Quantum Computing

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Review: Lecture 2

- Complex Vector Space
  - Transpose, conjugate and adjoint
- Basis and Dimension
  - Change of basis
- Inner Product and Hilbert Space
  - Inner product, norm and distance
- Eigenvalues and Eigenvectors
- Hermitian and Unitary Matrices
  - Properties and physical meaning

Lecture 3: The Leap from Classic to Quantum

1

Classic Deterministic Systems

- Deterministic state
- Deterministic dynamics: Boolean adjacency matrix
- keynotes

2

Probabilistic Systems

- Probabilistic state
- Stochastic dynamics: (doubly) stochastic matrix
- Example 1: the stochastic billiard ball
- Example 2: probabilistic double-slit experiment
- keynotes

3

Quantum Systems

- Interference
- Quantum state
- Quantum dynamics: unitary matrix
- Example 1: the quantum billiard ball
- Example 2: double-slit experiment
- Particle-wave duality
- Superposition and measurement
- keynotes

1. Classic Deterministic Systems

- State
  - Deterministic state

Example 3.1.1 Let there be 6 vertices in a graph and a total of 27 marbles. We might place 6 marbles on vertex 0, 2 marbles on vertex 1, and the rest as described by this picture.

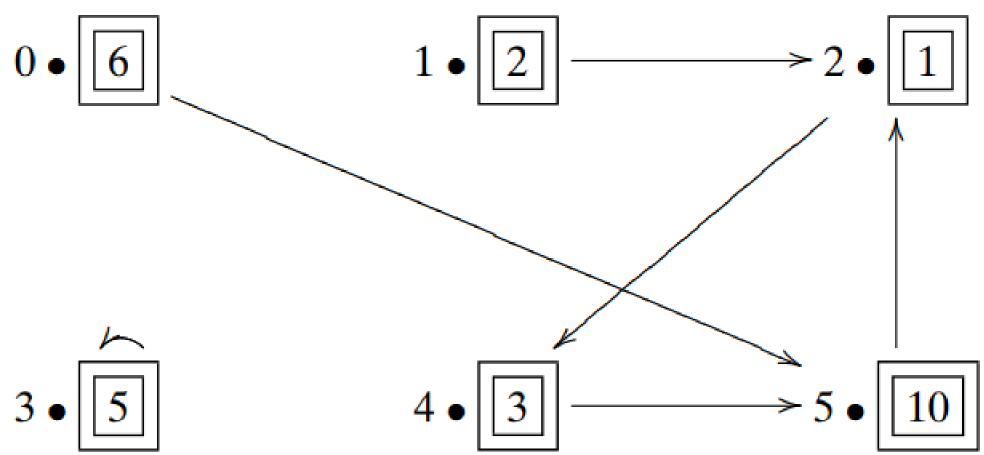
$$
\begin{array}{ccc}
0 \bullet \boxed{6} & 1 \bullet \boxed{2} & 2 \bullet \boxed{1} \\
 & & \\
3 \bullet \boxed{5} & 4 \bullet \boxed{3} & 5 \bullet \boxed{10}
\end{array}
\tag{3.1}
$$

We shall denote this state as $X = [6, 2, 1, 5, 3, 10]^T$. □

1. Classic Deterministic Systems

- Dynamics
  - Simple (unweighted) directed graph

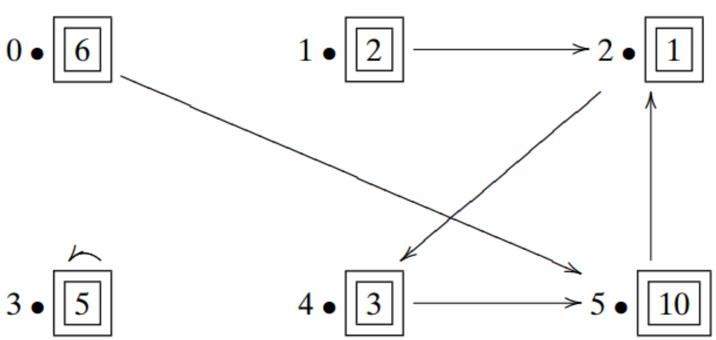
Example 3.1.3 An example of the dynamics might be described by the following directed graph:



(3.3)

1. Classic Deterministic Systems

- Dynamics
  - Boolean adjacency matrix



$$
M = \begin{array}{c|cccccc}
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\
\hline
\mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{2} & 0 & 1 & 0 & 0 & 0 & 1 \\
\mathbf{3} & 0 & 0 & 0 & 1 & 0 & 0 \\
\mathbf{4} & 0 & 0 & 1 & 0 & 0 & 0 \\
\mathbf{5} & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

$\mathbf{M}(i,j) = 1$ if and only if there is an arrow from vertex j to vertex i

1. Classic Deterministic Systems

- Dynamics
  - State evolvement: matrix * vector

Let's say that we multiply M by a state of the system $X = [6, 2, 1, 5, 3, 10]^T$. Then we have

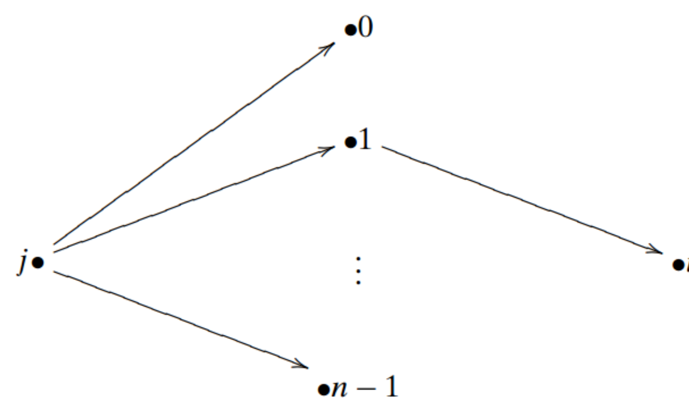
$$MX = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \\ 5 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 5 \\ 1 \\ 9 \end{bmatrix} = Y. \tag{3.5}$$

To what does this correspond? If X describes the state of the system at time t, then Y is the state of the system at time $t+1$, i.e., after one time click. We can see

1. Classic Deterministic Systems

- Dynamics
  - Multiple step dynamics
    - Boolean matrix multiplication

$$\mathbf{M}^2(i,j) = \bigvee_{k=0}^{n-1} \mathbf{M}(i,k) \wedge \mathbf{M}(k,j) \quad \Leftrightarrow$$



$\mathbf{M}^2(i,j) = 1$ if and only if there is an path of length 2 from vertex j to vertex i

1. Classic Deterministic Systems

- Keynotes
  - The states of a system correspond to column vectors (state vectors)
  - The dynamics of a system correspond to matrices
  - To progress from one state to another in one time step, one must multiply the state vector by a matrix
  - Multiple step dynamics are obtained via (Boolean) matrix multiplication

（感谢物理学院2020级云凡同学指出本页句点格式错误）

2. Probabilistic Systems

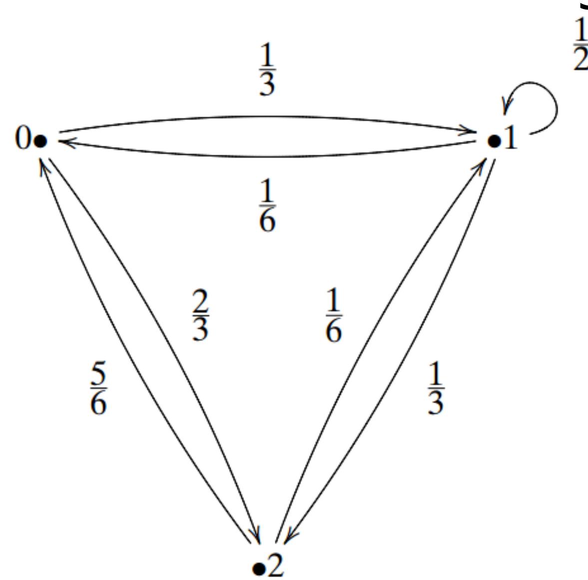
- State
  - Probabilistic entries
  - Sum of all entries be 1
- Example: a three-vertex graph

$$\boldsymbol{x} = \left[\frac{1}{5},\ \frac{3}{10},\ \frac{1}{2}\right]^{\mathrm{T}}$$

  - one-fifth chance that the marble is on vertex 0
  - three-tenths chance that the marble is on vertex 1
  - half chance that the marble is on vertex 2

2. Probabilistic Systems

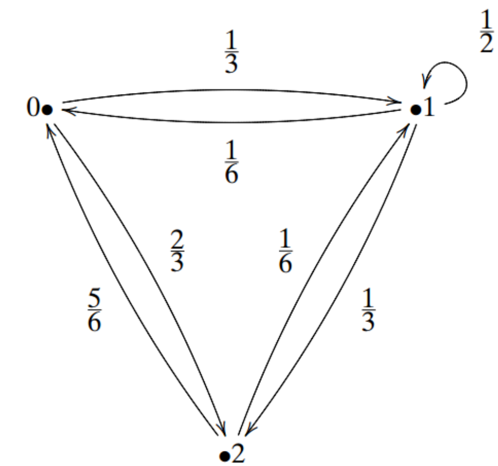
- Dynamics
  - Directed (probabilistic) weighted graph
    - several arrows shooting out of each vertex with real numbers between 0 and 1 as weights



2. Probabilistic Systems

- Dynamics
  - Doubly stochastic matrix
    - The column sum, i.e., the sum of all weights leaving a vertex, is 1
    - The row sum, i.e., the sum of all weights entering a vertex, is 1
  - Example

Example 3.2.1 An example of such a graph is



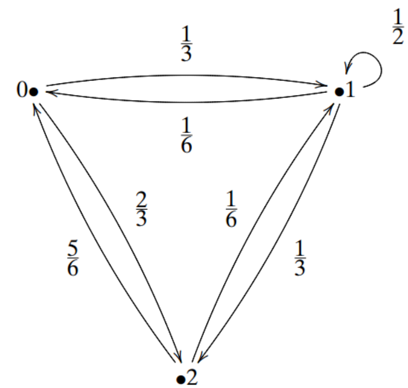
The adjacency matrix for this graph is

$$M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

2. Probabilistic Systems

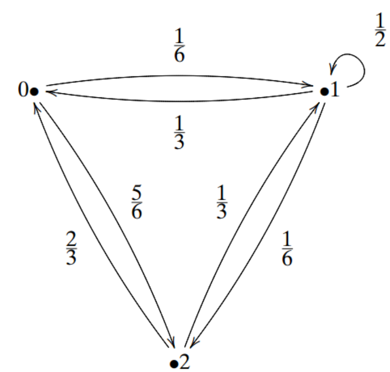
- Dynamics
  - Forward dynamics

$$M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$



  - Backward dynamics

$$M^T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{6} & 0 \end{bmatrix}$$

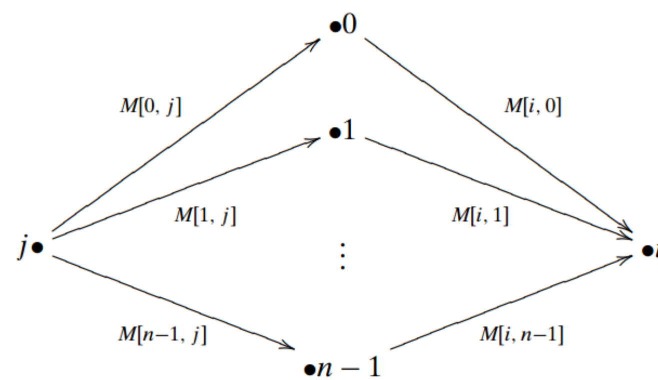


（感谢人工智能专业 2020 级何姜杉同学纠正此处左乘Backward dynamic矩阵无法实现undo操作的讲解错误）

2. Probabilistic Systems

- Dynamics
  - Multiple step dynamics
    - Matrix multiplication with probability entries (a.k.a., normal matrix multiplication)

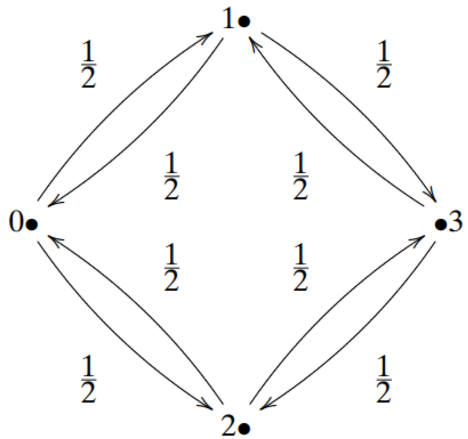
$$\mathbf{M}^2(i,j) = \sum_{k=0}^{n-1} \mathbf{M}(i,k)\mathbf{M}(k,j) \quad \Leftrightarrow$$



$\mathbf{M}^2(i,j) =$ the probability of going from vertex j to vertex i in 2 time clicks

2. Probabilistic Systems

- Example
  - Stochastic billiard ball

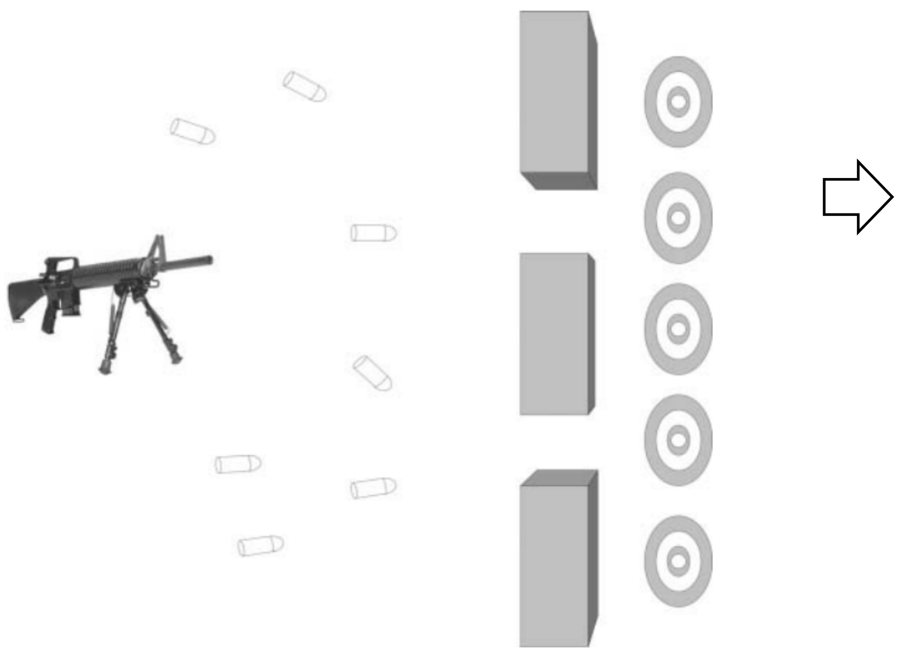


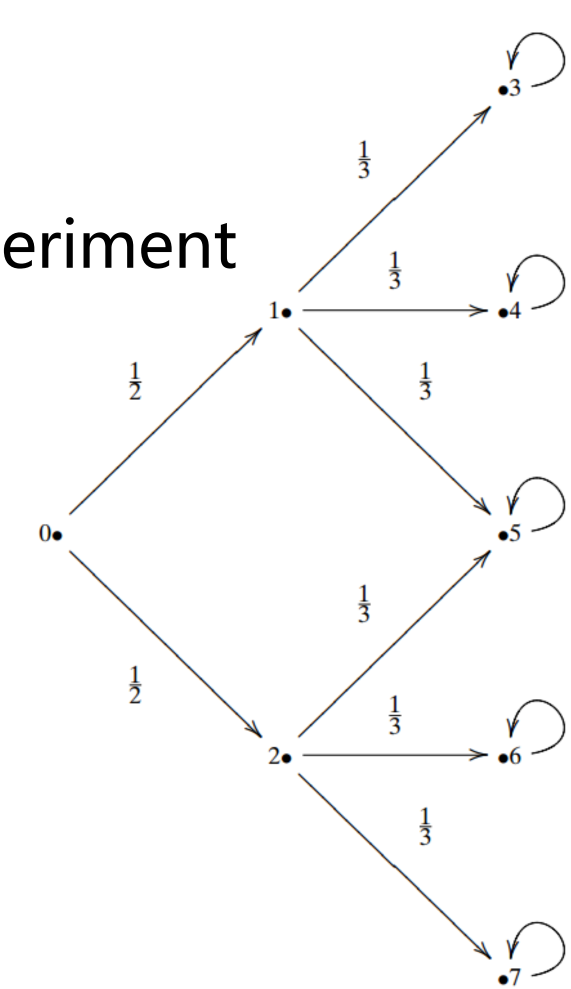
$$\Rightarrow \quad A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{x}^{(1)} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}^{(3)} = {\color{red}\boldsymbol{x}^{(1)}} \quad \Rightarrow \quad \boldsymbol{x}^{(4)} = {\color{blue}\boldsymbol{x}^{(2)}} \quad \Rightarrow \quad \cdots\cdots$$

2. Probabilistic Systems

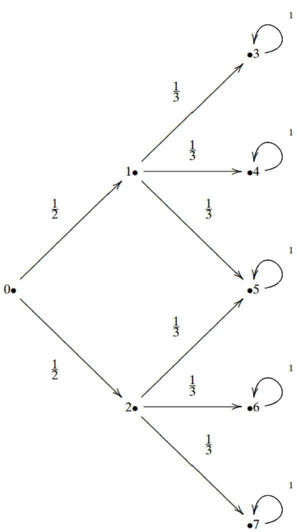
- Example
  - Probabilistic double-slit experiment





2. Probabilistic Systems

- Example
  - Probabilistic double-slit experiment



$$\Rightarrow \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad B \star B = B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad X = [1, 0, 0, 0, 0, 0, 0, 0]^T \quad \Rightarrow \quad B^2 X = \left[0, 0, 0, \frac{1}{6}, \frac{1}{6}, \boxed{\frac{1}{3}}, \frac{1}{6}, \frac{1}{6}\right]^T$$

2. Probabilistic Systems

■ Keynotes

- The vectors that represent states of a probabilistic physical system express a type of indeterminacy （不确定性） about the exact physical state of the system
- The matrices that represent the dynamics express a type of indeterminacy about the way the physical system will change over time. Their entries enable us to compute the likelihood of transitioning from one state to the next
- The way in which the indeterminacy progresses is simulated by matrix multiplication, <u>just as</u> in the deterministic scenario (note: normal matrix multiplication VS Boolean matrix multiplication)

3. Quantum Systems

- Real number weight: $p \in [0, 1]$

$$(p_1 + p_2) \geqslant p_1 \text{ and } (p_1 + p_2) \geqslant p_2$$

- Complex number weight: $c \in \mathbb{C}$ and $|c|^2 \in [0, 1]$

$$|c_1 + c_2|^2 \lesseqqgtr |c_1|^2 \text{ and } |c_1 + c_2|^2 \lesseqqgtr |c_2|^2$$

Example 3.3.1 For example,[3] if $c_1 = 5 + 3i$ and $c_2 = -3 - 2i$, then $|c_1|^2 = 34$ and $|c_2|^2 = 13$ but $|c_1 + c_2|^2 = |2 + i|^2 = 5$. 5 is less than 34, and 5 is less than 13. □

- Interference
  - complex numbers may cancel each other when added

3. Quantum Systems

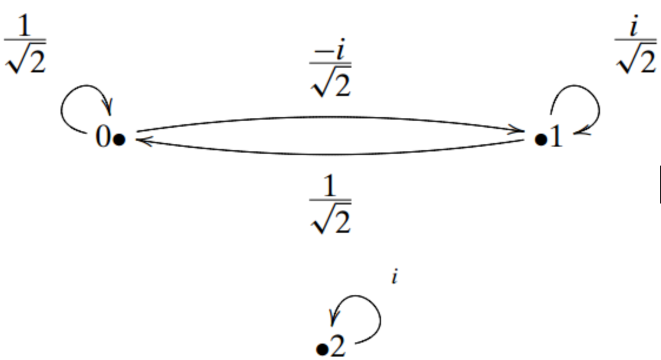
- State
  - quantum entries (complex values)
  - Modulus square represent the probability
    - sum of moduli squared of all entries is 1
- Example

$$\boldsymbol{x} = \left[\frac{1}{\sqrt{3}},\ \frac{2i}{\sqrt{15}},\ \sqrt{\frac{2}{5}}\right]^T$$

$$\boldsymbol{x}^\dagger \boldsymbol{x} = \frac{1}{3} + \frac{4}{15} + \frac{2}{5} = 1$$

3. Quantum Systems

- Dynamics
  - Graph
    - directed (complex) weighted graph
  - Matrix
    - Unitary matrix $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{U}\mathbf{U}^{\dagger} = \mathbf{I}$
    - Its modulus squares is a doubly stochastic matrix



$$\Rightarrow U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} \Rightarrow |U[i,j]|^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Quantum Systems

- Comparisons of three systems

		Classical Deterministic system	Probabilistic System	Quantum System
State		$\boldsymbol{x}=[x_1,x_2,x_3]^T$ $x_i \in \mathbb{N}$	$\boldsymbol{x}=[p_1,p_2,p_3]^T$ $p_i \in [0,1],\ \sum_i p_i=1$	$\boldsymbol{x}=[c_1,c_2,c_3]^T$ $c_i \in \mathbb{C},\ \sum_i \lvert c_i\rvert^2=1$
Dynamics	**Graph**	exactly one arrow leaving each vertex	several arrows shooting out of each vertex with probabilistic weights	several arrows shooting out of each vertex with complex weights
	Matrix	Boolean adjacency matrix	Doubly stochastic matrix	Unitary matrix whose modulus squares is a doubly stochastic matrix

3. Quantum Systems

- Dynamics
  - State evolvement

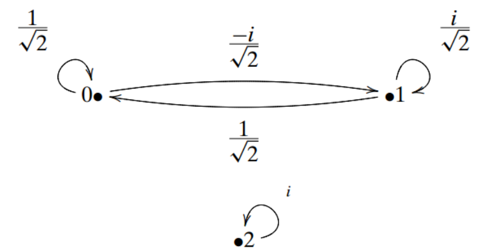
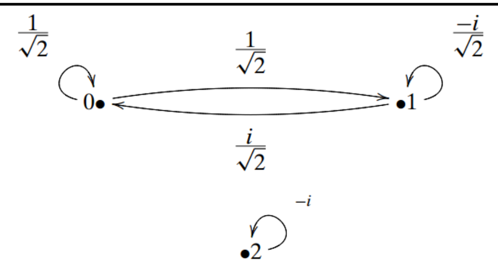
$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} \qquad \boldsymbol{x} = \left[\frac{1}{\sqrt{3}}, \frac{2i}{\sqrt{15}}, \sqrt{\frac{2}{5}}\right]^T$$

$$\Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{2i}{\sqrt{15}} \\ \sqrt{\frac{2}{5}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}+2i}{\sqrt{30}} \\ \frac{-2-\sqrt{5}\,i}{\sqrt{30}} \\ \sqrt{\frac{2}{5}}\,i \end{bmatrix}$$

the sum of result's modulus squares is 1

3. Quantum Systems

■ Dynamics

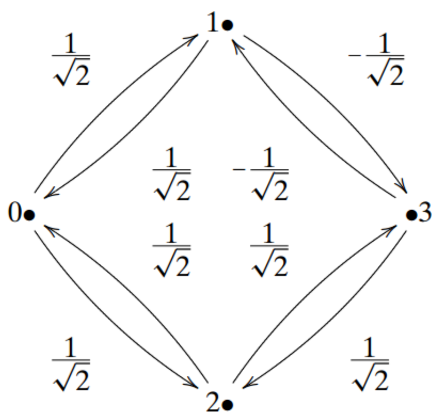
Forward dynamics		$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix}$
backward dynamics		$U^{\dagger} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & -i \end{bmatrix}$

- Time reversible: $\boldsymbol{x} \mapsto \mathbf{U}\boldsymbol{x} \mapsto \mathbf{U}^{\dagger}\mathbf{U}\boldsymbol{x} = \mathbf{I}\boldsymbol{x} = \boldsymbol{x}$

  This means that if you perform some operation and then "undo" the operation, you will find yourself (with probability 1) in the same state with which you began.

3. Quantum Systems

- Example
  - Quantum billiard ball

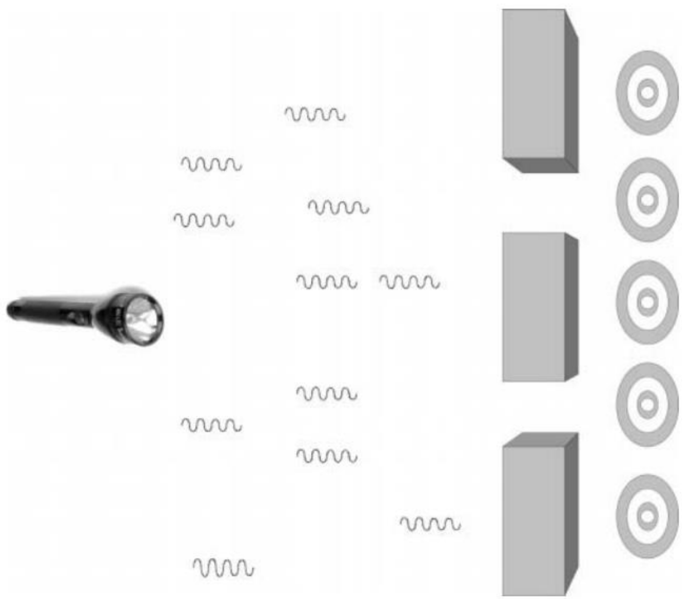


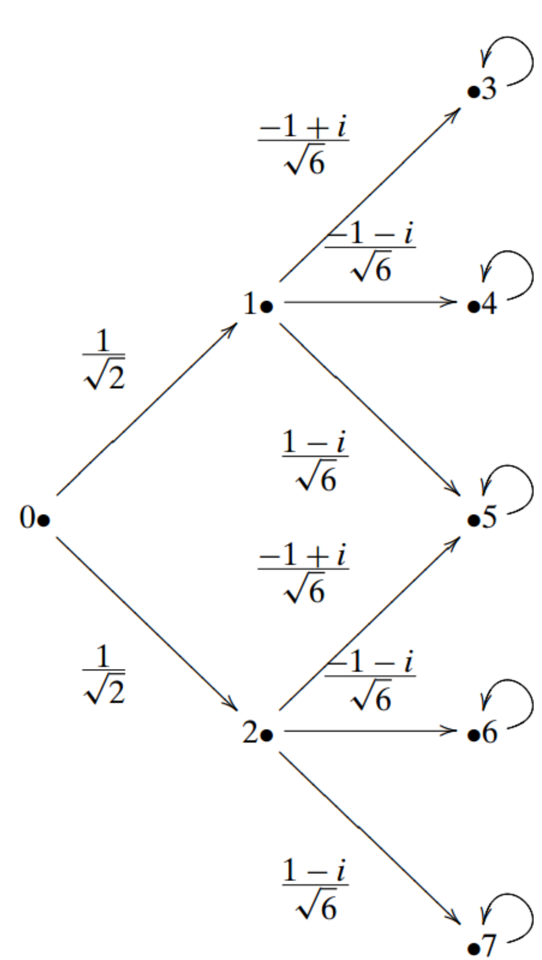
$$\Rightarrow \quad A = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{x}^{(1)} = \begin{bmatrix} 0 \\ \sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}^{(2)} = \boldsymbol{x}^{(0)} \quad \Rightarrow \quad \boldsymbol{x}^{(3)} = \boldsymbol{x}^{(1)} \quad \Rightarrow \quad \cdots\cdots$$

3. Quantum Systems

- Example
  - Double-slit experiment

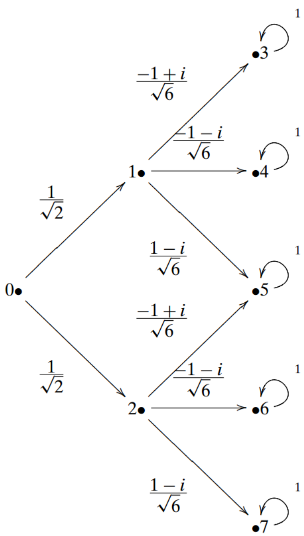




3. Quantum Systems

- Example
  - Double-slit experiment

$$\boxed{\frac{1}{\sqrt{2}}\left(\frac{-1+i}{\sqrt{6}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1-i}{\sqrt{6}}\right)=\frac{-1+i}{\sqrt{12}}+\frac{1-i}{\sqrt{12}}=\frac{0}{\sqrt{12}}=0}$$



$$P=\begin{bmatrix}0&0&0&0&0&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&0&0&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&0&0&0&0&0\\ 0&\frac{-1+i}{\sqrt{6}}&0&1&0&0&0&0\\ 0&\frac{-1-i}{\sqrt{6}}&0&0&1&0&0&0\\ 0&\frac{1-i}{\sqrt{6}}&\frac{-1+i}{\sqrt{6}}&0&0&1&0&0\\ 0&0&\frac{-1-i}{\sqrt{6}}&0&0&0&1&0\\ 0&0&\frac{1-i}{\sqrt{6}}&0&0&0&0&1\end{bmatrix}$$

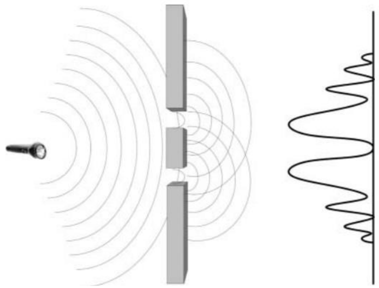
$$P^2=\begin{bmatrix}0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \frac{-1+i}{\sqrt{12}}&\frac{-1+i}{\sqrt{6}}&0&1&0&0&0&0\\ \frac{-1-i}{\sqrt{12}}&\frac{-1-i}{\sqrt{6}}&0&0&1&0&0&0\\ \boxed{0}&\frac{1-i}{\sqrt{6}}&\frac{-1+i}{\sqrt{6}}&0&0&1&0&0\\ \frac{-1-i}{\sqrt{12}}&0&\frac{-1-i}{\sqrt{6}}&0&0&0&1&0\\ \frac{-1+i}{\sqrt{12}}&0&\frac{1-i}{\sqrt{6}}&0&0&0&0&1\end{bmatrix}$$

➢ Interference: wave-like nature of light

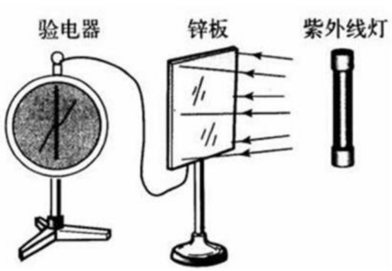
（感谢弘毅学堂 2021 级廖叶飞同学纠正P²矩阵中部分位置元素值的错误）

3. Quantum Systems

- Wave-particle duality
  - Double-slit experiment
    - Wave-like nature of light



  - Photoelectric effect experiment
    - Particle-like nature of light



  - Magic
    - Double-slit experiment can be done with *a single photon !!!*

3. Quantum Systems

- Superposition
  - Double-slit experiment can be done with *a single photon* !!!

The naive probabilistic interpretation of the position of the photon following the bullet metaphor of the previous section is thus not entirely adequate. Let the state of the system be given by $X = [c_0, c_1, \ldots, c_{n-1}]^T \in \mathbb{C}^n$. It is incorrect to say that the probability of the photon's being in position k is $|c_k|^2$. Rather, to be in state X means that the particle is in some sense in *all* positions simultaneously. The photon passes through the top slit *and* the bottom slit simultaneously, and when it exits both slits, it can cancel *itself* out. A photon is not in *a* single position, rather it is in *many* positions, a **superposition**.

Schrödinger's Cat

3. Quantum Systems

- Measurement
  - Common sense vs. superposition
    - How to explain?

This might generate some justifiable disbelief. After all, we do not see things in many different positions. Our everyday experience tells us that things are in one position or (exclusive or!) another. How can this be? The reason we see particles in one particular position is because we have performed a **measurement**. When we measure something at the quantum level, the quantum object that we have measured is no longer in a superposition of states, rather it collapses to a single classical state. So we have to redefine what the state of a quantum system is: a system is in state X means that *after measuring* it, it will be found in position i with probability $|c_i|^2$.

3. Quantum Systems

- Power of quantum computing
  - superposition

It is exactly this superposition of states that is the real power behind quantum computing. Classical computers are in one state at every moment. Imagine putting a computer in many different classical states simultaneously and then processing with *all* the states at once. This is the ultimate in parallel processing! Such a computer can only be conceived of in the quantum world.

3. Quantum Systems

■ Keynotes

- States in a quantum system are represented by column vectors of complex numbers whose sum of moduli squared is 1
- The dynamics of a quantum system is represented by unitary matrices and is therefore reversible. The "undoing" is obtained via the algebraic inverse, i.e., the adjoint of the unitary matrix representing forward evolution
- The probabilities of quantum mechanics are always given as the modulus square of complex numbers
- Quantum states can be superposed, i.e., a physical system can be in more than one basic state simultaneously

Conclusion

1. Classical Deterministic Systems
   - States, dynamics (transition graphs, adjacency matrices)
   - Evolvement
2. Probabilistic Systems
   - Probabilistic states and doubly stochastic matrices
3. Quantum Systems
   - Quantum states and unitary matrices
   - Comparison of three systems
   - Time reversible
   - Wave-particle duality
   - Superposition and measurement