COMP9417 - Machine Learning
Homework 0: Revision

Introduction The goal of this homework is to review some important mathematical concepts that are used regularly in machine learning, and which are assumed knowledge for the course. If you find yourself struggling significantly with any aspects of this homework, please reach out to course staff so that we can better help you prepare for the course. Please also note that we have posted some helpful resources under the Week 0 tab on Moodle which may be of use to you for this homework.

What to Submit

- A single PDF file which contains solutions to each question. For each question, provide your solution in the form of text and requested plots. For any question in which you use code, provide a copy of your code at the bottom of the relevant section.
- You are free to format your work in any way you think is appropriate. This can include using LaTeX, or taking pictures of handwritten work, or writing your solutions up using a tablet. Please ensure that your work is neat, and start each question on a new page.

When and Where to Submit

- Due date: Friday **June 4th**, 2021 by **5:00pm**.
- For this homework, we will **not** accept late submissions.
- Submissions must be through Moodle - email submissions will be ignored.

Question 1. (Calculus Review)

(a) Consider the function

$$f(x, y) = a_1 x^2 y^2 + a_4 xy + a_5 x + a_7$$

compute all first and second order derivatives of f with respect to x and y.

Solution:

We have the following results:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2a_1xy^2 + a_4y + a_5 \\
\frac{\partial f}{\partial y} &= 2a_1x^2y + a_4x \\
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} &= 4a_1xy + a_4 \\
\frac{\partial^2 f}{\partial x^2} &= 2a_1y^2 \\
\frac{\partial^2 f}{\partial y^2} &= 2a_1x^2
\end{aligned}$$

(b) Consider the function

$$f(x, y) = a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4xy + a_5x + a_6y + a_7$$

compute all first and second order derivatives of f with respect to x and y.

Solution:
We have the following results:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2a_1xy^2 + 2a_2xy + a_3y^2 + a_4y + a_5 \\
\frac{\partial f}{\partial y} &= 2a_1x^2y + a_2x^2 + 2a_3xy + a_4x + a_6 \\
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} &= 4a_1xy + 2a_2x + 2a_3y + a_4 \\
\frac{\partial^2 f}{\partial x^2} &= 2a_1y^2 + 2a_2y \\
\frac{\partial^2 f}{\partial y^2} &= 2a_1x^2 + 2a_3x
\end{aligned}$$

(c) Consider the logistic sigmoid:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

show that $\sigma'(x) = \frac{\partial \sigma}{\partial x} = \sigma(x)(1 - \sigma(x))$

Solution:

Rewriting $\sigma(x) = (1+e^{-x})^{-1}$, then using the chain rule gives

$$\begin{aligned}\sigma'(x) &= (-1)(-e^{-x})(1+e^{-x})^{-2} \\ &= \frac{e^{-x}}{1+e^{-x}}\frac{1}{1+e^{-x}} \\ &= (1-\sigma(x))\sigma(x).\end{aligned}$$

(d) Consider the following functions:

- $y_1 = 4x^2 - 3x + 3$
- $y_2 = 3x^4 - 2x^3$
- $y_3 = 4x + \sqrt{1-x}$
- $y_4 = x + x^{-1}$

Using the second derivative test, find all local maximum and minimum points.

Solution:
For the first function, we differentiate and set equal to zero to find the critical point(s) of the function:

$$y_1' := \frac{\partial y_1}{\partial x} = 8x - 3 \overset{\text{set}}{=} 0 \implies x = \frac{3}{8}.$$

To assess whether y_1 achieves a minimum or maximum at $x = 3/8$, we compute the second derivative:

$$y_1'' := \frac{\partial^2 y_1}{\partial x^2} = 8.$$

Since the second derivative is positive everywhere, it is positive at $x = 3/8$, and so by the second derivative test, y_1 has a minimum at $x = 3/8$. Another approach to this question that does not require the second derivative test would be to recognize that this function is a convex quadratic, and so must have a minimum. Next, note that

$$y_2' = 12x^3 - 6x^2 = 0 \implies x = \frac{1}{2}$$

and

$$y_2'' = 36x^2 - 12x.$$

Evaluating the second derivative of y_2 at $x = 1$ gives:

$$y_2''(1/2) = 36 - 12 = 24 > 0,$$

and so we can conclude that y_2 has a minimum at $x = 1/2$. Netx, we have

$$y_3' = 4 - \frac{1}{2\sqrt{1-x}} = 0 \implies x = \frac{63}{64},$$

and

$$y_3'' = \frac{-1}{4(1-x)^{3/2}},$$

and so $y_3''(\frac{63}{64}) < 0$, so x os a maximum of y_3. Finally,

$$y_4' = 1 - x^{-2} = 0 \implies x = \pm 1$$

and

$$y_4'' = 2x^{-3}.$$

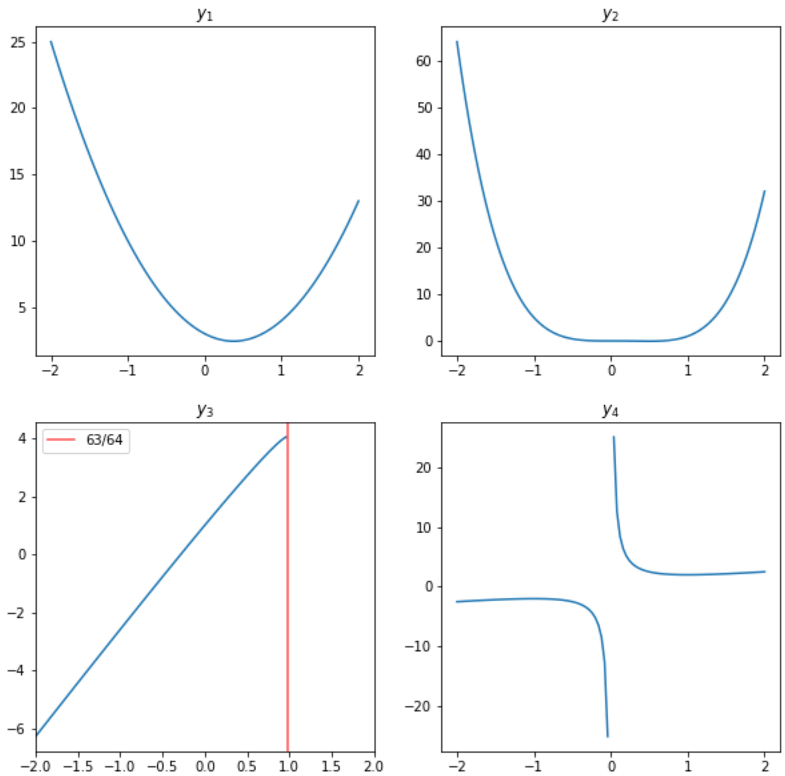
Therefore, $y_4''(1) < 0$ and $y_4''(-1) > 0$, so y_4 has a minimum at $y = 1$ and a maximum at $y = -1$. In matplotlib, we can plot these functions to verify:

```
import numpy as np
import matplotlib.pyplot as plt

y1 = lambda x: 4*x**2 - 3*x + 3
y2 = lambda x: 3*x**4 - 2*x**3
y3 = lambda x: 4*x + np.sqrt(1-x)
y4 = lambda x: x + x**(-1)

x = np.linspace(start=-2, stop=2, num=101)

fig, axs = plt.subplots(2,2, figsize=(10,10))
axs[0,0].plot(x, y1(x))
axs[0,0].set_title("$y_1$")
axs[0,1].plot(x, y2(x))
axs[0,1].set_title("$y_2$")
axs[1,0].plot(x[x<1], y3(x[x<1]))
axs[1,0].axvline(63/64, color='red', alpha=0.7, label='63/64')
axs[1,0].legend()
axs[1,0].set_xlim(-2,2)
axs[1,0].set_title("$y_3$")
axs[1,1].plot(x, y4(x))
axs[1,1].set_title("$y_4$")
plt.savefig("4funs.png")
plt.show()
```



Question 2. (Probability Review)

(a) A manufacturing company has two retail outlets. It is known that 20% of potential customers buy products from Outlet I alone, 10% buy from both I and II, and 40% buy from neither. Let A denote the event that a potential customer, randomly chosen, buys from outel I, and B the event that the customer buys from outlet II. Compute the following probabilities:

$$P(A), \quad P(B), \quad P(A \cup B), \quad P(\bar{A}\bar{B})$$

Solution:

Hint: Draw a Venn diagram.

$$\begin{aligned}
P(A) &= 0.3 \\
P(B) &= 0.4 \\
P(A \cup B) &= P(A) + P(B) - P(AB) = 0.3 + 0.4 - 0.1 = 0.6 \\
P(\bar{A}\bar{B}) &= P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.4.
\end{aligned}$$

Note that we have made regular use of the fact that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

and also made use of De Morgan's law which states that for any two sets A, B:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}, \quad \text{and} \quad \overline{A \cap B} = \bar{A} \cup \bar{B}.$$

Also note that it is customary to write:

$$A \cap B = AB.$$

(b) Let X, Y be two discrete random variables, with joint probability mass function $P(X = x, Y = y)$ displayed in the table below:

		y		
		1	2	3
	1	$1/6$	$1/12$	$1/12$
x	2	$1/6$	0	$1/6$
	3	0	r	0

Compute the following quantities:

(i) r
(ii) $P(X = 2, Y = 3)$
(iii) $P(X = 3)$ and $P(X = 3|Y = 2)$
(iv) $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[XY]$
(v) $\mathbb{E}[X^2]$, $\mathbb{E}[Y^2]$
(vi) $\text{Cov}(X, Y)$
(vii) $\text{Var}(X)$ and $\text{Var}(Y)$
(viii) $\text{Corr}(X, Y)$
(ix) $\mathbb{E}[X + Y]$, $\mathbb{E}[X + Y^2]$, $\text{Var}(X + Y)$ and $\text{Var}(X + Y^2)$.

Solution:

(i) Since this is a probability mass function, the probabilities must sum to 1, and so it follows that $r = 1/3$.

(ii) We can just read this off the table $P(X = 2, Y = 3) = 1/6$.

(iii) To compute $P(X = 3)$, we use the law of total probability, which allows us to sum over all possibilities for (X, Y) when $X = 3$, we can therefore write:

$$
\begin{aligned}
P(X = 3) &= \sum_{y=1}^{3} P(X = 3, Y = y) \\
&= P(X = 3, y = 1) + P(X = 3, y = 2) + P(X = 3, y = 3) \\
&= 1/3.
\end{aligned}
$$

Note that are basically just adding the values in the row corresponding to $X = 3$. Then, to compute the conditional probability, we can use Bayes rule:

$$
\begin{aligned}
P(X = 3|Y = 2) &= \frac{P(X = 3, Y = 2)}{P(Y = 2)} \\
&= \frac{P(X = 3, Y = 2)}{P(X = 1, Y = 2) + P(X = 2, Y = 2) + P(X = 3, Y = 2)} \\
&= \frac{1/3}{1/12 + 0 + 1/3} \\
&= \frac{1/3}{5/12} \\
&= \frac{4}{5}.
\end{aligned}
$$

(iv) Recall the formula for expectation is

$$
\mathbb{E}[X] = \sum_x xP(X = x)
$$

where the summation is over all possible values of X. So, for our case we have

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{3} xP(X = x) \\
&= 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) \\
&= 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} \\
&= 2.
\end{aligned}
$$

Similarly, we can show that

$$
\mathbb{E}[Y] = \frac{23}{12}.
$$

To compute XY, we can apply the same definition, which tells us that

$$
\mathbb{E}[XY] = \sum_{x,y} xyP(X = x, Y = y),
$$

so we are now summing over all possible value taken by X and Y:

$$\begin{aligned}\mathbb{E}[XY] = 1\times 1\times P(X=1,Y=1) + 1\times 2\times P(X=1,Y=2) + 1\times 3\times P(X=1,Y=3)\\ + 2\times 1\times P(X=2,Y=1) + 2\times 2\times P(X=2,Y=2) + 2\times 3\times P(X=2,Y=3)\\ + 3\times 1\times P(X=3,Y=1) + 3\times 2\times P(X=3,Y=2) + 3\times 3\times P(X=3,Y=3)\\ = \frac{47}{12}.\end{aligned}$$

(v) Similar to the previous question, we have

$$\mathbb{E}[X^2] = \sum_x x^2 P(X=x),$$

so this definition applies for any function of X. We therefore get

$$\begin{aligned}\mathbb{E}[X^2] &= 1^2\times P(X=1) + 2^2 P(X=2) + 3^2 P(X=3)\\ &= 1\times\frac{1}{3} + 4\times\frac{1}{3} + 9\times\frac{1}{3}\\ &= \frac{14}{3}.\end{aligned}$$

(vi) Recall the formula for covariance:

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y],$$

and using our results from previous questions we have

$$\text{Cov}(X,Y) = \frac{47}{12} - 2\times\frac{23}{12} = \frac{1}{12}.$$

(vii) Recall that variance is defined by:

$$\text{Var}(X) = \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

We will illustrate the usage of both definitions. First

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{14}{3} - (2)^2 = \frac{2}{3}.$$

Then, we compute the variance of Y using the standard definition of variance:

$$\begin{aligned}\text{Var}(Y) &= \mathbb{E}[(Y-\mathbb{E}[Y])^2]\\ &= \mathbb{E}\left[\left(Y - \frac{23}{12}\right)^2\right]\\ &= \sum_{y=1}^{3}\left(y - \frac{23}{12}\right)^2 P(Y=y)\\ &= \left(1-\frac{23}{12}\right)^2\times\frac{1}{3} + \left(2-\frac{23}{12}\right)^2\times\frac{5}{12} + \left(3-\frac{23}{12}\right)^2\times\frac{1}{4}\\ &= 0.5763889.\end{aligned}$$

Note that you can use either approach, they will give identical results.

(viii) We denote the standard deviation of X by SD(X), then

$$\begin{aligned}\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{\text{SD}(X)\text{SD}(Y)} \\ &= \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}},\end{aligned}$$

then simply plug in the values we computed earlier.

(ix) Recall that expectation is linear, regardless of whether the random variables are dependent or independent, so we have

(a) $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

(b) $\mathbb{E}[X+Y^2] = \mathbb{E}[X] + \mathbb{E}[Y^2]$

Note however that variance is **not** linear in general. We have

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$$

In general, for constants a, b and random variables X, Y,

$$\text{Var}(aX+bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X,Y).$$

Question 3. (Linear Algebra Review)

(a) Write down the dimensions of the following objects:

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 2 \\ 1 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 5 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \qquad A^T$$

Solution:
A is a matrix with 3 rows and 5 columns, so we say that A is a 3×5 matrix, or equivalently $A \in \mathbb{R}^{3\times5}$. Similarly, b is a matrix with a single column, also called a vector, and so $b \in \mathbb{R}^{6\times1}$ or sometimes shortened to $b \in \mathbb{R}^6$. A^T is the transpose of A. To find the tranpose of A, we take element (i,j) in A, and make that element (j,i) in A^T. In this case we get

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & 5 \\ 2 & 2 & 2 \end{bmatrix} \in \mathbb{R}^{5\times3},$$

note that the dimension of the transpose is just the reverse of the dimension of the original object. In Python, we can do:

```python
import numpy as np
A = np.array([[1,3,1,0,2], [1,1,4,1,2], [1,1,1,5,2]])
dimA = A.shape            # dimension of A
AT = A.T                  # A transpose
dimAT = AT.shape          # dimension of A transpose
```

(b) Consider the following objects:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Compute the following:

(i) AB and BA
(ii) AC and CA
(iii) AD and DA
(iv) DC and CD and D^TC
(v) Bu and uB
(vi) Au
(vii) Av and vA
(viii) $Av + Bv$

Solution:
Before computing any of the above, we recall that for any two matrices $X \in \mathbb{R}^{n\times p}$ and $Z \in \mathbb{R}^{q\times m}$, their matrix product XZ exists if and only if $p = q$, and the resulting matrix will have dimenion $n \times m$.

(i) Since $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{2\times 2}$ the matrix product AB is not defined, and neither is BA.

(ii) Both A and C are 3×3 matrices, so their product exists, and is given by

$$AC = \begin{bmatrix} 21 & 14 & 14 \\ 20 & 10 & 10 \\ 56 & 28 & 28 \end{bmatrix}$$

Further, CA is also defined and given by:

$$CA = \begin{bmatrix} 31 & 39 & 40 \\ 10 & 12 & 12 \\ 18 & 18 & 16 \end{bmatrix}$$

So note that in general, $AC \neq CA$, so order matters. In fact, a matrix product might exist in one order, but not the other, as we will see below.

(iii) $A \in \mathbb{R}^{3\times 3}$ and $D \in \mathbb{R}^{3\times 2}$, in this case, $p = q = 3$, and so the matrix product is defined, note that the resulting matrix will be of dimension 3×2 and is given by:

$$AD = \begin{bmatrix} 20 & 32 \\ 17 & 19 \\ 43 & 45 \end{bmatrix}$$

However, DA is not defined, since now $p = 2$ but $q = 3$.

(iv) DC is not defined, but CD is, and has dimension 3×2. $D^T C$ is also defined and has dimension 2×3.

(v) Bu exists and has dimension 2×1

$$Bu = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$$

but uB of course does not exist.

(vi) Au does not exist.

(vii) Av exists, but vA does not.

(viii)

$$Av + Cv = (A + C)v = \begin{bmatrix} 8 & 6 & 7 \\ 4 & 3 & 2 \\ 8 & 6 & 5 \end{bmatrix} v = \begin{bmatrix} 47 \\ 22 \\ 45 \end{bmatrix}$$

In Python, we could compute the above as follows:

```python
import numpy as np
A = np.array([[1,3,4], [2,2,1], [6,4,3]])
B = np.array([[2,4], [1,1]])
C = np.array([[7,3,3], [2,1,1], [2,2,2]])
D = np.array([[4,2], [4,6], [1,3]])
u = np.array([1,3]).reshape(2,1)    # ensures correct shape for vector
v = np.array([2,4,1]).reshape(3,1)

AB = A @ B       # will not work
BA = B @ A       # will not work
AC = A @ C
CA = C @ A
AD = A @ D
DA = D @ A       # will not work
DC = D @ C       # will not work
CD = C @ D
DTC = D.T @ C
Bu = B @ u
A@v + C@v
```

(c) Consider the following objects:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Compute the following:

(i) $\|u\|_1, \|u\|_2, \|u\|_2^2, \|u\|_\infty$

(ii) $\|v\|_1, \|v\|_2, \|v\|_2^2, \|v\|_\infty$

(iii) $\|v+w\|_1, \|v+w\|_2, \|v+w\|_\infty$

(iv) $\|Av\|_2, \|A(v-w)\|_\infty$

Solution:

Recall that for a general vector $x \in \mathbb{R}^n$, the p-norm is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

and for $p = \infty$, this becomes:

$$\|x\|_\infty = \max_{i=1,\dots,n} |x_i|,$$

i.e. the ∞-norm is just the largest (absolute) element in the vector. Note that in the special case $p = 2$, we have $\|x\|_2^2 = x^T x$. Let's work through (ii) explicitly:

$$\begin{aligned} \|v\|_1 &= \sum_{i=1}^{3} |v_i| = |2| + |4| + |1| = 7 \\ \|v\|_2 &= \sqrt{\sum_{i=1}^{3} |v_i|^2} = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21} \\ \|v\|_2^2 &= v^T v = 21 \\ \|v\|_\infty &= \max\{|2|, |4|, |1|\} = 4 \end{aligned}$$

Note that a norm is a way of measuring the size of an object, and different choices of p correspond to different ways of measuring the size. In Python we would compute:

```
v_1norm = np.linalg.norm(v, ord=1)
v_2norm = np.linalg.norm(v, ord=2)
v_2norm_sq = np.linalg.norm(v, ord=2)**2
v_infnorm = np.linalg.norm(v, ord=np.inf)
```

(d) Consider the following vectors in $\mathbb{R}^2$

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$$

Compute the dot products between all pairs of vectors. Note that the dot product may be written using the following equivalent forms:

$$\langle x, y \rangle = x \cdot y = x^T y.$$

Then compute the angle between the vectors and plot.

Solution:

$$\begin{aligned} \langle u, v \rangle &= 1 \times 1 + 2 \times 1 = 3 \\ \langle u, w \rangle &= 0 \\ \langle v, w \rangle &= -1/2 \end{aligned}$$

To compute the angle between vectors x and y, which we will label as θ_{xy}, we recall the following important formula (we will explore this result in more depth in the next part):

$$\langle x, y \rangle = \|x\|_2 \|y\|_2 \cos(\theta_{xy}),$$

and so rearranging yields:

$$\theta_{xy} = \arccos\left(\frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2}\right).$$

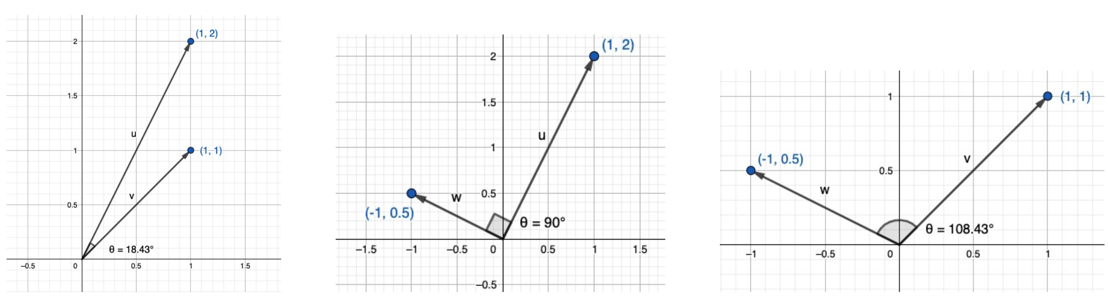
Then, for example, we have

$$\theta_{uv} = \arccos\left(\frac{3}{\sqrt{5} \times \sqrt{2}}\right) = 18.43°.$$

We similarly have

$$\theta_{uw} = 90°, \qquad \theta_{vw} = 108.43°.$$

The following plots (created using Geogebra) depict the three scenarios:



In python, we can do:

```
u = np.array([1,2])
v = np.array([1,-1])
w = np.array([-1,-1])
dotuv = np.dot(u, v)

```

```python
def angle_bw(x,y):
    dotprod = np.dot(x,y)
    norm_x = np.linalg.norm(x, ord=2)
    norm_y = np.linalg.norm(y, ord=2)
    # arccos returns radians, we want degrees
    return np.degrees(np.arccos(dotprod/(norm_x*norm_y)))

theta_uv = angle_bw(u,v)
```

(e) Dot products are extremely important in machine learning, explain what it means for a dot product to be zero, positive or negative.

Solution:
Recall the following angle types:

1. acute angle: $\theta \in [0^\circ, 90^\circ)$
2. right angle: $\theta = 90^\circ$
3. obtuse angle: $\theta \in [90^\circ, 180^\circ)$

Continuing from the previous question, we saw that when a dot product was positive, the resulting angle was acute, when the dot-product was zero, then the angle was a right angle, and when the dot-product was negative, the result was an obtuse angle. In machine learning, we use dot-products to measure the strength of relationship between two objects (where the objects are represented as vectors). If we take the dot product between two vectors and it is positive, then we know that the objects point in the same direction (acute). The right angle is the special case where the objects neither agree or disagree, and the obtuse angle tells us they are pointing in different directions. We will often just be concerned with the **sign** of the dot product (is it positive or negative?) and not the exact angle.

(f) Consider the 2×2 matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$$

Compute the inverse of A.

Solution:
Recall that the inverse matrix of A is denoted A^{-1}, and is defined as the matrix that satisfies $AA^{-1} = I$, where I is the identity matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

. Now, for the 2×2 case, there is a simple formula for the determinant. For a general 2×2

matrix

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse is

$$X^{-1} = \frac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where det denotes the matrix determinant. Using this formula, we have

$$A^{-1} = \frac{1}{1 \times 1 - 4 \times 3} \begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1/11 & 3/11 \\ 4/11 & 1/11 \end{bmatrix}$$

To check that this is indeed the inverse, we can verify that $A^{-1}A = I$. Note that in Python, we can do the following:

```
import numpy as np
from numpy.linalg import det, inv

A = np.array([[1,3], [4,1]])
detA = det(A)
invA = inv(A)
```

(g) Consider the 2×2 matrix

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$$

Compute its inverse A^{-1}.

Solution:
Note that in this case, $\det(A) = 12 - 12 = 0$, so in order to use the formula from the previous part, we would have to divide by zero. In other words, the inverse does not exist in this case.

(h) Let X be a matrix (of any dimension), show that $X^T X$ is always symmetric.

Solution:
Recall that a symmetric matrix is a matrix that is equal to its transpose, i.e. A is symmetric if and only if $A = A^T$. Now, it follows easily that

$$(X^T X)^T = X^T (X^T)^T = X^T X,$$

so $X^T X$ is symmetric, and this holds for any matrix X. Note that we used two results in the previous calculation, namely that that the transpose of a transpose just gives back the original matrix:

$$(A^T)^T = A$$

and that taking a transpose of a product reverses the order:

$$(AB)^T = B^T A^T.$$