FRE 6233 Option Pricing and Stochastic Calculus in Finance

Final Project: Two Assets Correlation Options

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1. Problem review

In this project, we are going to price an exotic option called Two Assets Correlation Call Option, a call option with the following payoff:

$$\max[S_2 - K_2, 0]\ if\ S_1 > K_1, and\ 0\ otherwise$$

We can rewrite this as:

$$(S_2(T) - K_2)^+ * \mathbb{I}(S_1(T) > K_1)$$

The prices of the two assets follow two GBMs with constant volatilities σ_1, σ_2 and correlation ρ. Also, there is no dividend, the constant interest rate is r.

2. Analytical solution

2.1 Pricing Value

Underlying assets $S_1(t), S_2(t)$ set up: $\sigma_{11} = \sigma_1,\ \sigma_{21} = \rho\sigma_2,\ \sigma_{22} = \sqrt{1-\rho^2} * \sigma_2$

$$\frac{dS_1(t)}{S_1(t)} = \mu_1 * dt + \sigma_{11} * dW_1(t)$$

$$\frac{dS_2(t)}{S_2(t)} = \mu_2 * dt + \sigma_{21} * dW_1(t) + \sigma_{22} * dW_2(t)$$

Where $W_1(t)$ and $W_2(t)$ are independent BM under P

Now we want to change measure to Q^* so $S_1(t)$ and $S_2(t)$ are both martingales with respect to Q^* [risk-neutral measure]

Applying 2 variate Girsanov theorem:

$$\begin{cases} dW_1^*(t) = \gamma_1 dt + dW_1(t) \\ dW_2^*(t) = \gamma_2 dt + dW_2(t) \end{cases}$$

Where $W_1^*(t)$ and $W_2^*(t)$ are independent BM under Q^*

Rewrite dynamics for $S_1(t)$ and $S_2(t)$ with $dW_1^*(t)$ and $dW_2^*(t)$:

$$dS_1(t) = (\mu_1 - \gamma_1\sigma_{11})S_1(t)dt + \sigma_{11}S_1(t)dW_1^*(t)$$

$$\begin{aligned} dS_2(t) &= \mu_2 S_2(t)dt + \sigma_{21}S_2(t)[-\gamma_1 dt + dW_1^*(t)] + \sigma_{22}S_2(t)[-\gamma_2 dt + dW_2^*(t)] \\ &= (\mu_2 - \gamma_1\sigma_{21} - \gamma_2\sigma_{22})S_2(t)dt + \sigma_{21}S_2(t)dW_1^*(t) + \sigma_{22}S_2(t)dW_2^*(t) \end{aligned}$$

Find risk neutral measure Q^*:

$$\begin{cases} \mu_1 - \gamma_1\sigma_{11} = r \\ \mu_2 - \gamma_1\sigma_{21} - \gamma_2\sigma_{22} = r \end{cases}$$

Solve it we get:

$$\begin{cases} \gamma_1 = \frac{\mu_1 - r}{\sigma_{11}} \\ \gamma_2 = \dfrac{\mu_2 - \gamma_1\sigma_{21} - r}{\sigma_{22}} = \dfrac{\sigma_{11}\mu_2 - (\mu_1 - r) * \sigma_{21} - r * \sigma_{11}}{\sigma_{11} * \sigma_{22}} \end{cases}$$

The solution is unique if σ_{11} *and* $\sigma_{22} \neq 0$

Therefore, under this risk neutral measure Q^*, The discounted stocks are both martingales

$$dS_1(t) = rS_1(t)dt + \sigma_{11}S_1(t)dW_1^*(t)$$

$$dS_2(t) = rS_2(t)dt + \sigma_{21}S_2(t)dW_1^*(t) + \sigma_{22}S_2(t)dW_2^*(t)$$

So,

$$S_1(t) = S_1(0) * e^{\sigma_{11}W_1^*(t) + \left(r - \frac{1}{2}\sigma_{11}{}^2\right)t}$$

$$S_2(t) = S_2(0) * e^{\sigma_{21}W_1^*(t) + \sigma_{22}W_2^*(t) + \left(r - \frac{1}{2}(\sigma_{21}{}^2 + \sigma_{22}{}^2)\right)t}$$

Since $\sigma_{11} = \sigma_1$, $\sigma_{21} = \rho\sigma_2$, $\sigma_{22} = \sqrt{1-\rho^2} * \sigma_2$, we can rewrite $S_1(t)$ and $S_2(t)$ as:

$$S_1(t) = S_1(0) * e^{\sigma_1 W_1^*(t) + \left(r - \frac{1}{2}\sigma_1{}^2\right)t}$$

$$S_2(t) = S_2(0) * e^{\rho\sigma_2 W_1^*(t) + \sqrt{(1-\rho^2)}\sigma_2 W_2^*(t) + \left(r - \frac{1}{2}\sigma_2{}^2\right)t}$$

Hence, we have: (assume T – t = τ)

$$S_1(T) = S_1(t) * e^{\sigma_1(W_1^*(T) - W_1^*(t)) + \left(r - \frac{1}{2}\sigma_1{}^2\right)\tau}$$

$$S_2(T) = S_2(t) * e^{\rho\sigma_2 * (W_1^*(T) - W_1^*(t)) + \sqrt{(1-\rho^2)}\sigma_2(W_2^*(T) - W_2^*(t)) + \left(r - \frac{1}{2}\sigma_2{}^2\right)\tau}$$

Define two standard normal random variables:

$$X = \frac{(W_1^*(T) - W_1^*(t))}{\sqrt{\tau}},\ Y = \frac{(W_2^*(T) - W_2^*(t))}{\sqrt{\tau}} \quad (1)$$

Then rewrite:

$$S_1(T) = S_1(t) * e^{\left(r - \frac{1}{2}\sigma_1{}^2\right)\tau + \sigma_1\sqrt{\tau} * X}$$

$$S_2(T) = S_2(t) * e^{\left(r - \frac{1}{2}\sigma_2{}^2\right)\tau + (\rho * X + \sqrt{(1-\rho^2)} * Y)\sigma_2\sqrt{\tau}}$$

From payoff function: $(S_2(T)-K_2)^+ * \mathbb{1}(S_1(T)>K_1)$

Set $S_1(t)=S_1,\ S_2(t)=S_2$

Then we have:

$$V(S_1,S_2,K_1,K_2,t)$$
$$= E^Q[e^{-r\tau} * (S_2(t)e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau+\left(\rho*x+\sqrt{(1-\rho^2)}*y\right)\sigma_2\sqrt{\tau}} - K_2) * \mathbb{1}(S_1(t) * e^{\left(r-\frac{1}{2}\sigma_1{}^2\right)\tau+\sigma_1\sqrt{\tau}*x} > K_1)]$$
$$= \frac{1}{2\pi}e^{-r\tau} * \int_{-d_-(S_2)}^{\infty}\int_{-d_-(S_1)}^{\infty}(S_2(t)e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau+\left(\rho*x+\sqrt{(1-\rho^2)}*y\right)\sigma_2\sqrt{\tau}} - K_2) * e^{-\frac{1}{2}x^2-\frac{1}{2}y^2})dxdy \qquad (2)$$

Where:

$$-d_-(S_1):\ S_1(t) * e^{\left(r-\frac{1}{2}\sigma_1{}^2\right)\tau+\sigma_1\sqrt{\tau}*x} - K_1 > 0$$

$$so, \qquad x > \frac{\ln\left(\frac{K_1}{S_1}\right)-\left(r-\frac{1}{2}\sigma_1{}^2\right)\tau}{\sigma_1\sqrt{\tau}} = -d_-(S_1)$$

$$-d_-(S_2):\ (S_2(t)e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau+\left(\rho*x+\sqrt{(1-\rho^2)}*y\right)\sigma_2\sqrt{\tau}} - K_2 > 0$$

$$so, \qquad y > \frac{\left(\frac{\ln\left(\frac{K_2}{S_2}\right)-\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau}{\sigma_2\sqrt{\tau}} - \rho x\right)}{\sqrt{(1-\rho^2)}} = -d_-(S_2)$$

So we can rewrite (2) as:

$$V(S_1,S_2,K_1,K_2,t)$$
$$= \frac{1}{2\pi}e^{-r\tau} * \int_{-d_-(S_2)}^{\infty} e^{-\frac{1}{2}x^2}dx \int_{-d_-(S_1)}^{\infty}(S_2(t)e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau+\left(\rho*x+\sqrt{(1-\rho^2)}*y\right)\sigma_2\sqrt{\tau}} - K_2) * e^{-\frac{1}{2}y^2}dy$$

Define Inner $= \frac{1}{\sqrt{2\pi}}e^{-r\tau} * \int_{-d_-(S_2)}^{\infty}(S_2(t)e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau+\left(\rho*x+\sqrt{(1-\rho^2)}*y\right)\sigma_2\sqrt{\tau}} - K_2) * e^{-\frac{1}{2}y^2}dy$

$$= I_1 - I_2$$

Where

$$I_2 = \frac{1}{\sqrt{2\pi}}e^{-r\tau} * \int_{-d_-(S_2)}^{\infty} K_2 * e^{-\frac{1}{2}y^2}dy = K_2 * e^{-r\tau} * N(d_-(S_2))$$
$$I_1 = S_2(t) * \frac{1}{\sqrt{2\pi}}\int_{-d_-(S_2)}^{\infty} e^{\left(-\frac{1}{2}\sigma_2{}^2\right)\tau+\left(\rho*x+\sqrt{(1-\rho^2)}*y\right)\sigma_2\sqrt{\tau}-\frac{1}{2}y^2}dy$$

Suppose $Z_1 = y-\sqrt{(1-\rho^2)}*\sigma_2\sqrt{\tau}$, then $\frac{-1}{2}(Z_1)^2 = \frac{-1}{2}y^2+\sqrt{(1-\rho^2)}*y*\sigma_2\sqrt{\tau}-\frac{1}{2}(1-\rho^2)*\sigma_2^2\sqrt{\tau}$

So,

$$I_1 = S_2(t) * \frac{1}{\sqrt{2\pi}} \int_{-d_-(S_2)}^{\infty} e^{\left(-\frac{1}{2}Z_1{}^2\right)+\rho\sigma_2\sqrt{\tau}x-\frac{1}{2}\rho^2\sigma_2^2\tau} dZ_1$$

$$= S_2(t) * e^{\rho\sigma_2\sqrt{\tau}x-\frac{1}{2}\rho^2\sigma_2^2\tau} * N(d_-(Z_1))$$

Where $-d_-(Z_1) = -d_-(S_2) - \sqrt{(1-\rho^2)} * \sigma_2\sqrt{\tau}$

So,

$$V(S_1, S_2, K_1, K_2, t)$$

$$= \frac{1}{2\pi} e^{-r\tau} * \int_{-d_-(S_1)}^{\infty} e^{-\frac{1}{2}x^2} * [S_2(t)e^{\left(\rho*\sigma_2\sqrt{\tau}*x-\frac{1}{2}*\rho^2\sigma_2^2\tau\right)} * N\big(d_-(Z_1)\big) - K_2 e^{-r\tau} * N\big(d_-(S_2)\big)]dx$$

Where,

$$-d_-(S_1) = \frac{\ln\left(\frac{K_1}{S_1}\right) - \left(r - \frac{1}{2}\sigma_1{}^2\right)\tau}{\sigma_1\sqrt{\tau}}$$

$$d_-(S_2) = \frac{\left(\rho x - \frac{\ln\left(\frac{K_2}{S_2}\right) - \left(r-\frac{1}{2}\sigma_2{}^2\right)\tau}{\sigma_2\sqrt{\tau}}\right)}{\sqrt{(1-\rho^2)}}$$

$$d_-(Z_1) = -d_-(S_2) - \sqrt{(1-\rho^2)} * \sigma_2\sqrt{\tau}$$

While this formula can't be easily solved by our program, because the term $-d_-(S_2)$ contains "x"

So then we try on the correlated normal variable from (1):

Redefine above at (1)

$$X = -\frac{(W_1^*(T) - W_1^*(t))}{\sqrt{\tau}},\ Y = -\frac{(W_2^*(T) - W_2^*(t))}{\sqrt{\tau}}$$

Then we set $Z = \rho X + \sqrt{(1-\rho^2)} * Y$, so X and Z are correlated normal variables with correlation equals to ρ

We can write here

$$S_1(T) = S_1(t) * e^{\left(r-\frac{1}{2}\sigma_1{}^2\right)\tau - \sigma_1\sqrt{\tau}*X}$$

$$S_2(T) = S_2(t) * e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau - \sigma_2\sqrt{\tau}*Z}$$

Rewrite function V from (2) we now have:

$$V(S_1, S_2, K_1, K_2, t)$$

$$= E^Q[e^{-r\tau} * (S_2(t)e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau-\sigma_2\sqrt{\tau}*Z} - K_2) * \mathbb{1}(S_1(t) * e^{\left(r-\frac{1}{2}\sigma_1{}^2\right)\tau-\sigma_1\sqrt{\tau}*x} > K_1)]$$

$$= \frac{1}{2\pi(1-\rho^2)} e^{-r\tau} * \int_{-\infty}^{d_Z}\int_{-\infty}^{d_x} (S_2(t) * e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau-\sigma_2\sqrt{\tau}*Z} - K_2) * e^{-\frac{x^2-2xZ\rho+Z^2}{2(1-\rho^2)}} dxdz$$

$$= I_1{}' - I_2{}'$$

Where $d_x = \frac{\ln\left(\frac{S_1}{K_1}\right)+\left(r-\frac{1}{2}\sigma_1{}^2\right)\tau}{\sigma_1\sqrt{\tau}}$, $d_z = \frac{\ln\left(\frac{S_2}{K_2}\right)+\left(r-\frac{1}{2}\sigma_2{}^2\right)\tau}{\sigma_2\sqrt{\tau}}$

Then $I_1' = \frac{S_2(t)}{2\pi*(1-\rho^2)} \int_{-\infty}^{d_Z} \int_{-\infty}^{d_x} e^{-\frac{1}{2}{\sigma_2}^2\tau - \sigma_2\sqrt{\tau}*Z - \frac{x^2-2xZ\rho+Z^2}{2(1-\rho^2)}} dxdz$

Where by change var:

$$x' = x + \rho\sigma_2\sqrt{\tau}$$
$$Z' = Z + \sigma_2\sqrt{\tau}$$

We have $I_1' = \frac{S_2(t)}{2\pi*(1-\rho^2)} \iint e^{-\frac{1}{2}{\sigma_2}^2\tau - \sigma_2\sqrt{\tau}*(Z'-\sigma_2\sqrt{\tau}) - \frac{x'^2+Z'^2-2x'Z'\rho-2(1-\rho^2)*\sigma_2\sqrt{\tau}*Z'+(1-\rho^2)*{\sigma_2}^2\tau}{2(1-\rho^2)}} dx'dZ'$

$$= \frac{S_2(t)}{2\pi*(1-\rho^2)} \int_{dZ'} \int_{dx'} e^{-\frac{x'^2+Z'^2-2x'Z'\rho}{2(1-\rho^2)}} dx'dZ'$$

Where

$$dx' = dx + \rho\sigma_2\sqrt{\tau}$$
$$dZ' = dZ + \sigma_2\sqrt{\tau}$$

For I_2':

$$I_2 = K_2 * \frac{1}{2\pi*(1-\rho^2)} e^{-r\tau} * \int_{d_x} \int_{d_Z} e^{-\frac{x^2-2xZ\rho+Z^2}{2(1-\rho^2)}} dxdz$$

So if we define $\frac{1}{2\pi*(1-\rho^2)} \int_{-\infty}^{a} \int_{-\infty}^{b} e^{-\frac{x^2-2xZ\rho+Z^2}{2(1-\rho^2)}} dxdz$ as $M(a,b,\rho)$

Then we get the Pricing Formula:

$$V(S_1,S_2,K_1,K_2,t) = S_2(t) * M\left(d_Z + \sigma_2\sqrt{\tau}, d_x + \rho\sigma_2\sqrt{\tau}, \rho\right) - Ke^{-r\tau} * M(d_Z, d_x, \rho) \qquad (3)$$

Where $$d_Z = \frac{\ln\left(\frac{S_2}{K_2}\right) + \left(r - \frac{1}{2}{\sigma_2}^2\right)\tau}{\sigma_2\sqrt{\tau}}, \quad d_x = \frac{\ln\left(\frac{S_1}{K_1}\right) + \left(r - \frac{1}{2}{\sigma_1}^2\right)\tau}{\sigma_1\sqrt{\tau}}$$

$$M(a,b,\rho) = \frac{1}{2\pi*(1-\rho^2)} \int_{-\infty}^{a} \int_{-\infty}^{b} e^{-\frac{x^2-2xZ\rho+Z^2}{2(1-\rho^2)}} dxdz$$

2.2 Calculate Analytical Delta

According to the formula (3) above,

$$Delta1 = \frac{\partial V}{\partial S_1} = S_2 \frac{\partial M(d_z + \sigma_2\sqrt{t}, d_x + \rho\sigma_2\sqrt{t}, \rho)}{\partial S_1} - K_2 e^{-rt} \frac{\partial M(d_z, d_x, \rho)}{\partial S_1}$$

From

$$M(a,b,\rho) = \frac{1}{2\pi*(1-\rho^2)} \int_{-\infty}^{a} \int_{-\infty}^{b} e^{-\frac{x^2-2xz\rho+z^2}{2(1-\rho^2)}} dxdz,$$

with a is a function of S_2, we write $a = a(S_2)$, and b is a function of S_1, so $b = b(S_1)$.
Then we can get

$$\frac{\partial M}{\partial S_1} = \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{a} e^{-\frac{b^2(S_1)-2\rho z b(S_1)+z^2}{2(1-\rho^2)}} * b'(S_1) dz$$

Where

$$b'(S_1) = \frac{1}{\sigma_1\sqrt{t} * S_1}$$

$$S_0 = \frac{\partial M(d_z + \sigma_2\sqrt{t}, d_x + \rho\sigma_2\sqrt{t}, \rho)}{\partial S_1}$$

$$= \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{d_z+\sigma_2\sqrt{t}} e^{\frac{(d_x(S_1)+\rho\sigma_2\sqrt{t})^2 - 2\rho z(d_x(S_1)+\rho\sigma_2\sqrt{t}) + z^2}{-2(1-\rho^2)}} * \frac{1}{\sigma_1\sqrt{t} * S_1} dz \triangleq A$$

$$\frac{\partial M(d_z, d_x, \rho)}{\partial S_1} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_z} e^{\frac{d_x^2(S_1) - 2\rho z d_x(S_1) + z^2}{-2(1-\rho^2)}} * \frac{1}{\sigma_1\sqrt{t} * S_1} dz \triangleq B$$

To sum up, for Delta 1:

$$\frac{\partial V}{\partial S_1} = S_2 * A - K_2 e^{-rt} * B \tag{4}$$

Where

$$A = \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{d_z+\sigma_2\sqrt{t}} e^{\frac{(d_x(S_1)+\rho\sigma_2\sqrt{t})^2 - 2\rho z(d_x(S_1)+\rho\sigma_2\sqrt{t}) + z^2}{-2(1-\rho^2)}} * \frac{1}{\sigma_1\sqrt{t} * S_1} dz$$

$$B = \frac{\partial M(d_z, d_x, \rho)}{\partial S_1} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_z} e^{\frac{d_x^2(S_1) - 2\rho z d_x(S_1) + z^2}{-2(1-\rho^2)}} * \frac{1}{\sigma_1\sqrt{t} * S_1} dz$$

Similar, for Delta 2, we assume term C and D here:

$$Delta2 = \frac{\partial V}{\partial S_2} = M(d_z + \sigma_2\sqrt{t}, d_x + \rho\sigma_2\sqrt{t}, \rho) + S_2 C + K_2 e^{-rt} D$$

Since

$$\frac{\partial M(a(S_2), b, \rho)}{\partial S_2} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{b} e^{-\frac{x^2 - 2\rho x a(S_2) + a^2(S_2)}{2(1-\rho^2)}} * a'(S_2) dx$$

and

$$a'(S_2) = \frac{1}{\sigma_2\sqrt{t} * S_2}$$

So C and D are:

$$C = \frac{\partial M(d_z + \sigma_2\sqrt{t}, d_x + \rho\sigma_2\sqrt{t}, \rho)}{\partial S_2}$$

$$= \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_x+\rho\sigma_2\sqrt{t}} e^{-\frac{x^2 - 2\rho x(d_z+\sigma_2\sqrt{t}) + (d_z+\sigma_2\sqrt{t})^2}{2(1-\rho^2)}} * \frac{1}{\sigma_2\sqrt{t} * S_2} dx$$

$$D = \frac{\partial M(d_z, d_x, \rho)}{\partial S_2} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_x} e^{-\frac{x^2 - 2\rho x d_z + d_z^2}{2(1-\rho^2)}} * \frac{1}{\sigma_2\sqrt{t} * S_2} dx$$

Hence, to sum up, for Delta2:

$$\frac{\partial V}{\partial S_2} = M\left(d_z + \sigma_2\sqrt{t}, d_x + \rho\sigma_2\sqrt{t}, \rho\right) + S_2 C + K_2 e^{-rt} D \tag{5}$$

where

$$C = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_x + \rho\sigma_2\sqrt{t}} e^{-\frac{x^2 - 2\rho x (d_z + \sigma_2\sqrt{t}) + (d_z + \sigma_2\sqrt{t})^2}{2(1-\rho^2)}} * \frac{1}{\sigma_2\sqrt{t} * S_2} dx$$

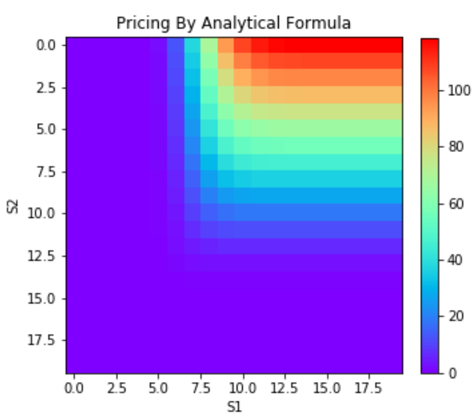
$$D = \frac{\partial M(d_z, d_x, \rho)}{\partial S_2} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_x} e^{-\frac{x^2 - 2\rho x d_z + d_z^2}{2(1-\rho^2)}} * \frac{1}{\sigma_2\sqrt{t} * S_2} dx$$

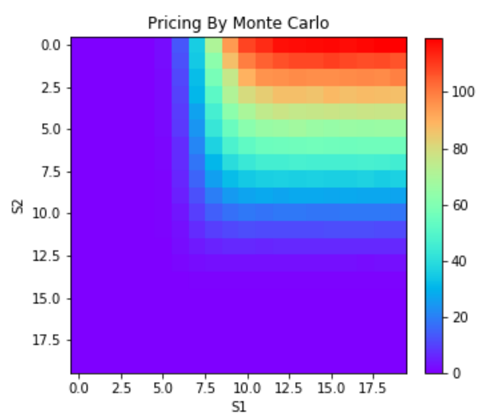
3. Validation

3.1 Validate Pricing formula with Monte Carlo

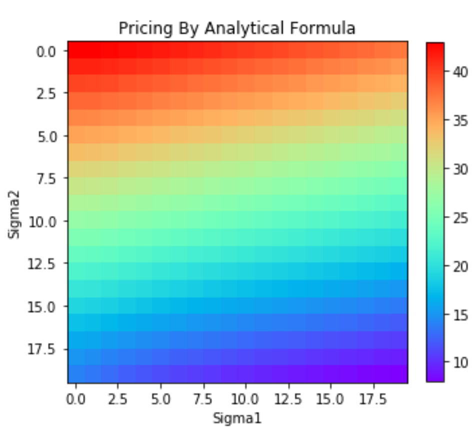
In this part, we do three control experiments to validate our analytical formula of pricing.

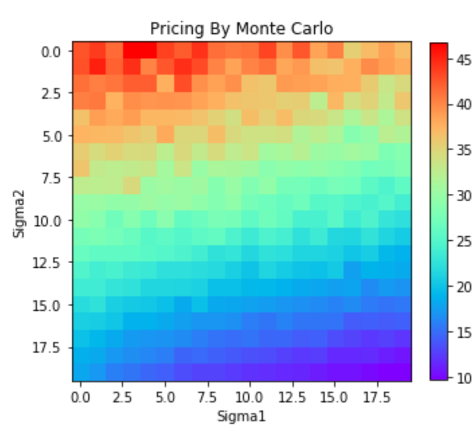
Firstly, we fixed all the other parameters, then did pricing with different initial price of Stock1 and Stock2. Here is what we got, the graphs look very similar, meaning that the analytical formula is verify by Monte Carlo method within the change of stock prices.





Secondly, we did pricing with different volatilities of Stock1 and Stock2:





When we change the pair (Sigma1, Sigma2), the graphs also look similar, meaning that the analytical formula is verify by Monte Carlo method within the change of volatility pairs.

3.2 Validate Analytical Deltas with Numerical Deltas

In this part, we do three control experiments to validate our analytical formula of calculating delta1 and delta2 with Numerical Deltas.

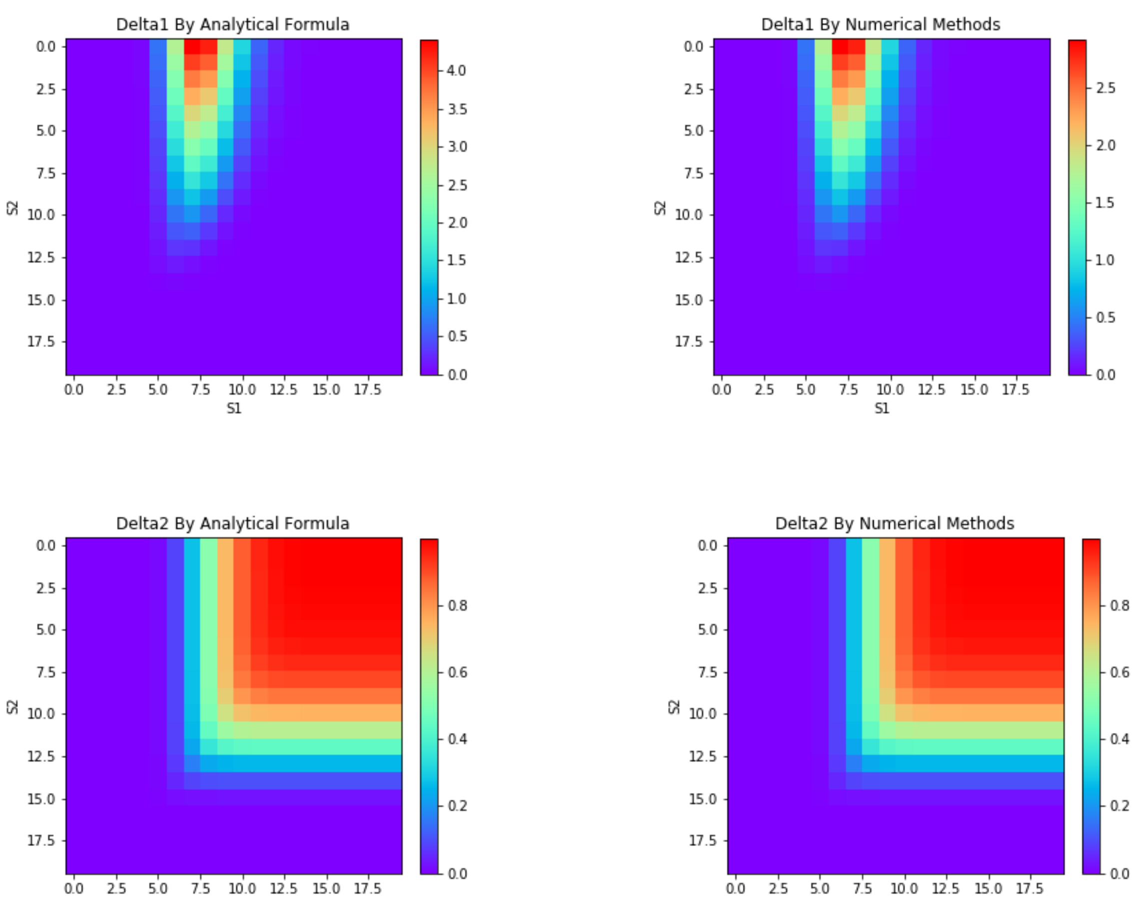
We use the definition of Delta $\Delta_i = \frac{\partial V(S_1, S_2, t)}{\partial S_i}, i = 1,2$. To calculate Delta numerically, we apply finite difference by

$$\Delta_{1,i} = \frac{V\left(S_{1,i} + \Delta, S_{2,i}, t\right) - V\left(S_{1.i} - \Delta, S_{2,i}, t\right)}{2\Delta}$$

$$\Delta_{2,i} = \frac{V\left(S_{1,i}, S_{2,i} + \Delta, t\right) - V\left(S_{1,i}, S_{2,i} - \Delta, t\right)}{2\Delta}$$

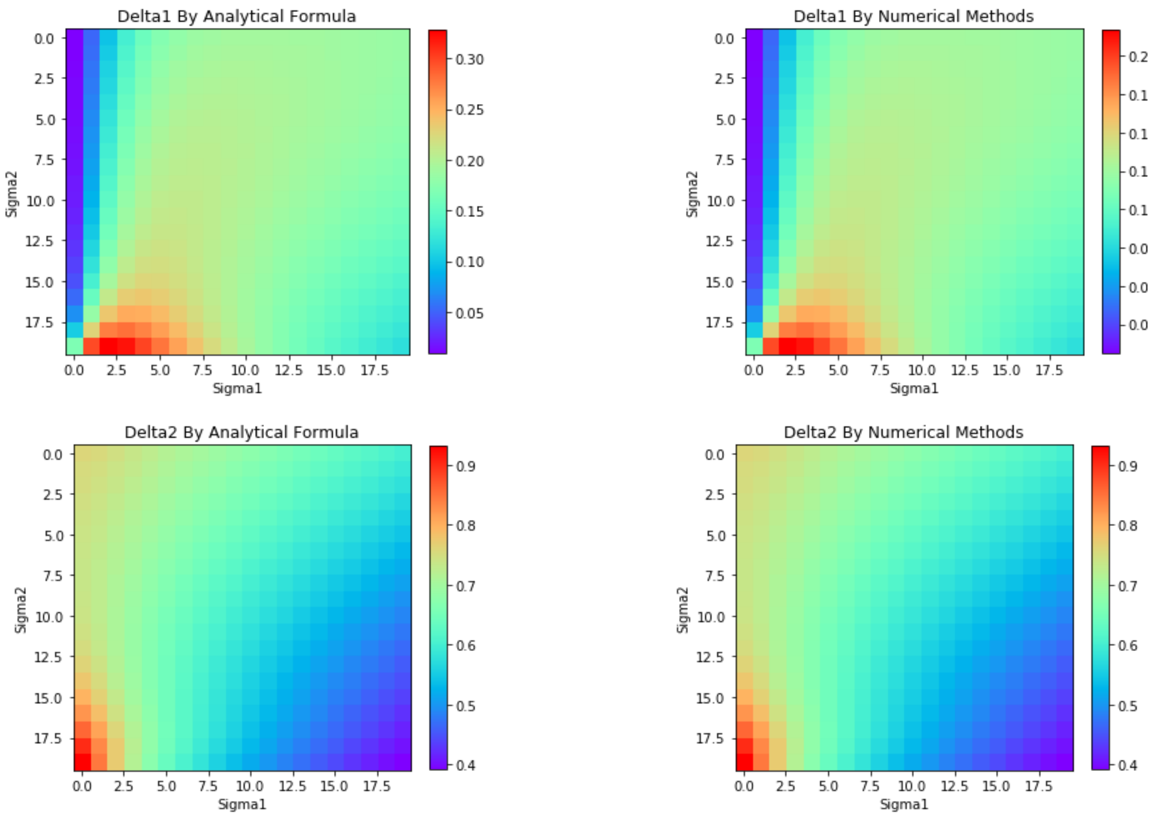
$$\Delta = 10\ cents$$

Firstly, we fixed all the other parameters, then did pricing with different initial price of Stock1 and Stock2. Here is what we got:



The graphs above show that those two deltas are similar in these two methods.

Secondly, we did pricing with different volatilities of Stock1 and Stock2:



Here we again generate some similar graphs. Comparing left ones (with analytical formula) with right ones (with numerical methods), we may conclude that our analytical formulas for pricing and deltas calculation is approved by Monte Carlo and Numerical Deltas

3.3 Verify model in some Extreme Cases

During this process, we use the data from sample real stocks (HP as stock1, DLTR as stock2).

We consider four extreme cases. The first case is a "zero correlation" situation when correlation equals zero. The second case is a perfect correlation situation when correlation equals one. The third case is extremely large strike price $\boldsymbol{K_1}$, and the last case is extremely small strike price $\boldsymbol{K_1}$.

Case 1: Zero Correlation

The original correlation equals 0.37, and when correlation goes down 0, the option price goes down from 10 dollars to 6 dollars, Delta 1 goes down from 0.16 to 0.24, and Delta 2 goes down from 0.28 to 0.19.

Case 2: Perfect Correlation

When correlation goes up to 1, the option price goes up from 10 dollars to 18 dollars, Delta 1 goes up from 0.16 to 0.6, and Delta 2 goes up from 0.28 to 0.5. From the first and second cases we find that option value, Delta 1, and Delta 2 are increasing as correlation increase. Notice that when correlation equals 1, the formula is divided by 0 and return NaN, so we try 0.99 on the second case.

Date	First	firlogret	Second	seclogret	tao	OptionPrice	Delta1	Delta2
2019-05-15	57.930000	NaN	100.199997	NaN	1.000000	10.410965	0.155785	0.296651
2019-05-16	58.080002	0.002586	100.889999	0.006863	0.996032	10.624699	0.158773	0.299488
2019-05-17	55.480000	-0.045799	100.389999	-0.004968	0.992063	10.042789	0.164530	0.284728
2019-05-20	55.009998	-0.008508	100.059998	-0.003293	0.988095	9.856477	0.164313	0.281367
2019-05-21	55.529999	0.009408	99.769997	-0.002902	0.984127	9.843518	0.161112	0.283304

Original correlation = 0.37

Date	First	firlogret	Second	seclogret	tao	OptionPrice	Delta1	Delta2	rho
2019-05-15	57.930000	NaN	100.199997	NaN	1.000000	6.351866	0.138233	0.202095	0
2019-05-16	58.080002	0.002586	100.889999	0.006863	0.996032	6.506398	0.141150	0.204798	0
2019-05-17	55.480000	-0.045799	100.389999	-0.004968	0.992063	6.032042	0.142100	0.191900	0
2019-05-20	55.009998	-0.008508	100.059998	-0.003293	0.988095	5.895573	0.141194	0.188996	0
2019-05-21	55.529999	0.009408	99.769997	-0.002902	0.984127	5.906352	0.139343	0.190661	0

Correlation = 0

Date	First	firlogret	Second	seclogret	tao	OptionPrice	Delta1	Delta2	rho
2019-05-15	57.930000	NaN	100.199997	NaN	1.000000	18.275267	0.530100	0.515126	0.99
2019-05-16	58.080002	0.002586	100.889999	0.006863	0.996032	18.607971	0.557782	0.516585	0.99
2019-05-17	55.480000	-0.045799	100.389999	-0.004968	0.992063	18.093454	0.680801	0.495948	0.99
2019-05-20	55.009998	-0.008508	100.059998	-0.003293	0.988095	17.850428	0.690245	0.491931	0.99
2019-05-21	55.529999	0.009408	99.769997	-0.002902	0.984127	17.721354	0.640717	0.496008	0.99

Correlation = 1

Case 3: $K_1 \gg S_1$

On the third case, we assume that when the strike price K_1 is extremely large, our option is hard to be activated, and it will be nearly worthless. The result confirms our hypothesis, and we can see that Delta 1, Delta 2, and option price are all tend to 0.

Date	First	firlogret	Second	seclogret	tao	OptionPrice	Delta1	Delta2
2019-05-15	57.930000	NaN	100.199997	NaN	1.000000	0.009175	0.000726	0.000176
2019-05-16	58.080002	0.002586	100.889999	0.006863	0.996032	0.009179	0.000727	0.000174
2019-05-17	55.480000	-0.045799	100.389999	-0.004968	0.992063	0.007283	0.000615	0.000139
2019-05-20	55.009998	-0.008508	100.059998	-0.003293	0.988095	0.006802	0.000582	0.000130
2019-05-21	55.529999	0.009408	99.769997	-0.002902	0.984127	0.006874	0.000583	0.000132

Extremely large K_1

Case 4: $K_1 \ll S_1$

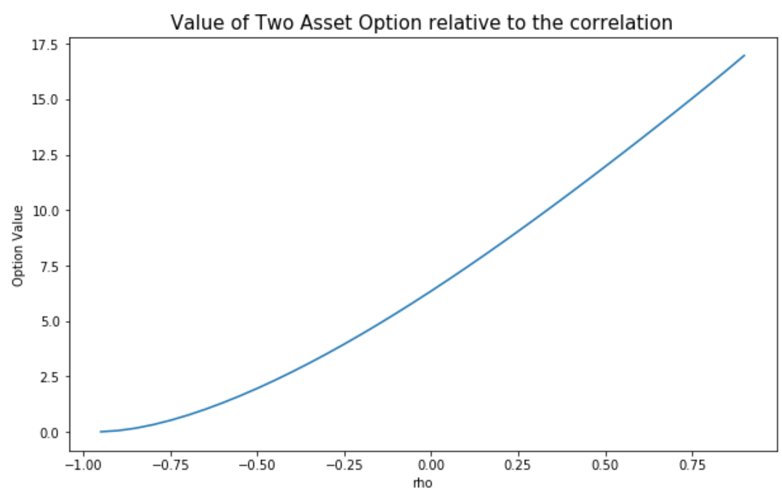
On the last case, when the strike price K_1 is extremely small, we assume that the barrier will become useless, and our option will become a vanilla option based on S_2. The result confirms our hypothesis, and we can see that Delta 2 of our option is pretty close to Delta 2 of a vanilla call option.

Date	First	firlogret	Second	seclogret	tao	OptionPrice	Delta1	Delta2	VanillaOptionPrice	VanillaOptionDelta2
2019-05-15	57.930000	NaN	100.199997	NaN	1.000000	18.814356	0.001330	0.598290	18.831622	0.599159
2019-05-16	58.080002	0.002586	100.889999	0.006863	0.996032	19.192353	0.001330	0.603790	19.209607	0.604651
2019-05-17	55.480000	-0.045799	100.389999	-0.004968	0.992063	18.851077	0.001585	0.599353	18.870928	0.600348
2019-05-20	55.009998	-0.008508	100.059998	-0.003293	0.988095	18.616131	0.001592	0.596418	18.635869	0.597414
2019-05-21	55.529999	0.009408	99.769997	-0.002902	0.984127	18.407299	0.001469	0.593860	18.425556	0.594789

Extremely small K_1

3.4 Option Values vs Correlation

Here we take the data from sample real stocks on the first day to plot the graph. It demonstrates that the option value is monotonic increasing as correlation increase, the curve is convex.



4. Back Test model with Historical Data

We choose sample real stocks (HP as stock1, DLTR as stock2) price to conduct to test the accuracy of our pricing model. Firstly, we process data by calculating it log returns and daily volatility. Next, we choose Strike Prices as S_1, S_2 to make Two-Asset Option ATM on T=1 (yearly). We used the simulation function derived above to simulate the price of Two-Asset Option. Then, we calculate Delta by infinite difference. At the evaluation of our testing results, we hedge with that Delta and calculate the Residual PnL on each day of the history.

4.1 Quantitative Method Description

A. Data Processing

Taking HP and DLTR into example. We choose data from May 15, 2019 - May 14, 2020, with 252 numbers each. We calculate log return by

$$r_i = \frac{\log\left(\frac{S_i}{S_{i-1}}\right)}{\sqrt{t_i - t_{i-1}}}$$

$$i = 2, \dots, 252$$

Then, we calculate daily volatility of log returns.

B. Parameter Designing

S_i: Stock 1 HP, Stock 2 DLTR

K_1:Stock 1 Price on Day 1 (May 15, 2020)

K_2:Stock 2 Price on Day 1 (May 15, 2020)

r: One-year LIOBR rate

$$V(S_1, S_2, K_1, K_2, t) = S_2(t) * M\left(d_Z + \sigma_2\sqrt{\tau}, d_x + \rho\sigma_2\sqrt{\tau}, \rho\right) - Ke^{-r\tau} * M(d_Z, d_x, \rho)$$

$$\text{Where } d_z = \frac{\ln\left(\frac{S_2}{K_2}\right) + \left(r - \frac{1}{2}{\sigma_2}^2\right)\tau}{\sigma_2\sqrt{\tau}},\ d_x = \frac{\ln\left(\frac{S_1}{K_1}\right) + \left(r - \frac{1}{2}{\sigma_1}^2\right)\tau}{\sigma_1\sqrt{\tau}}$$

C. Analytical Deltas Calculation

We take advantage of the derived formula from 2.2, including formula (4) and (5):

$$Delta1: \quad \frac{\partial V}{\partial S_1} = S_2 * A - K_2 e^{-rt} * B$$

Where

$$A = \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{d_z + \sigma_2\sqrt{t}} e^{\frac{(d_x(S_1)+\rho\sigma_2\sqrt{t})^2 - 2\rho z\left(d_x(S_1)+\rho\sigma_2\sqrt{t}\right) + z^2}{-2(1-\rho^2)}} * \frac{1}{\sigma_1\sqrt{t} * S_1} dz$$

$$B = \frac{\partial M(d_z, d_x, \rho)}{\partial S_1} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_z} e^{\frac{d_x^2(S_1) - 2\rho z d_x(S_1) + z^2}{-2(1-\rho^2)}} * \frac{1}{\sigma_1\sqrt{t} * S_1} dz$$

$$And\ Delta2: \quad \frac{\partial V}{\partial S_2} = M\left(d_z + \sigma_2\sqrt{t}, d_x + \rho\sigma_2\sqrt{t}, \rho\right) + S_2 C + K_2 e^{-rt} D$$

where

$$C = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_x + \rho\sigma_2\sqrt{t}} e^{-\frac{x^2 - 2\rho x(d_z + \sigma_2\sqrt{t}) + (d_z + \sigma_2\sqrt{t})^2}{2(1-\rho^2)}} * \frac{1}{\sigma_2\sqrt{t} * S_2} dx$$

$$D = \frac{\partial M(d_z, d_x, \rho)}{\partial S_2} = \frac{1}{2\pi * (1-\rho^2)} \int_{-\infty}^{d_x} e^{-\frac{x^2 - 2\rho x d_z + d_z^2}{2(1-\rho^2)}} * \frac{1}{\sigma_2\sqrt{t} * S_2} dx$$

D. Hedge with Two Delta and calculate the residual PnL on each day of the history

$$PnL_i = \left[V\left(S_{1,i}, S_{2,i}, t_i, T\right) - V\left(S_{1,i-1}, S_{2,i-1}, t_i, T\right)\right] - \Delta_{1,i-1}\left(S_{1,i} - S_{1,i-1}\right) - \Delta_{2,i-1}\left(S_{2,i} - S_{2,i-1}\right)$$

$$i = 2, 3, \ldots$$

4.2 Result Analysis

We calculate the daily PnL and plot the result.

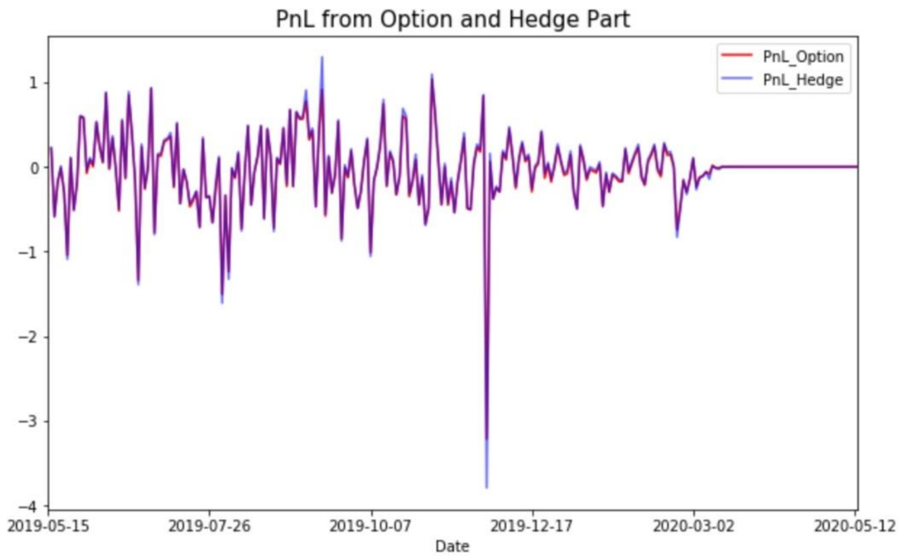


Diagram 1: Different side of Position Of HP&DLTR and Two-Asset options

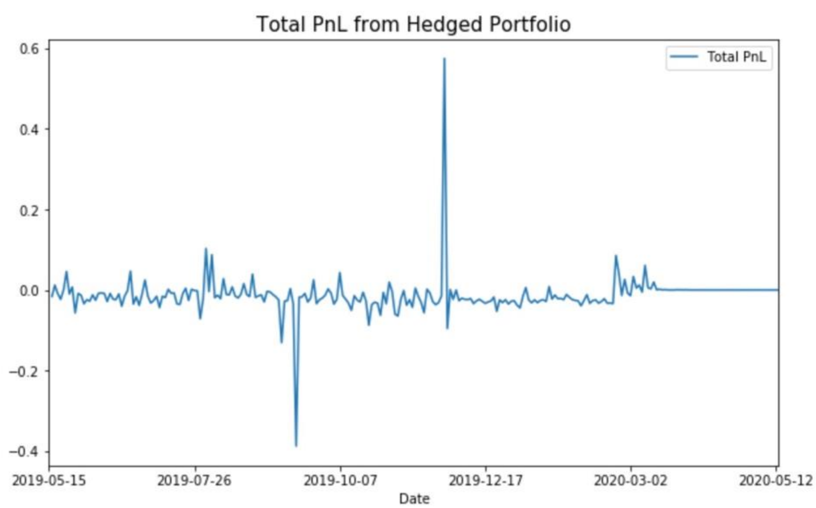


Diagram 2: PnL Curve of HP and DLTR

In the diagram 1, we plot the different side of position into red (Two-Asset Option) and blue (Two Stocks). In the diagram 2, we draw out the curve of the difference, which refers to PnL. It is apparent that the PnL is close to line zero. That means the PnL is very small and could be ignored. The PnL refers to the difference between Long and Short Position. If we Long Two-Asset options and Short with Stocks, we should use simulated delta to hedge it. To test the comparative accuracy of Delta and Two-Asset Option simulation results, we use PnL to find the difference of opposite positions. The conclusion is our Pricing Model is proved to be accurate.

We may detect an interesting point that, after March 2020, when the market was volatile due to the COVID-19, our Total PnL stays at (or very close) to zero as a straight line. Hence, we check in the data and found that Stock1 HP, crashed in March, leading to the fact that our option is hard to be activated after that. So, the option value, delta1, delta2 are all close to zero after the melt down in March 2020, that leads to the fact that the Total PnL and also PnL from both side are all close to zero after March 2020.

5. Results and Conclusion

In this project, we found the Analytical Solution for pricing the two assets correlation option in the method of risk-neutral pricing, with the help of Girsanov theorem and change of measure, we derive the pricing formula as a discounted double integral.
Once we obtain the pricing formula, we derive the Analytical Deltas (delta1 and delta2) for this option, which is written in a format of combination of some single integrals.

Then we use Monte Carlo to validate derived pricing formula, and utilize Numerical Delta to validate derived analytical deltas. Results show that our analytical solutions work for both parts. We also verify our models on some extreme cases, and results show that as the correlation goes up, option price, Delta 1, and Delta 2 also go up. Option value is monotonic increasing as correlation increase. When the strike price K_1 is extremely large, Delta 1, Delta 2, and price of our option are all tend to 0, showing that our option is hard to be activated and becomes nearly worthless. When the strike price K_1 is extremely small, the barrier will become useless. Our option will become a vanilla option based on S_2, and Delta 2 of our option is pretty close to Delta 2 of a vanilla call option.

Eventually, we test our models on historical data with sample stocks (HP as stock1, DLTR as stock2), the Profit and Loss (PnL) result shows that daily PnLs are all around zero, which means our model works fairly well with historical data. We may detect an interesting point that, after March 2020, when the market was volatile due to the COVID-19, our Total PnL stays at (or very close) to zero as a straight line. Hence, we check in the data and found that Stock1 HP, crashed in March, leading to the fact that our option is hard to be activated after that. So, the option value, delta1, delta2 are all close to zero after the melt down in March 2020, that leads to the fact that the Total PnL and also PnL from both side are all close to zero after March 2020.