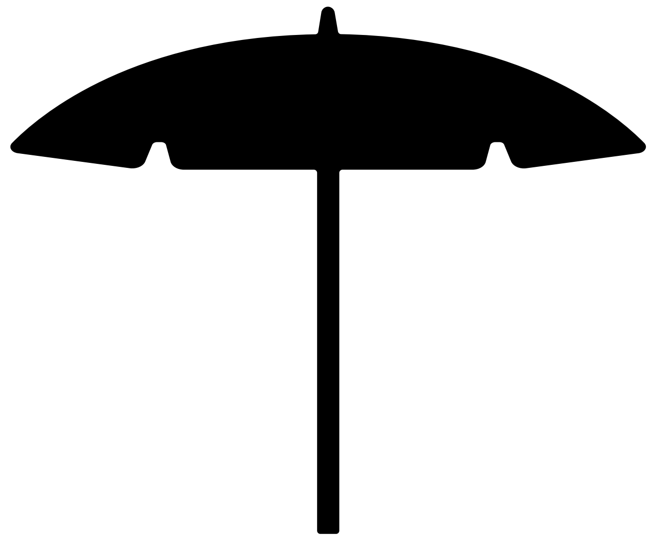

If $w \approx p$, it is a reliable prediction at prediction w.

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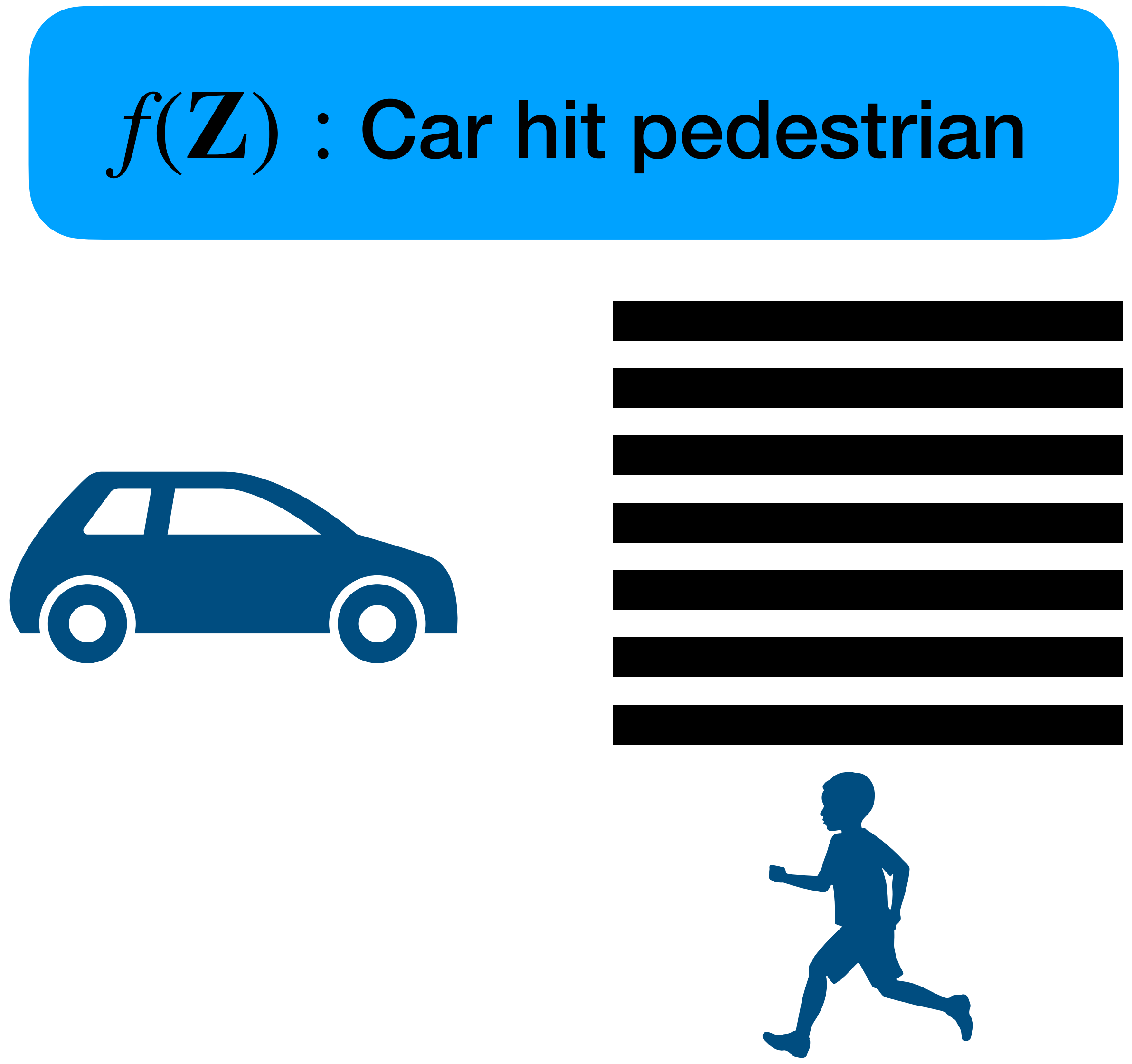

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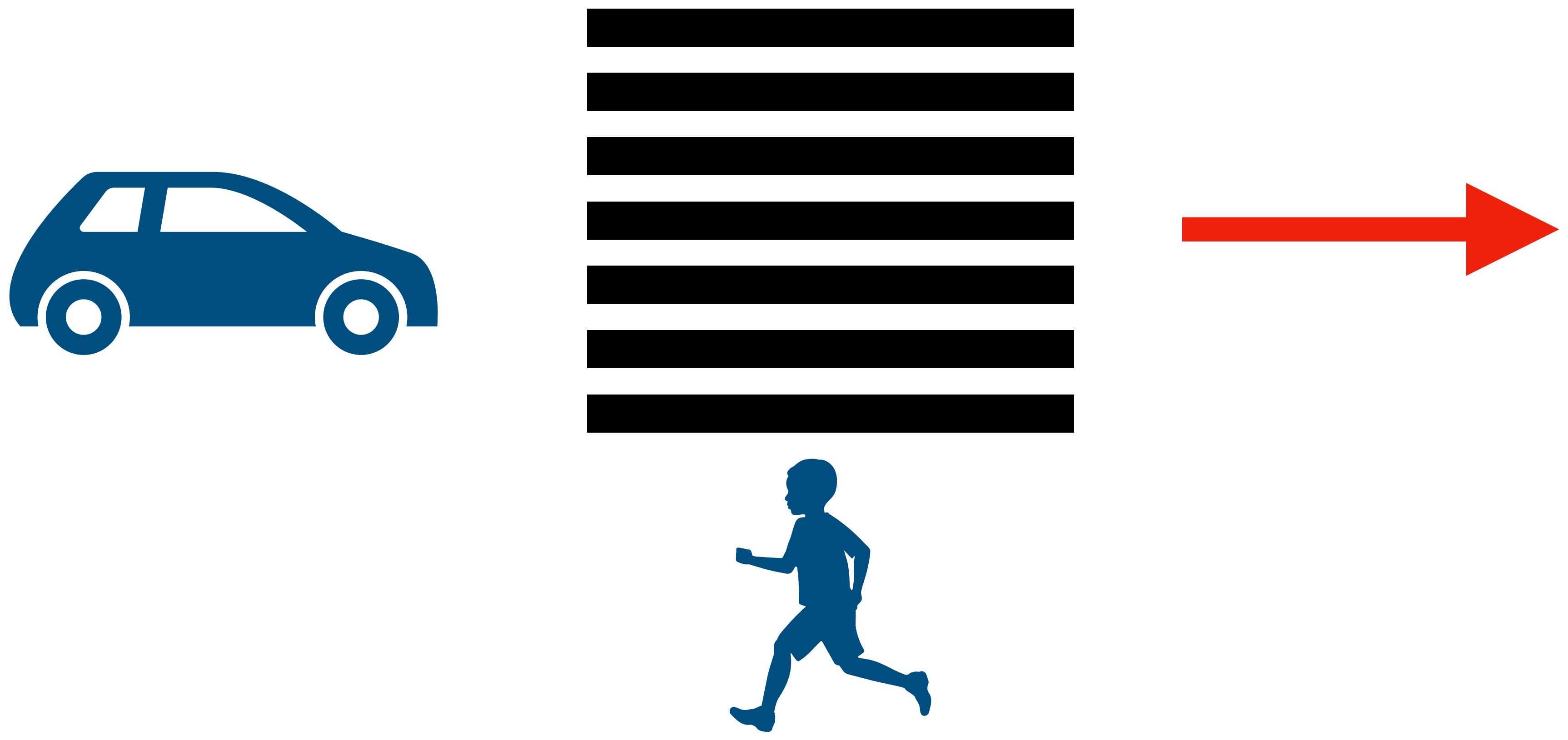

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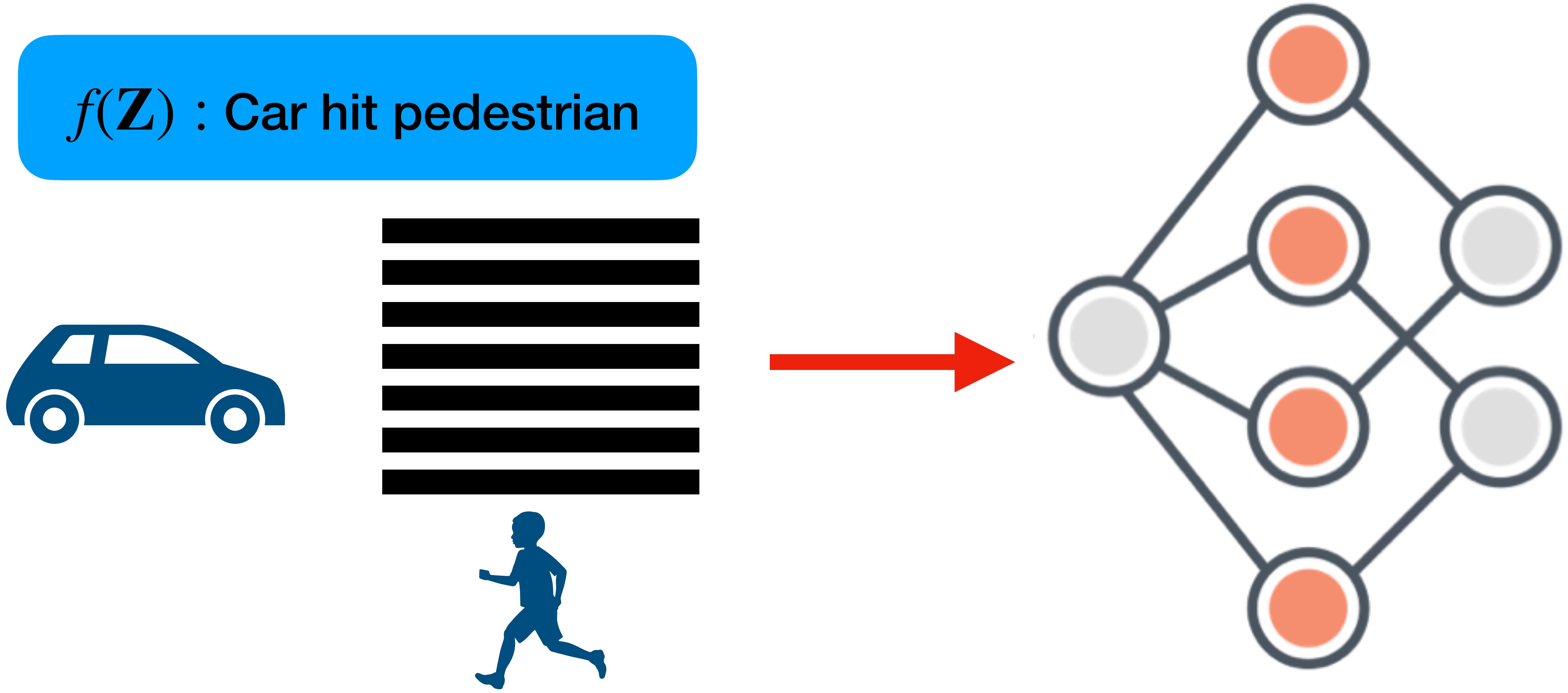

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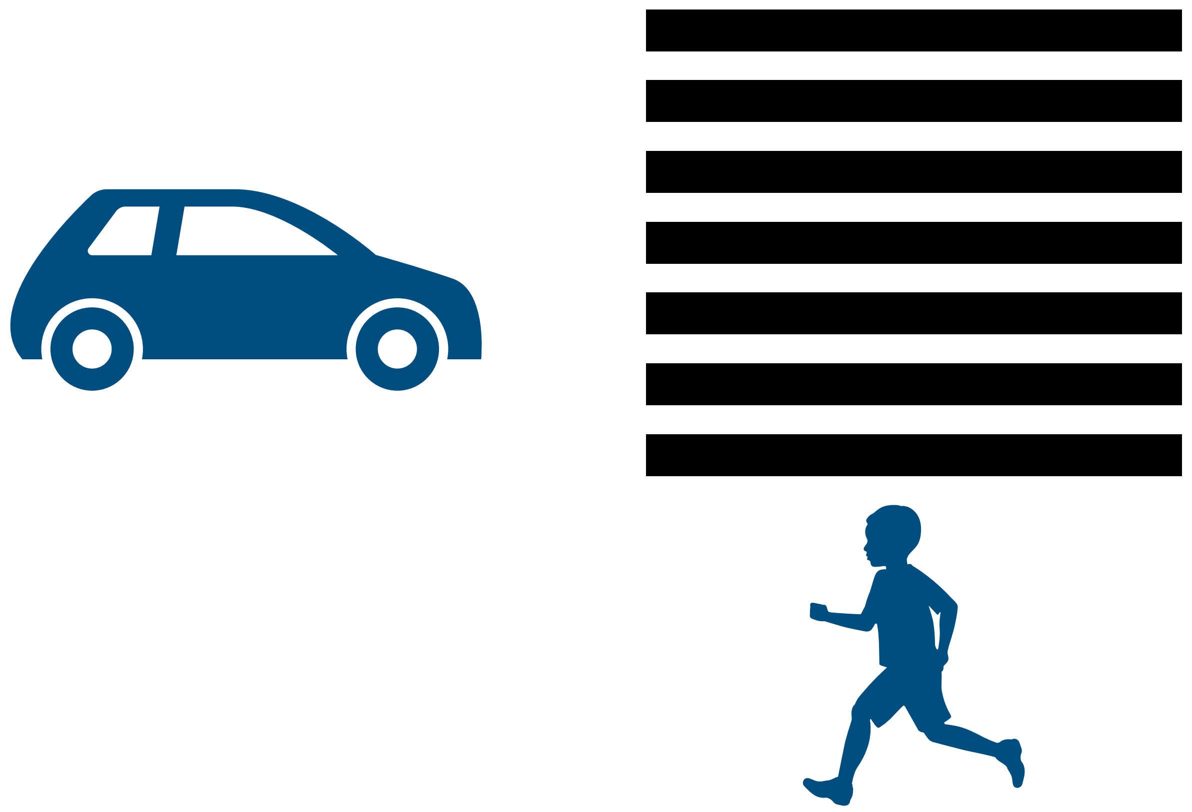

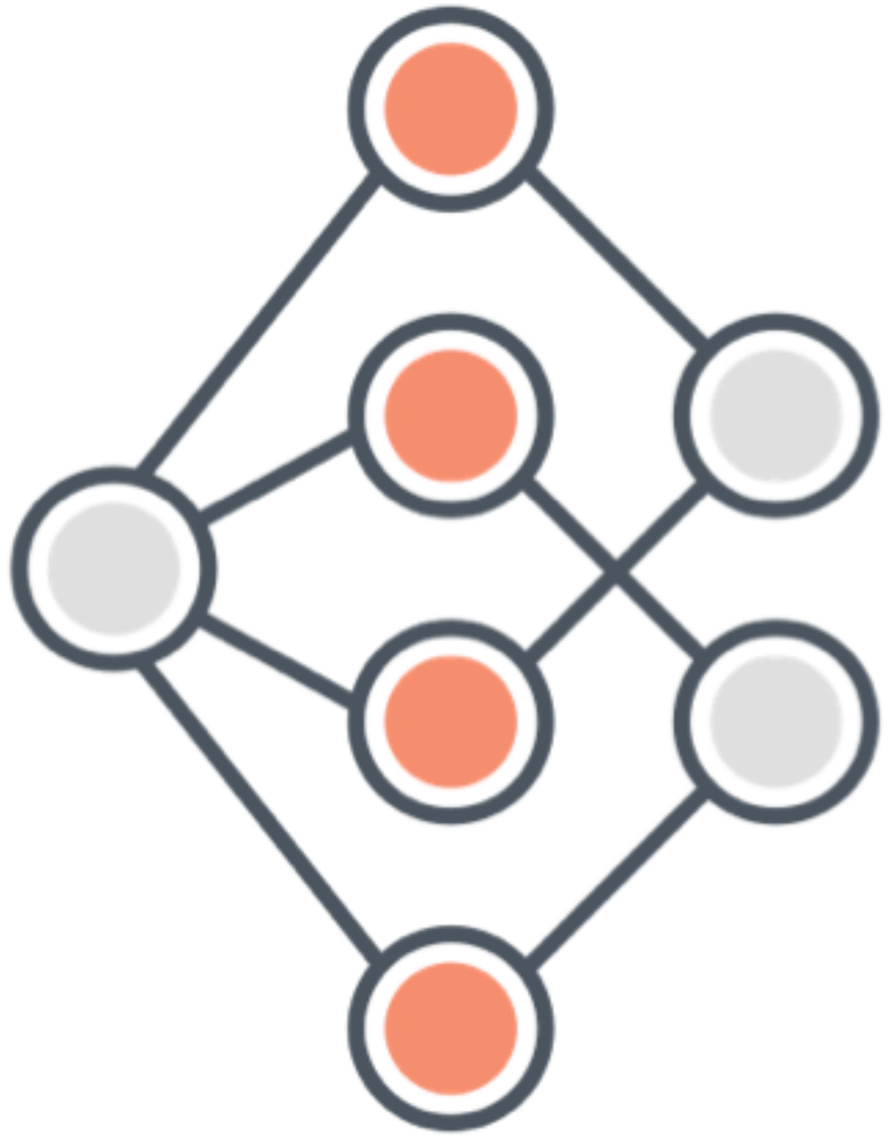

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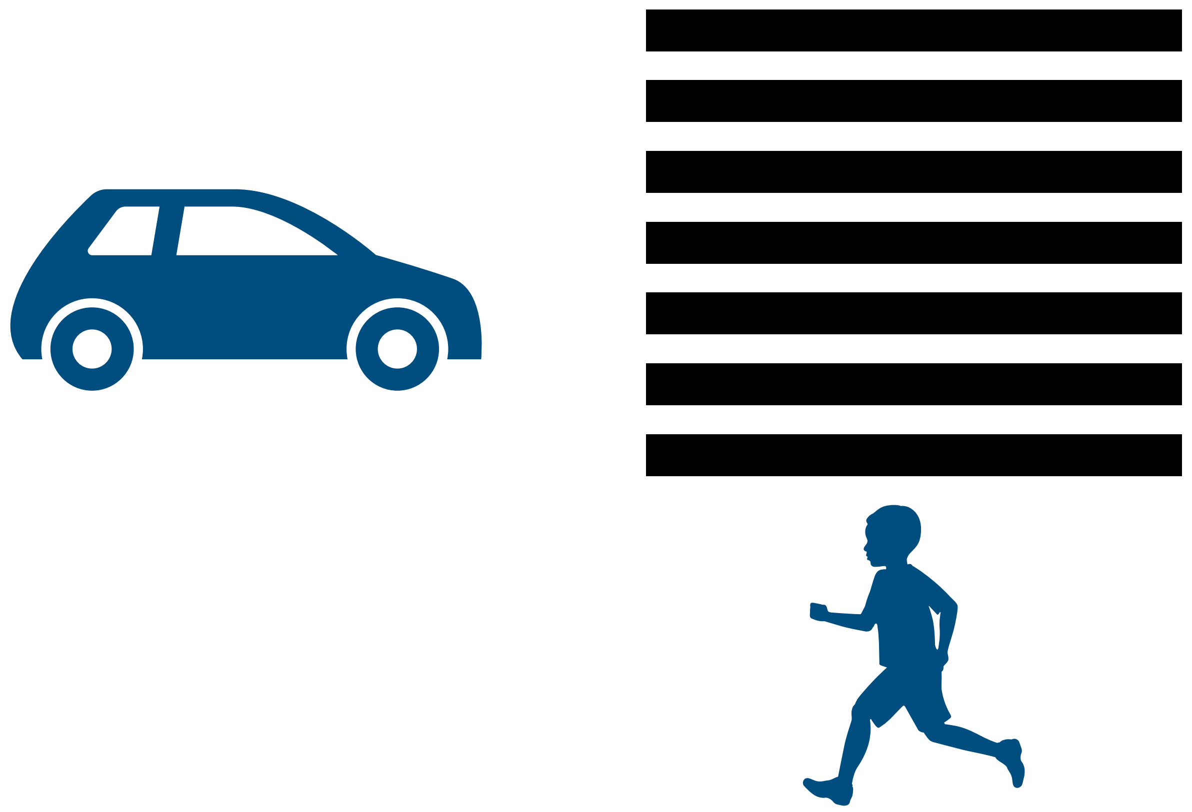

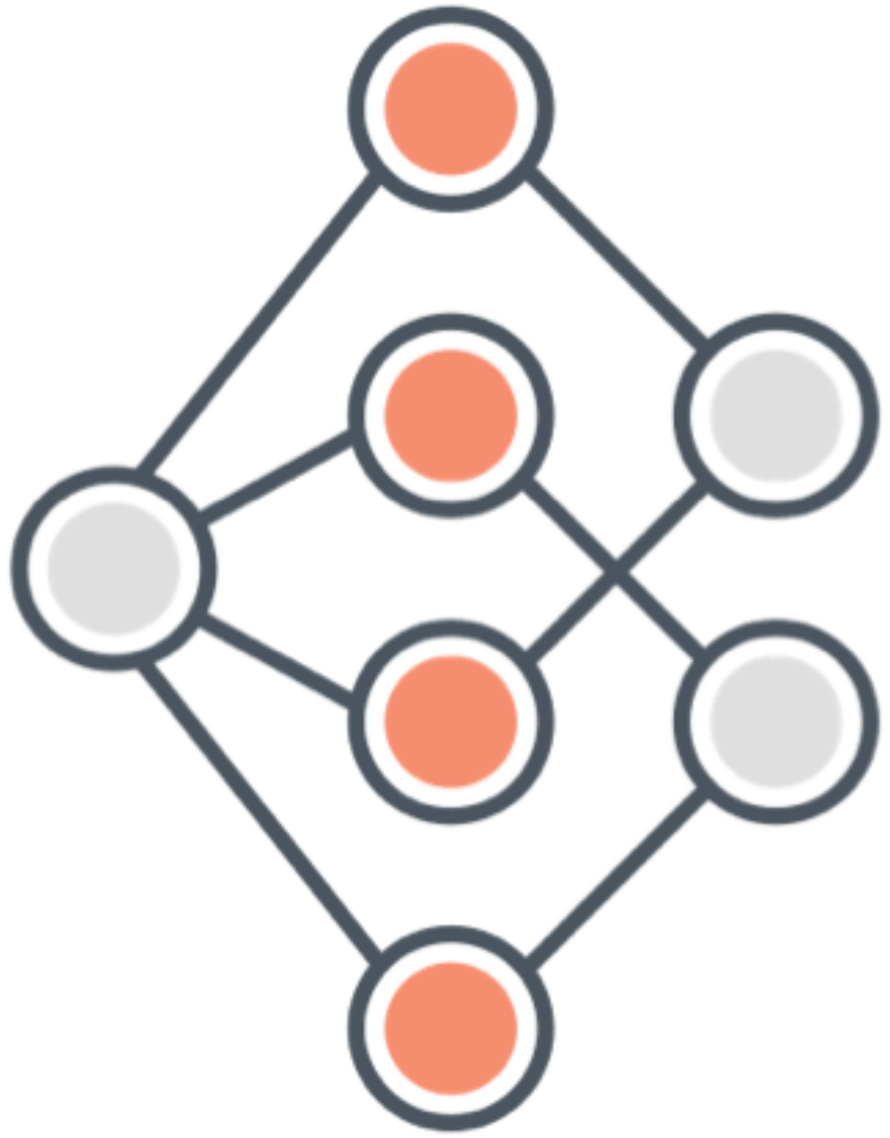

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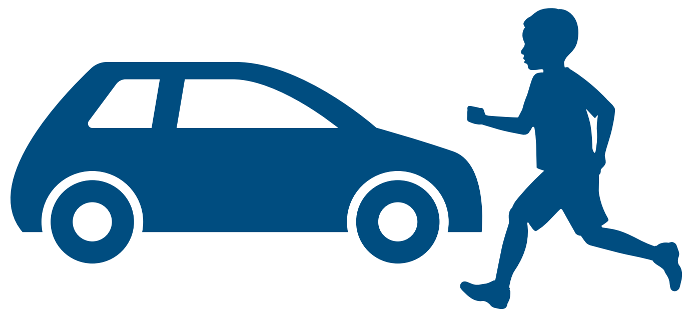

$\mathbb{E}[\mathbf{Y} \mid \hat{f}(\mathbf{Z}) = 0.0001] = 0.1$

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(Widmann et. al. 2019, NeurIPS; Widmann et. al. 2021, ICLR) SKCE method

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Intractable distribution $\sum_{m=1}^{\infty} \lambda_m (Z_k^2 - 1)$ **versus "nice" distribution** $N(0,1)$**.**

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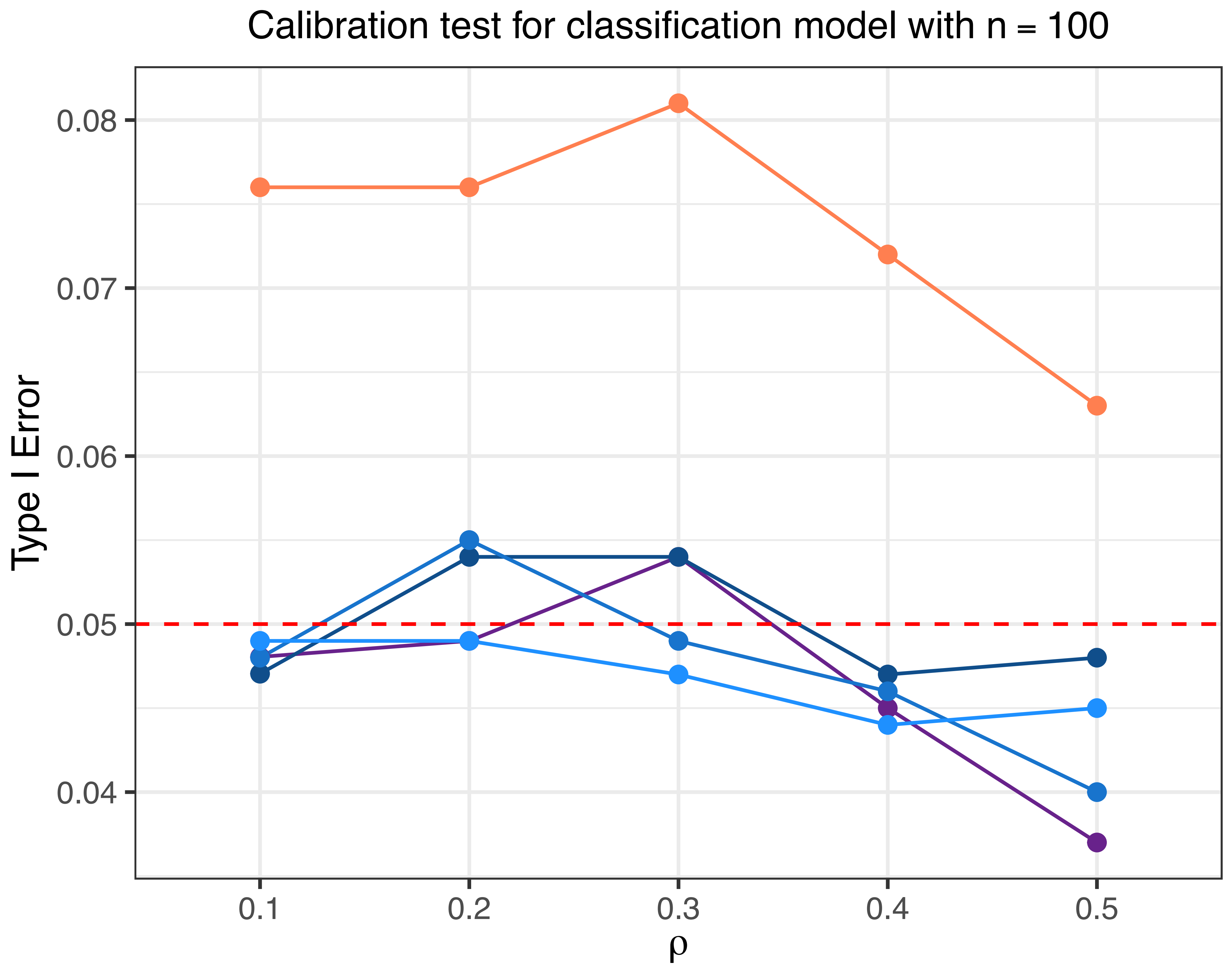

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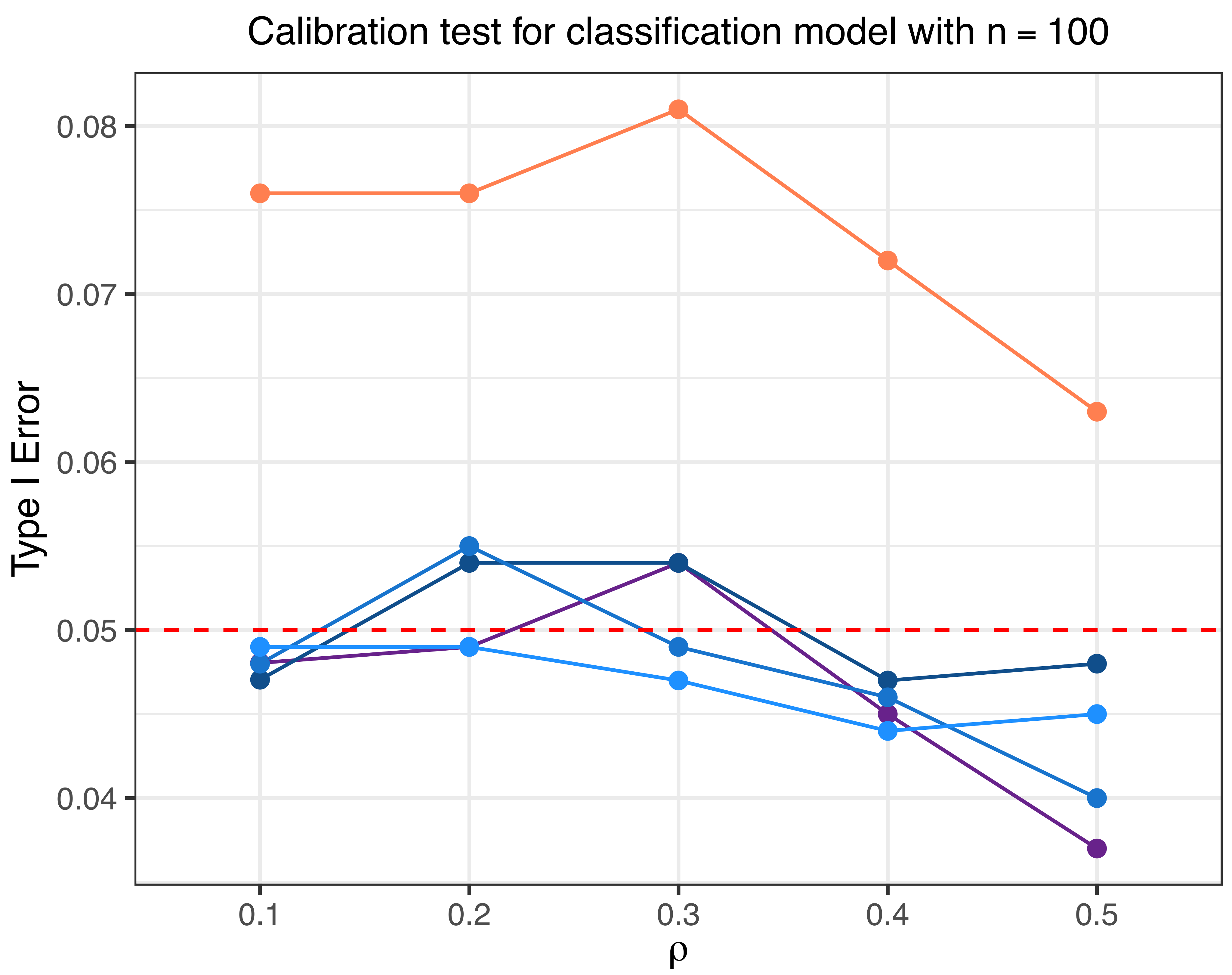

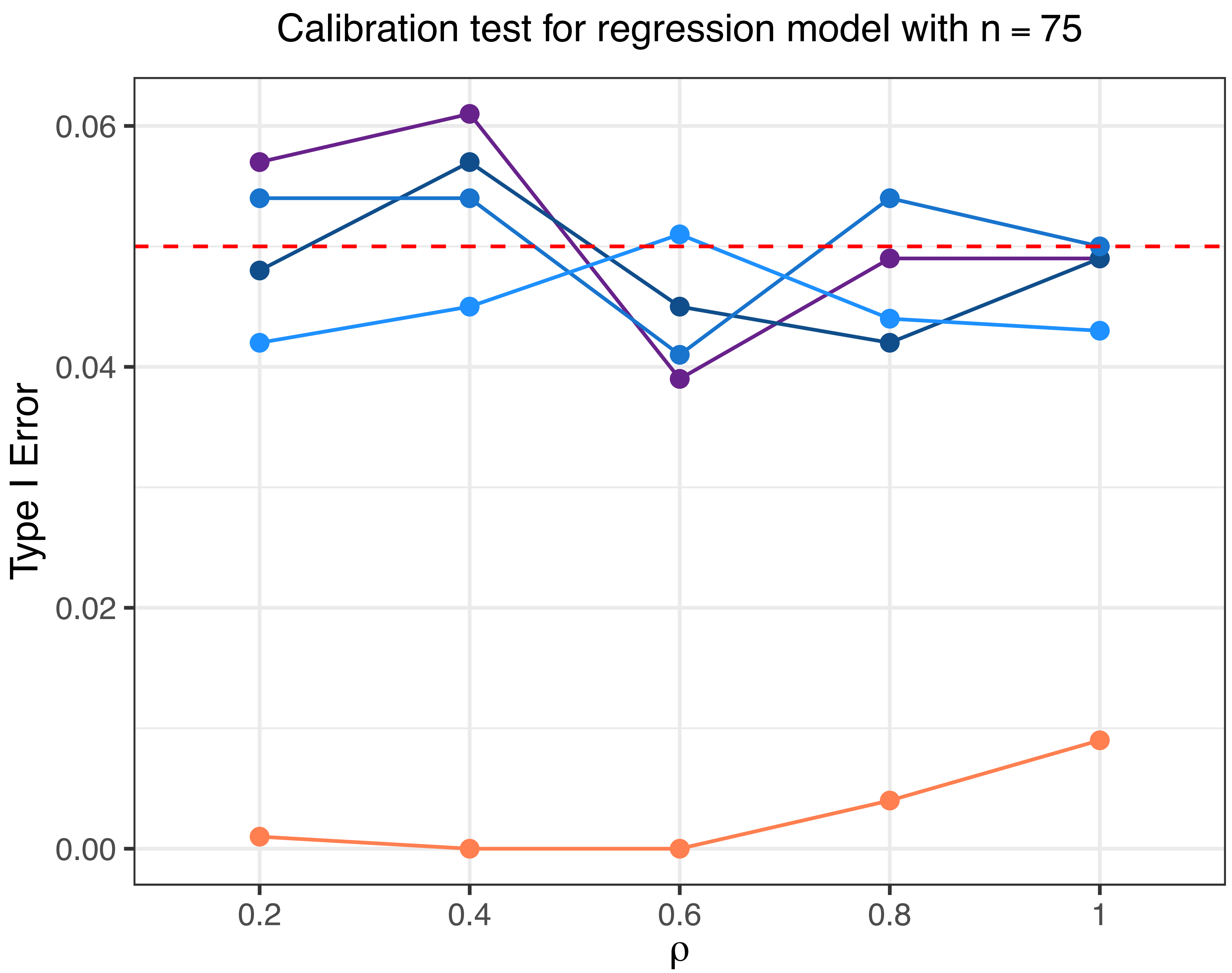

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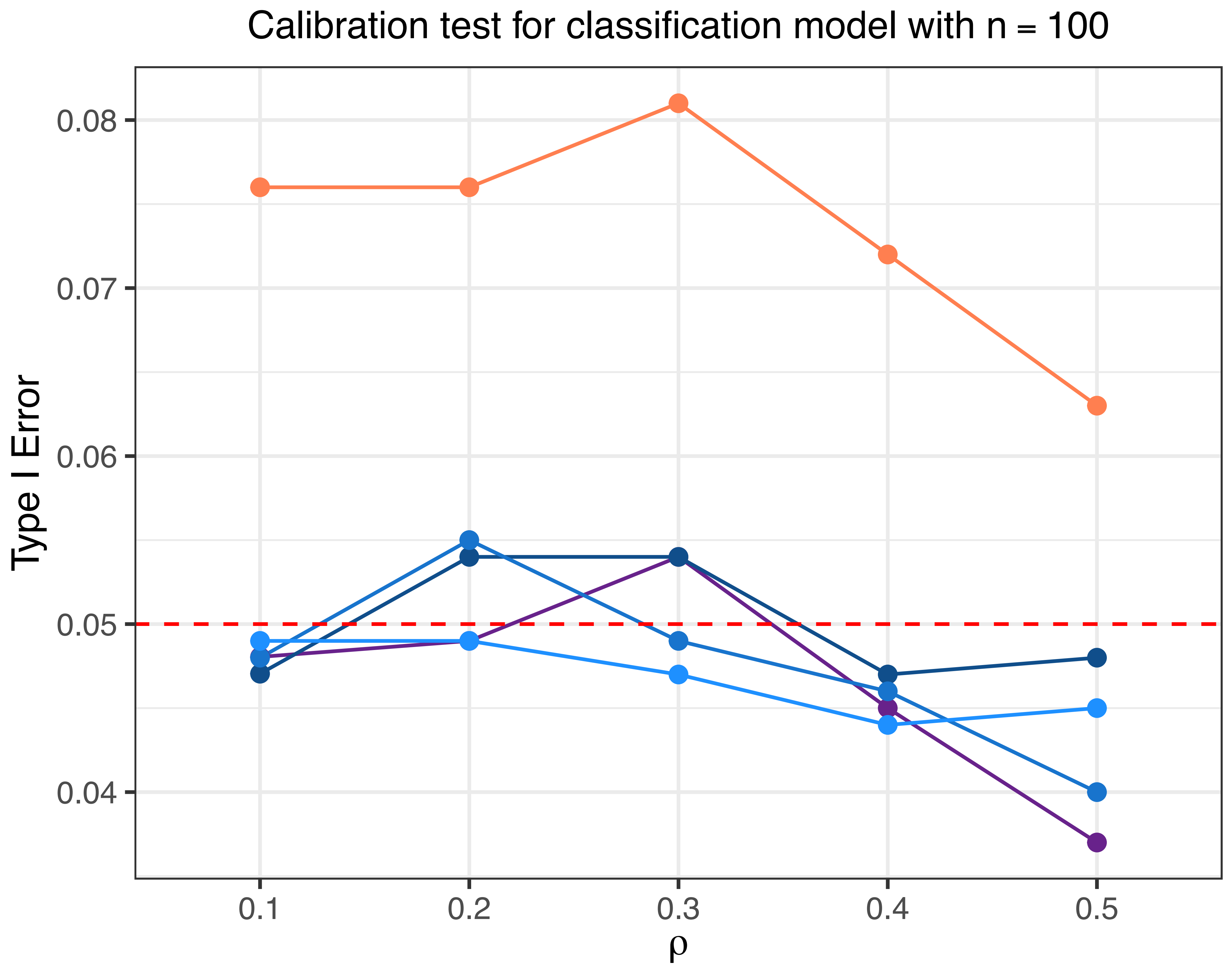

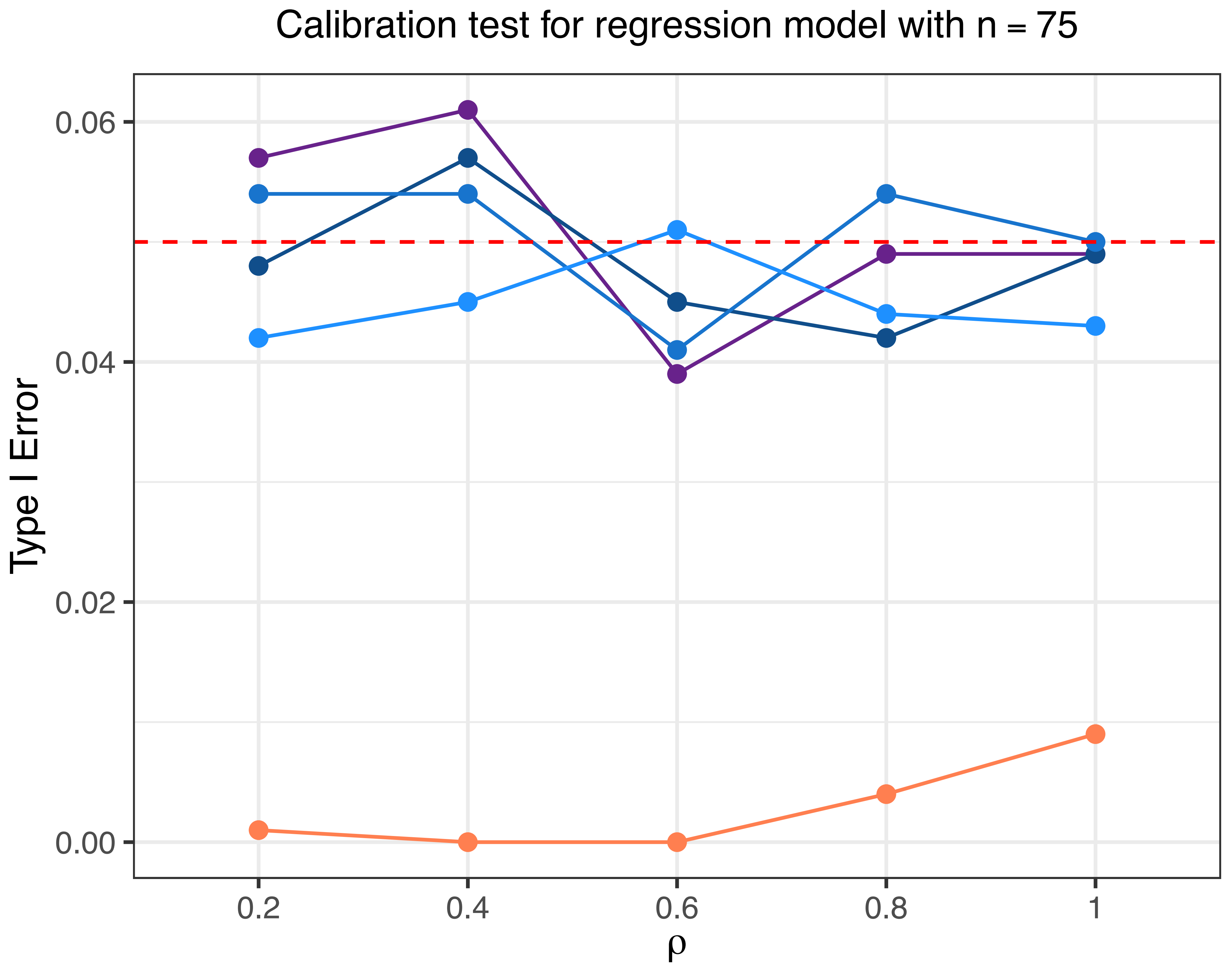

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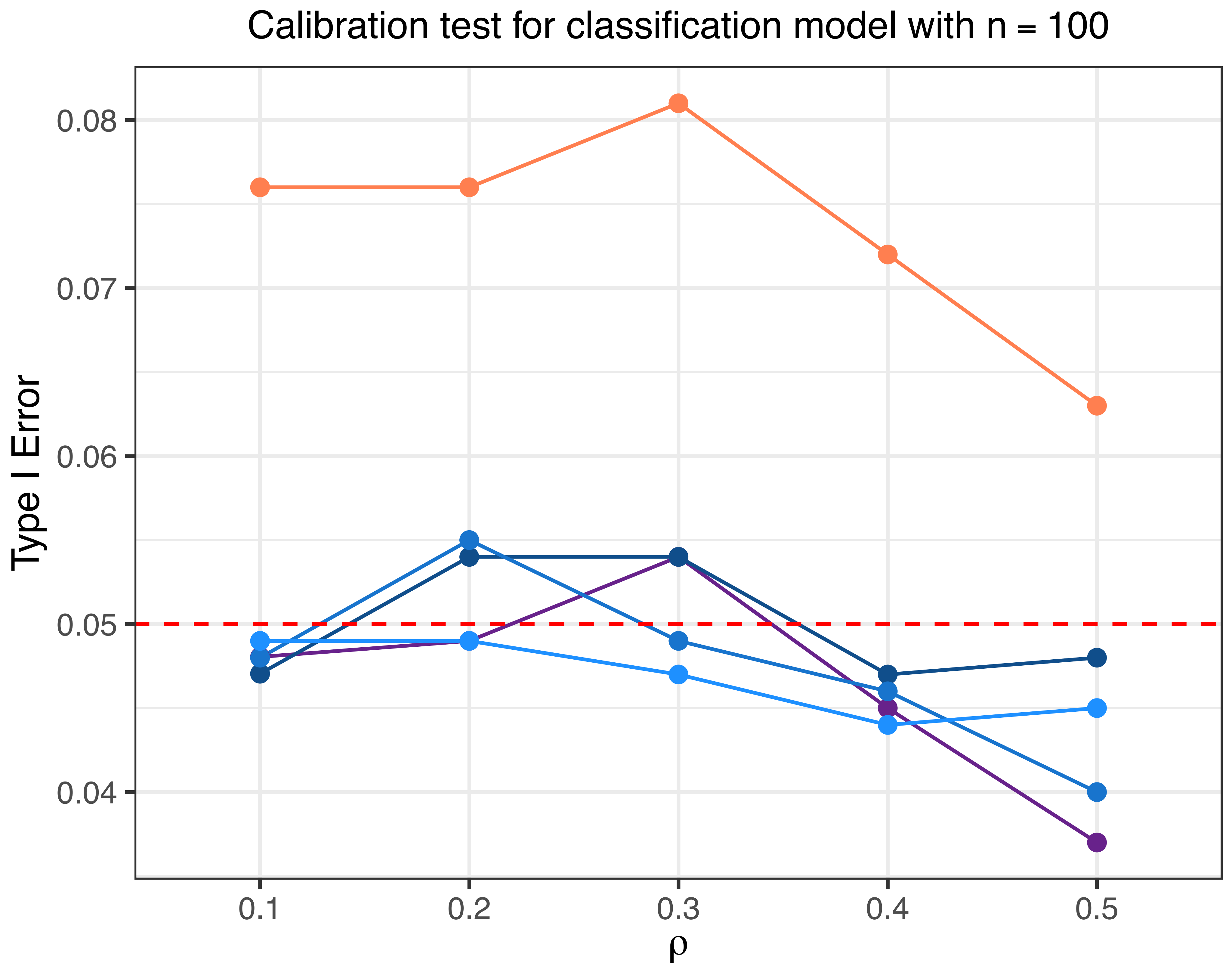

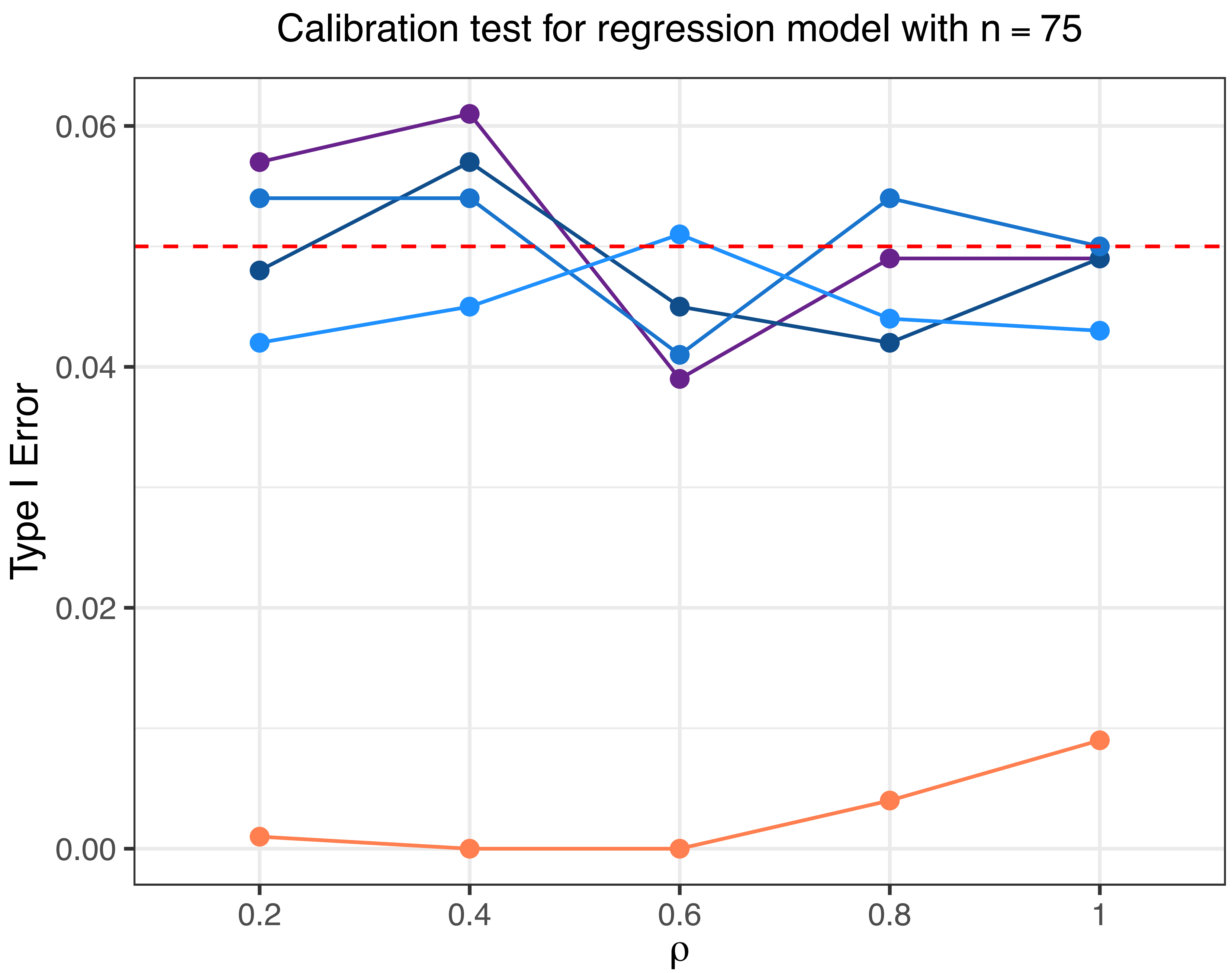

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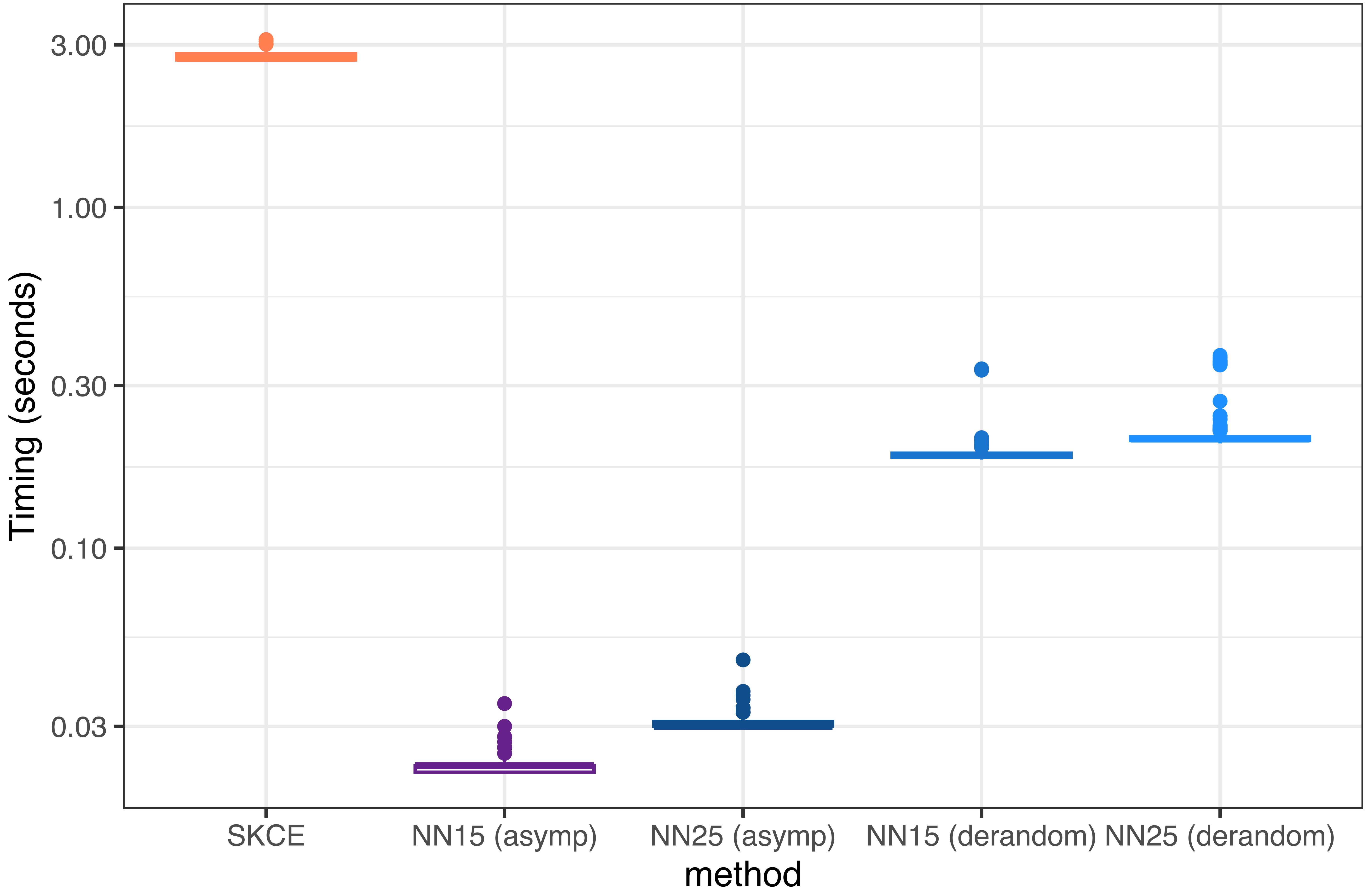

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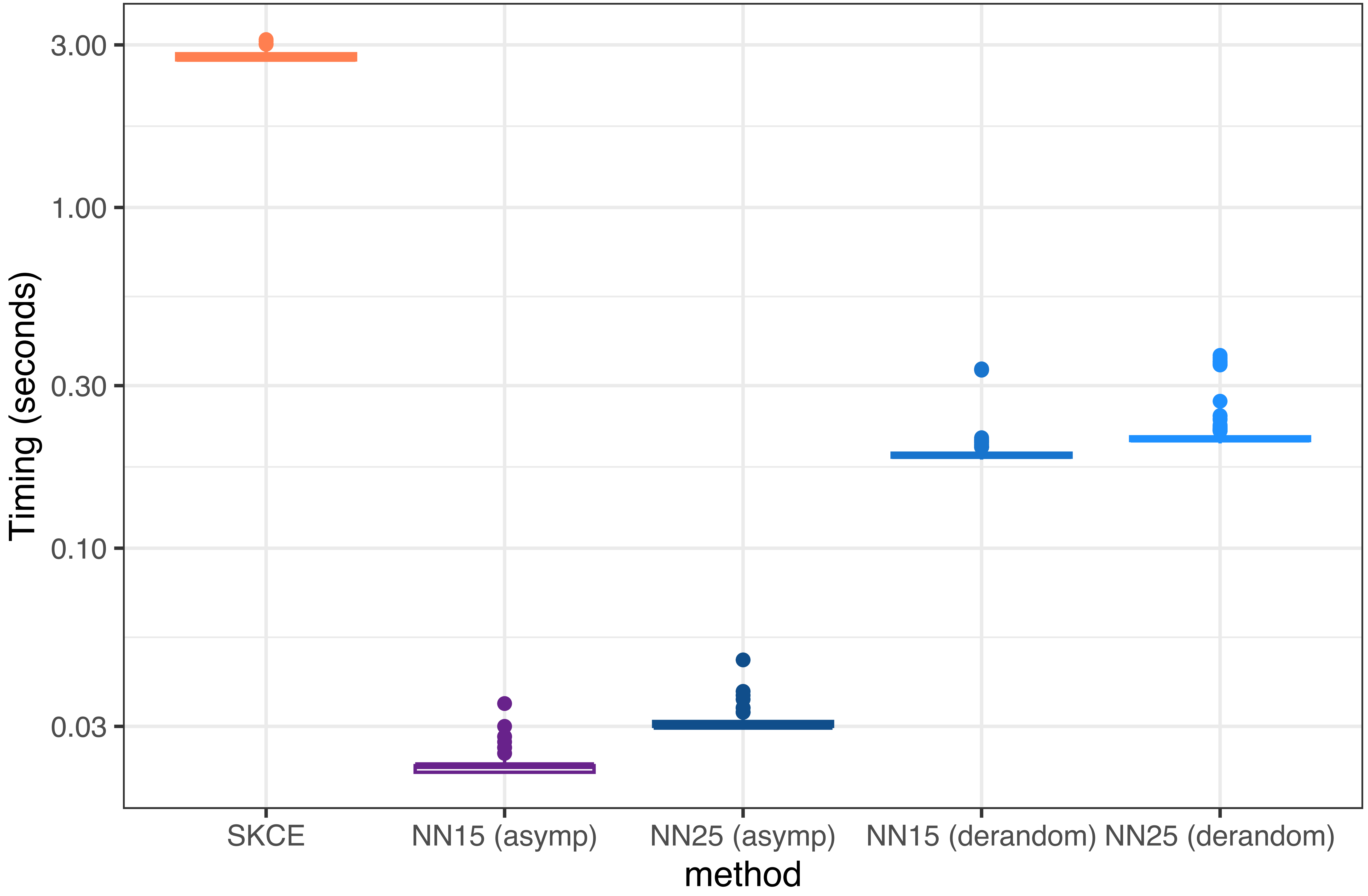

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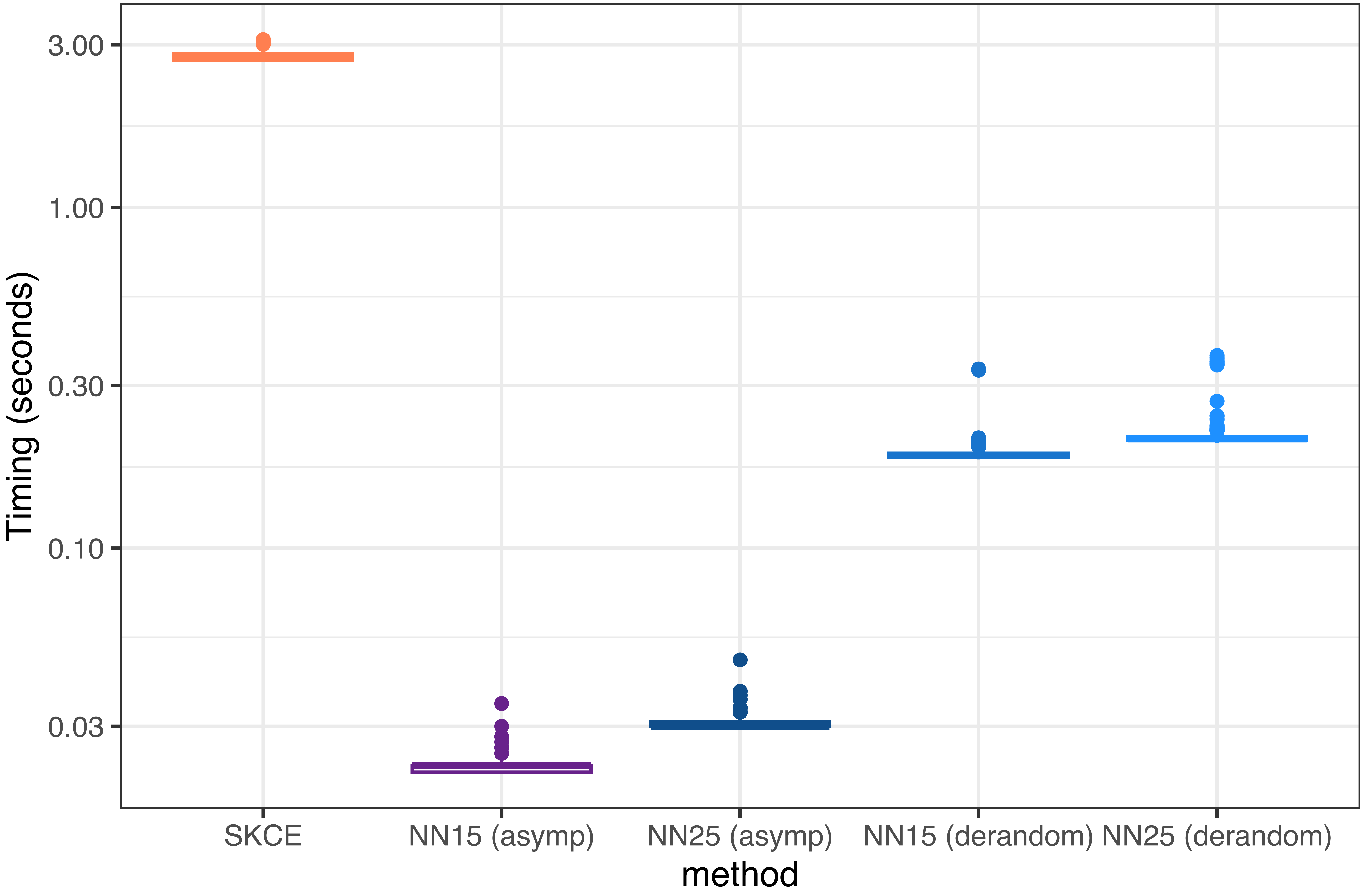

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Linear kernel: $K(x, y) = x \cdot y$ and $\mu_{\mathbb{P}}(y) = \mathbb{E}_{X\sim\mathbb{P}}[X] \cdot y$.

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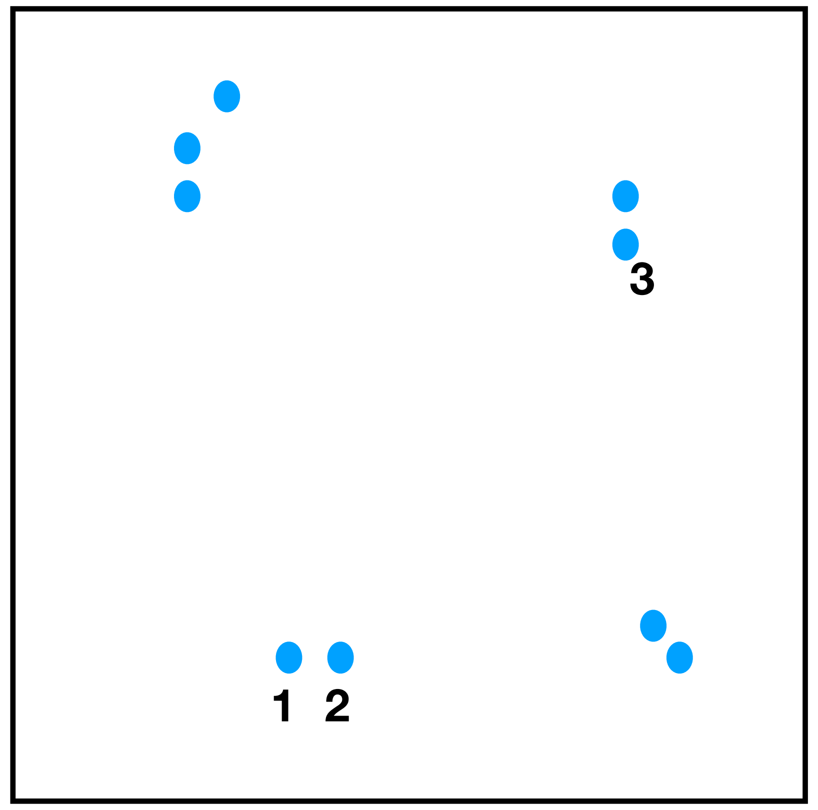

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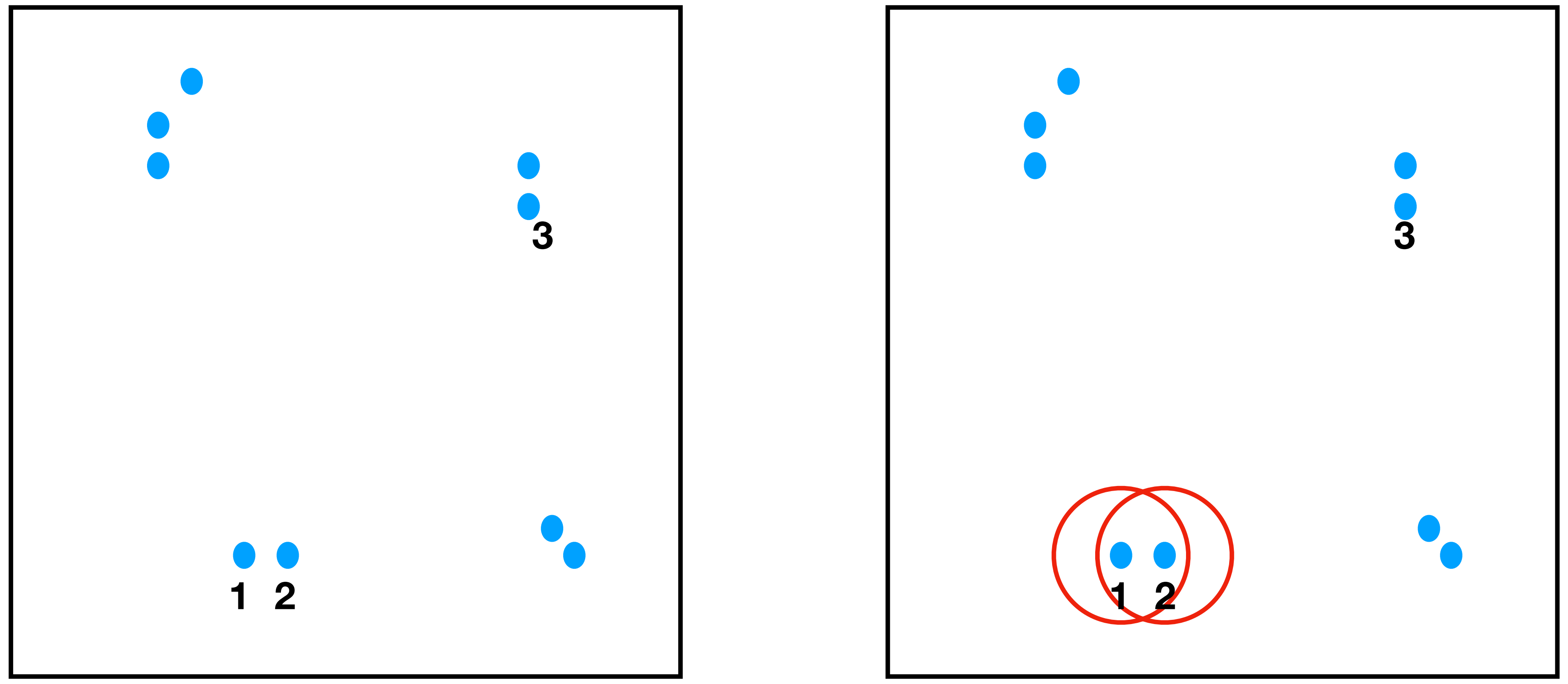

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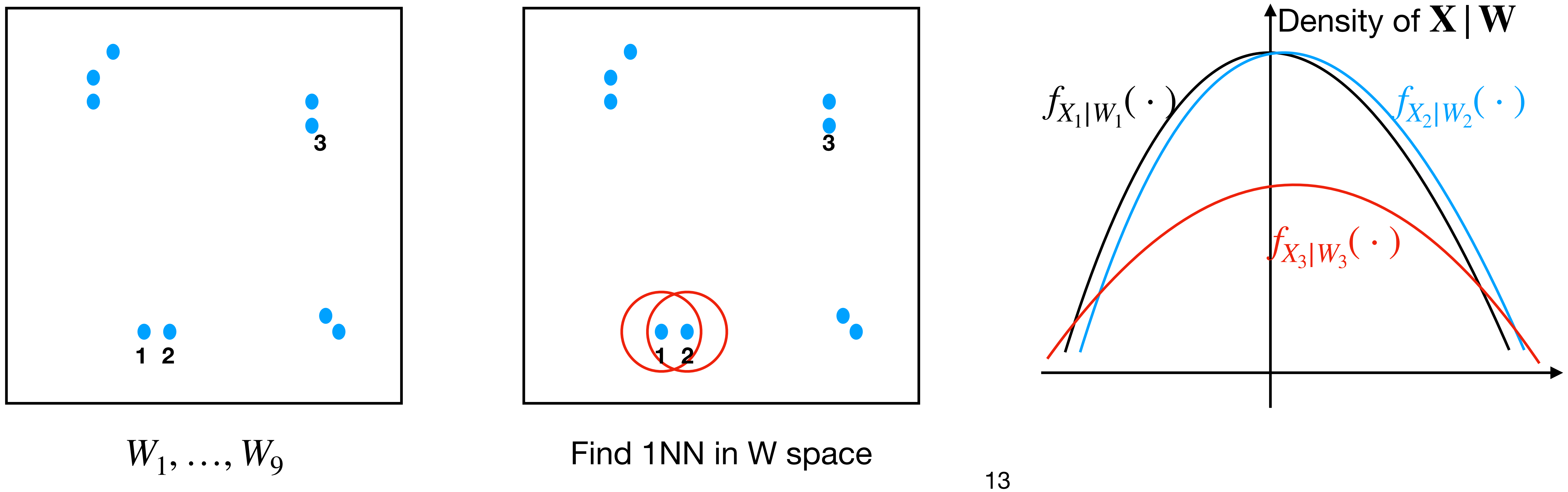

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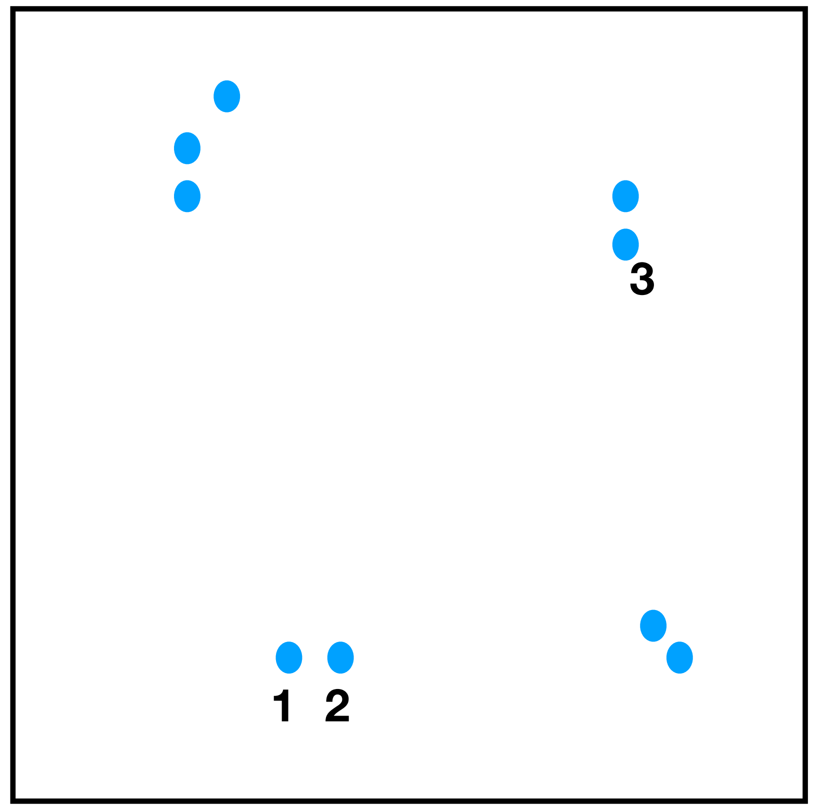

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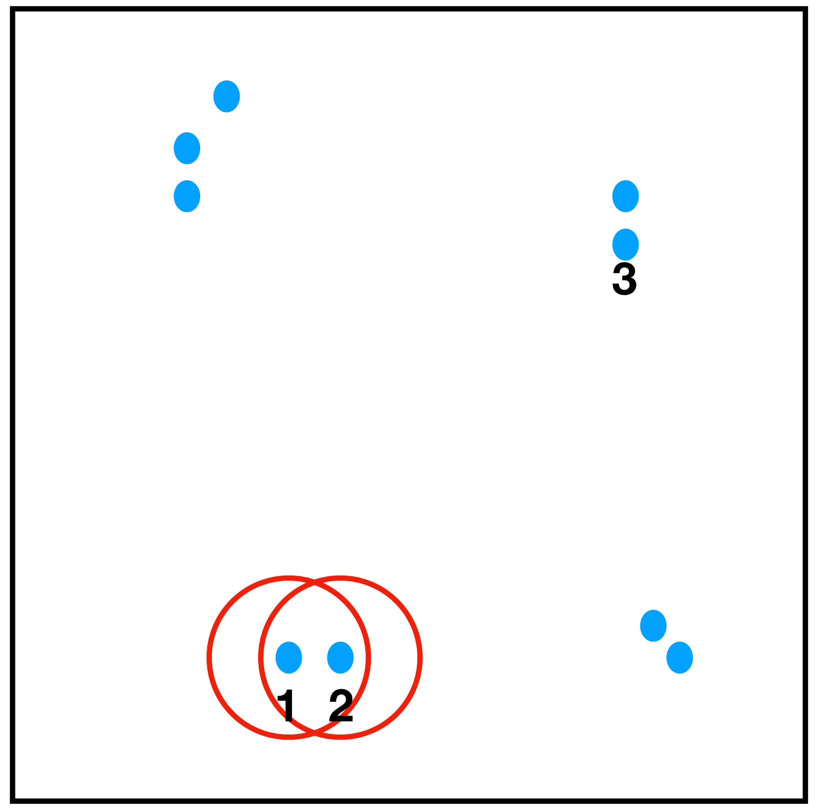

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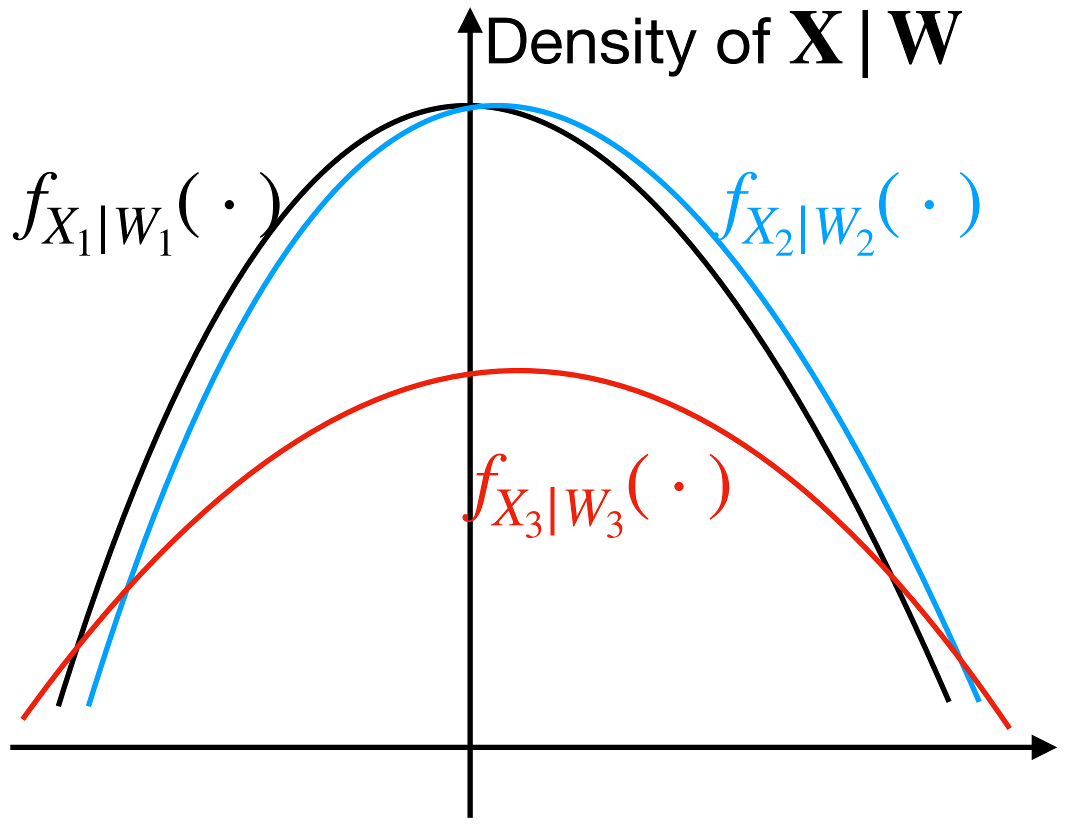

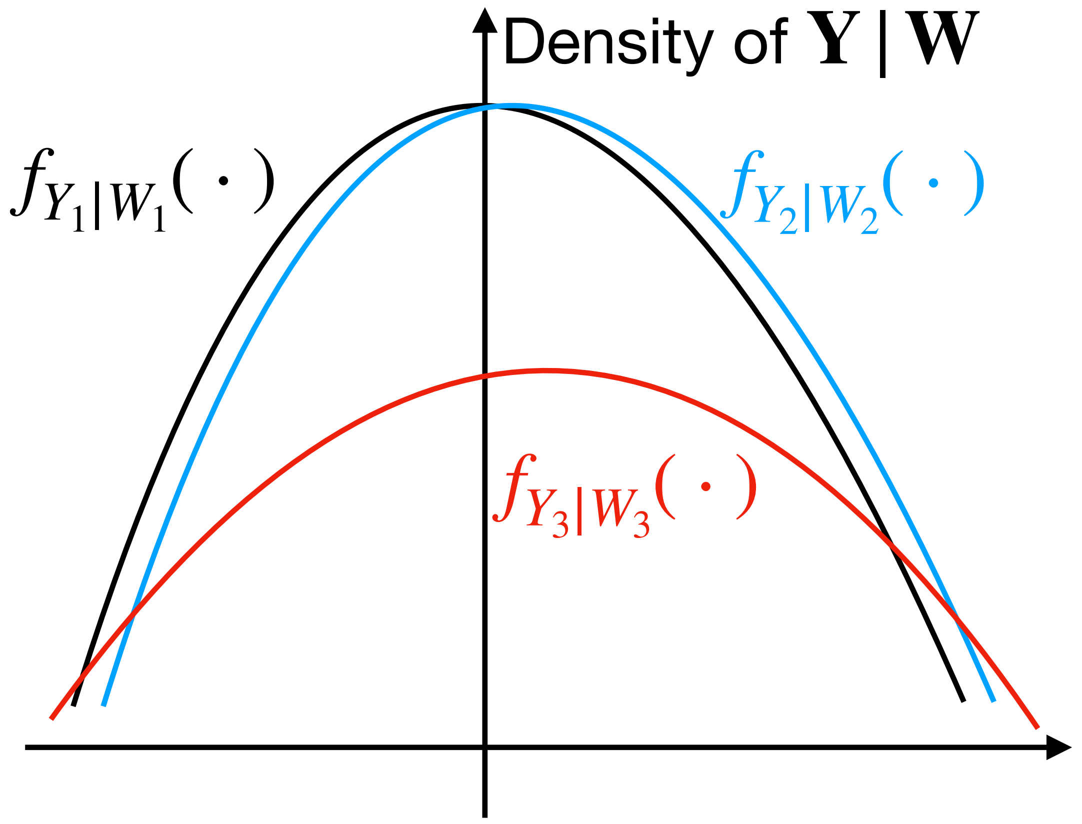

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Stein's method for dependency graph + dedicate analysis on $\hat{\sigma}_n$!

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Sampling $X_i \sim \mathbb{P}_{\mathbf{X}_i|\mathbf{W}_i}$ **will induce a random test!**

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2. Get M_n samples $(\widetilde{X}_i^{(1)}, W_i)_{i=1,\ldots,n}, \ldots, (\widetilde{X}_i^{(M_n)}, W_i)_{i=1,\ldots,n}$ from $\mathbb{P}_{X_i|W_i}$

$$T^{(m)} \equiv \frac{1}{n}\sum_{i=1}^{n}\frac{1}{k_n}\sum_{j\in\mathcal{N}(i)} H((Y_i, \widetilde{X}_i^{(m)}), (Y_j, \widetilde{X}_j^{(m)})), \; \widetilde{T} = \frac{1}{M_n}\sum_{m=1}^{M_n} T^{(m)};$$

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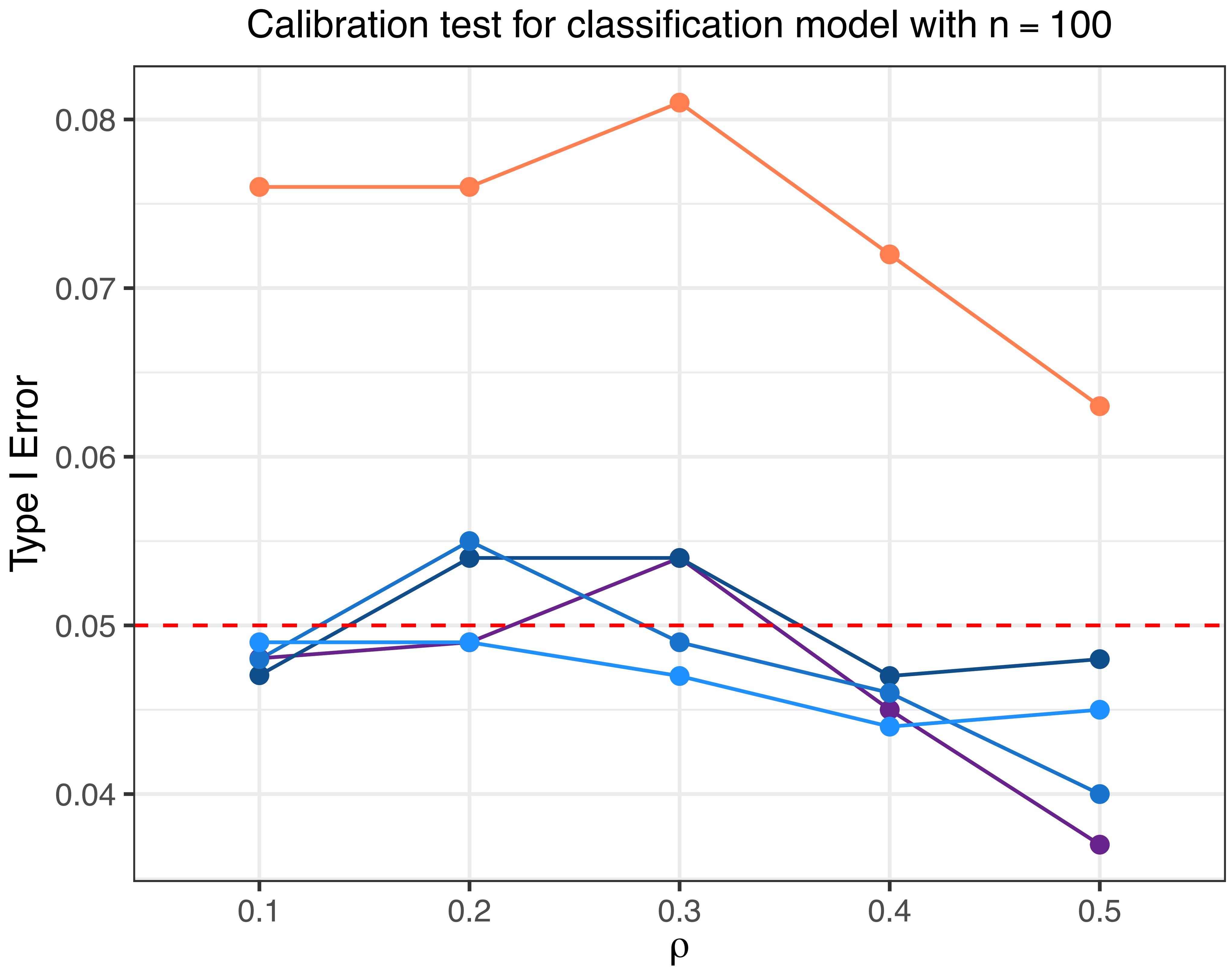
Classification calibration

Classification calibration



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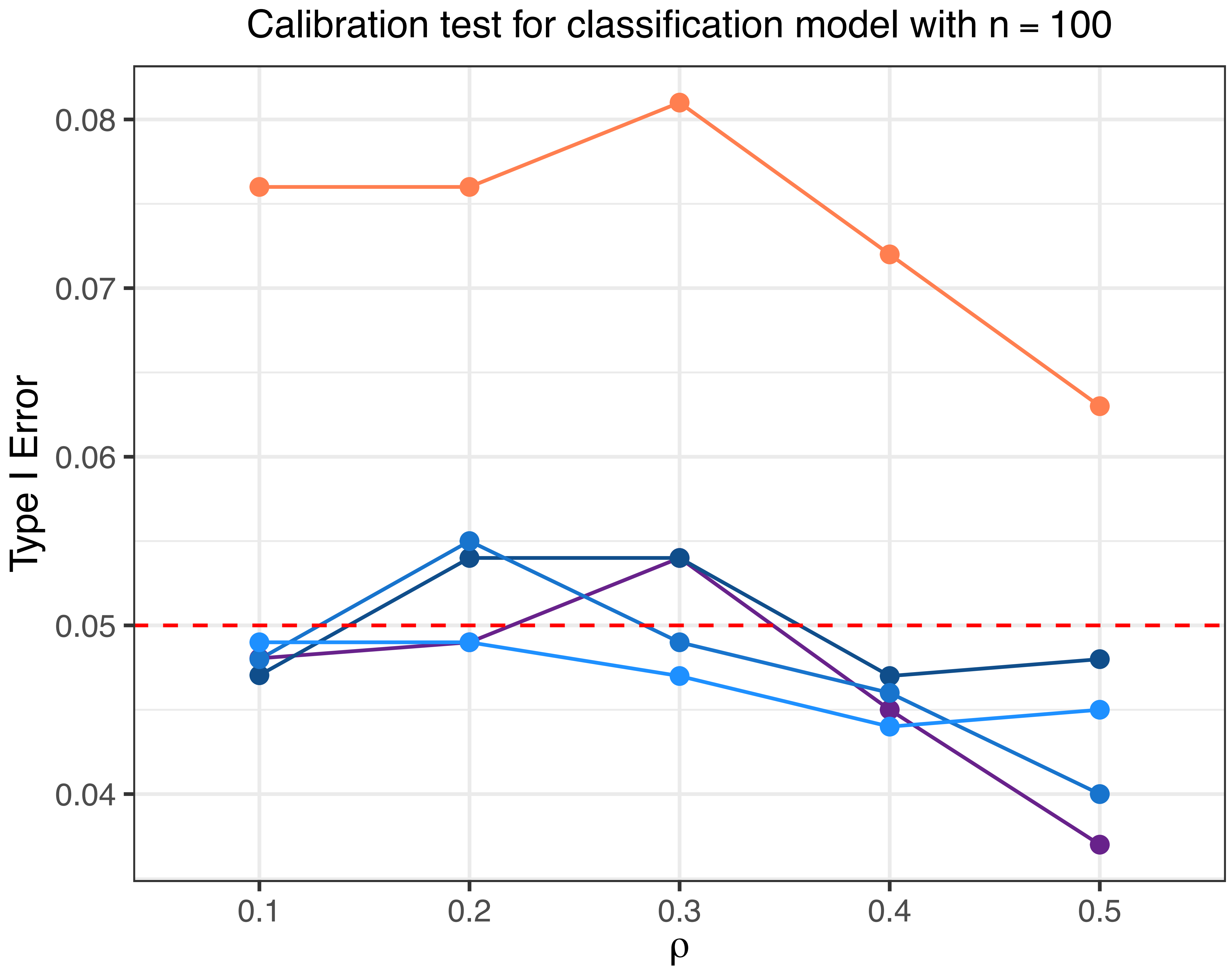

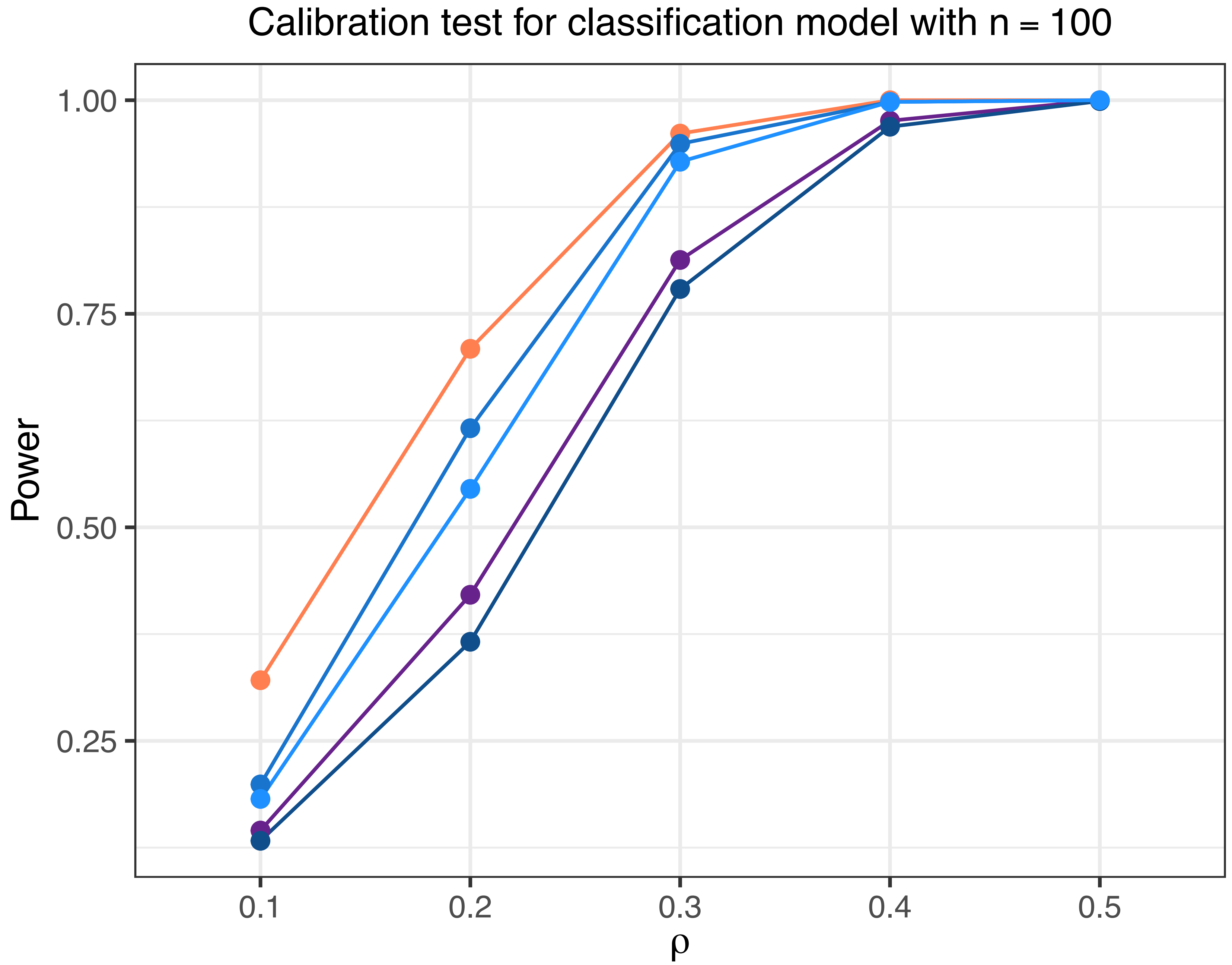

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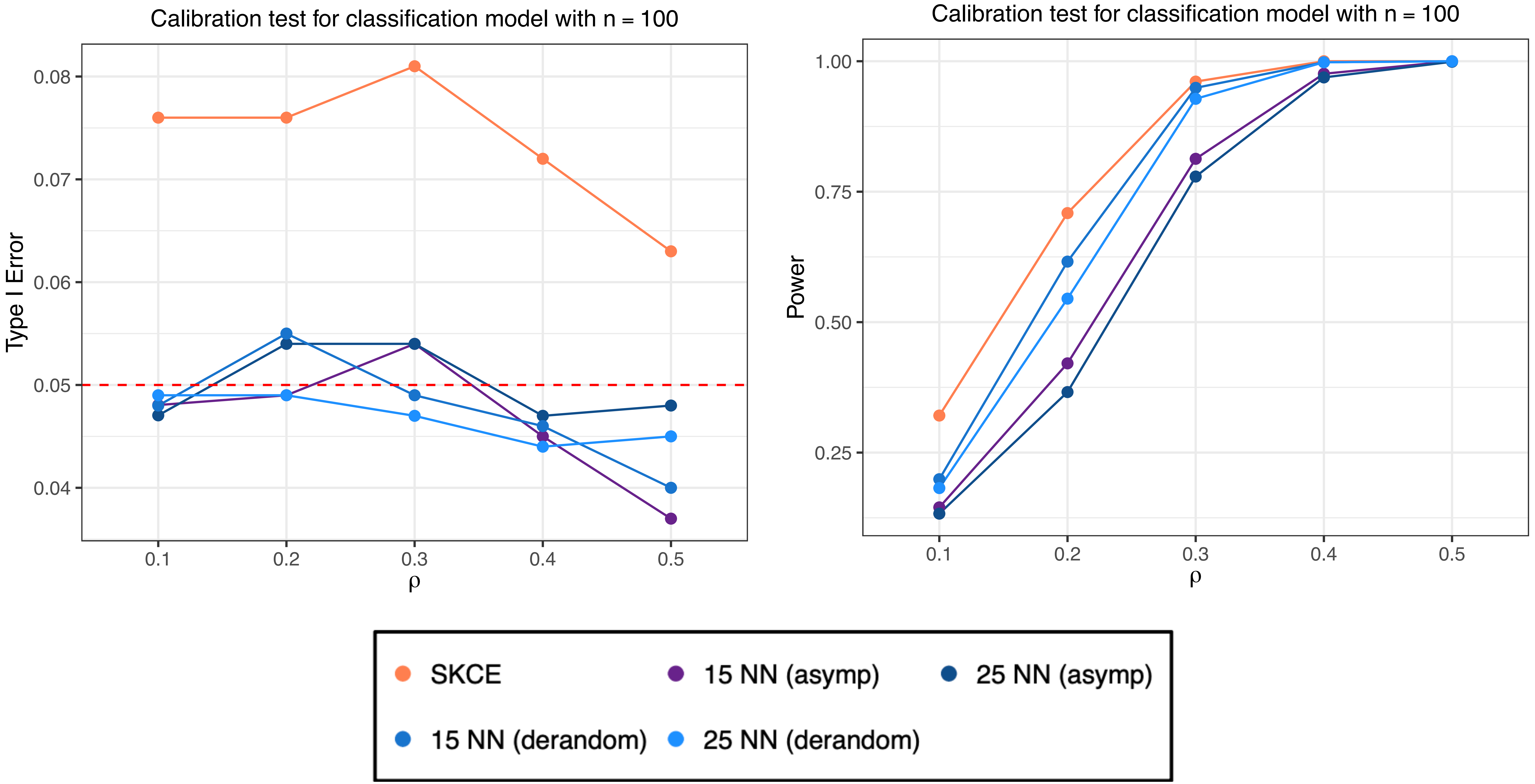

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- High-stakes application with the proposed method?

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