Notes on Algebraic Geometry [3]

1. Chapter I

1.1.

Definition 1.1.1. Let $A = k[x_1, \ldots, k_n]$, where k is algebraically closed. Then a subset Y of $\mathbb{A}^n$ is an *algebraic set* if there exists a subset $T \subset A$ such that $Y = \text{Zero}(T)$.

Proposition 1.1.2. The collection of algebraic sets satisfies the axioms for closed sets.

Definition 1.1.3. We define the *Zariski topology* on $\mathbb{A}^n$ by declaring that the open sets are the complements of algebraic sets.

Example 1.1.4 (What are the non-trivial open sets of the affine line $\mathbb{A}^1$?).
Open sets are complements of closed sets, and every closed set is the zero set of an ideal $\mathfrak{a}$ of $A = k[x]$. Since A is principal in this case, we have that $\mathfrak{a} = (f(x))$ for some f. Since k is algebraically closed, $f = c(x - a_1) \cdots (x - a_s)$ for some $c \neq 0, a_1, \ldots, a_s \in k$. Therefore, $\text{Zero}(\mathfrak{a}) = \text{Zero}(f) = \{a_1, \ldots, a_s\}$ is a finite set of points.[[This topology is not Hausdorff, since any algebraically closed field is necessarily infinite.]]

Definition 1.1.5. A nonempty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper closed subsets (with respect to induced topology). The empty set is not considered to be irreducible. [[Equivalently, any two nonempty open subset of Y intersects. Or equivalently, any nonempty open set of an irreducible space is dense.]]

Exercise 1.1.6. [[Any nonempty open subset of an irreducible space is irreducible.]]

Proof. Let $U \subset Y$ be a nonempty open subset of the irreducible Y.
If U is irreducible, then $U = (K_1 \cap U) \cup (K_2 \cap U)$, where K_1, K_2 are two closed sets of Y such that $K_1 \cap U, K_2 \cap U$ are nonempty proper subsets of U. Then K_1, K_2 are necessarily proper closed subset of Y. Thus, $Y = K_1 \cup (K_2 \cup (Y - U))$ shows that Y is not irreducible. □

Exercise 1.1.7. [[If Y is an irreducible subset of X, so is its closure $\overline{Y}$.]]

Proof. Let U, V be two nonempty open subsets of $\overline{Y}$. Then, $U \cap Y, V \cap Y$ are nonempty, and $U \cap V \cap Y \neq \emptyset$ because Y is irreducible. □

Definition 1.1.8. An *affine variety* is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a *quasi-affine variety*.

Proposition 1.1.9. For any subset $Y \subset \mathbb{A}^n$ we define the *ideal of Y* in A as

$$I(A) = \{f \in A | f(P) = 0 \text{ for all } P \in Y\}.$$

Then,

(1) If $T_1 \subset T_2 \subset A$, then $\text{Zero}(T_1) \supset \text{Zero}(T_2)$.
(2) If $Y_1 \subset Y_2 \subset \mathbb{A}^n$, then $I(Y_1) \supset I(Y_2)$.
(3) For any $Y_1, Y_2 \subset \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(4) For any ideal $\mathfrak{a} \subset A$, $I(\mathrm{Zero}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(5) For any $Y \subset \mathbb{A}^n$, $\mathrm{Zero}(I(Y)) = \overline{Y}$.

Proof. (1) Trivial.
(2) Of course.
(3) Sure.
(4) This is Hilbert's Nullstellensatz, which applies only when k is algebraically closed.
(5) Since the closure is the smallest closed set containing Y, it is enough to show that $\mathrm{Zero}(I(Y)) \subset K$ whenever $K = \mathrm{Zero}(\mathfrak{a})$ is a closed set containing Y. But we clearly have $\mathfrak{a} \subset I(Y)$, for Y is inside K, the zero set of $\mathfrak{a}$. Now use (1). □

Remark 1.1.10. From the proposition above, we see that $I : \mathbb{A}^n \to A$ and $\mathrm{Zero} : A \to \mathbb{A}^n$ induce a *1-1 correspondence* between the set of algebraic sets and the set of radical ideals.

Exercise 1.1.11. [[An algebraic set is an affine variety if and only if its ideal is prime.]]

Proof. Suppose that Y is a variety and $fg \in I(Y)$. Then

$$Y \subset \mathrm{Zero}(fg) = \mathrm{Zero}(f) \cup \mathrm{Zero}(g).$$

Now Y equals the union of $(Y \cap \mathrm{Zero}(f))$ and $(Y \cap \mathrm{Zero}(g))$, while both of these two are closed subsets of Y. Since Y is irreducible, either $Y \subset \mathrm{Zero}(f)$ or $Y \subset \mathrm{Zero}(g)$, which means exactly that either $f \in I(Y)$ or $g \in I(Y)$, proving that $I(Y)$ is prime.

Conversely, suppose that $I(Y) = \mathfrak{p}$ is prime. Let $Y_1, Y_2 \subset Y$ be two closed subsets such that $Y_1 \cup Y_2 = Y$. Then, $I(Y_1) \cap I(Y_2) = I(Y) = \mathfrak{p}$, so $I(Y_1) = \mathfrak{p}$ or $I(Y_2) = \mathfrak{p}$. Therefore, $Y = \mathrm{Zero}(\mathfrak{p}) = Y_1$ or Y_2, proving that Y is irreducible. □

Definition 1.1.12. An *affine curve of degree d* is the affine variety determined by an irreducible polynomial $f(x, y) \in k[x, y]$ of total degree d. [[The ideal (f) is prime since $k[x, y]$ is a UFD.]] More generally, we have *surfaces (or hypersurfaces)*, which are the variety determined by an irreducible $f \in k[x_1, \dots, x_n]$ (surface when $n = 3$, and hypersurface when $n > 3$).

Exercise 1.1.13. [[Every maximal ideal of $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.]]

Proof. Since a maximal ideal is of course prime and radical, by the correspondence, we see that a maximal ideal corresponds to a minimal irreducible algebraic set, which is a point $P = (a_1, \dots, a_n) \in \mathbb{A}^n$. Thus, the maximal ideal is $I(P) = (x_1 - a_1, \dots, x_n - a_n)$. □

Definition 1.1.14. If $Y \subset \mathbb{A}^n$ is an affine algebraic set, the *affine coordinate ring $A(Y)$* of Y is defined to be $A(Y) := A/I(Y) = k[x_1, \dots, x_n]/I(Y)$.

Remark 1.1.15. If Y is an affine coordinate ring, then $A(Y)$ is a finitely generated k-algebra which is also an integral domain. Conversely if B is a finitely generated k-algebra that is also an integral domain, then B is the affine coordinate ring for some affine variety.

Definition 1.1.16. A topological space X is *Noetherian* if any of the following equivalent condition holds:

(a). Every descending chain of closed sets stabilizes.
(b). Every nonempty collection of closed sets has a minimal element.

Proposition 1.1.17. X is Noetherian if and only if every open set in X is quasi-compact. [[if and only if every subset is quasi-compact, because the proof shows that (X is Noetherian $\implies$ every subset of X is quasi-compact).]]

Proof. Suppose that X is Noetherian and let U be a nonempty open subset of X.

Assume, for the sake of contradiction, that there exists $(V_i)_{i\in I}$, an open cover of U having no finite subcover. Then, we can inductively extract an ascending chain from finite unions of sets in $(V_i)_{i\in I}$ that never stabilizes, the complement of which is a descending chain of closed sets that never stabilizes, a contradiction.

Conversely, suppose that X is such that every open subset $U \subseteq X$ is compact. Let

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$$

be a descending chain of closed sets. Then with $U_i := X \setminus K_i$, the following

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$$

is an ascending chain of open sets. We have that $X \setminus \bigcap_{i=1}^{\infty} K_i = \bigcup_{i=1}^{\infty} U_i$.

Since $X \setminus \bigcap_{i=1}^{\infty} K_i$ is compact, there exists n such that $\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{n} U_i = U_n$, meaning that the U_i-chain (and hence the K_i-chain) stabilizes. $\square$

Example 1.1.18. $\mathbb{A}^n$ is a Noetherian space since any descending chain of closed subsets corresponds to an ascending chain of ideals of A, which has to stabilize since A is a Noetherian ring.

Proposition 1.1.19. In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \dots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. They are called the *irreducible components* of Y.

Corollary 1.1.20. Every algebraic set in $\mathbb{A}^n$ is expressed uniquely as a union of varieties, no one containing another. [[by Example 1.1.18 and Proposition 1.1.19]]

An algebraic proof. It is well-known that in a Noetherian ring, any radical ideal $\mathfrak{a} = \sqrt{\mathfrak{a}}$ equals the intersection of its (finitely many) minimal primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_{\ell_\mathfrak{a}}\}$.

Then

$$\mathrm{Zero}(\mathfrak{a}) = \bigcup_{i=1}^{\ell_\mathfrak{a}} \mathrm{Zero}(\mathfrak{p}_i).$$

Since the prime ideals are minimal, none of these varieties contain another.
[[In this proof, we used the fact that

$$\mathrm{Zero}(\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{\ell_\mathfrak{a}}) = \mathrm{Zero}\left(\prod_{i=1}^{\ell_\mathfrak{a}} \mathfrak{p}_i\right),$$

which follows from the observations that

$$\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{\ell_\mathfrak{a}} \supseteq \prod_{i=1}^{\ell_\mathfrak{a}} \mathfrak{p}_i \quad \text{and} \quad \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{\ell_\mathfrak{a}} \subseteq \sqrt{\prod_{i=1}^{\ell_\mathfrak{a}} \mathfrak{p}_i}$$

.]] $\square$

Definition 1.1.21. If X is a topological space, the *dimension* of X is

$$\dim(X) = \sup\{n \in \mathbb{Z}_{\geq 0} | \exists \text{ a chain of irreducible closed subsets in } X\text{: } Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n\}.$$

In a ring A, the *height* of a prime ideal $\mathfrak{p}$ is

$$\mathrm{ht}(\mathfrak{p}) = \sup\{n \in \mathbb{Z}_{\geq 0} | \exists \text{ a chain of prime ideals contained in } \mathfrak{p}\text{: } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of a ring A is then

$$\begin{aligned}\dim(A) &= \sup\{\mathrm{ht}(\mathfrak{p}) | \mathfrak{p} \subset A \text{ is prime.}\} \\ &= \sup\{n \in \mathbb{Z}_{\geq 0} | \exists \text{ a chain of prime ideals in } A\text{: } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}.\end{aligned}$$

Proposition 1.1.22. The dimension of an affine algebraic set equals to Krull dimension of its coordinate ring.

Proof. Let Y be an affine algebraic set in $\mathbb{A}^n$. Any irreducible closed subset of Y corresponds to a prime ideal in A containing $I(Y)$, which then corresponds to a prime ideal in $A(Y) = A/I(Y)$. Hence, every strictly ascending chain of irreducible closed sets of Y corresponds to a strictly descending chain of prime ideals. Thus, $\dim(Y) = \dim(A(Y))$. □

Proposition 1.1.23. Let k be a field, and B be an integral domain that is also a finitely generated k-algebra. Then,

(1) The Krull dimension of B equals $\mathrm{tr.deg.}_k(\mathrm{Quot}(B))$.
(2) For a prime ideal $\mathfrak{p}$ of B, one has

$$\mathrm{ht}(\mathfrak{p}) + \dim(B/\mathfrak{p}) = \dim(B),$$

which says (?) that a maximal chain of prime ideals in B is the composition of a maximal chain of prime ideals contained in $\mathfrak{p}$ and a maximal chain of prime ideals containing $\mathfrak{p}$.

As an application, we have

Example 1.1.24 (The dimension of $\mathbb{A}^n$ is n)**.**
Indeed, $\mathrm{Quot}(A(\mathbb{A}^n)) = k(x_1, \ldots, x_n)$ has transcendence degree n over k, for $\{x_1, \ldots, x_n\} \subset \mathrm{Quot}(A(\mathbb{A}^n))$ is algebraically independent over k and $\mathrm{Quot}(A(\mathbb{A}^n))$ is algebraic over $k(\{x_1, \ldots, x_n\})$. (we have shown that $\{x_1, \ldots, x_n\}$ is a *transcendental basis* of $k(x_1, \ldots, x_n)$ over k. For details, cf. [2])

Proposition 1.1.25. If Y is a quasi-affine variety, then $\dim(Y) = \dim(\overline{Y})$.

Theorem 1.1.26 (Krull's Hauptidealsatz)**.** Let A be a Noetherian ring, and let $f \in A$ be an element which is neither a unit nor a zero-divisor. Then, every minimal prime ideal $\mathfrak{p}$ containing f has height 1.

Theorem 1.1.27. A Noetherian domain A is a UFD $\iff$ every prime ideal of height 1 is principal.

Theorem 1.1.28. A variety Y in $\mathbb{A}^n$ has dimension $n-1$ $\iff$ $Y = \mathrm{Zero}(f)$, where f is an irreducible polynomial in $k[x_1, \ldots, x_n]$.

Proof. If f is irreducible, then $\text{Zero}(f)$ is a variety whose coordinate ring is $A/(f)$. By Theorem 1.1.26, $\text{ht}((f)) = 1$. By applying Proposition 1.1.23 to A, we have $1+\dim(A/(f)) = n$, so $\dim(A/(f)) = n-1$, which is just $\dim(\text{Zero}(f))$.

Conversely, a variety Y of dimension $n-1$ corresponds to a prime ideal of height 1 since its ideal is prime and by Proposition 1.1.23,

$$\text{ht}(I(Y)) + n - 1 = \text{ht}(I(Y)) + \dim(Y) = \text{ht}(I(Y)) + \dim(A/I(Y)) = \dim(A) = n.$$

Since $A = k[x_1, \ldots, x_n]$ is a Noetherian domain that is also a UFD, by Theorem 1.1.27 we have that $I(Y) = (f)$ for some $f \in A$ that is necessarily irreducible. So, $Y = \text{Zero}(f)$. □

Problem 1 (Ex 1.1).

(1) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.
(2) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.
(3) * Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof.

(1) We have a morphism $\varphi : k[x] \to k[x, y]/(y - x^2) = A(Y)$ sending $x \mapsto x$, which is clearly surjective. Since the morphism $\psi : A(Y) \to k[x]$ defined by $x \mapsto x, y \mapsto x^2$ is well-defined and $\psi \circ \varphi = \text{Id}_{k[x]}$, φ is injective and hence an isomorphism.
(2) In $A(Z)$ x, y are units, so the image of any morphism from $A(Z)$ to $k[t]$ lies in k.
(3) Zan Shi Bu Hui!

□

Problem 2 (Ex 1.2, The Twisted Cubic Curve). Let $Y \subset \mathbb{A}^3$ be the set $\{(t, t^2, t^3) | t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Proof. We claim that $(y - x^2, z - x^3) = I(Y)$. Clearly $(y - x^2, z - x^3) \subset I(Y)$. Conversely, $\text{Zero}(y - x^2, z - x^3) \subset Y$. So, $\text{Zero}(y - x^2, z - x^3) = Y$ and $I(Y) = \sqrt{(y - x^2, z - x^3)} = (y - x^2, z - x^3)$, where the last equality follows from the fact that $(y - x^2, z - x^3)$ is prime. To see that $(y - x^2, z - x^3)$ is prime, we just observe that $k[x, y, z]/(y - x^2, z - x^3) \cong k[x]$, which also shows that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. □

Problem 3 (Ex 1.3). Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Proof. Solve the system of equations $\begin{cases} x^2 - yz = 0 \\ xz - x = 0 \end{cases}$. We get that:

If $x = 0$, then $yz = 0$ so at least one of y, z is 0. We are getting the y-axis and the z-axis. If $x \neq 0$, then $z = 1$ and $y = x^2$. We are getting a parabola in the plane $z = 1$.

So, Y is the union of two straight lines and one parabola.
The prime ideals are respectively $(x, z), (x, y), (x^2 - y, z - 1)$. $\square$

Problem 4 (Ex 1.4). If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$, show that the Zariski topology does not coincide with the product topology.

Proof. The nonempty open sets in $\mathbb{A}^1$ are of the form $\mathbb{A}^1 \setminus M$, where $M \subset \mathbb{A}^1$ is finite. So, every nonempty open set in $\mathbb{A}^2$ with respect to the product topology contains $\mathbb{A}^2$ with finitely many straight lines (i.e., a grid with finitely many points of intersection) removed.

But then the complement of $y = x$ in $\mathbb{A}^2$ is a set that is open with respect to the Zariski topology but not open with repsect to the product topology (since $y = x$ pass through infinitely many straight lines, for example, of the form $x = \alpha$ for $\alpha \in k$). $\square$

Problem 5 (Ex 1.5). Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Proof. If B is the coordinate ring of an algebraic set, say $B = A/I(Y)$, then $I(Y) = \sqrt{I(Y)}$ by Hilbert's Nullstellensatz, which means that B has no nilpotent elements.

Conversely, if B is a finitely generated k-algebra with no nilpotent elements, say $B = k[b_1, \ldots, b_n]$ for some $b_i \in B$, then $B = k[x_1, \ldots, x_n]/\ker(\pi)$, where $\pi : k[x_1, \ldots, x_n] \to B$ is the canonical projection. The assumption that B has no nilpotent elements means that $\ker(\pi)$ is radical. So $\ker(\pi) = I(\text{Zero}(\ker(\pi)))$; B is the coordinate ring of $\text{Zero}(\ker(\pi))$. $\square$

Problem 6 (Ex 1.6). Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof. Proved in Definition 1.1.5, Exercises 1.1.6 and 1.1.7. $\square$

Problem 7 (Ex 1.7).

(1) Show that the following conditions are equivalent for a topological space X:
 (a) X is Noetherian;
 (b) Every nonempty family of closed subsets has a minimal element;
 (c) X satisfies the ascending chain condition for open sets;
 (d) every nonempty family of open subsets has a maximal element.
(2) A Noetherian topological space is quasi-compact.
(3) Any subset of a Noetherian space is Noetherian in its induced topology.
(4) A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Proof. (1) Recall that X is Noetherian if it satisfies the descending chain condition for closed sets. The equivalence of (a), (b), (c), (d) is easy to see.
(2) Proved in Proposition 1.1.17.
(3) Let X be Noetherian and $Y \subset X$ be a subspace. Suppose that $K_1 \supset K_2 \supset K_3 \supset \ldots$ is a descending chain of closed subsets of Y. Then, $\overline{K_1} \supset \overline{K_2} \supset \ldots$ is a descending chain of closed sets in X. Since X is Noetherian, this stabilizes. Since $K_n = \overline{K_n} \cap Y$, the K_n-chain also stabilizes, showing that Y is Noetherian.

(4) Suppose that X is Noetherian and Hausdorff. It suffices to prove that every point of X is open (the finiteness of X also follows since all the singletons form an open cover of the quasi-compact X).

Since the cases where $X = \emptyset$ or $X = \{x\}$ are trivial, we may assume that X contains at least two distinct elements. Let $x_0 \in X$ be fixed. For any $x \neq x_0$, we can find neighborhoods U_x of x and V_x of x_0 such that $U_x \cap V_x = \emptyset$. Therefore $X \setminus \{x_0\} = \bigcup_{x \neq x_0} U_x = \bigcup_{i=1}^n U_{x_i}$ for finitely many points $x_1, \ldots, x_n$ since every subset of Noetherian space is compact by Proposition 1.1.17, or by (2) and (3). But then $\{x_0\} = \bigcap_{i=1}^n V_{x_i}$ is open. □

Problem 8 (Ex 1.8). Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible components of $Y \cap H$ has dimension $r - 1$.

Proof. To find the irreducible components of $Y \cap H$, we are really looking for the minimal primes of the radical ideal $I(Y \cap H)$. (cf. the algebraic proof of Corollary 1.1.20).

Let $I(H) = (f)$ where $f \in A$ is irreducible, and $I(Y) = \mathfrak{p}$. Then $I(Y \cap H) = \sqrt{\mathfrak{p} + (f)}$. To find the minimal primes of $\sqrt{\mathfrak{p} + (f)}$, it is equivalent to find the minimal primes containing f in $A(Y) = A/\mathfrak{p}$.

Since $Y \not\subseteq H$, $f \notin \mathfrak{p}$ and hence f is not a zero-divisor in $A/\mathfrak{p}$ because $\mathfrak{p}$ is prime.

If f is a unit in $A/\mathfrak{p}$, then f is nowhere zero on Y, showing that $Y \cap H = \emptyset$. In this case, the statement about irreducible components of $Y \cap H$ is vacuously true.

So we may assume that f is neither a zero-divisor nor a unit in $A/\mathfrak{p}$. By Theorem 1.1.26, every minimal prime $\mathfrak{q}$ containing f has height 1. Then by Proposition 1.1.23,

$$\operatorname{ht}(\mathfrak{q}) + \dim(A(Y)/\mathfrak{q}) = \dim(A(Y))$$

, from which we read off that

$$\dim(\operatorname{Zero}(\pi^{-1}(\mathfrak{q}))) = \dim(A/\pi^{-1}(\mathfrak{q})) = \dim\left(\frac{A/\mathfrak{p}}{\pi^{-1}(\mathfrak{q}))/\mathfrak{p}}\right) = \dim(A(Y)/\mathfrak{q}) = r - 1$$

, where $\pi : A \to A/\mathfrak{p}$ is the canonical projection. □

Problem 9 (Ex 1.9). Let $\mathfrak{a} \subset A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $\operatorname{Zero}(\mathfrak{a})$ has dimension $\geq n - r$.

Proof. Since every irreducible component of $\operatorname{Zero}(\mathfrak{a})$ has as its ideal a minimal prime of $\mathfrak{a}$, by Proposition 1.1.23 we just need to show that for any minimal prime $\mathfrak{p}$ containing $\mathfrak{a}$, $\operatorname{ht}\mathfrak{p} \leq r$.
The magical theorem that comes to help is Dimension Theorem for Noetherian local rings, which implies that

$$\operatorname{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) = \text{ the least number of generators of an } \mathfrak{p}A_{\mathfrak{p}}\text{-primary ideal of } A_{\mathfrak{p}}$$

, while the radical of the extension of $\mathfrak{a}$ is $\mathfrak{p}A_{\mathfrak{p}}$-primary:

$$\sqrt{S^{-1}\mathfrak{a}} = \sqrt{S^{-1}\mathfrak{p}_1 \cap \cdots \cap S^{-1}\mathfrak{p}_{\ell_{\mathfrak{a}}}} = \sqrt{S^{-1}\mathfrak{p}} = S^{-1}\sqrt{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$$

because localization commutes with finite intersections and radicals.
So, $r =$ a number of generators of a $\mathfrak{p}A_{\mathfrak{p}}$-primary ideal $S^{-1}\mathfrak{a} \geq \dim(A_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p})$. □

Problem 10 (Ex 1.10).

(1) If Y is any subset of a topological space X, then $\dim(Y) \leq \dim(X)$.
(2) If X is a topological space which is covered by a family of open subsets $\{U_i\}_i$ then $\dim(X) = \sup \dim(U_i)$.
(3) Give an example of a topological space X and a dense open subset U with $\dim(U) < \dim(X)$.
(4) Give an example of a Noetherian space of infinite dimension.

Proof. (1) Let $K_1 \supset K_2 \supset \dots$ be a chain of irreducible closed sets in Y. Then we have $\overline{K_1} \supset \overline{K_2} \supset \dots$, a chain of irreducible (cf. Exercise 1.1.7) closed sets in X. Since $K_i = \overline{K_i} \cap Y$, the $\overline{K_i}$-chain is strictly decreasing if the K_i-chain is.

(2) The topological space X is covered by its irreducible components. [[I.e., the maximal irreducible subsets, which are automatically closed and cover X. (cf. [1] Exercise 20 (iii), Chapter 1)]] Hence, every descending chain of irreducible closed sets is contained in some irreducible components.

Let $X_0 \subset X$ be an irreducible components of X. Since $\dim(\emptyset) = \infty$ for which the statement is true, we may assume that $X \neq \emptyset$ and hence $X_0 \neq \emptyset$. Since $\{U_i\}_i$ covers X, we can find some U_i such that $U_i \cap X_0 \neq \emptyset$. Then $U_i \cap X_0$ is dense and irreducible in X_0 by Exercise 1.1.6.

For any strictly descending chain of irreducible closed subsets of X_0

$$C_1 \supset C_2 \supset C_3 \supset \dots$$

the chain

$$(C_1 \cap U_i) \supset (C_2 \cap U_i) \supset (C_3 \cap U_i) \supset \dots$$

consists of irreducible closed subsets of U_i and decreases strictly since $\overline{C_n \cap U_i} = C_n$, where the closure is taken in X_0. We conclude that $\dim(X_0) \leq \dim(U_i)$.

We now that

$$\dim(X) = \sup_{X_j \in \{\text{irreducible components of } X\}} \dim(X_j) \leq \sup \dim(U_i).$$

On the other hand, since $U_i \subset X$, $\sup \dim(U_i) \leq \dim(X)$ and hence the equality.

(3) Let $X = \{P_1, P_2\}$ and the topology on X be $\{\emptyset, \{P_1\}, X\}$. Then the collection of all closed sets is $\{\emptyset, \{P_2\}, X\}$, and $X, \{P_2\}$ are the only irreducible closed subsets of X. So, $\dim(X) = 2$. On the other hand, $\{P_1\}$ is a dense open subset of X whose dimension is 1.

(4) Let $X = \mathbb{Z}_{\geq 1}$ and the topology be $\{\emptyset, X, L_1, L_2, L_3, \dots\}$, where $L_n = \{k \in X | k \geq n\}$. Then the non-empty proper irreducible closed sets are of the form $\{1, \dots, n\}$, from which we see that X is Noetherian of infinite dimension. ☐

Problem 11 (Ex 1.11). Let $Y \subset \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection.*

Proof. Zan Shi Bu Hui! ☐

Problem 12 (Ex 1.12). Give an example of an irreducible polynomial in $\mathbb{R}[x, y]$ whose zero set in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible.

Proof. Consider $f = x^2 + 1$, an irreducible polynomial whose zero set is $\emptyset$, which is not irreducible by definition.

A non-trivial example is $f(x, y) = y^2 + x^2(x-1)^2$. Then, $\mathrm{Zero}(f) = \{(0,0),(1,0)\}$ is not irreducible. To show that f is irreducible, we think of $\mathbb{R}[x, y] = (\mathbb{R}[x])[y]$. By considering $\deg_y(f)$, the only possible factorization is that $f(x, y) = (uy + f(x))(u^{-1}y + g(x))$ for some $u \in \mathbb{R}^\times$ and $f, g \in \mathbb{R}[x]$. We may assume that $u = 1$. Then

$$y^2 + (f+g)y + fg = y^2 + (x(x-1))^2$$

from which we see that $g = -f$ and $-f^2 = (x(x-1))^2$, which is impossible. Hence, f is irreducible. $\square$

References

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