Efficient Computation of Centroidal Voronoi Tessellations

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Outline

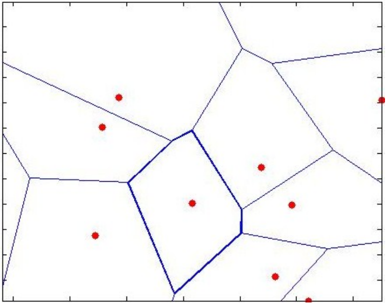
- *Introduction to Centroidal Voronoi Tesselations (CVT)*
  - CVT:concepts
  - List of applications
- Commonly used CVT construction algorithm
  - Lloyd iteration
  - Some results concerning Lloyd
- *Multigrid-based Optimization for CVT construction*
  - Multigrid Optimization (MG/OPT) algorithm: background
  - Application of MG/OPT to 1-D CVT problem
- Summary and discussion

Concept of the Voronoi tessellation

- *Given*
  - a set S
  - *elements* $z_i, i = 1, 2, \ldots, K$
  - *a distance function* $d(z, w), \forall z, w \in S$
- *The Voronoi set* V_j *is the set of all elements belonging to* S *that are closer to* z_j *than to any of the other elements* z_i*, that is*

  $$V_j = \{w \in S \,|\, d(w, z_j) < d(w, z_i),\ i = 1, \ldots, K, i \neq j\}$$

- $\{V_1, V_2, \ldots, V_k\}$ *is a Voronoi tessellation of* S
- $\{z_i\}$ *are generators of the Voronoi tessellation*



CVT: facts and definitions

- Center of Mass: $C = \frac{\int_v \rho(y) y dy}{\int_v \rho(y) dy}$, where $\rho(y)$ is a density function
- Define the Voronoi sets V_i, $i = 1, \ldots, K$ corresponding to the given $\{z_i\}$ generators
  - we can define the associated centroids

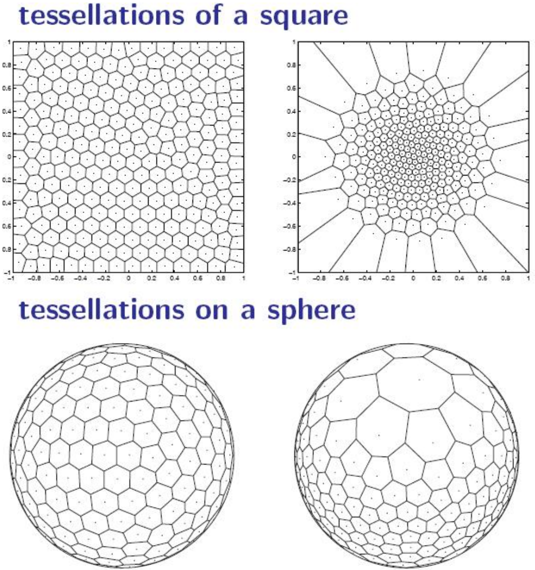
$$z_i^*,\ i = 1, \ldots, K$$

- In general, the centroids of the Voronoi sets don't coincide with the generators of the Voronoi sets, but if they do, i.e.

$$z_i = z_i^*,\ i = 1, \ldots, K$$

we call this kind of tessellation Centroidal Voronoi Tessellation (CVT)

Examples of CVT



Range of applications

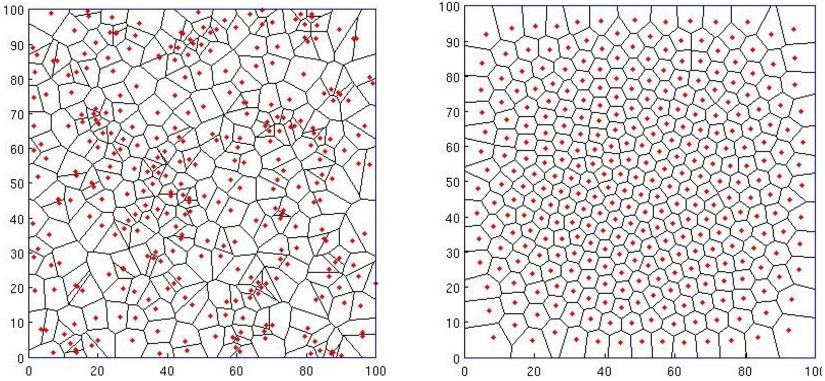
- Location optimization:
  - optimal allocation of resources
  - mailboxes, bus stops, etc. in a city
  - distribution/manufacturing centers
- Grain/cell growth
- Crystal structure
- Territorial behavior of animals
- Data analysis:
  - image compression, computer graphics, sound denoising etc
  - clustering gene expression data, stock market data
- Engineering:
  - vector quantization etc
  - Statistics (k-means):
  - classification, minimum variance clustering
  - data mining
- Numerical methods
  - Atmospheric and ocean modeling
  - Various other PDE solvers

Lloyd's algorithm to construct CVT's

1. Start with the initial set of points $\{z_i\}_{i=1}^K$
2. Construct the Voronoi tessellation $\{V_i\}_{i=1}^K$ of Ω associated with the points $\{z_i\}_{i=1}^K$
3. Construct the centers of mass of the Voronoi regions $\{V_i\}_{i=1}^K$ found in Step 2; take centroids as the new set of points $\{z_i\}_{i=1}^K$
4. Go back to Step 2. Repeat until some convergence criterion is satisfied

Note:Steps 2 and 3 can both be costly to effect

Illurstration of Lloyd's method



Convergence result of Lloyd's method

- Lloyd method has linear convergence rate: $\|error_{k+1}\| \approx r\|error_k\|$
- For strongly log-concave densities,

$$r \approx 1 - \frac{C}{K^2}$$

- very slow if K large.

Is speedup possible?

Multilevel approach to construct CVT

- Given generators $\{\mathbf{z}_i\}_{i=1}^k$ and the corresponding tessellation $\{V_i\}_{i=1}^k$, define the *energy functional*

$$\mathcal{G}\left(\{\mathbf{z}_i\}_{i=1}^k\right) = \sum_{i=1}^{k} \int_{V_i} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_i|^2 \, d\mathbf{y}.$$

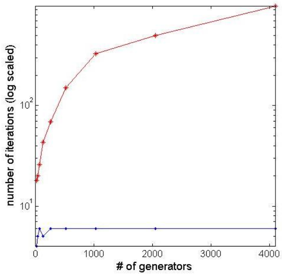
- The minimizer of $\mathcal{G}$ necessarily forms a CVT
- We treat CVT as a minimization problem and apply a multilevel optimization framework called MG/OPT to this functional
- The multilevel framework uses coarse approximations to $\mathcal{G}$ to accelerate a traditional optimization algorithm (OPT)

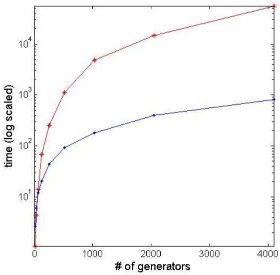
Multilevel Algorithm: MG/OPT [S.G.Nash 2000]

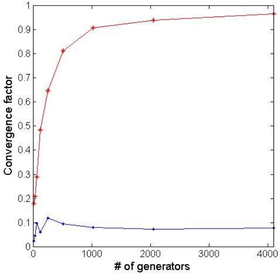
- Given:
  - Traditional optimization algorithm OPT
  - Downdate and update operators
  - Integers k_1 and k_2 satisfying $k_1 + k_2 > 0$
- One iteration of MG/OPT:
  - Pre-smoothing: Apply k_1 iterations of OPT to the fine energy function
  - Recursion:
    - Downdate the generators
    - Apply MG/OPT to a shifted version of the coarse energy function
    - Use result to update the generators on the fine level
  - Post-smoothing: Apply k_2 iterations of OPT to the fine energy function

Convergence result of MG/OPT on 1-D CVT

Red: Opt; Blue: MG/OPT; $\rho(x) = 1$







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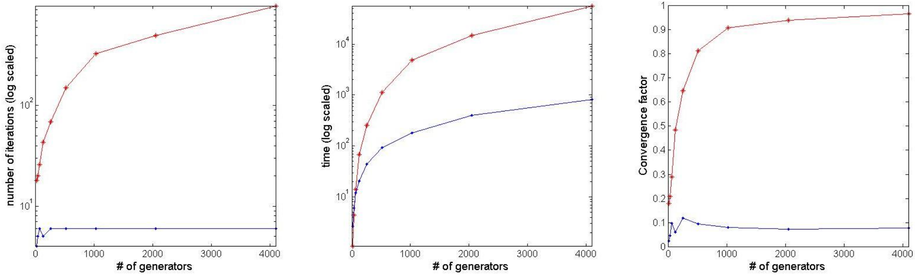

For more information, please read our paper: "Truncated Newton-based multigrid algorithm for centroidal Voronoi calculation", Z. Di, M. Emelianenko and S. Nash

Discussion

Results and challenges:

- CVT is in the heart of many applications and the number is growing: computer science, physics, social sciences, biology, engineering ...
- The main advantage of MG/OPT is its superior convergence speed when compared to other existing approaches.
- The simplicity of its design and the results of preliminary tests suggest that the method is generalizable to higher dimensions, which is the subject of current investigations
- Future work also includes application of this technique to various scientific and engineering applications, including image analysis and grid generation.

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THANKS!