CHANGE OF WEIGHTS OPERATIONS FOR TRIANGULATED (φ, Γ)-MODULES

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ABSTRACT. Zhixiang Wu has shown the existence of "change of weights" operation on (φ, Γ)-modules in families, [Wu, Prop. 3.16]. We interpret it in the trianguline case as pullbacks with a discussion on related stacks. Finally, we prove that it intertwines well with translation functors via a 1-1 correspondence defined by Yiwen Ding [Din25] in the non-critical crystabelline case.

CONTENTS

1. INTRODUCTION

1.1. **Background and main results.** Under the locally analytic p-adic Langlands correspondence of $\mathrm{GL}_2(\mathbb{Q}_p)$ established in [Col16], the papers [JLS24] and more generally [Dina] studied certain "change of weights" operations on 2-dimensional p-adic $G_{\mathbb{Q}_p}$-representations (or rather, rank 2 (φ, Γ)-modules over the Robba ring) that were shown to correspond to translation functors ([Hum08]) on the side of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then, such operations were generalized by Zhixiang Wu to families of rank 2 (φ, Γ)-modules over rigid spaces in [Wu].

The paper [Col19] can be seen as a precursor of these techniques and results (even though the connection to translation functors is not considered there).

1.1.1. *Change of weights for triangulated (φ, Γ_K)-modules.* It is natural to ask for a general theory of "change of weights" for (φ, Γ_K)-modules in families. For $\mathrm{GL}_2(\mathbb{Q}_p)$, using the basic constructions in the p-adic Langlands correspondence of Colmez, [Dina] equipped $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules D of rank 2 over the Robba ring $\mathcal{R}_{\mathbb{Q}_p,E}$ with $\mathfrak{gl}_2(\mathbb{Q}_p)$-module structures, equipped $D \otimes_E \mathrm{Sym}^k(E^2)$ with natural $\mathfrak{gl}_2(\mathbb{Q}_p)$-module and $(\varphi, \Gamma_{\mathbb{Q}_p})$-module structures, and developed a theory of change of weights.

For $n > 2$ or base field $K \neq \mathbb{Q}_p$, it is unclear to us how to equip (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$ with $\mathfrak{gl}_n(K)$-module structures, but we always have pullback and pushforward operations in the category of pairs $(D, \mathrm{Fil}^\bullet(D))$, where D is a trianguline (φ, Γ_K)-modules and $\mathrm{Fil}^\bullet(D)$ is a triangulation on D. The invertibility of the corresponding maps on the extension groups in [Col08, Theorem 2.22(i)] is reminiscent of the fact that translation functors between dominant integral weights of the same regularity induce equivalences of categories, cf. [Hum08, §7.8].

More precisely, given a triangulated (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ of rank n over $\mathcal{R}_{K,A}$

$$\mathrm{Fil}^0(D) = 0 \subsetneq \mathrm{Fil}^1(D) \subsetneq \cdots \subsetneq \mathrm{Fil}^n(D) = D$$

where A is any affinoid algebra over a finite extension $E/\mathbb{Q}_p$ containing a Galois closure of K and $\mathrm{Fil}^i(D)$ are saturated (φ, Γ_K)-submodules of D, we pick $0 \leq j \leq n$ and consider the extension

$$0 \to \mathrm{Fil}^j(D) \to D \to D/\,\mathrm{Fil}^j(D) \to 0.$$

Let Σ_K be the set of embeddings from K to E. We assume always that $|\Sigma_K| = [K : \mathbb{Q}_p]$. For any $\mathbf{k} = (k_\sigma)_\sigma \in \mathbb{N}^{\Sigma_K}$, let $t^{\mathbf{k}} := \prod_{\sigma: K \to E} t_\sigma^{k_\sigma} \in \mathcal{R}_{K,E}$, cf. [KPX14, Notation 6.2.7]. Then we can pullback D along $t^{\mathbf{k}}(D/\,\mathrm{Fil}^j(D)) \subset D/\,\mathrm{Fil}^j(D)$ to get a subobject $p_{\mathbf{k}}(D, \mathrm{Fil}^j(D))$ of D:

$$0 \to \mathrm{Fil}^j(D) \to p_{\mathbf{k}}(D, \mathrm{Fil}^j(D)) \to t^{\mathbf{k}}(D/\,\mathrm{Fil}^j(D)) \to 0.$$

Note that if $(h_{i,\sigma})_{1 \leq i \leq n, \sigma} \in (A^n)^{\Sigma_K}$ are the Sen weights of D, with $(h_{i,\sigma})_\sigma \in A^{\Sigma_K}$ being the Sen weights of the rank 1 (φ, Γ_K)-module $\mathrm{Fil}^i(D)/\,\mathrm{Fil}^{i-1}(D)$ for $1 \leq i \leq n$, then

$$h'_{i,\sigma} = \begin{cases} h_{i,\sigma} & \text{if } i \leq j \\ h_{i,\sigma} + k_\sigma & \text{if } i > j \end{cases}$$

are the Sen weights of $p_{\mathbf{k}}(D, \mathrm{Fil}^j(D))$ of D.

One can then ask whether these weight-shifting operations descend to the category of trianguline (φ, Γ_K)-modules (without fixing a filtration). But it is easy to see that in general they depend on the choice of triangulation, as the following example[1] shows.

Let $D = \bigoplus_{i=1}^2 \mathcal{R}_{\mathbb{Q}_p,E}(x^i)$ be split of rank 2. Then we have two triangulations

$$0 \to \mathcal{R}(x) \to D \to \mathcal{R}(x^2) \to 0,$$
$$0 \to \mathcal{R}(x^2) \to D \to \mathcal{R}(x) \to 0.$$

Applying pullbacks with $k \in \mathbb{Z}_{>0}$, we get $\mathcal{R}(x) \oplus \mathcal{R}(x^{k+2})$ and $\mathcal{R}(x^2) \oplus \mathcal{R}(x^{k+1})$, respectively, which are of different Hodge-Tate weights $(k+2, 1)$ and $(k+1, 2)$.

[1] See Remark 3.6 for a non-split example.

1.1.2. *Using Sen polynomials.* However, [Wu, Proposition 3.16] implies that this weight issue is the only obstruction for the pullback to be independent of the triangulation. Let $K = \mathbb{Q}_p$ here for simplicity, and let A be any affinoid E-algebra. Wu showed that if the Sen polynomial $P_{\mathrm{Sen},D}(T) \in A[T]$ of D admits a factorization

$$P_{\mathrm{Sen},D}(T) = Q(T)S(T)$$

with monic $Q(T), S(T) \in A[T]$, with $(Q, S) = 1$, then there exists a unique (φ, Γ)-submodule D' of D satisfying

$$tD \subset D' \subset D$$

and having Sen polynomial $Q(T-1)S(T)$. In other words, under the equivalence of (φ, Γ_K)-modules and Γ_K-equivariant vector bundles on the Fargues-Fontaine curve $X_{K_\infty,A}$ [EGH, Theorem 5.1.5], with respect to a given canonical decomposition of the Sen module

$$D_{\mathrm{Sen}}(D) = \ker(Q(\Theta_{\mathrm{Sen}})) \oplus \ker(S(\Theta_{\mathrm{Sen}}))$$

there is a unique modification of the (φ, Γ)-module D at ∞ on the Fargues-Fontaine curve such that the resulting subbundle D' has the Sen polynomial $P_{\mathrm{Sen},D'}(T) = Q(T-1)S(T)$.

If D is a trianguline (φ, Γ_K)-module over $\mathcal{R}_{K,A}$ and $\mathrm{Fil}^\bullet(D)$ is a triangulation on D, then for any $1 \leq j \leq n$, the extension

$$0 \to \mathrm{Fil}^j(D) \to D \to D/\mathrm{Fil}^j(D) \to 0$$

induces a splitting of the Sen polynomial

$$P_{\mathrm{Sen},D}(T) = P_{\mathrm{Sen},D/\mathrm{Fil}^j(D)}(T) \cdot P_{\mathrm{Sen},\mathrm{Fil}^j(D)}(T), \tag{1.1.1}$$

and $p_1(D, \mathrm{Fil}^j(D))$ is a (φ, Γ)-submodule of D satisfying

$$tD \subset p_1(D, \mathrm{Fil}^j(D)) \subset D$$

and having Sen polynomial $P_{\mathrm{Sen},D/\mathrm{Fil}^j(D)}(T-1)P_{\mathrm{Sen},\mathrm{Fil}^j(D)}(T)$. If (1.1.1) is a coprime factorization, then $p_1(D, \mathrm{Fil}^j(D))$ is the unique (φ, Γ)-submodule of D containing tD of Sen polynomial $P_{\mathrm{Sen},D/\mathrm{Fil}^j(D)}(T-1)P_{\mathrm{Sen},\mathrm{Fil}^j(D)}(T)$. Moreover, in this case, for any triangulation $\mathrm{Fil}^\bullet(D)'$ on D inducing the same splitting as (1.1.1), by the uniqueness, we have

$$p_1(D, \mathrm{Fil}^j(D)') = p_1(D, \mathrm{Fil}^j(D)).$$

Hence, we may consider a (φ, Γ_K)-modules D over $\mathrm{Sp}(A)$ that (at least after passing to a Tate-fpqc cover of $\mathrm{Sp}(A)$) admits a triangulation over $\mathcal{R}_{K,A}$ from which we obtain a factorization of its Sen polynomial $P_{\mathrm{Sen},D}(T)$, and such that, over every geometric point x of $\mathrm{Sp}(A)$, the fiber D_x is "weight-uniform trianguline" in the sense that all triangulations of D_x over $\mathcal{R}_{K,\overline{k(x)}}$ induce the same ordering on Sen weights of D_x in $\overline{k(x)}$. After imposing necessary restrictions on its Sen weights at geometric fibers, we can perform[2] such "modifications at ∞" iteratively on D in an invertible way, whenever the resulting movement of our ordered Sen weights in the weight space does not meet any of the relevant walls, cf. Theorem 4.10 for our exposition on Wu's result and Theorem 4.12 for discussion in the weight-uniform trianguline case.

[2]Order does not matter.

1.1.3. *The point of view of the analytic Emerton-Gee stacks.* Recall from [EGH, §5.3] that over the category Rig_E of rigid analytic spaces over E equipped with the Tate-fpqc topology, we have the moduli stack $\mathfrak{X}_n$ of rank n G_K-equivariant vector bundles over the Fargues-Fontaine curve $X_{\overline{K}}$, and the stack $\mathfrak{X}_B$ of G_K-equivariant B-bundles on $X_{\overline{K}}$, where B denotes the Borel subgroup of $G = \mathrm{GL}_n$ consisting of upper triangular invertible matrices. Then, by the equivalence [EGH, Theorem 5.1.5], a triangulated (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ of rank n over $\mathcal{R}_{K,A}$ defines a point in $\mathfrak{X}_B(A)$, and $D \in \mathfrak{X}_n(A)$.

We introduce in Definition 4.4 "weight-uniform trianguline substack" $\mathfrak{X}_n^{\mathrm{wu}}$ of $\mathfrak{X}_n$, and some additional substacks $\mathfrak{X}_n^{\sigma\text{-wu},i} \subset \mathfrak{X}_n^{\sigma\text{-wu}}$ characterized by the property that, for any triangulation on D, the first $n-i$ σ-Sen weights are distinct from the last i σ-Sen weights at any $x \in \mathrm{Sp}(A)$. The pullback operator $p_{i,\sigma}$ (§4) that increases the last i σ-Sen weights by 1 and leaves all other Sen weights invariant descends to a map from $\mathfrak{X}_n^{\sigma\text{-wu},i}$ to $\mathfrak{X}_n$. Then, we deduce the following theorem, cf. Definition 4.6 (and Remark 4.7) for the precise meaning of our notation.

Theorem A. *The pullback maps $p_{i,\sigma} : \mathfrak{X}_B \to \mathfrak{X}_B$ descend to canonical morphisms of stacks*
$$p_{i,\sigma} : \mathfrak{X}_n^{\sigma\text{-wu},i} \longrightarrow \mathfrak{X}_n$$
such that for $S \subset \Sigma_K$, $I = \prod_{\sigma \in S} I_\sigma \subset \{1, \dots, n\}^S$ and $\mathbf{k} = (k_{i,\sigma})_{\sigma \in S, i \in I_\sigma} \in \mathbb{N}^I$,
$$p_{\mathbf{k}} := \prod_{\sigma} (p_{i,\sigma})^{k_{i,\sigma}} : \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}} \xrightarrow{\sim} \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}$$
are isomorphisms between these weight-uniform trianguline substacks, where the change of Sen weights does not change the regularity of the weights.

Question. What can be said about the geometry of $\mathfrak{X}_n^{\mathrm{wu}}$?

1.1.4. We also mention that, for such directions of changing the weights, Wu obtained in [Wu, §3] general geometric results: the stack $\mathfrak{X}_n$ of rank n (φ, Γ_K)-modules at integral Hodge-Tate weights $\mathbf{h} = (h_{i,\sigma}) \in (\mathbb{Z}^n)^{\Sigma_K}$ are described using a "product formula" of the form (if $K = \mathbb{Q}_p$)
$$(\mathfrak{X}_n)^\wedge_{\mathbf{h}} \cong (\mathfrak{X})^\wedge_0 \times_{\mathfrak{g}/\mathrm{GL}_n} \widetilde{\mathfrak{g}}_{P_{\mathbf{h}}}/\mathrm{GL}_n,$$
which then induces change of weights maps
$$f_{\mathbf{h},\mathbf{h}'} : (\mathfrak{X}_n)^\wedge_{\mathbf{h}} \to (\mathfrak{X}_n)^\wedge_{\mathbf{h}'}$$
and this can be formulated for non-integral weights using local models developed in [Wu22, Ch. 5]. Moreover, change of weights maps $f_{\mathbf{h},\mathbf{h}'}$ exist at arbitrary weights $\mathbf{h}$ whenever changing from $\mathbf{h}$ to $\mathbf{h}'$ does not increase the regularity, cf. [Wu, §1.3] for a discussion and [Wu, §3.3] for details.

1.1.5. *Relation with translation functor under Ding's crystabelline correspondence.* A class of points of $\mathfrak{X}_n^{\mathrm{wu}}(E)$ are those non-critical crystabelline (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$. Recently, for non-critical crystabelline (φ, Γ_K)-modules D over $\mathcal{R}_{K,E}$ of regular Hodge-Tate weights, Ding constructed in [Din25] locally $\mathbb{Q}_p$-analytic representations $\pi_{\min}(D) \subset \pi_{\mathrm{fs}}(D)$ of $\mathrm{GL}_n(K)$, which are extensions of locally algebraic representations $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by locally analytic representations $\pi(\phi, \mathbf{h})$ that only depend on $D[1/t]$ and the Hodge-Tate weights $\mathbf{h}$. If $K = \mathbb{Q}_p$, these extensions can "recover the Hodge filtration" and hence determines D, cf. [Din25, Theorem 3.34].

Note that the pullback D' is a non-critical crystabelline (φ, Γ_K)-submodule of D whenever this change of weights preserves the regularity. It is then natural to expect that under $\pi := \pi_{\min}$ or π_{fs}, pulling back from D to D' corresponds to translating from $\pi(D)$ to $\pi(D')$. For $K = \mathbb{Q}_p$, we prove the expected intertwining (§5.3) of the two kinds of weight-shifting operators:

Theorem B. *Let D be a non-critical crystabelline $(\varphi, \Gamma_{\mathbb{Q}_p})$-module with regular Sen weights $\mathbf{h}$. Let $p_{\mathbf{k}}(D) = f_{\mathbf{h},\mathbf{h}'}(D)$ be the module obtained from D by applying a sequence of pullback operators $p_{\mathbf{k}} = p_1^{k_1} \cdots p_n^{k_n}$ such that its weights $\mathbf{h}'$ are still regular. Let $\lambda' := \mathbf{h}' - \theta$ and $\lambda := \mathbf{h} - \theta$ be the corresponding "automorphic weights" with $\theta := (0, -1, \dots, -(n-1)) \in \mathbb{Z}^n$. Then,*

$$T_\lambda^{\lambda'}(\pi_\bullet(D)) = \pi_\bullet(p_{\mathbf{k}}(D)) = \pi_\bullet(f_{\mathbf{h},\mathbf{h}'}(D))$$

for $\bullet \in \{\min, \mathrm{fs}\}$.

Since translation functors between regular weights are equivalence of categories, Theorem B follows straightforwardly from various results in [Din25], [JLS24] and [Wu].

1.2. **Structure of the paper.** In §2, we recall basics of (φ, Γ_K)-modules over affinoid algebras and their cohomology, generalizing a pointwise result [Col08, Theorem 2.22(i)] to the affinoid coefficients in Lemma 2.9(i), leading to Theorem 3.1 for trianguline families. In §3, we review triangulations on generic crystabelline (φ, Γ_K)-modules over fields, and observe that "generically, a trianguline (φ, Γ_K)-module has a unique non-split triangulation." In §4, we discuss Wu's change of weights maps in general and in the weight-uniform trianguline families (cf. Theorem 4.10 = [Wu, Proposition 3.16], and Theorem 4.12 = Theorem A), which is followed by a discussion on when étaleness can be preserved up to twist for local Artinian E-algebras $A \in \mathcal{C}_E$ of residue field E. Finally, the entire §5 is devoted to Theorem B, which could be read independently of the previous sections, except for some notations taken from [Din25].

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2. Preliminaries

2.1. **Notations and conventions.** Let K be a finite extension of $\mathbb{Q}_p$ of ramification index e and inertial degree f, and let Σ_K be the set of all $\mathbb{Q}_p$-algebra embeddings of K into $\overline{\mathbb{Q}_p}$. Let E be a finite extension of $\mathbb{Q}_p$ such that all the $\mathbb{Q}_p$-algebra embeddings of K into $\overline{\mathbb{Q}_p}$ factor through E. We allow E to be enlarged at will.

Let Rig_E be the category of rigid E-analytic spaces, and Aff_E the category of affinoid E-algebras. For $X \in \mathrm{Rig}_E$, let $\mathcal{R}_{K,X}$ be the relative Robba ring of K over X, and write $\mathcal{R}_{K,A} := \mathcal{R}_{K,\mathrm{Sp}(A)}$ for affinoid $A \in \mathrm{Aff}_E$. Let $\Gamma_K := \mathrm{Gal}(K(\mu_{p^\infty})|K)$. For a review of the (generalized) (φ, Γ_K)-modules over the Robba ring $\mathcal{R}_{K,A}$, we refer the reader to [Ber17, §2.1].

We use the notation $D = (Q - P)$ to indicate that $D \in \mathrm{Ext}^1(P, Q)$ is an extension of P by Q.

Our conventions are that $\mathbb{N} := \mathbb{Z}_{\geq 0}$ and the p-adic cyclotomic character

$$\varepsilon : K^\times \longrightarrow E^\times, \quad x \mapsto N_{K|\mathbb{Q}_p}(x)|N_{K|\mathbb{Q}_p}(x)|_p$$

has σ-Sen weight $+1$ for all $\sigma \in \Sigma_K$.

For $\mathbf{k} = (k_\sigma)_{\sigma \in \Sigma_K} \in \mathbb{Z}^{[K:\mathbb{Q}_p]}$, set $t^{\mathbf{k}} := \prod_{\sigma \in \Sigma_K} t_\sigma^{k_\sigma} \in \mathcal{R}_{K,E}$, where $t_\sigma \in \mathcal{R}_{K,E}$ are the Lubin-Tate elements defined up to units in [KPX14, Notation 6.2.7]. Let x_σ be the embedding $\sigma : K^\times \to E^\times$ viewed as a character of $K^\times$, and set $x^{\mathbf{k}} := \prod_{\sigma \in \Sigma_K} x_\sigma^{k_\sigma}$. Then, the (φ, Γ_K)-module $t^{\mathbf{k}}\mathcal{R}_{K,A} \cong \mathcal{R}_{K,A}(x^{\mathbf{k}})$ is free of rank 1 with σ-Sen weight k_σ for each $\sigma \in \Sigma_K$.

2.2. **Extensions of (φ, Γ_K)-modules.**

2.2.1. We recall operations on extensions of (φ, Γ_K)-modules and their cohomology.

Definition 2.1. For $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$ and (φ, Γ_K)-modules D_1, D_2 over $\mathcal{R}_{K,A}$, there are two natural A-linear maps between Yoneda Ext groups:

- pushing out along $D_1 \hookrightarrow t^{-\mathbf{k}} D_1$ defines a map
$$\iota_{\mathbf{k}} : \mathrm{Ext}^1(D_2, D_1) \longrightarrow \mathrm{Ext}^1(D_2, t^{-\mathbf{k}} D_1),$$
- pulling back along $t^{\mathbf{k}} D_2 \hookrightarrow D_2$ defines a map
$$p_{\mathbf{k}} : \mathrm{Ext}^1(D_2, D_1) \longrightarrow \mathrm{Ext}^1(t^{\mathbf{k}} D_2, D_1).$$

The following lemma shows that these two maps are related by the "twisting" isomorphism
$$x^{\mathbf{k}} : \mathrm{Ext}^1(D_2, t^{-\mathbf{k}} D_1) \xrightarrow{\simeq} \mathrm{Ext}^1(t^{\mathbf{k}} D_2, D_1)$$
induced by twisting by the character $x^{\mathbf{k}} : K^\times \to A^\times$, so that $x^{\mathbf{k}} \circ \iota_{\mathbf{k}} = p_{\mathbf{k}}$.

Lemma 2.2. *Let $\mathbf{k} = (k_\sigma)_\sigma \in \mathbb{N}^{[K:\mathbb{Q}_p]}$ and let*
$$0 \to D_1 \xrightarrow{i} D \xrightarrow{\pi} D_2 \to 0$$
be an exact sequence of (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$. Then, the pushout $\iota_{\mathbf{k}}(D)$ of D along $D_1 \hookrightarrow t^{-\mathbf{k}} D_1$ and the pullback $p_{\mathbf{k}}(D)$ of D along $t^{\mathbf{k}} D_2 \hookrightarrow D_2$ are related by a commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & t^{-\mathbf{k}} D_1 \arrow[r] & \iota_{\mathbf{k}}(D) \arrow[r] & D_2 \arrow[r] & 0 \\
0 \arrow[r] & D_1 \arrow[r, "i"] \arrow[u, hook] \arrow[d, equal] & D \arrow[r, "\pi"] \arrow[u, dashed] & D_2 \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & D_1 \arrow[r] & p_{\mathbf{k}}(D) \arrow[r] \arrow[u, dashed] & t^{\mathbf{k}} D_2 \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$
with exact rows and injective columns, from which we deduce $p_{\mathbf{k}}(D) = t^{\mathbf{k}} \iota_{\mathbf{k}}(D) \cong \iota_{\mathbf{k}}(D)(x^{\mathbf{k}})$.

Proof. Explicitly, $p_{\mathbf{k}}(D) = \pi^{-1}(t^{\mathbf{k}} D_2)$, and $\iota_{\mathbf{k}}(D) = (t^{-\mathbf{k}} D_1) \oplus_{D_1} D$ is an amalgamated sum over D_1. By the first row, we see that $t^{\mathbf{k}} \iota_{\mathbf{k}}(D)$ is a (φ, Γ_K)-submodule of $D[1/t]$ containing D_1 with the associated quotient being $t^{\mathbf{k}} D_2$, so it equals $p_{\mathbf{k}}(D)$ by the third row. $\square$

By Lemma 2.2, the pullback $p_{\mathbf{k}}$ is injective/surjective/zero if and only if the pushout $\iota_{\mathbf{k}}$ is. We recall the following cohomological interpretation of the extensions.

Lemma 2.3. *Let D_1, D_2 be (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$. Then,*
$$H^1(D_2^\vee \otimes D_1) \cong \mathrm{Ext}^1(D_2, D_1)$$
where H^1 is computed using the Herr complex
$$\mathcal{C}^\bullet(D) : \quad D \xrightarrow{(\varphi-1, \gamma-1)} D \oplus D \xrightarrow{(\gamma-1)\oplus(1-\varphi)} D$$
for $D := D_2^\vee \otimes D_1 \cong \mathrm{Hom}_{\mathcal{R}}(D_2, D_1)$, where $\gamma \in \Gamma$ is a topological generator.

Proof. This is essentially [EG23, Lemma 5.1.2], but adapted to our convention and notation.

Let M be an extension of D_2 by D_1 as (φ, Γ_K)-modules over $\mathcal{R} := \mathcal{R}_{K,A}$ given by

$$0 \to D_1 \xrightarrow{i} M \xrightarrow{\pi} D_2 \to 0.$$

It splits on the level of $\mathcal{R}$-modules. We choose any $\mathcal{R}$-linear section $s : D_2 \to M$, which is unique up to an element $h \in D = \mathrm{Hom}_{\mathcal{R}}(D_2, D_1)$. Using the section s, we can write

$$\varphi_M = \begin{pmatrix} \varphi_{D_1} & f \circ \varphi_{D_2} \\ & \varphi_{D_2} \end{pmatrix} \quad \text{and} \quad \gamma_M = \begin{pmatrix} \gamma_{D_1} & g \circ \gamma_{D_2} \\ & \gamma_{D_2} \end{pmatrix}$$

for uniquely determined $f, g \in \mathrm{Hom}_{\mathcal{R}}(D_2, D_1) = D$ since D_2 has an $\mathcal{R}$-basis in $\varphi(D_2)$.

That is, for any $x \in D_2$, we have

$$f(\varphi_{D_2}(x)) = (\varphi_M - s \circ \varphi_{D_2} \circ \pi)(s(x)) = \varphi_M(s(x)) - s(\varphi_{D_2}(x)) \in \ker(\pi) = D_1.$$

Similarly, since γ acts invertibly on $\mathcal{R}$, we have

$$g = (\gamma_M - s \circ \gamma_{D_2} \circ \pi) \circ s \circ \gamma_{D_2}^{-1} = \gamma_M \circ s \circ \gamma_{D_2}^{-1} - s \in \mathrm{Hom}_{\mathcal{R}}(D_2, D_1).$$

The commutativitiy $\varphi_M \circ \gamma_M = \gamma_M \circ \varphi_M$ is then equivalent to the equality

$$\varphi_{D_1} g \gamma_{D_2} + f \varphi_{D_2} \gamma_{D_2} = \gamma_{D_1} f \varphi_{D_2} + g \gamma_{D_2} \varphi_{D_2} \in \mathrm{Hom}_{\mathcal{R}}(D_2, D_1)$$

which, by precomposing with $\gamma_{D_2}^{-1}$ on D_2, is equivalent to

$$\varphi_{D_1} g - g \varphi_{D_2} = \gamma_{D_1} f \gamma_{D_2}^{-1} \varphi_{D_2} - f \varphi_{D_2}.$$

By the definition of $\mathrm{Hom}_{\mathcal{R}}(D_1, D_2)$ as (φ, Γ_K)-module, this last displayed equation is the same as $(\varphi_D - 1).g = (\gamma_D - 1).f \in D$, hence $(f, g) \in \ker((\gamma - 1) \oplus (1 - \varphi))$ is a 1-cocycle.

Modifying the section s by $h \in D$ results in another 1-cocycle (f', g') such that

$$\begin{pmatrix} \varphi_{D_1} & f' \circ \varphi_{D_2} \\ & \varphi_{D_2} \end{pmatrix} = \begin{pmatrix} 1 & -h \\ & 1 \end{pmatrix} \begin{pmatrix} \varphi_{D_1} & f \circ \varphi_{D_2} \\ & \varphi_{D_2} \end{pmatrix} \begin{pmatrix} 1 & h \\ & 1 \end{pmatrix}$$

(and a similar identity involving g, g', γ_{D_1} and γ_{D_2}) which is equivalent to

$$f' \varphi_{D_2} = f \varphi_{D_2} - h \varphi_{D_2} + \varphi_{D_1} h \quad (\text{resp. } g' = g - h + \gamma_{D_1} h \gamma_{D_2}^{-1})$$

Thus, $f' = f + (\varphi_D - 1).h$ and $g' = g + (\gamma_D - 1).h$. So, $(f', g') - (f, g)$ is a 1-coboundary. □

Given Lemma 2.3, we see that $\iota_{\mathbf{k}}$ and $p_{\mathbf{k}}$ induce the same map

$$H^1(D) \longrightarrow H^1(t^{-\mathbf{k}} D)$$

for $D := D_2^\vee \otimes D_1$, if we identify the codomains $H^1(D_2^\vee \otimes t^{-\mathbf{k}} D_1) \simeq H^1((t^{\mathbf{k}} D_2)^\vee \otimes D_1)$ via $x^{\mathbf{k}}$.

Example 2.4. When $D_2 = \mathcal{R}_{\mathbb{Q}_p}$, Lemma 2.3 was proven by Colmez [Col08, §2.1]: take $D_2 = \mathcal{R}$ and $D_1 = \mathcal{R}(\delta)$; then the isomorphism

$$\mathrm{Ext}^1_{(\varphi, \Gamma)}(\mathcal{R}, \mathcal{R}(\delta)) \xrightarrow{\simeq} H^1(\delta)$$

given in [Col08, §2.1] is as follows: given an extension M, let $e \in M$ be a lift of the vector $1 \in \mathcal{R}$, and we associate to M (the class of) the 1-cocycle $[(\varphi_M - 1)e, (\gamma_M - 1)e] \in R(\delta) \oplus R(\delta)$. This is the map constructed in Lemma 2.3. Indeed, we have $s : 1 \mapsto e$, $\varphi_{D_2} = \gamma_{D_2} = \mathrm{id}$, so we deduce $f = \varphi_M(e) - e = (\varphi_M - 1)e$. Likewise, $g = (\gamma_M - 1)e$, as desired.

2.2.2. We review some results on rank 1 (φ, Γ_K)-modules and their cohomology.

Theorem 2.5. *For any rank 1 (φ, Γ_K)-module D over a rigid analytic space X, there exist a unique continuous character $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and a unique up to isomorphism line bundle $\mathcal{L}$ on X such that*

$$\mathcal{R}_{K,X}(\delta) \otimes_{\mathcal{O}_X} \mathcal{L} \cong D.$$

We can choose $\mathcal{L} := H^0_{\varphi,\Gamma_K}(D(\delta^{-1})) = \mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,X}(\delta), D)$ so that the canonical map

$$\mathcal{R}_{K,X}(\delta) \otimes_{\mathcal{O}_X} H^0_{\varphi,\Gamma_K}(D(\delta^{-1})) \longrightarrow D$$

is an isomorphism.

Proof. This is [KPX14, Theorem 6.2.14]. For the definition of the free of rank 1 (φ, Γ_K)-module $\mathcal{R}_{K,X}(\delta)$, see [KPX14, Construction 6.2.4]. □

Definition 2.6. Let $\delta : K^\times \to A^\times$ be a continuous character. Then its derivative at 1 defines a K-linear map $d\delta : K \to A$, and hence an A-linear map $K \otimes_{\mathbb{Q}_p} A \to A$. Through the map

$$K \otimes_{\mathbb{Q}_p} A \xrightarrow{\sim} \prod_{\sigma \in \Sigma_K} A, \quad x \otimes y \mapsto (\sigma(x)y)_{\sigma \in \Sigma_K} \tag{2.2.1}$$

we may view $d\delta$ as a $[K : \mathbb{Q}_p]$-tuple $\mathrm{wt}(\delta) := (\mathrm{wt}_\sigma(\delta))_{\sigma \in \Sigma_K} \in A^{[K:\mathbb{Q}_p]}$, which we call the **weight** of the character δ. We also call $\mathrm{wt}_\sigma(\delta)$ the **σ-weight** of δ. By [KPX14, Lemma 6.2.12], for any continuous character $\delta : K^\times \to A^\times$, the ($\sigma$-)Sen weight of $\mathcal{R}_{K,A}(\delta)$ is the (σ-)weight of δ.

Lemma 2.7. *Let $\delta : K^\times \to E^\times$ be a continuous character.*

(i) For $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$, if $\mathrm{wt}_\sigma(\delta) \notin \{1, \ldots, k_\sigma\}$ for each $\sigma \in \Sigma_K$, then

$$\iota_{\mathbf{k}} : H^1(\delta) \to H^1(x^{-\mathbf{k}}\delta)$$

is an isomorphism.

(ii) We have

- $\dim_E H^0(\delta) = \begin{cases} 1 & \text{if } \delta = x^{-\mathbf{k}} \text{ for some } \mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}, \\ 0 & \text{otherwise.} \end{cases}$
- $\dim_E H^2(\delta) = \begin{cases} 1 & \text{if } \delta = \left(N_{K|\mathbb{Q}_p} |N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \text{ for some } \mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}, \\ 0 & \text{otherwise.} \end{cases}$
- $\dim_E H^1(\delta) = \begin{cases} [K : \mathbb{Q}_p] + 1 & \text{if either } H^0 \text{ or } H^2 \text{ does not vanish,} \\ [K : \mathbb{Q}_p] & \text{otherwise.} \end{cases}$

(iii) Any nonzero (φ, Γ_K)-submodule of $\mathcal{R}_{K,E}(\delta)$ must be of the form $t^{\mathbf{k}}\mathcal{R}_{K,E}(\delta)$ for some $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$.

Proof. The first statement is [BHS19, Lemma 3.3.3], the second statement is [KPX14, Proposition 6.2.8] and the third statement is [KPX14, Corollary 6.2.9]. □

Definition 2.8. Let $\mathcal{T}$ denote the rigid analytic space of continuous characters of $K^\times$. Let $\mathcal{T}_{\mathrm{wreg}}$ denote the open complement in $\mathcal{T}$ to the points

$$\left\{ \left(N_{K|\mathbb{Q}_p} |N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]} \right\},$$

and let $\mathcal{T}_{\mathrm{reg}}$ denote the open complement in $\mathcal{T}$ to the points

$$\left\{ x^{-\mathbf{k}}, \left(N_{K|\mathbb{Q}_p} |N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]} \right\}.$$

For $n \geq 1$, we denote by $\mathcal{T}^n_{\text{wreg}}$ the space of characters $\delta = (\delta_1, \ldots, \delta_n)$ such that $\delta_i/\delta_j \in \mathcal{T}_{\text{wreg}}$ for all $i \neq j$ and by $\mathcal{T}^n_{\text{reg}}$ the space of characters $\delta = (\delta_1, \ldots, \delta_n)$ such that $\delta_i/\delta_j \in \mathcal{T}_{\text{reg}}$ for all $i \neq j$.

Lemma 2.9. *Let $\delta : K^\times \to A^\times$ be a continuous character.*

(i) For $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$, if $\mathrm{wt}_\sigma(\delta_x) \notin \{1, \ldots, k_\sigma\}$ for each $\sigma \in \Sigma_K$ and for all $x \in \mathrm{Sp}(A)$, then

$$\iota_{\mathbf{k}} : H^1(\delta) \to H^1(x^{-\mathbf{k}}\delta)$$

is an isomorphism.

(ii) If $\delta \in \mathcal{T}_{\text{wreg}}(A)$, then $H^2(\delta) = 0$, and $H^1(\delta) \otimes_A k(x) \cong H^1(\delta_x)$ for all $x \in \mathrm{Sp}(A)$.

(iii) If $\delta \in \mathcal{T}_{\text{reg}}(A)$, then $H^1(\delta)$ is locally free of rank 1 over A and $H^0(\delta) = H^2(\delta) = 0$. Moreover, $H^i(\delta) \otimes_A k(x) \cong H^i(\delta_x)$ for all i and $x \in \mathrm{Sp}(A)$.

Proof. By [KPX14, Corollary 6.3.3], for any (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$ and any $\sigma \in \Sigma_K$, $H^i(D)$ and $H^i(D/t_\sigma)$ are coherent sheaves over $\mathrm{Sp}(A)$; moreover, they are the cohomology of complexes of locally free sheaves on $\mathrm{Sp}(A)$ concentrated in degrees $[0, 2]$. Thus as in [Stacks, Tag 061Z], we have the base change spectral sequence

$$E_2^{j,-i} = \mathrm{Tor}_i^A\left(H^j_{\varphi,\Gamma}(\heartsuit), k(x)\right) \Rightarrow H^{j-i}_{\varphi,\Gamma}(\heartsuit \otimes_A k(x)) \tag{2.2.2}$$

for $\heartsuit \in \{D, D/t_\sigma\}$. As the (φ, Γ_K)-cohomology are concentrated in $[0, 2]$, from (2.2.2) we see

$$H^2(\heartsuit) \otimes_A k(x) \xrightarrow{\sim} H^2(\heartsuit_x) \tag{2.2.3}$$

is an isomorphism for every $x \in \mathrm{Sp}(A)$.

(i) Since $\iota_{\mathbf{k}}$ is induced by the inclusion $\mathcal{R}_{K,A} \hookrightarrow t^{-k}\mathcal{R}_{K,A}(\delta) = \mathcal{R}_{K,A}(x^{-\mathbf{k}}\delta)$, which factors as

$$\mathcal{R}_{K,A} \subset t_{\sigma_1}^{-1}\mathcal{R}_{K,A} \subset \cdots \subset t_{\sigma_1}^{-k_{\sigma_1}}\mathcal{R}_{K,A} \subset t_{\sigma_2}^{-1}t_{\sigma_1}^{-k_{\sigma_1}}\mathcal{R}_{K,A} \subset \cdots \subset t_{\sigma_2}^{-k_{\sigma_2}}t_{\sigma_1}^{-k_{\sigma_1}}\mathcal{R}_{K,A} \subset \cdots \subset t^{-\mathbf{k}}\mathcal{R}_{K,A}$$

where we enumerate $\Sigma_K = \{\sigma_1, \sigma_2, \ldots, \sigma_{[K:\mathbb{Q}_p]}\}$, it suffices to prove that if $\mathrm{wt}_\sigma(\delta_x) \neq 1$ for all $x \in \mathrm{Sp}(A)$, then $\iota_\sigma : H^1(\delta) \to H^1(x_\sigma^{-1}\delta)$ is an isomorphism.

By the long exact sequence in cohomology attached to the exact sequence

$$0 \to \mathcal{R}_{K,A}(\delta) \to t_\sigma^{-1}\mathcal{R}_{K,A}(\delta) \to t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta) \to 0,$$

it suffices to show the vanishing of H^0 and H^2 of $t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)$.

For each $x \in \mathrm{Sp}(A)$, the fiber $t_\sigma^{-1}\mathcal{R}_{K,k(x)}(\delta_x)/\mathcal{R}_{K,k(x)}(\delta_x)$ has vanishing H^2 by [Liu08, Theorem 3.7(ii)] since it is a torsion (φ, Γ_K)-module. By base change (2.2.3) and that

$$t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta) \cong \mathcal{R}_{K,A}(\delta)/t_\sigma\mathcal{R}_{K,A}(\delta)$$

has coherent cohomology, we have $H^2(t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)) = 0$ by Nakayama's lemma. From the spectral sequence (2.2.2), we see that for all $x \in \mathrm{Sp}(A)$

$$H^1(t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)) \otimes_A k(x) \cong H^1(t_\sigma^{-1}\mathcal{R}_{K,k(x)}(\delta_x)/\mathcal{R}_{K,k(x)}(\delta_x))$$

which is zero since $\dim_{k(x)} H^0(t_\sigma^{-1}\mathcal{R}_{K,k(x)}(\delta_x)/\mathcal{R}_{K,k(x)}(\delta_x)) = 0$ by [Nak09, Lemma 2.16] and $\dim_{k(x)} H^0(t_\sigma^{-1}\mathcal{R}_{K,k(x)}(\delta_x)/\mathcal{R}_{K,k(x)}(\delta_x)) = \dim_{k(x)} H^1(t_\sigma^{-1}\mathcal{R}_{K,k(x)}(\delta_x)/\mathcal{R}_{K,k(x)}(\delta_x))$ by Euler-Poincare formula [Liu08, Theorem 4.3] and vanishing of H^2. Again by Nakayama's lemma, we deduce the vanishing of H^1 and hence from the spectral sequence (2.2.2)

$$H^0(t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)) \otimes_A k(x) \cong H^0(t_\sigma^{-1}\mathcal{R}_{K,k(x)}(\delta_x)/\mathcal{R}_{K,k(x)}(\delta_x))$$

from which we deduce the vanishing of H^0 by Nakayama's lemma.

(ii) For $\delta \in \mathcal{T}_{\mathrm{wreg}}(A)$, $\delta_x : K^\times \to k(x)^\times$ does not belong to $\left\{ \left(N_{K|\mathbb{Q}_p} |N_{K|\mathbb{Q}_p}|_p \right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]} \right\}$ for every $x \in \mathrm{Sp}(A)$, which implies that $H^2(\delta_x) = H^2(\delta) \otimes_A k(x) = 0$ for each $x \in \mathrm{Sp}(A)$ by (2.2.3) and Lemma 2.7(ii). By Nakayama's lemma, $H^2(\delta) = 0$. Hence, (2.2.2) shows

$$H^1(\mathcal{R}_{K,A}(\delta)) \otimes_A k(x) \cong H^1(\mathcal{R}_{K,k(x)}(\delta_x))$$

for all $x \in \mathrm{Sp}(A)$.

(iii) This is [HS16, Proposition 2.3]. $\square$

2.2.3. Let D be a (φ, Γ_K)-module of rank n over $\mathcal{R}_{K,A}$ for affinoid A equipped with a filtration

$$\mathrm{Fil}^\bullet(D): \quad \mathrm{Fil}^0(D) = 0 \subsetneq \mathrm{Fil}^1(D) \subsetneq \cdots \subsetneq \mathrm{Fil}^n(D) = D$$

by saturated (φ, Γ_K)-submodules $\mathrm{Fil}^i(D)$ such that $\mathrm{gr}^i(\mathrm{Fil}^\bullet(D)) := \mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)$ is locally free rank 1 over $\mathcal{R}_{K,A}$ for $1 \le i \le d$. By Theorem 2.5 there exists a unique continuous character $\delta_i : K^\times \to A^\times$ such that the canonical map

$$\mathcal{R}_{K,A}(\delta_i) \otimes_A \mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_i), \mathrm{gr}^i(\mathrm{Fil}^\bullet(D))) \longrightarrow \mathrm{gr}^i(\mathrm{Fil}^\bullet(D))$$

is an isomorphism. In this case, we say D is **trianguline**, the filtration $\mathrm{Fil}^\bullet(D)$ is a **triangulation** of D. and $\delta = (\delta_i)_{1 \le i \le n}$ is the **parameter** of $(D, \mathrm{Fil}^\bullet(D))$.

If the triangulation on D is clear, we write $D_i := \mathrm{Fil}^i(D)$ and $D^i := D/\mathrm{Fil}^{n-i}(D)$ so that the ranks of the subobject D_i and the quotient D^i are always i, for $1 \le i \le n$.

2.2.4. We discuss various notions of "non-split" for (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$.

Definition 2.10. (i) Given a (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$ with triangulation $\mathrm{Fil}^\bullet(D)$, if

$$0 \to \mathrm{Fil}^{i-1}(D_x) \to \mathrm{Fil}^i(D_x) \to \mathcal{R}(\delta_{i,x}) \to 0 \tag{2.2.4}$$

are non-split as (φ, Γ_K)-module for all $1 \le i \le n$ and all geometric points x of $\mathrm{Sp}(A)$, then $(D, \mathrm{Fil}^\bullet(D))$ is called **non-split**.

(ii) Given a (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$ with triangulation $\mathrm{Fil}^\bullet(D)$, if the exact sequences

$$0 \to \mathcal{R}(\delta_{i-1,x}) \cong \mathrm{Fil}^{i-1}(D_x)/\mathrm{Fil}^{i-2}(D_x) \to \mathrm{Fil}^i(D_x)/\mathrm{Fil}^{i-2}(D_x) \to \mathcal{R}(\delta_{i,x}) \to 0 \tag{2.2.5}$$

are non-split as (φ, Γ_K)-module for all $1 \le i \le n$ and all geometric points x of $\mathrm{Sp}(A)$, then $(D, \mathrm{Fil}^\bullet(D))$ is called **strongly non-split**.

Remark 2.11. (i) If $(D, \mathrm{Fil}^\bullet(D))$ is non-split (2.2.4) or strongly non-split (2.2.5), then

$$0 \to \mathrm{Fil}^i(D) \to D \to D/\mathrm{Fil}^i(D) \to 0$$

is a non-split extension of (φ, Γ_K)-modules, for all i.

(ii) If $D \in \mathrm{Ext}^1(D_2, D_1)$ is a non-split extension of trianguline (φ, Γ_K)-modules $(D_i, \mathrm{Fil}^\bullet(D_i))$ of rank r_i for $i = 1, 2$, then $(D, \mathrm{Fil}^\bullet(D))$ need *not* be non-split in the sense of (2.2.4), where $\mathrm{Fil}^\bullet(D)$ is the triangulation induced on D by $\mathrm{Fil}^\bullet(D_1)$ and $\mathrm{Fil}^\bullet(D_2)$.

Proof. (i) For (i), the image of $0 \to \mathrm{Fil}^i(D) \to D \to D/\mathrm{Fil}^i(D) \to 0$ under pullback

$$\mathrm{Ext}^1(D/\mathrm{Fil}^i(D), \mathrm{Fil}^i(D)) \xrightarrow{\mathrm{Fil}^{i+1}/\mathrm{Fil}^i(D) \subset D/\mathrm{Fil}^i(D)} \mathrm{Ext}^1(\delta_{i+1}, \mathrm{Fil}^i(D))$$

is the extension $0 \to \mathrm{Fil}^i(D) \to \mathrm{Fil}^{i+1}(D) \to \mathcal{R}(\delta_{i+1}) \to 0$ in (2.2.4), which is a non-split extension whose image under the pushout

$$\mathrm{Ext}^1(\delta_{i+1}, \mathrm{Fil}^i(D)) \xrightarrow{\mathrm{Fil}^i(D) \twoheadrightarrow \mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)} \mathrm{Ext}^1(\delta_{i+1}, \delta_i)$$

is the extension $0 \to \mathcal{R}(\delta_i) \to \mathrm{Fil}^{i+1}(D)/\mathrm{Fil}^{i-1}(D) \to \mathcal{R}(\delta_{i+1}) \to 0$ in (2.2.5).

So in either case, D represents a nonzero extension class in $\mathrm{Ext}^1(D/\mathrm{Fil}^i(D), \mathrm{Fil}^i(D))$. We also see that strongly non-split (2.2.5) implies non-split (2.2.4).

(ii) For (ii), we give a counterexample for $n = 3$ and $\mathcal{R} = \mathcal{R}_{\mathbb{Q}_p,E}$. Consider $D_1 = \mathcal{R}(\delta_1)$ and $D_2 = (\mathcal{R}(\delta_2) - \mathcal{R}(\delta_3))$ the unique non-split extension with $\delta_1 = x$, $\delta_2 = |x|x$ and $\delta_3 = 1$. Consider the long exact sequence

$$H^0_{\varphi,\Gamma}(\delta_1\delta_2^{-1}) \to H^1_{\varphi,\Gamma}(\delta_1\delta_3^{-1}) \to H^1_{\varphi,\Gamma}(D_2^\vee(\delta_1)) \to H^1_{\varphi,\Gamma}(\delta_1\delta_2^{-1}) \to H^2_{\varphi,\Gamma}(\delta_1\delta_3^{-1})$$

associated to $0 \to \mathcal{R}(\delta_2) \to D_2 \to \mathcal{R}(\delta_3) \to 0$, where the maps between H^1 are pullbacks. By Lemma 2.7(ii), this sequence becomes $0 \to E \to E^2 \to E \to 0$, which in concrete terms means that if we take the non-split extension of $\mathcal{R}(\delta_3)$ by $\mathcal{R}(\delta_1)$, and pullback along $D_2 \twoheadrightarrow \mathcal{R}(\delta_3)$, then we get a non-split extension D of $\mathcal{R}(\delta_3)$ by D_1 whose pullback along $\mathcal{R}(\delta_2) \hookrightarrow D_2$ is a split extension of $\mathcal{R}(\delta_2)$ by $\mathcal{R}(\delta_1)$, which is $\mathrm{Fil}^2(D)$. Hence, although D is non-split as an extension of D_2 by D_1, it is not non-split in the sense of (2.2.4). $\square$

3. Triangulations on trianguline (φ, Γ)-modules

Theorem 3.1. *Let $(D, \mathrm{Fil}^\bullet(D))$ be trianguline with parameters $(\delta_1, \ldots, \delta_n) : (K^\times)^n \to A^\times$ over $\mathcal{R}_{K,A}$. For any fixed $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$ and $i \in \{1, \ldots, n\}$, suppose that*

$$\mathrm{wt}_\sigma(\delta_{j,x}/\delta_{k,x}) \notin \{1, \ldots, k_\sigma\}$$

for all $1 \le j \le n-i$, $n-i+1 \le k \le n$, for all $\sigma \in \Sigma_K$ and for all $x \in \mathrm{Sp}(A)$. Then

$$p_{\mathbf{k}} : \mathrm{Ext}^1(D/\mathrm{Fil}^i(D), \mathrm{Fil}^{n-i}(D)) \longrightarrow \mathrm{Ext}^1(t^{\mathbf{k}}(D/\mathrm{Fil}^i(D)), \mathrm{Fil}^{n-i}(D))$$

is an isomorphism.

Proof. By induction on $\mathbf{k}$ it suffices to prove the statement for a fixed $\sigma \in \Sigma_K$ with $\mathbf{k} = k_\sigma = 1$ and $k_\tau = 0$ for all $\tau \in \Sigma_K \setminus \{\sigma\}$, under the assumption that $\mathrm{wt}_\sigma(\delta_{j,x}/\delta_{k,x}) \neq 1$ for $j \le n-i < k$.

Recall our notation that $D_{n-i} := \mathrm{Fil}^{n-i}(D)$ and $D^i := D/\mathrm{Fil}^{n-i}(D)$. By Theorem 2.5, D is a successive extension of rank 1 (φ, Γ_K)-modules $\mathrm{gr}_m(\mathrm{Fil}^\bullet(D)) \cong \mathcal{R}_{K,A}(\delta_m)\hat{\otimes}_A\mathcal{L}_m$ for some line bundles $\mathcal{L}_m$ over A. There is an finite admissible cover $\{\mathrm{Sp}(A_l)\}_{l=1}^r$ of $\mathrm{Sp}(A)$ trivializing them. Since H^*_{φ,Γ_K} commutes with flat base change by [KPX14, Theorem 4.4.3(2)], passing to $\mathrm{Sp}(A_l)$ we may assume that the line bundles $\mathcal{L}_m$ are all trivial.

After the reductions above, we show that

$$p_{i,\sigma} : \mathrm{Ext}^1(D^i, D_{n-i}) \cong H^1(D^{i^\vee} \otimes D_{n-i}) \to \mathrm{Ext}^1(t_\sigma D^i, D_{n-i}) \cong H^1(t_\sigma^{-1}(D^{i^\vee} \otimes D_{n-i}))$$

is bijective, where we have by assumption that

$$D_{n-i} = (\mathcal{R}_{K,A}(\delta_1) - \cdots - \mathcal{R}_{K,A}(\delta_{n-i})) \quad \text{and} \quad D^i = (\mathcal{R}_{K,A}(\delta_{n-i+1}) - \cdots - \mathcal{R}_{K,A}(\delta_n)).$$

Thus, putting $M := D^{i^\vee} \otimes D_{n-i}$, we see that M is trianguline over $\mathcal{R}_{K,A}$ with parameters

$$\left\{\delta_j\delta_k^{-1} \middle| 1 \le j \le n-i \text{ and } n-i+1 \le k \le n\right\}.$$

Note that $\mathrm{wt}_\sigma(\delta_j\delta_k^{-1}) \neq 1$ for these parameters by the assumption. By the short exact sequence

$$0 \to M \to t_\sigma^{-1}M \to t_\sigma^{-1}M/M \to 0$$

it is enough to establish the vanishing of $H^i(t_\sigma^{-1}M/M)$ as coherent sheaves for $i = 0, 1, 2$.

This follows from a dévissage argument on the above triangulation on M, using the proof of Lemma 2.9(i). Indeed, we know that $M = (\mathcal{R}_{K,A}(\eta_1) - \cdots - \mathcal{R}_{K,A}(\eta_m))$ of parameters η_s whose weights are not 1 at all $x \in \mathrm{Sp}(A)$, so that

$$t_\sigma^{-1}M/M \cong M/t_\sigma = (\mathcal{R}_{K,A}(\eta_1)/t_\sigma - \cdots - \mathcal{R}_{K,A}(\eta_m)/t_\sigma)$$

We induct on the rank of M to show the vanishing of cohomologies of M/t_σ. The base case where $m = 1$ was shown in the proof of Lemma 2.9(i). For $m \geq 2$, we have

$$\cdots \to H^i(\mathcal{R}_{K,A}(\eta_1)/t_\sigma) \to H^i(M/t_\sigma) \to H^i((M/\mathcal{R}_{K,A}(\eta_1))/t_\sigma) \to \dots$$

from which we know $H^i(M/t_\sigma) = 0$ by inductive hypothesis. □

3.1. **Very generic case.**

Definition 3.2. Let $(D, \mathrm{Fil}^\bullet(D))$ be a trianguline (φ, Γ)-module over $\mathcal{R}_{K,E}$ with parameter $\delta = (\delta_1, \dots, \delta_n)$ and Sen weights $(h_{i,\sigma})_{i,\sigma} \in E^{n[K:\mathbb{Q}_p]}$, where $h_{i,\sigma} := \mathrm{wt}_\sigma(\delta_i)$. We say that D is **very generic** if for each $\sigma \in \Sigma_K$, $h_{i,\sigma} - h_{j,\sigma} \notin \mathbb{Z}$ for all $i \neq j$.

Definition 3.3. Let $\mathcal{T}_\circ^n$ be the open subspace of $\mathcal{T}^n$ such that for $A \in \mathrm{Aff}_E$, $\mathcal{T}_\circ^n(A)$ is the set of all the continuous $A^\times$-valued characters of $(K^\times)^n$

$$\delta = (\delta_1, \dots, \delta_n) : (K^\times)^n \to A^\times$$

satisfying $\mathrm{wt}_\sigma(\delta_{i,x}/\delta_{j,x}) \notin \mathbb{Z}_{\geq 1}$ for all $1 \leq i < j \leq n$, all $\sigma \in \Sigma_K$, and all $x \in \mathrm{Sp}(A)$.

Proposition 3.4. *For trianguline (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ of parameters $\delta = (\delta_i)_{1 \leq i \leq n} \in \mathcal{T}_\circ^n(E)$ over $\mathcal{R}_{K,E}$ and a fixed index $i \in \{1, \dots, n\}$, suppose*
(a). $\mathrm{Fil}^{n-i}(D)$ with the induced triangulation is strongly non-split, and
(b). $D/\mathrm{Fil}^{n-i-1}(D)$ with the induced triangulation is non-split.
Then, for each $1 \leq j \leq n-i$, $\mathrm{Fil}^j(D)$ is the unique saturated rank j (φ, Γ_K)-submodule of D.

Proof. Note that similar to Remark 2.11, (a) and (b) imply that $(D, \mathrm{Fil}^\bullet(D))$ is non-split.

We proceed by induction. Let D' be a saturated (φ, Γ_K)-submodule of D. We claim that D' contains $\mathrm{Fil}^1(D)$. Indeed, let m be the largest integer such that $\mathrm{Fil}^{m-1}(D) \cap D' = 0$. Then $D' \cap \mathrm{Fil}^m(D)$ is an $\mathcal{R}_{K,E}$-submodule of D' stable under the (φ, Γ_K)-action such that the quotient $D'/(D' \cap \mathrm{Fil}^m(D))$ is torsion-free as it injects into $D/\mathrm{Fil}^m(D)$. As

$$\mathcal{R}_{K,E} \simeq \prod_{\tau: K_0 \hookrightarrow E} \mathcal{R}_{K,\mathbb{Q}_p} \otimes_{K_0,\sigma} E$$

is a product of Bézout domains on which φ acts transitively, it follows from the torsion-freeness that $D' \cap \mathrm{Fil}^m(D)$ is free over $\mathcal{R}_{K,E}$ (cf. [Ber17, §2.6]) and hence is a (φ, Γ_K)-module.

Now if $m > 1$, then $D' \cap \mathrm{Fil}^m(D)$ is a nonzero (φ, Γ_K)-submodule of $\mathrm{Fil}^m(D)$ injecting to the quotient $\mathrm{Fil}^i(D)/\mathrm{Fil}^{m-1}(D) \cong \mathcal{R}_{K,E}(\delta_m)$, which is of the form $t^{\mathbf{k}}\mathcal{R}\delta_m = \mathcal{R}(x^{\mathbf{k}}\delta_m)$ for some $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$ by Lemma 2.7(iii). Then, $\mathrm{Fil}^m(D)$ contains the split (φ, Γ_K)-submodule

$$\mathrm{Fil}^{m-1}(D) \oplus (D' \cap \mathrm{Fil}^m(D)) \cong \mathrm{Fil}^{m-1}(D) \oplus t^{\mathbf{k}}\mathcal{R}(\delta_m)$$

which is the image of $\mathrm{Fil}^m(D)$ under the pullback map

$$p_{\mathbf{k}} : \mathrm{Ext}^1(\delta_m, \mathrm{Fil}^{m-1}(D)) \to \mathrm{Ext}^1(t^{\mathbf{k}}\delta_m, \mathrm{Fil}^{m-1}(D))$$

which is bijective by Theorem 3.1, contrary to our assumptions (a) and (b). Hence, $m = 1$. Since $D' \cap \mathrm{Fil}^1(D) = D' \cap \mathcal{R}\delta_1$ is nonzero and saturated in $\mathcal{R}\delta_1$, it has to be $\mathcal{R}\delta_1 = \mathrm{Fil}^1(D)$.

In particular, $\mathrm{Fil}^1(D)$ is the unique saturated rank 1 (φ, Γ_K)-submodule of D. If $n-i > 1$, we pass to the quotient by $\mathrm{Fil}^1(D)$ and claim that every saturated (φ, Γ_K)-submodule of $D/\mathrm{Fil}^1(D)$ contains $\mathrm{Fil}^2(D)/\mathrm{Fil}^1(D)$. By (a) and (b), $D/\mathrm{Fil}^1(D)$ with its induced filtration is non-split and satisfies the analogous (a) and (b), so the argument above works. We can iterate this argument until $n-i$, hence the conclusion. □

Corollary 3.5. *Let $(D, \mathrm{Fil}^\bullet(D)$ be strongly non-split of parameters $\delta \in \mathcal{T}_\circ^n(E)$ over $\mathcal{R}_{K,E}$. Then, any saturated (φ, Γ_K)-submodule D' of D is such that $D' = \mathrm{Fil}^i(D)$ for some $0 \leq i \leq n$. In particular, the given triangulation $\mathrm{Fil}^\bullet(D)$ is the unique triangulation of D.*

Proof. This follows from Proposition 3.4 by taking $i = 1$. □

Remark 3.6. One cannot expect uniqueness result for non-split (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ with parameters in $\mathcal{T}_\circ^n(E)$ without the strongly non-split assumption.

Indeed, for regular $(\delta_1, \delta_2, \delta_3) \in \mathcal{T}_{\mathrm{reg}}^3(E) \cap \mathcal{T}_\circ^3(E)$, we have an exact sequence

$$\cancel{\mathrm{Ext}^0(\delta_3,\delta_2)} \to \mathrm{Ext}^1(\delta_3,\delta_1) = E \to \mathrm{Ext}^1(\delta_3,(\delta_1 - \delta_2)) = E^2 \to \mathrm{Ext}^1(\delta_3,\delta_2) = E \to \cancel{\mathrm{Ext}^2(\delta_3,\delta_1)}$$

where $(\delta_1 - \delta_2)$ denotes the non-split extension of $\mathcal{R}(\delta_2)$ by $\mathcal{R}(\delta_1)$. The image of any nonzero class of $\mathrm{Ext}^1(\delta_3, \delta_1)$ in $\mathrm{Ext}^1(\delta_3, (\delta_1 - \delta_2))$ is a trianguline (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D), (\delta_1, \delta_2, \delta_3))$ that is non-split in the sense of (2.2.4), but $D/\mathrm{Fil}^1(D)$ is a split extension of δ_2 by δ_3. So, D has another triangulation whose parameter is $(\delta_1, \delta_3, \delta_2)$.

3.2. **Crystabelline case.** We discuss triangulations on a class of trianguline (φ, Γ_K)-modules that is closer to p-adic Hodge theory. Recall that we have Fontaine's functors D_{cris}, D_{dR} defined for (φ, Γ_K)-modules as well, cf. [HS16, Definition 2.5] or [Nak13, Definition 2.3].

Definition 3.7.

(i) A (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$ is **crystabelline** if there exists a finite abelian extension L/K such that $\mathcal{R}_{L,E} \otimes_{\mathcal{R}_{K,E}} D$ is crystalline, i.e., the $(L_0 \otimes_{\mathbb{Q}_p} E)$-rank of
$$D_{\mathrm{cris}}^L(D) := (\mathcal{R}_{L,E} \otimes_{\mathcal{R}_{K,E}} D)[1/t]^{\Gamma_L}$$
is equal to $n := \mathrm{rank}_{\mathcal{R}_{K,E}}(D)$.

(ii) A crystabelline (φ, Γ_K)-module D as in (i) is **generic** if the Weil-Deligne representation associated to the $(\varphi, \mathrm{Gal}(L/K))$-module $D_{\mathrm{st}}^L(D) = D_{\mathrm{cris}}^L(D)$ via the equivalence in [BS07, Proposition 4.1] is "generic", i.e., it is a direct sum of characters $\phi_1, \ldots, \phi_n$ of W_K that are trivial on I_L such that if $\phi_1, \ldots, \phi_n$ are viewed as smooth characters of $K^\times$ via the local reciprocity map, then $\phi_i/\phi_j \notin \{1, |\cdot|_K^{-1}, |\cdot|_K\}$ for all $i \neq j$, cf. [Din25, §2.1].

(iii) For a crystabelline generic (φ, Γ_K)-module D as in (ii), a **refinement** of D is any ordering of the characters $\phi_1, \ldots, \phi_n$ appearing in the associated Weil-Deligne representation.

Definition 3.8 ([Nak13, Definition 2.4])**.** Let L be a finite Galois extension of K with Galois group $G(L/K)$. We say that D is an E**-filtered $(\varphi, G(L/K))$-module over** K if

(i) D is a finite free $(L_0 \otimes_{\mathbb{Q}_p} E)$-module with a Frobenius semilinear operator $\varphi : D \xrightarrow{\simeq} D$, and a semilinear action by $G(L/K)$ that commutes with φ.

(ii) $D_L := (L \otimes_{L_0} D)$ has a separated and exhaustive descending filtration $(\mathrm{Fil}^i(D_L))_{i \in \mathbb{Z}}$ by $G(L/K)$-stable $(L \otimes_{\mathbb{Q}_p} E)$-submodules $\mathrm{Fil}^i(D_L)$.

Theorem 3.9. *The functor $D \mapsto D_{\mathrm{cris}}^L(D)$ induces a $\otimes$-equivalence of categories from (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$ that become crystalline over L to E-filtered $(\varphi, \mathrm{Gal}(L/K))$-modules over K.*

The saturated submodules of D correspond to $(\varphi, \mathrm{Gal}(L/K))$-submodules of $D^L_{\mathrm{cris}}(D)$ with their filtrations induced by the Hodge filtration on $D^L_{\mathrm{cris}}(D)$.

Proof. This is [Nak13, Theorem 2.5] and [Ber17, Proposition 3.3(a)]. Also see [Ber08]. □

Remark 3.10. Recall that this Hodge filtration is induced by

$$L \otimes_{L_0} D^L_{\mathrm{cris}}(D) \xrightarrow{\simeq} L \otimes_K D_{\mathrm{dR}}(D)$$

where $D_{\mathrm{dR}}(D)$ has its Hodge filtration $\{\mathrm{Fil}^i(D_{\mathrm{dR}}(D))\}_{i\in\mathbb{Z}}$ by $K \otimes_{\mathbb{Q}_p} E$-submodules. Via (2.2.1) we may write $D_{\mathrm{dR}(D)} = \prod_\sigma D_{\mathrm{dR}}(D)_\sigma$ and $\mathrm{Fil}^i(D_{\mathrm{dR}}(D)) = \prod_\sigma \mathrm{Fil}^i(D_{\mathrm{dR}}(D)_\sigma)$. Tensor $L\otimes_K(-)$ gives a filtration $\{L \otimes_{\mathbb{Q}_p} \mathrm{Fil}^i(D_{\mathrm{dR}}(D))\}_{i\in\mathbb{Z}}$ on $L \otimes_{L_0} D^L_{\mathrm{cris}}(D)$ by $\mathrm{Gal}(L/K)$-stable $L \otimes_{\mathbb{Q}_p} E$-submodules, making D^L_{cris} an E-filtered $(\varphi, \mathrm{Gal}(L/K))$-module over K. Using (2.2.1) again,

$$L \otimes_K \mathrm{Fil}^i(D_{\mathrm{dR}}(D)) = \prod_{\sigma\in\Sigma_K} L \otimes_{K,\sigma} \mathrm{Fil}^i(D_{\mathrm{dR}}(D)_\sigma),$$

and jumps in the free $L \otimes_{K,\sigma} E$-modules $L \otimes_{K,\sigma} \mathrm{Fil}^i(D_{\mathrm{dR}}(D)_\sigma)$ are $(-1)\cdot(\sigma$-Sen weights of $D)$.

Definition 3.11. Let $\mathbf{h} = (h_{1,\sigma} \geq \cdots \geq h_{n,\sigma})_\sigma \in \mathbb{Z}^{n[K:\mathbb{Q}_p]}$ be the Sen weights of a crystabelline generic (φ, Γ_K)-module D of rank n.

(i) We say $\mathbf{h}$ is **regular** if $h_{1,\sigma} > \cdots > h_{n,\sigma}$ for each $\sigma \in \Sigma_K$.

(ii) A refinement of the crystabelline generic (φ, Γ_K)-module D

$$w(\phi) = (\phi_{w(1)}, \ldots, \phi_{w(n)}), \text{ for } w \in S_n,$$

is **non-critical** if for all $1 \leq i \leq n$, the weights of the saturated (φ, Γ_K)-submodule $D_{w,i} := \mathrm{Fil}^i_w(D)$ of D of Weil-Deligne representation $\bigoplus_{s=1}^i \phi_{w(s)}$ are precisely $(h_{1,\sigma} \geq \cdots \geq h_{i,\sigma})$. Then, D is called **non-critcal** if all its refinements are non-critical.

Denote by $\Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ the class of all crystabelline non-critical (φ, Γ_K)-modules of rank n over $\mathcal{R}_{K,E}$ with Weil-Deligne representation $\phi = \bigoplus_{i=1}^n \phi_i$ and Sen weights $\mathbf{h}$.

Proposition 3.12. *Let D be a crystabelline generic (φ, Γ_K)-module of rank n over $\mathcal{R}_{K,E}$ that is crystalline over L, with Weil-Deligne representation $\phi = \bigoplus_{i=1}^n \phi_i$ and Sen weights $\mathbf{h}$. Then,*

(i) D is trianguline and has $n!$ triangulations

$$\mathrm{Fil}^\bullet_w(D): \quad 0 \subsetneq D_{w,1} \subsetneq \cdots \subsetneq D_{w,n} = D,$$

which are indexed by refinements $w \in S_n$, and are of parameters $\delta_w = (\delta_{w,1}, \ldots, \delta_{w,n})$ with

$$\delta_{w,i} = \left(\prod_{\sigma\in\Sigma_K} x_\sigma^{k^w_{i,\sigma}}\right) \phi_{w(i)}$$

for some integer $k^w_{i,\sigma} \in \mathbb{Z}$ such that $\{h_{1,\sigma}, \ldots, h_{n,\sigma}\} = \{k^w_{1,\sigma}, \ldots, k^w_{n,\sigma}\}$ as sets, for all σ.

(ii) $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ is non-critical if and only if for all $1 \leq i \leq n$, $w \in S_n$ and $\sigma \in \Sigma_K$,

$$\delta_{w,i} = \left(\prod_{\sigma\in\Sigma_K} x_\sigma^{h_{i,\sigma}}\right) \phi_{w(i)}.$$

Proof. Let π_K be a uniformizer of K. Then, there are $m \in \mathbb{Z}_{\geq 1}$ and a finite unramified extension L'/K such that $L \subset K_m.L'$, where $K_m := K([\pi^m])$ is the m-th Lubin-Tate extension of K obtained by adjoining all π^m-torsion points of the Lubin-Tate formal $\mathcal{O}_K$-module for π.

We give the proof in the language of B-pairs following [Nak13]. Consider the exact sequence

$$1 \to G_{K_m.L'} \to G_{K_m} \to G_m := G(K_m.L'/K_m) \to 1.$$

Since $L \subset K_m.L'$, it follows that $G_{K_m.L'} \subset G_L$, and D becomes crystalline over $K_m.L'$. Denote by $W = (W_e, W_{\mathrm{dR}}^+)$ the B-pair with G_K-action corresponding to the (φ, Γ_K)-module D. Then,

$$D_{\mathrm{cris}}^{K_m.L'}(W) := (B_{\max} \otimes_{B_e} W_e)^{G_{K_m.L'}}$$

is a semilinear G_m-module over $(K_m.L')_0 = L'$. By Galois descent, we have an isomorphism

$$D_{\mathrm{cris}}^{K_m}(W) = (B_{\max} \otimes_{B_e} W_e)^{G_{K_m}} \otimes_{\mathbb{Q}_p} L' \xrightarrow{\simeq} (B_{\max} \otimes_{B_e} W_e)^{G_{K_m.L'}}$$

showing that D becomes crystalline over K_m. From now on, we assume $L = K_m$ for some m.

(i) The genericity of D implies that there are $n!$ complete flags consisting of subobjects in the Weil-Deligne representation $\bigoplus_{i=1}^n \phi_i$ attached to D. By the equivalences in Theorem 3.9 and [BS07, Proposition 4.1], the flags give rise to $n!$ distinct triangulations on D, indexed by permutations $w \in S_n$ of the set $\{\phi_1, \ldots, \phi_n\}$.

As for the description of parameters, it suffices to prove one refinement, say $w = \mathrm{id}$. Let

$$0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_n = D$$

be the corresponding triangulation. Then, D_i/D_{i-1} is the (φ, Γ_K)-module, corresponding to the Weil character ϕ_i, of rank 1 that becomes crystalline over K_m. So it is de Rham, and by [Nak13, Theorem 1.45] we have, for $\mathcal{R} := \mathcal{R}_{K,E}$,

$$D_{w,i}/D_{w,i-1} \cong \mathcal{R}(\delta_i)$$

for a unique continuous character $\delta_i : K^\times \to E^\times$. By [Nak09, Lemma 4.1], we know

$$\delta_i = \widetilde{\delta_i} \prod_{\sigma \in \Sigma_K} x_\sigma^{k_{i,\sigma}}$$

for some integers $k_{i,\sigma} \in \mathbb{Z}$ and a smooth character $\widetilde{\delta_i}$ of $K^\times$. We claim that $\widetilde{\delta_i} = \phi_i$.

By the ($\otimes$-)equivalence in Theorem 3.9, it remains to show that the Weil-Deligne representation attached to $D_{\mathrm{cris}}^{K_m}(\delta_i)$ via the equivalence in [BS07, Proposition 4.1] is $\widetilde{\delta_i}$.

We first understand $D_{\mathrm{cris}}^{K_m}(\delta_i)$ as a (φ, G_m)-module. We have as (φ, G_m)-modules

$$\begin{aligned} D_{\mathrm{cris}}^{K_m}(\mathcal{R}(\delta_i)) &= D_{\mathrm{cris}}^{K_m}(\widetilde{\delta_i}) \otimes \bigotimes_{\sigma \in \Sigma_K} D_{\mathrm{cris}}^{K_m}\left(x_\sigma^{k_{i,\sigma}}\right) \\ &= D_{\mathrm{cris}}^{K_m}(\widetilde{\delta_i}^{\mathrm{unr}}) \otimes D_{\mathrm{cris}}^{K_m}(\widetilde{\delta_i}^{\mathrm{wt}}) \otimes \bigotimes_{\sigma \in \Sigma_K} D_{\mathrm{cris}}^{K_m}\left(x_\sigma^{k_{i,\sigma}}\right) \\ &= D_{\mathrm{cris}}^{K_m}(\widetilde{\delta_i}^{\mathrm{wt}}) \otimes D_{\mathrm{cris}}^{K_m}\left(\widetilde{\delta_i}^{\mathrm{wt}} \prod_\sigma x_\sigma^{k_{i,\sigma}}\right) \end{aligned}$$

where we put $\widetilde{\delta_i}^{\mathrm{unr}}|_{\mathcal{O}_{K^\times}} = 1$ and $\widetilde{\delta_i}^{\mathrm{wt}}|_{\mathcal{O}_K^\times} = \widetilde{\delta_i}|_{\mathcal{O}_K^\times}$ such that $\widetilde{\delta_i} = \widetilde{\delta_i}^{\mathrm{unr}} \widetilde{\delta_i}^{\mathrm{wt}}$. Then by [KPX14, Example 6.2.6(3)],

$$D_{\mathrm{cris}}\left(\widetilde{\delta_i}^{\mathrm{wt}} \prod_\sigma x_\sigma^{k_{i,\sigma}}\right) = D_{f, \widetilde{\delta_i}(\pi_K)}$$

in the notation of [KPX14, Lemma 6.2.3]: $D_{f,\widetilde{\delta}_i(\pi_K)}$ is the rank 1 $(K_0 \otimes_{\mathbb{Q}_p} E)$-module with a $(\varphi \otimes \mathrm{id}_E)$-semilinear endomorphism φ such that $\varphi^f = \widetilde{\delta}_i(\pi_K)$, and its σ-Sen weight is $k_{i,\sigma}$ for each σ. Then,

$$D_{\mathrm{cris}}^{K_m}\left(\widetilde{\delta}_i^{\mathrm{wt}} \prod_\sigma x_\sigma^{k_{i,\sigma}}\right) = (K_m)_0 \otimes_{K_0} D_{\mathrm{cris}}\left(\widetilde{\delta}_i^{\mathrm{wt}} \prod_\sigma x_\sigma^{k_{i,\sigma}}\right) = D_{\mathrm{cris}}\left(\widetilde{\delta}_i^{\mathrm{wt}} \prod_\sigma x_\sigma^{k_{i,\sigma}}\right)$$

with the trivial G_m-action as $K_m|K$ is totally ramified. The character $\widetilde{\delta}_i^{\mathrm{wt}}$ is unitary and of weight 0 by definition. We compute its Fontaine module via its Galois representation. Taking an larger m if necessary, we may assume that $\widetilde{\delta}_i^{\mathrm{wt}}|_{1+\pi_K^m\mathcal{O}_K} = 1$, i.e., it restricts to the trivial character of G_{K_m}. Then,

$$D_{\mathrm{cris}}^{K_m}(\widetilde{\delta}_i^{\mathrm{wt}}) = \left(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E(\widetilde{\delta}_i^{\mathrm{wt}})\right)^{G_{K_m}} = K_0 \otimes_{\mathbb{Q}_p} E(\widetilde{\delta}_i^{\mathrm{wt}})$$

is isomorphic to $K_0 \otimes_{\mathbb{Q}_p} E$ as φ-module, and G_m acts via the character $\widetilde{\delta}_i^{\mathrm{wt}}$. Hence,

$$D_{\mathrm{cris}}^{K_m}(\mathcal{R}(\delta_i)) \cong D_{f,\widetilde{\delta}_i(\pi_K)}$$

as φ-modules over $K_0 \otimes_{\mathbb{Q}_p} E$, with a linear action by G_m via the character $\widetilde{\delta}_i^{\mathrm{wt}}$.

We then show that the Weil-Deligne representation $\phi_i : W_K \to \mathrm{GL}_1(E) = E^\times$ corresponding to $D_{\mathrm{cris}}^{K_m}(\mathcal{R}(\delta_i))$ is precisely $\widetilde{\delta}_i$. We follow the equivalence of categories described in [BS07, above Proposition 4.1]. For $w \in W_K$, let $\alpha(w) \in f\mathbb{Z}$ be the integer such that the image of w in $\mathrm{Gal}(\overline{\mathbb{F}_p}|\mathbb{F}_p)$ is the $\alpha(w)$-th power of the absolute arithmetic Frobenius. Then,

$$\phi_i(w) := \overline{w} \circ \varphi^{-\alpha(w)} = \widetilde{\delta}_i^{\mathrm{wt}}(\mathrm{rec}_K^{-1}(w)) \cdot \varphi^{-\alpha(w)},$$

where $\overline{w}$ denotes the image of w in $\mathrm{Gal}(K_m|K) = G_m$, and $\mathrm{rec}_K : K^\times \to W_K^{\mathrm{ab}}$ is the local reciprocity map sending uniformizers to geometric Frobenii. Then,

$$\phi_i(\pi_K) = \widetilde{\delta}_i^{\mathrm{wt}}(\pi_K)\varphi^{-(-f)} = \widetilde{\delta}_i^{\mathrm{wt}}(\pi_K)\widetilde{\delta}_i(\pi_K) = \widetilde{\delta}_i(\pi_K)$$

because $\mathrm{rec}_K(\pi_K)$ is a geometric Frobenius. For $u \in \mathcal{O}_K^\times \cong I_K$, $\alpha(\mathrm{rec}_K(u)) = 0$ and

$$\phi_i(u) = \widetilde{\delta}_i^{\mathrm{wt}}(u)\varphi^{-(0)} = \widetilde{\delta}_i(u).$$

Hence, $\widetilde{\delta}_i = \phi_i$ as desired.

(ii) This follows immediately from the previous part and Definition 3.11(ii). ☐

Proposition 3.13. *For $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ of regular Sen weight ($h_{1,\sigma} > \cdots > h_{n,\sigma}$ for each $\sigma \in \Sigma_K$), D is indecomposable. In particular, all of its triangulations are strongly non-split.*

Proof. This is a direct generalization of [Che11, Lemma 3.21] from the case $K = \mathbb{Q}_p$. We recall the argument here. Suppose $D = D' \oplus D''$ is the direct sum of two saturated (φ, Γ_K)-submodules over $\mathcal{R}_{K,E}$. By Theorem 3.9, $D_{\mathrm{cris}}^{K_m}(D) = D_{\mathrm{cris}}^{K_m}(D') \oplus D_{\mathrm{cris}}^{K_m}(D'')$ decomposes as a direct sum of filtered (φ, G_m)-modules, and so does the associated Weil-Deligne representations $\bigoplus_{i=1}^n \phi_i$.

Since $\mathbf{h}$ is regular, we have a partition of $\{1, \ldots, n\} = A \sqcup B$ such that a number i belongs to A if and only if $\mathbf{h}_i := (h_{i,\sigma})_{\sigma\in\Sigma_K}$ is a Sen weight of D'.

For any $w \in S_n$, the refinement $(\phi_{w(1)}, \ldots, \phi_{w(n)})$ is non-critical by assumption, which means that $\phi_{w(i)}$ appears in the Weil-Deligne representation attached to $D_{\mathrm{cris}}^{K_m}(D')$ if and only if $(h_{i,\sigma})_\sigma$

is a Sen weight of D'. As the action of S_n on $\{1, \dots, n\}$ is transitive, we conclude that $A = \emptyset$ or $A = \{1, \dots, n\}$. Hence, D is indecomposable.

If D is non-critical crystabelline of regular weight, so is the subquotient $\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-2}(D)$. Hence, (2.2.5) is always non-split. $\square$

Proposition 3.14. *Let $K = \mathbb{Q}_p$, and D be a generic crystabelline $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_{\mathbb{Q}_p,E}$ of regular Sen weight $\mathbf{h} = (h_1 > \cdots > h_n)$ and Weil-Deligne representation $\bigoplus_{i=1}^n \phi_i$. If D has a critical refinement w, then $(D, \mathrm{Fil}_w^\bullet(D))$ is not strongly non-split in the sense of* (2.2.5).

Proof. By [Nak13, Remark 2.43], D becomes crystalline over $K_m := \mathbb{Q}_p(\mu_{p^m})$ for some $m \geq 0$. Then because $K_m(\mu_{p^\infty}) = \mathbb{Q}_p(\mu_{p^\infty})$, the Robba ring $\mathcal{R}_{K_n,E}$ equals $\mathcal{R} := \mathcal{R}_{\mathbb{Q}_p,E}$ with the "usual" actions by φ and the open subgroup

$$\Gamma_m := \mathrm{Gal}(K_n(\mu_{p^\infty})|K_n) = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})|K_n)$$

of $\Gamma \cong \mathbb{Z}_p^\times$. For all i, let $e_i \in D_{\mathrm{cris}}^{K_m}(D) = D[1/t]^{\Gamma_m}$ be an (φ, G_m)-eigenvector of eigenvalue ϕ_i, where $G_m := \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^m})|\mathbb{Q}_p)$. By [BC09, Proposition 2.4.1], a refinement $(\alpha_{w(1)}, \dots, \alpha_{w(n)})$, for $w \in S_n$, of D is non-critical if and only if the induced ordering $(k_1^w, \dots, k_n^w)$ of $\{h_1, \dots, h_n\}$, which is defined by saying that the jumps of the induced Hodge filtration on $\bigoplus_{s=1}^i E.e_{w(s)} \subset D_{\mathrm{cris}}(D)$ are $-k_1^w, \dots, -k_i^w$ for all $1 \leq i \leq n$, is decreasing.

By Lemma 3.15 below, if some critical refinement $w \in S_n$ for D induces the ordering

$$k_1^w, \dots, k_{i-1}^w, k_i^w, k_{i+1}^w, \dots, k_n^w$$

then there is some $i \in \{1, \dots, n-1\}$ such that $w' := w \circ (i, i+1) \in S_n$ induces the ordering

$$k_1^w, \dots, k_{i-1}^w, k_{i+1}^w, k_i^w, \dots, k_n^w$$

on the Sen weights of D. This means that $(D, \mathrm{Fil}_w^\bullet(D))$ has the subquotient

$$\mathrm{Fil}_w^{i+1}(D)/\mathrm{Fil}_w^{i-1}(D) = \mathrm{Fil}_{w'}^{i+1}(D)/\mathrm{Fil}_{w'}^{i-1}(D)$$

carrying two triangulations

$$(\mathcal{R}(\phi_{w(i)}x^{k_i^w}) - \mathcal{R}(\phi_{w(i+1)}x^{k_{i+1}^w})) = (\mathcal{R}(\phi_{w(i+1)}x^{k_{i+1}^w}) - \mathcal{R}(\phi_{w(i)}x^{k_i^w}))$$

which, by genericity of ϕ_i, must be split. Hence, $(D, \mathrm{Fil}_w^\bullet(D))$ is not strongly non-split. $\square$

Lemma 3.15. *Let V be an n-dimensional E-vector space with basis $e_1, \dots, e_n$ equipped with an exhaustive decreasing filtration $(\mathrm{Fil}^i(V))_{i \in \mathbb{Z}}$ of V with jumps at $h_1 < \cdots < h_n$, i.e.,*

$$\dim_E(\mathrm{Fil}^{h_i}(V)/\mathrm{Fil}^{h_i+1}(V)) = 1.$$

For each $w \in S_n$ and $1 \leq i \leq n$, we let $V_i^w := \mathrm{Span}_E(e_{w(1)}, \dots, e_{w(i)})$ and define an ordering $(j_i(w), \dots, j_n(w))$ on $\{h_1, \dots, h_n\}$ such that $j_i(w)$ is the unique jump for the induced filtration on the line V_i^w/V_{i-1}^w.

If $(j_1(w), \dots, j_n(w)) \neq (h_1, \dots, h_n)$, then there exists $i \in \{1, \dots, n-1\}$ such that

$$j_k(w \circ (i, i+1)) = \begin{cases} j_k(w) & \text{if } k \notin \{i, i+1\} \\ j_{i+1}(w) & \text{if } k = i \\ j_i(w) & \text{if } k = i+1 \end{cases}$$

Proof. Assume, for the sake of contradiction that $j_i(w \circ (i, i+1)) = j_i(w)$ for all $1 \leq i \leq n-1$. For each i, both $e_{w(i)}$ and $e_{w(i+1)}$ span the line $\mathrm{Fil}^{j_i(w)}(V/V_{i-1}^w)/\mathrm{Fil}^{j_i(w)+1}(V/V_{i-1}^w)$, so $j_{i+1}(w) > j_i(w)$. Hence, $(j_1(w) < \cdots < j_n(w)) = (h_1 < \cdots < h_n)$, contradiction. $\square$

Remark 3.16. For critical generic crystalline $(\varphi, \Gamma_{\mathbb{Q}_p})$-module D over $\mathbb{Q}_p$, it could be strongly non-split for a non-critical refinement $\mathrm{Fil}^\bullet(D)$, or it could have no strongly non-split non-critical refinement at all. Hence, "strongly non-split" depends on the choice of triangulation.

3.3. **More general cases.**

Proposition 3.17. *(i) Suppose $(D, \mathrm{Fil}^\bullet(D))$ with parameters δ is strongly non-split in the sense of (2.2.5), with D_1 strongly non-split very generic with non-integer Sen weights, and D_2 crystabelline non-critical such that $D \in \mathrm{Ext}^1(D_2, D_1)$ with*

$$D = \underbrace{(\mathcal{R}\delta_1 - \cdots - \mathcal{R}\delta_m)}_{D_1} - \underbrace{(\mathcal{R}\delta_{m+1} - \cdots - \mathcal{R}\delta_n)}_{D_2}$$

for some $1 \le m \le n$, then any strongly non-split triangulation of D has to be of the form

$$D = \underbrace{(\mathcal{R}\delta_1 - \cdots - \mathcal{R}\delta_m)}_{D_1} - \underbrace{(\mathcal{R}\delta_{w,m+1} - \cdots - \mathcal{R}\delta_{w,n})}_{D_2}$$

for a permutation w of the set $\{m+1, \dots, n\}$, where $\delta_{w,i} := x^{\mathbf{h}_i}\phi_{w(i)}$ as in Proposition 3.12.

(ii) The analogous description holds when $(D, \mathrm{Fil}^\bullet(D)) \in \mathrm{Ext}^1(D_1, D_2)$ is strongly non-split, where D_1 is very generic with non-integer Sen weights and D_2 is crystabelline non-critical.

Proof. (i) For any other strongly non-split triangulation $(D, \mathrm{Fil}^\bullet(D)')$, we write

$$D = (\underbrace{\mathcal{R}\eta_1 - \cdots - \mathcal{R}\eta_i}_{\mathrm{Fil}^i(D)'} - \cdots - \mathcal{R}\eta_n)$$

for all $1 \le i \le n$. We claim $\mathrm{Fil}^i(D)' = \mathrm{Fil}^i(D)$ for $1 \le i \le m$. We proceed by induction, and we first show $\mathrm{Fil}^1(D)' = \mathrm{Fil}^1(D)$. Let j be the largest integer such that the intersection $\mathrm{Fil}^{j-1}(D)' \cap \mathrm{Fil}^1(D) = 0$ is zero. Then, we have an injection

$$\mathcal{R}(\delta_1) = \mathrm{Fil}^1(D) \hookrightarrow \mathrm{Fil}^j(D)'/\mathrm{Fil}^{j-1}(D)' = \mathcal{R}(\eta_j).$$

By Lemma 2.7(iii), there is $\mathbf{k} \in \mathbb{N}^{[K:\mathbb{Q}_p]}$ such that $x^{\mathbf{k}}\delta_1 = \eta_j$. So, $\mathrm{wt}(\delta_1) - \mathrm{wt}(\eta_j) \in \mathbb{Z}^{[K:\mathbb{Q}_p]}$. By our assumption, this can happen only if $\mathrm{wt}(\delta_1) = \mathrm{wt}(\eta_j)$, and thus $\mathbf{k} = 0$ and $\delta_1 = \eta_j$. Since $(D, \mathrm{Fil}^\bullet(D)')$ is non-split, we have $j = 1$. Taking the quotient by $\mathrm{Fil}^1(D) = \mathrm{Fil}^1(D)'$, we see that $\mathrm{Fil}^i(D)' = \mathrm{Fil}^i(D)$ for $1 \le i \le m$ by induction. For the induction to work, the strongly non-split assumption ensures that $D/\mathrm{Fil}^1(D)$ and $D/\mathrm{Fil}^1(D)'$ are non-split.

We have shown that $D_1 = \mathrm{Fil}^m(D)'$. Then,

$$D_2 = D/D_1 = (\mathcal{R}\eta_{m+1} - \cdots - \mathcal{R}\eta_n)$$

which has $(n-m)!$ triangulations of the desired form by Proposition 3.12.

(ii) Taking dual $D^\vee$ of D reduces this case to the previous case (i) treated above. ∎

Remark 3.18. Let us discuss how one can find triangulations on any (φ, Γ_K)-modules D over $\mathcal{R}_{K,E}$, where E is a finite extension of $\mathbb{Q}_p$ containing the Galois closure K^{norm} of K. Let $\mathcal{T}_E$ be the rigid E-space parameterizing continuous characters of $K^\times$, with universal character δ.

As the first step, consider the locus X_1 in $\mathcal{T}_E(E)$ where the character $\delta_1 = \delta$ makes the space

$$\left\{ f_1 \in H^0(D^\vee(\delta_1)) \middle| \forall\sigma \in \Sigma_K, \overline{f_1} \neq 0 \in H^0(D^\vee(\delta_1)/t_\sigma) \right\}$$

nonempty, where $\overline{f_1}$ denotes the image of f under the mod-t_σ reduction map

$$\mathrm{Hom}_{\varphi,\Gamma_K}(D, \mathcal{R}_{K,E}(\delta)) \to \mathrm{Hom}_{\varphi,\Gamma_K}(D/t_\sigma, \mathcal{R}_{K,E}(\delta)/t_\sigma).$$

By the classification of (φ, Γ_K)-submodules of $\mathcal{R}_{K,E}(\delta_1)$ in Lemma 2.7(iii), any f_1 in the space above corresponds to a map $f_1 : D \to \mathcal{R}_{K,E}(\delta_1)$ of (φ, Γ_K)-modules that is surjective, and vice versa. To the data of (δ_1, f_1), we associate a (φ, Γ_K)-submodule of D of rank $n-1$ as

$$0 \to D_{n-1} := \ker(f_1) \to D \xrightarrow{f_1} \mathcal{R}_{K,E}(\delta_1) \to 0.$$

The next step is to consider, for all (f_1, δ_1) found in Step 1, the locus X_2 in $\mathcal{T}_E(E)$ where the character $\delta_2 = \delta$ makes the space

$$\left\{ f_2 \in H^0(D_{n-1}^\vee(\delta_2)) \middle| \forall \sigma \in \Sigma_K, \overline{f_2} \neq 0 \in H^0(D_{n-1}^\vee(\delta_2)/t_\sigma) \right\}$$

nonempty. Any f_2 from this space corresponds to a surjective map from D_{n-1} to $\mathcal{R}_{K,E}(\delta_2)$ of (φ, Γ_K)-modules, whose kernel (we denote by D_{n-2}) satisfies

$$0 \to D_{n-2} := \ker(f_2) \to D_{n-1} \xrightarrow{f_2} \mathcal{R}_{K,E}(\delta_2) \to 0.$$

Continuing this way, we get data $(\delta_1, f_1, \delta_2, f_2, \dots, \delta_{n-1}, f_{n-1}, \delta_n)$, where for $D_{n-i} := \ker(f_i)$ with $D_n = D$, f_i are chosen from

$$\left\{ f_i \in H^0(D_{n-i+1}^\vee(\delta_i)) \middle| \forall \sigma \in \Sigma_K, \overline{f_i} \neq 0 \in H^0(D_{n-i+1}^\vee(\delta_i)/t_\sigma) \right\},$$

from which we know there is a triangulation $\mathrm{Fil}^\bullet(D) = D_\bullet$ on D with parameters $(\delta_n, \dots, \delta_1)$.

4. Pullback operations

Consider the stack $\mathfrak{X}_n = \mathfrak{X}_{\mathrm{GL}_n}$ of G_K-equivariant vector bundles on rank n on the Fargues-Fontaine curve $X_{\overline{K}}$, the stack $\mathfrak{X}_P$ of G_K-equivariant P-bundles on $X_{\overline{K}}$ for a standard parabolic subgroup $P \subset \mathrm{GL}_n$ containing the upper triangular Borel subgroup $B \subset \mathrm{GL}_n$ with Levi quotient $M \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$, and the stack $\mathfrak{X}_M$ of G_K-equivariant M-bundles on $X_{\overline{K}}$, all over the category of rigid E-analytic spaces Rig_E equipped with the Tate-fpqc topology, cf. [EGH, §5.1, §5.3]. Note that $\mathfrak{X}_M \cong \mathfrak{X}_{n_1} \times \cdots \times \mathfrak{X}_{n_r}$.

For P_i the parabolic subgroup containing B of Levi quotient $M_i \cong \mathrm{GL}_{n-i} \times \mathrm{GL}_i$, we may consider the pullback map $p_{i,\sigma} : \mathfrak{X}_{P_i} \to \mathfrak{X}_{P_i}$, sending a P_i-bundle

$$(0 \to D_{n-i} \to D \to D^i := D/D_{n-i} \to 0),$$

where D_{n-i} is a saturated (φ, Γ_K)-submodule of D of rank $n-i$, to the P_i-subbundle

$$(0 \to D_{n-i} \to p_{i,\sigma}(D) \to t_\sigma D^i \to 0).$$

This makes sense on $\mathfrak{X}_Q$ for any parabolic Q such that $B \subset Q \subset P_i$, where $p_{i,\sigma}$ acts similarly.

Remark 4.1. For $\mathfrak{X}_B$, by the argument of [EGH, Lemma 5.3.10], using affine Grassmannian one can show that $p_{i,\sigma} : \mathfrak{X}_B \to \mathfrak{X}_B$ is relatively representable.

Over the sublocus of $\mathcal{T}^n$ where $\mathrm{wt}_\sigma(\delta_j/\delta_k) \neq 1$ for all $1 \leq j \leq n-i < k \leq n$ with fixed i, and for non-split pair $(D, \mathrm{Fil}^\bullet(D)) \in \mathfrak{X}_n(B)$, it follows from Theorem 3.1 that from $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))$ together with its induced triangulation, we can recover D together with its triangulation.

Lemma 4.2. *For any $1 \leq i < j \leq n$ and $\sigma, \tau \in \Sigma_K$ (with possibly $\sigma = \tau$), we have that*

$$p_{i,\sigma} \circ p_{j,\tau} = p_{j,\tau} \circ p_{i,\sigma}$$

as maps from $\mathfrak{X}_Q$ to $\mathfrak{X}_Q$, for any parabolic Q such that $B \subset Q \subset P_i \cap P_j$.

Proof. We may assume $Q = P_i \cap P_j$. Then, a Q-bundle is $D \in \mathfrak{X}_n(A)$ together with a filtration

$$0 \subset D_{n-j} := \mathrm{Fil}^{n-j}(D) \subset D_{n-i} := \mathrm{Fil}^{n-i}(D) \subset D_n = D$$

by (φ, Γ_K)-submodule over $\mathcal{R}_{K,A}$ such that $D^\bullet := D/D_{n-\bullet}$ is locally free of rank $\bullet \in \{i, j\}$, and

$$p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D)) = (D_{n-i} - t_\sigma D^i)$$

which has the induced Q-filtration

$$0 \subset D_{n-j} \subset D_{n-i} \subset p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D)),$$

from which we see, because the twist by t_τ commutes with pullback via t_σ, that

$$\begin{aligned} p_{j,\tau}(p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))) &= (D_{n-j} - t_\tau(D_{n-i}/D_{n-j} - t_\sigma D^i)) \\ &= (D_{n-j} - t_\tau(D_{n-i}/D_{n-j}) - t_\tau t_\sigma D^i) \\ &= (D_{n-j} - t_\tau(D_{n-i}/D_{n-j}) - t_\sigma(t_\tau D^i)) \\ &= p_{i,\sigma}(p_{j,\tau}(D, \mathrm{Fil}^\bullet(D))) \end{aligned}$$

which completes the proof. □

4.1. **Substacks.** We define several substacks of $\mathfrak{X}_n$ to which $p_{i,\sigma}$ descends, and the reason why we can do so is given by [Wu, Proposition 3.16].

Definition 4.3. (i) Recall we have the morphisms of stacks

$$\begin{tikzcd} & \mathfrak{X}_P \arrow[dl, "\beta_P"'] \arrow[dr, "\alpha_P"] & \\ \mathfrak{X}_n & & \mathfrak{X}_M \end{tikzcd}$$

where for a paraboline (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D)) \in \mathfrak{X}_P(A)$, we have the forgetful map

$$\beta_P(D, \mathrm{Fil}^\bullet(D)) = D$$

and the map

$$\alpha_P(D, \mathrm{Fil}^\bullet(D)) = (\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D))_i$$

taking the successive quotients of the P-structure $\mathrm{Fil}^\bullet(D)$ on D.

(ii) For $P = B$, we define the **weight maps**

$$\omega_T : \mathfrak{X}_T \to \mathrm{Res}_{K/\mathbb{Q}_p}(\mathbb{A}^{n,\mathrm{rig}})_E \cong \mathbb{A}_E^{n[K:\mathbb{Q}_p],\mathrm{rig}}, \quad (\delta_1, \dots, \delta_n) \mapsto (\mathrm{wt}(\delta_1), \dots, \mathrm{wt}(\delta_n))$$

and

$$\omega_B : \mathfrak{X}_B \xrightarrow{\alpha_B} \mathfrak{X}_T \xrightarrow{\omega_T} \mathbb{A}_E^{n[K:\mathbb{Q}_p],\mathrm{rig}}, \quad (D, \mathrm{Fil}^\bullet(D)) \mapsto (\mathrm{wt}(\delta_1), \dots, \mathrm{wt}(\delta_n))$$

where δ_i are the unique characters determined by the rank 1 subquotients

$$\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D) \cong \mathcal{R}_{K,A}(\delta_i) \otimes_{\mathcal{O}_{\mathrm{Sp}(A)}} \mathcal{L}_i.$$

With the projection

$$\mathrm{pr}_\sigma : \mathbb{A}_E^{n[K:\mathbb{Q}_p],\mathrm{rig}} \to \mathbb{A}_E^{n,\mathrm{rig}}$$

onto the σ-component for $\sigma \in \Sigma_K$, we define the **σ-weight maps** by

$$\omega_{T,\sigma} := \mathrm{pr}_\sigma \circ \omega_T : \mathfrak{X}_T \to \mathbb{A}_E^{n,\mathrm{rig}}, \quad (\delta_1, \dots, \delta_n) \mapsto (\mathrm{wt}_\sigma(\delta_1), \dots, \mathrm{wt}_\sigma(\delta_n))$$

and

$$\omega_{B,\sigma} := \omega_{T,\sigma} \circ \alpha_B : \mathfrak{X}_B \to \mathbb{A}_E^{n,\mathrm{rig}}, \quad (D, \mathrm{Fil}^\bullet(D)) \mapsto (\mathrm{wt}_\sigma(\delta_1), \ldots, \mathrm{wt}_\sigma(\delta_n)).$$

Definition 4.4. Let $S \subset \Sigma_K$ be a subset of embeddings of K into E.

(i) For a trianguline (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$, we say D is **S-weight-uniform trianguline** if for all geometric points $x \to \mathrm{Sp}(A)$ and for all $\sigma \in S$, the fiber D_x is such that the σ-weight map $\omega_{B,\sigma}$ is constant on $\beta_B^{-1}(D_x)$. That is, all triangulations on D_x induce the same ordering on its σ-Sen weights, for all geometric points x of $\mathrm{Sp}(A)$ and for all $\sigma \in S$.

(ii) As the S-weight-uniform trianguline condition is defined via geometric fibers, it is stable under base change. We denote by $\mathfrak{X}_B^{S\text{-wu}}$ this substack of $\mathfrak{X}_B$, and by $\mathfrak{X}_n^{S\text{-wu}}$ the sheafification of its image under β_B in $\mathfrak{X}_n$ with respect to the Tate-fpqc topology on Rig_E. We refer to $\mathfrak{X}_B^{S\text{-wu}}$ and $\mathfrak{X}_n^{S\text{-wu}}$ as the **S-weight-uniform substacks**.

(iii) If $S = \Sigma_K$, we simply write $\mathfrak{X}_B^{\mathrm{wu}}$ and $\mathfrak{X}_n^{\mathrm{wu}}$, and call them **weight-uniform substacks**.

Remark 4.5. Sheafification is necessary for descent reason. [EGH, Example 5.3.2] gives a filtered φ-module on $\mathbb{P}^1$ defining a rank 2 $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules on $\mathbb{P}^1$, which has no global triangulation on $\mathbb{P}^1$, but is globally trianguline over the two copies of $\mathbb{A}^1$.

Definition 4.6. (i) Fix $\sigma \in \Sigma_K$ and $i \in \{1, \ldots, n\}$.

(a) Let U_i be the Zariski-open subspace of $\mathbb{A}_E^{n,\mathrm{rig}}$ that is complement to the hyperplanes $\{T_j - T_k = 0\}$ for all $1 \leq j \leq n-i$ and $n-i+1 \leq k \leq n$, where we denote by $\{T_l | 1 \leq l \leq n\}$ the standard coordinates on $\mathbb{A}_E^{n,\mathrm{rig}}$.

(b) Let $\mathfrak{X}_n^{\sigma\text{-wu},i}$ be the sheafification of the image $\beta_B(\mathfrak{X}_B^{\sigma\text{-wu}} \cap \omega_{B,\sigma}^{-1}(U_i))$, that is the substack of $\mathfrak{X}_n^{\sigma\text{-wu}}$ characterized by the property that, for any triangulation on $D \in \mathfrak{X}_n^{\sigma\text{-wu},i}(A)$, the first $n-i$ ordered σ-Sen weights are disjoint from the last i ordered σ-Sen weights in $k(x)$ for all $x \in \mathrm{Sp}(A)$.

(c) Generally, for $a \leq 0 \leq b \in \mathbb{Z}$, let $\mathfrak{X}_n^{\sigma\text{-wu},i,[a,b]}$ be the substack of $\mathfrak{X}_n^{\mathrm{wu}}$ characterized by the property that for any triangulation on $D \in \mathfrak{X}_n^{\sigma\text{-wu},i,[a,b]}(A)$, with parameters $(\delta_1, \ldots, \delta_n)$, we have

$$\{\mathrm{wt}_\sigma(\delta_{j,x}) | 1 \leq j \leq n-i\} \cap \{\mathrm{wt}_\sigma(\delta_{k,x}) + h | n-i+1 \leq k \leq n, h \in [a,b] \cap \mathbb{Z}\} = \emptyset$$

in $k(x)$ for all $x \in \mathrm{Sp}(A)$. In particular, $\mathfrak{X}_n^{\sigma\text{-wu},i}$ defined in (b) is equal to $\mathfrak{X}_n^{\sigma\text{-wu},i,[0,0]}$.

(ii) Fix $S \subset \Sigma_K$, and for each $\sigma \in S$, choose a subset

$$I_\sigma := \{1 \leq i_1 < \cdots < i_{d_\sigma} \leq n\}$$

of $\{1, \ldots, n\}$ of size d_σ. Set $i_0 = 0$ and $i_{d_\sigma+1} = n$. For $\mathbf{k} = (k_{\sigma,i})_{\sigma \in S, i \in I_\sigma} \in \mathbb{N}^{\sum_{\sigma \in S} d_\sigma}$, let

$$\bigcap_{\sigma \in S} \bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,\left[0,\sum_{r=1}^m k_{\sigma,i_r}\right]} \subset \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}} \subset \bigcap_{\sigma \in S} \bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,[0,k_{\sigma,i_m}]}$$

be the substack of $\mathfrak{X}_n^{S\text{-wu}}$ characterized by the property that for any triangulation on $D \in \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}}(A)$, with parameters $(\delta_1, \ldots, \delta_n)$, setting $h_{i,\sigma} := \mathrm{wt}_\sigma(\delta_i)$ we have

$$\forall 0 \leq m \leq d_\sigma, \forall n+1-i_{m+1} \leq j < n+1-i_m, \forall 0 \leq m' < m, \forall n+1-i_{m'+1} \leq k < n+1-i_{m'}$$

$$(h_{j,\sigma})_x \notin \left\{ (h_{k,\sigma})_x + a \,\middle|\, a \in \mathbb{N}, 0 \leq a \leq \sum_{r=m'+1}^{m} k_{\sigma,i_r} \right\}$$

in $k(x)$ for all $x \in \mathrm{Sp}(A)$ and for all $\sigma \in S$. Conversely, for $-\mathbf{k} = (-k_{\sigma,i}) \in \mathbb{Z}_{\leq 0}^{\sum_{\sigma\in S} d_\sigma}$, let

$$\bigcap_{\sigma\in S}\bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,[-\sum_{r=1}^m k_{\sigma,i_r},0]} \subset \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}} \subset \bigcap_{\sigma\in S}\bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,[-k_{\sigma,i_m},0]}$$

be the substack of $\mathfrak{X}_n^{S\text{-wu}}$ characterized by the property that for any triangulation on $D \in \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}(A)$, with parameters $(\delta_1,\dots,\delta_n)$, setting $h_{i,\sigma} := \mathrm{wt}_\sigma(\delta_i)$ we have

$$\forall 0 \leq m \leq d_\sigma, \forall n+1-i_{m+1} \leq j < n+1-i_m, \forall 0 \leq m' < m, \forall n+1-i_{m'+1} \leq k < n+1-i_{m'}$$

$$(h_{j,\sigma})_x \notin \left\{ (h_{k,\sigma})_x - a \,\middle|\, a \in \mathbb{N}, 0 \leq a \leq \sum_{r=m'+1}^{m} k_{\sigma,i_r} \right\}$$

in $k(x)$ for all $x \in \mathrm{Sp}(A)$ and for all $\sigma \in S$.

(iii) When $S = \Sigma_K$ and $I_\sigma = \{1,\dots,n\}$ for all σ, we simply write $\mathfrak{X}_n^{\mathrm{wu},\mathbf{k}}$ to ease the notation.

Remark 4.7. One should think of the conditions defining $\mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}$ as the requirements on weights such that changing the weights through the pullback $\prod_{\sigma\in S, i\in I_\sigma} p_{i,\sigma}^{k_{\sigma,i}}$ do not meet any of the relevant walls in the weight space.

Corollary 4.8. *We know two types of* $\mathrm{Sp}(A)$*-valued points of* $\mathfrak{X}_n^{\mathrm{wu}}$*:*

(i) if $D \in \mathfrak{X}_n(A)$ *is trianguline and* D_x *is non-critical crystabelline at all* $x \in \mathrm{Sp}(A)$*.*

(ii) if $D \in \mathfrak{X}_n(A)$ *has a strongly non-split triangulation* $\mathrm{Fil}^\bullet(D)$ *with parameters in* $\mathcal{T}_\circ^n(A)$*.*

Proof. (i) follows from Definition 3.11, and (ii) follows from Corollary 3.5. □

4.2. **Operation $p_{i,\sigma}$ on $\mathfrak{X}_n^{\sigma\text{-wu},i}$.** We begin by recalling Wu's result obtained in [Wu] that we rely on, and then explain how it extends $p_{i,\sigma}$ and allows us to define morphisms on our stacks.

Lemma 4.9. *(i) Let R be any commutative unital ring, and monic polynomials $Q, S, Q', S' \in R[T]$ be such that $\deg(Q) = \deg(Q'), \deg(S) = \deg(S')$, $Q(T)S(T) = Q'(T)S'(T)$ and $(Q,S) = (Q',S) = (Q,S') = (Q',S') = (1)$. Then, we have $Q = Q'$ and $S = S'$.*

(ii) Let k be a field of characteristic 0, and monic polynomials $Q, S, Q', S' \in k[T]$ be such that $Q(T)S(T) = Q'(T)S'(T)$, $Q(T-1)S(T) = Q'(T-1)S'(T)$ and $(Q,S) = (1)$. Then, we have $Q = Q'$ and $S = S'$.

(iii) Let A be an affinoid $\mathbb{Q}_p$-algebra, and polynomials $Q, S \in A[T]$ be given. Then, $(Q(T), S(T)) = (1)$ if and only if for any $x \in \mathrm{Sp}(A)$, the sets of roots of $Q(T) \otimes_A k(x)$ and of $S(T) \otimes_A k(x)$ in $\overline{k(x)}$ have empty intersection.

Proof. (i) Subtracting $Q'(T)S(T)$ from both sides of $Q(T)S(T) = Q'(T)S'(T)$, we get

$$(Q - Q')S = Q'(S' - S).$$

Since $(Q', S') = 1$, we find polynomials $A, B \in R[T]$ such that $AQ' + BS = 1$. Multiplying both sides of $AQ' + BS = 1$ by $(S' - S)$, we get

$$AQ'(S' - S) + BS(S' - S) = S' - S.$$

Each of the two terms on the left is divisible by S, so is the right-hand side $S' - S$. Thus there exists $C \in R[T]$ such that $S' - S = CS$, or equivalently,

$$S' = (1 + C)S.$$

But by assumption, S and S' are monic polynomials of the same degree $\deg(S) = \deg(S')$. So, $C = 0$ and $S = S'$. Then $(Q - Q')S = 0$ in $R[T]$. Since S is monic, we have $Q = Q'$.

(ii) Since $k[T]$ is a PID, we may denote by $g_Q(T)$ the monic generator of $(Q(T), Q'(T))$ and by $g_S(T)$ the monic generator of $(S(T), S'(T))$. From $QS = Q'S'$, we get
$$\frac{Q}{g_Q} \cdot \frac{S}{g_S} = \frac{Q'}{g_Q} \cdot \frac{S'}{g_S}$$
where each of the four factors is monic in $k[T]$. Our assumption $(Q, S) = 1$ together with the coprimeness $(\frac{Q}{g_Q}, \frac{Q'}{g_Q}) = 1 = (\frac{S}{g_S}, \frac{S'}{g_S})$ imply two equalities of monics
$$A := \frac{Q}{g_Q} = \frac{S'}{g_S} \in k[T] \quad \text{and} \quad B := \frac{S}{g_S} = \frac{Q'}{g_Q} \in k[T].$$
Dividing both sides of $Q(T-1)S(T) = Q'(T-1)S'(T)$ by $g_Q(T-1)g_S(T)$, we get
$$A(T-1)B(T) = \frac{Q(T-1)S(T)}{g_Q(T-1)g_S(T)} = \frac{Q'(T-1)S'(T)}{g_Q(T-1)g_S(T)} = B(T-1)A(T)$$
which shows that $C(T) := A(T)/B(T) \in k(T)$ satisfies $C(T) = C(T-1)$, and hence
$$C(T) = C(T-1) = C(T-2) = C(T-3) = \cdots .$$
Since k has characteristic 0, the rational function $C(T) = A(T)/B(T)$ must be a constant, which equals 1 since both A and B are monic. Hence, $A = B$ from which we deduce
$$Q = \frac{Q}{g_Q} g_Q = A.g_Q = B.g_Q = \frac{Q'}{g_Q} Q = Q'$$
$$S = \frac{S}{g_S} g_S = B.g_S = A.g_S = \frac{S'}{g_S} g_S = S'.$$
as desired.

(iii) This is [Wu, Lemma 3.15]. □

Let D be a (φ, Γ_K)-module of rank n over $\mathcal{R}_{K,A}$. Fix an embedding $\sigma \in \Sigma$ and denote by $P_{\mathrm{Sen}}(T) \in \left(K \otimes_{\mathbb{Q}_p} A\right)[T]$ the Sen polynomial of D, and by $P_{\mathrm{Sen},\sigma}(T)$ the σ-Sen polynomial, i.e., the σ-component of $P_{\mathrm{Sen}}(T)$ via $(K \otimes_{\mathbb{Q}_p} A)[T] \simeq \prod_{\sigma\in\Sigma} A[T]$.

Theorem 4.10 ([Wu, Proposition 3.16])**.** *Suppose that the σ-Sen polynomial $P_{Sen,\sigma}(T)$ of D admits a decomposition $P_{Sen,\sigma}(T) = Q(T)S(T)$ in $A[T]$ by co-maximal monic polynomials.*

(i) There exists a unique (φ, Γ_K)-module D' over $\mathcal{R}_{K,A}$ contained in D and containing $t_\sigma D$ such that the Sen polynomial of D' is equal to
$$Q(T-1)S(T) \prod_{\tau\neq\sigma} P_{\mathrm{Sen},\tau}(T) \in \prod_{\sigma\in\Sigma} A[T].$$

(ii) There exists a unique (φ, Γ_K)-module D'' over $\mathcal{R}_{K,A}$ contained in $t_\sigma^{-1}D$ and containing D such that the Sen polynomial of D'' is equal to
$$Q(T+1)S(T) \prod_{\tau\neq\sigma} P_{\mathrm{Sen},\tau}(T) \in \prod_{\sigma\in\Sigma} A[T].$$

(iii) If the image of each difference between the roots of $Q(T)$ and $S(T)$ in $\overline{k(x)}$ never belongs to $\{-1, 0, 1\}$ for all $x \in \mathrm{Sp}(A)$, then the operations in (i) and (ii) as mutual inverses.

Proof. (i) This part is [Wu, Proposition 3.16] except that it is stated there that the submodule D' is containing tD, instead of $t_\sigma D$. Let us sketch Wu's construction and the proof of its uniqueness, and then indicate why his proof shows that $t_\sigma D \subset D' \subset D$, which is *a priori* stronger than $tD \subset D' \subset D$.

By Beauville-Laszlo gluing [Wu, Proposition A.3], it suffices to look at the σ-component $D^+_{\mathrm{dif},\sigma}(D)$ of the "localization" $D^+_{\mathrm{dif}}(D) := D \otimes_{\mathcal{R}_{K,A}} (K_\infty \otimes_{\mathbb{Q}_p} A)[[t]]$, which is finite projective rank n over $(K_\infty \otimes_{K,\sigma} A)[[t]]$ with a semilinear Γ_K-action, and show that $D^+_{\mathrm{dif},\sigma}(D)$ contains a unique Γ_K-stable projective submodule M containing $tD^+_{\mathrm{dif},\sigma}(D)$ such that the Sen operator Θ_σ on $D_{\mathrm{Sen},\sigma}(M) = M/tM$ has σ-Sen polynomial $Q(T-1)S(T)$. Then, the (φ, Γ_K)-submodule D' corresponds to the Γ_K-stable lattice in $D^+_{\mathrm{dif}}(D)[1/t]$ such that

$$D^+_{\mathrm{dif},\tau}(D') = \begin{cases} D^+_{\mathrm{dif},\tau}(D) & \text{if } \tau \in \Sigma_K \setminus \{\sigma\}, \\ M & \text{if } \tau = \sigma. \end{cases}$$

This submodule M is explicitly given by

$$M := \ker(D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/t \to \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))$$

where we have the canonical decomposition as $A[T]$-module, where T acts by Θ_σ,

$$D_{\mathrm{Sen},\sigma}(D) = \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$$

thanks to the assumption that $P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T)$ and $(Q(T), S(T)) = 1$.

The uniqueness also follows from this canonical decomposition of $D_{\mathrm{Sen},\sigma}(D)$. Indeed, let $R := K_\infty \otimes_{K,\sigma} A$ and let M be any Γ_K-stable projective $R[[t]]$-submodule M of $D^+_{\mathrm{dif},\sigma}(D)$ containing $tD^+_{\mathrm{dif},\sigma}(D)$ so that the Sen operator Θ_σ on $D_{\mathrm{Sen},\sigma}(M) = M/tM$ has characteristic polynomial $Q(T-1)S(T)$. Then from

$$t^2 D^+_{\mathrm{dif},\sigma}(D) \subset tM \subset tD^+_{\mathrm{dif},\sigma}(D) \subset M \subset D^+_{\mathrm{dif},\sigma}(D)$$

we obtain two exact sequences of Γ_K-stable R-modules

$$0 \to tD^+_{\mathrm{dif},\sigma}(D)/tM \to M/tM \to M/tD^+_{\mathrm{dif},\sigma}(D) \to 0 \tag{4.2.1}$$

$$0 \to M/tD^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/M \to 0 \tag{4.2.2}$$

which, as we now show, following [Zhu17, Lemma 1.1.5], consist of projective R-modules. Since M is projective over $R[[t]]$, M/tM is projective over $R = R[[t]]/tR[[t]]$. Similarly, $D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D)$ is projective over R. Since M is an $R[[t]]$-lattice in the rank n projective $R((t))$-module $D_{\mathrm{dif},\sigma}(D) := D^+_{\mathrm{dif},\sigma}(D)[1/t]$, it follows that

$$D_{\mathrm{dif},\sigma}(D)/M = \bigoplus_{k\geq 0} t^{-(k+1)}M/t^{-k}M$$

is projective over R since M/tM and so $t^{-(k+1)}M/t^{-k}M$ are projective. Similarly,

$$D_{\mathrm{dif},\sigma}(D)/D^+_{\mathrm{dif},\sigma}(D) = \bigoplus_{k\geq 0} t^{-(k+1)}D^+_{\mathrm{dif},\sigma}(D)/t^{-k}D^+_{\mathrm{dif},\sigma}(D)$$

is projective over R because $D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D)$ is projective over R. Hence, from

$$0 \to D^+_{\mathrm{dif},\sigma}(D)/M \to D_{\mathrm{dif},\sigma}(D)/D^+_{\mathrm{dif},\sigma}(D) \to D_{\mathrm{dif},\sigma}(D)/M \to 0$$

we conclude that $D^+_{\mathrm{dif},\sigma}(D)/M$ is projective. By (4.2.2), $M/tD^+_{\mathrm{dif},\sigma}(D)$ is projective. For the sequences (4.2.1) and (4.2.2) of finite projective Γ_K-stable R-modules, we look at the derivative of Γ_K-action, i.e., the Sen operator Θ_σ, to get the factorizations

$$P_{\mathrm{Sen},M/tM}(T) = P_{\mathrm{Sen},tD^+_{\mathrm{dif},\sigma}(D)/tM}(T) \cdot P_{\mathrm{Sen},M/tD^+_{\mathrm{dif},\sigma}(D)}(T)$$
$$P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D)}(T) = P_{\mathrm{Sen},M/tD^+_{\mathrm{dif},\sigma}(D)}(T) \cdot P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/M}(T)$$

Since the derivative of Γ_K on $D^+_{\mathrm{dif},\sigma}(D)$ is a derivation $\nabla : D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)$ over the derivation $t\frac{\partial}{\partial t}$ on the coefficient $R[[t]]$, cf. [Fon04, §3.4], it follows that

$$\nabla(ty) = ty + t\nabla y$$

for any $y \in D^+_{\mathrm{dif},\sigma}(D)$, from which we see

$$P_{\mathrm{Sen},tD^+_{\mathrm{dif},\sigma}(D)/tM}(T) = P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/M}(T-1).$$

Thus $Q'(T) := P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/M}(T)$, $S'(T) := P_{\mathrm{Sen},M/tD^+_{\mathrm{dif},\sigma}(D)}(T)$ are monic polynomials in $A[T]$ satisfying the equalities

$$Q'(T)S'(T) = P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T),$$
$$Q'(T-1)S'(T) = P_{\mathrm{Sen},M/tM}(T) = Q(T-1)S(T).$$

For each $x \in \mathrm{Sp}(A)$, we can specialize to the polynomial ring $k(x)[T]$ over the residue field $k(x)$ and apply Lemma 4.9(ii) to deduce that $Q_x(T) = Q'_x(T)$ and $S_x(T) = S'_x(T)$. Hence, $\deg(Q') = \deg(Q)$ and $\deg(S) = \deg(S')$, and by Lemma 4.9(iii), we deduce comaximality $(Q,S) = (Q',S) = (Q,S') = (Q',S') = A[T]$, and that $Q'(T) = Q(T)$ and $S'(T) = S(T)$ in $A[T]$ by Lemma 4.9(i). By Chinese remainder theorem, we have a natural isomorphism

$$A[T] \xrightarrow[\sim]{f} A[T]/Q(T) \times A[T]/S(T)$$

and we let $e_Q = f^{-1}(1,0)$ and $e_S = f^{-1}(0,1)$ be the idempotents in $A[T]$. Then, for any R-linear section $s' : D^+_{\mathrm{dif},\sigma}(D)/M \to D^+_{\mathrm{dif},\sigma}(D)/t$ that splits (4.2.2) as R-modules, we get an $R[\Theta_\sigma]$-linear section $s := e_Q(\Theta_\sigma) \circ s'$ that splits (4.2.2) as Sen-modules by the choice of e_Q and the fact that $Q' = Q$ and $S = S'$. This must be the canonical decomposition

$$D_{\mathrm{Sen},\sigma}(D) = \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)).$$

So, $M/tD^+_{\mathrm{dif},\sigma}(D) = \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$ and

$$M = \ker(D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/t \to \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))),$$

which proves the uniqueness of such Γ_K-stable projective submodule.

By construction, $tD^+_{\mathrm{dif}}(D) \subset D^+_{\mathrm{dif}}(D') \subset D^+_{\mathrm{dif}}(D)$. Hence, we know that $tD \subset D' \subset D$ by Beauville-Laszlo. To show that $t_\sigma D \subset D' \subset D$, it suffices to note[3] that

- t_σ acts as a unit on the components $D^+_{\mathrm{dif},\tau}(D)$ for $\tau \neq \sigma$, and

[3] A quick way to see this is to apply [Wu, Proposition 3.16] to $\mathcal{R}_{K,E}$, which yields $t_\sigma\mathcal{R}_{K,E}$ as it has the correct Sen polynomial and satisfies $t\mathcal{R}_{K,E} \subset t_\sigma\mathcal{R}_{K,E} \subset \mathcal{R}_{K,E}$. Comparing $D^+_{\mathrm{dif},\bullet}$ yields that

$$t_\sigma(K_\infty \otimes_{K,\tau} E[[t]]) = \begin{cases} K_\infty \otimes_{K,\tau} E[[t]] & \text{if } \tau \neq \sigma, \\ t(K_\infty \otimes_{K,\tau} E[[t]]) & \text{if } \tau = \sigma. \end{cases}$$

which implies the desired result.

- t_σ acts as t on the σ-component $D^+_{\mathrm{dif},\sigma}(D)$.

This means that in the construction of $M \subset D^+_{\mathrm{dif},\sigma}(D)$ above and the proof of uniqueness of such M, we may replace t by t_σ and obtain the same result.

(ii) This is a corollary to (i), after replacing D by $t_\sigma^{-1}D$ and switching the roles played by $Q(T)$ and $S(T)$ in the statement of (i): the Sen polynomial of $t_\sigma^{-1}D$ is exactly

$$P_{\mathrm{Sen},\sigma}(T+1)\prod_{\tau\neq\sigma}P_{\mathrm{Sen},\tau}(T) = Q(T+1)S(T+1)\prod_{\tau\neq\sigma}P_{\mathrm{Sen},\tau}(T) \in \prod_{\sigma\in\Sigma}A[T]$$

The σ-component of $D^+_{\mathrm{dif}}(t_\sigma^{-1}D) = t_\sigma^{-1}D^+_{\mathrm{dif}}(D)$ contains the Γ_K-subrepresentation

$$M = \ker(D^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D) \to D^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D)/t \to \ker(S(\Theta_\sigma+1)|D_{\mathrm{Sen},\sigma}(t_\sigma^{-1}D)))$$

which is the unique Γ_K-stable projective submodule containing

$$tD^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D) = t_\sigma D^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D) = D^+_{\mathrm{dif}}(D)$$

such that Θ_σ acts on $D_{\mathrm{Sen},\sigma}(M)$ with Sen polynomial

$$Q(T+1)S(T+1-1) = Q(T+1)S(T)$$

by (i). The result then follows.

(iii) The assumption ensures $(Q(T), S(T)) = 1 = (Q(T-1), S(T)) = (Q(T+1), S(T))$. The desired conclusion follows from the uniqueness part of (i) and (ii). $\square$

Proposition 4.11. *Let $(D, \mathrm{Fil}^\bullet(D)) \in \mathfrak{X}_B(A)$ be a triangulated (φ,Γ_K)-module over $\mathcal{R}_{K,A}$ with parameters $(\delta_1,\dots,\delta_n)$. Then,*

$$P_{\mathrm{Sen},\sigma}(T) = \prod_{i=1}^{n}(T - \mathrm{wt}_\sigma(\delta_i)) \in A[T]$$

factors naturally. Suppose there exists a nonempty subset $I \subset \{1,\dots,n\}$ such that

$$Q_I(T) := \prod_{m\in I}(T-\mathrm{wt}_\sigma(\delta_m)), \quad S_I(T) := \prod_{m\in\{1,\dots,n\}\setminus I}(T-\mathrm{wt}_\sigma(\delta_m))$$

are comaximal in $A[T]$. Then, applying Theorem 4.10(i) (resp. Theorem 4.10(ii)) to

$$P_{\mathrm{Sen},\sigma}(T) = Q_I(T)S_I(T)$$

produces a (φ,Γ_K)-module $D' \subset D$ (resp. $D'' \supset D$) that is again trianguline.

Proof. Passing to a finite admissible cover $\{\mathrm{Sp}(A_l)\}_{l=1}^r$ of $\mathrm{Sp}(A)$, we assume that $\mathcal{L}_i$ are trivial. For rank 1 (φ,Γ_K)-module $R_{K,A}(\delta)$, $D^+_{\mathrm{dif}}(\delta)$ and $D_{\mathrm{Sen}}(\delta) = D^+_{\mathrm{dif}}(\delta)/t$ are free of rank 1. By the functoriality of D^+_{dif}, $D^+_{\mathrm{dif}}(D)$ is free of rank n over $(K_\infty\otimes_{\mathbb{Q}_p}A)[[t]]$. The triangulation $\mathrm{Fil}^\bullet(D)$ induces a basis $\{e_1,\dots,e_n\}$ of $D^+_{\mathrm{dif}}(D)$ with $\mathrm{Span}(e_1,\dots,e_m) = D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D))$ for $1\le m\le n$.

Recall the canonical decomposition

$$D_{\mathrm{Sen},\sigma}(D) = \ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)).$$

We claim that one may modify the basis such that $\mathrm{Span}(e_1,\dots,e_m) = D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D))$ for all $1\le m\le n$, and such that, after modulo t_σ, the set $\{\overline{e_m}|m\in I\}$ spans $\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$ while $\{\overline{e_m}|m\notin I\}$ spans $\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$. Indeed, e_1 reduces to an eigenvector of Θ_σ with eigenvalue being a root of, say Q_I. Then, inside the rank 2 module

$$\mathrm{Span}\{\overline{e_1},\overline{e_2}\} = D_{\mathrm{Sen},\sigma}(\mathrm{Fil}^2(D)) = \ker(Q_I(\Theta_\sigma)|\,\mathrm{Fil}^2(D)) \oplus \ker(S_I(\Theta_\sigma)|\,\mathrm{Fil}^2(D))$$

we either have $\overline{e_2} \in \ker(Q_I(\Theta_\sigma))$ which is good, or $\overline{e_2} \in \ker(Q_I(\Theta_\sigma))$ spans $\ker(S_I(\Theta_\sigma)|\operatorname{Fil}^2(D))$ modulo $\operatorname{Span}(\overline{e_1})$, in which case after subtracting a multiple of e_1 from e_2 we may assume that $\operatorname{Span}(\overline{e_2}) = \ker(S_I(\Theta_\sigma)|\operatorname{Fil}^2(D))$. Having modified e_1, e_2, we consider Θ_σ restricted on the rank 3 Sen module $D_{\mathrm{Sen},\sigma}(\operatorname{Fil}^3(D))$, and after modifying e_3 by $\operatorname{Span}\{e_1, e_2\}$, we may assume that the reduction $\overline{e_3}$ belongs to either $\ker(Q_I(\Theta_\sigma)|\operatorname{Fil}^3(D))$ or $\ker(S_I(\Theta_\sigma)|\operatorname{Fil}^3(D))$. Continuing until we reach $\operatorname{Fil}^n(D) = D$, we finish the modification of the basis.

By the construction of $D'' \supset D$, it suffices to consider the submodule $D' \subset D$. By Wu's construction given in the proof of Theorem 4.10, the submodule D' has $D^+_{\mathrm{dif},\tau}(D') = D^+_{\mathrm{dif},\tau}(D')$ for all $\tau \in \Sigma_K \setminus \{\sigma\}$, and using our modified basis, it is easy to see that

$$D^+_{\mathrm{dif},\sigma}(D') = \ker(D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/t \to \ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))) = \operatorname{Span}\{e'_1, \dots, e'_n\},$$

where $e'_m = t_\sigma e_m$ if $m \in I$ and $e'_m = e_m$ if $m \notin I$. Since $\operatorname{Span}(e_1, \dots, e_m) = D^+_{\mathrm{dif}}(\operatorname{Fil}^m(D))$ are Γ_K-stable for $1 \le m \le n$, the only way for $D^+_{\mathrm{dif},\sigma}(D')$ to be Γ_K-stable is that $\operatorname{Span}(e'_1, \dots, e'_m)$ are Γ_K-stable for $1 \le m \le n$. Using the triangulation $\operatorname{Fil}^\bullet(D)$ on D, Beauville-Laszlo gluing [Wu, Proposition A.3] implies that the Γ_K-stable flag on $D^+_{\mathrm{dif}}(D')$

$$\{\operatorname{Span}(e'_1, \dots, e'_m) | 1 \le m \le n\}$$

determines a triangulation $\operatorname{Fil}^\bullet(D')$ on D' such that $D^+_{\mathrm{dif}}(\operatorname{Fil}^m(D')) = \operatorname{Span}(e'_1, \dots, e'_m)$ and

$$\operatorname{Fil}^m(D')/\operatorname{Fil}^{m-1}(D') \cong \begin{cases} \mathcal{R}_{K,A}(x_\sigma \delta_m) & \text{if } m \in I, \\ \mathcal{R}_{K,A}(\delta_m) & \text{if } m \notin I, \end{cases}$$

for all $1 \le m \le n$. $\square$

Theorem 4.12. *(i) Let $D \in \mathfrak{X}^{\sigma\text{-wu},i}_n(A)$. Choose any triangulation $\operatorname{Fil}^\bullet(D)$ on D over $\mathcal{R}_{K,A}$ with parameters $\delta = (\delta_1, \dots, \delta_n)$ so that*

$$\operatorname{Fil}^i(D)/\operatorname{Fil}^{i-1}(D) \cong \mathcal{R}_{K,A}(\delta_i) \otimes_{\mathcal{O}_{\mathrm{Sp}(A)}} \mathcal{L}_i$$

for some line bundles $\mathcal{L}_i$ on $\operatorname{Sp}(A)$. Then, $p_{i,\sigma}(D, \operatorname{Fil}^\bullet(D))$ equals the unique submodule D' of D defined in Theorem 4.10(i) and is independent of the choice of $\operatorname{Fil}^\bullet(D)$.

(ii) Denote the operation of passing to a (φ, Γ_K)-supermodule in Theorem 4.10(ii) by $q_{i,\sigma}$. Then for $D \in \mathfrak{X}^{\sigma\text{-wu},i,1}(A) = \mathfrak{X}^{\sigma\text{-wu},i,[0,1]}(A)$, we have $p_{i,\sigma}(D) \in \mathfrak{X}^{\sigma\text{-wu},i,-1}_n(A) = \mathfrak{X}^{\sigma\text{-wu},i,[-1,0]}(A)$.

(iii) For any subset $S \subset \Sigma_K$, and subsets $I_\sigma \subset \{1, \dots, n\}$ for $\sigma \in S$, and $\mathbf{k} = (k_{i,\sigma})_{\sigma \in S, i \in I_\sigma} \in \mathbb{N}^{\sum_{\sigma \in S} |I_\sigma|}$ any tuple of nonnegative integers, there is an isomorphism

$$p_{\mathbf{k}} : \mathfrak{X}^{S\text{-wu},I,\mathbf{k}}_n \longrightarrow \mathfrak{X}^{S\text{-wu},I,-\mathbf{k}}_n$$

given by the composite of elements from $\{p_{i,\sigma} | 1 \le i \le n, \sigma \in \Sigma_K\}$ in any order in which $p_{i,\sigma}$ appears with multiplicity $k_{i,\sigma}$, whose inverse is given by

$$q_{\mathbf{k}} : \mathfrak{X}^{S\text{-wu},I,-\mathbf{k}}_n \longrightarrow \mathfrak{X}^{S\text{-wu},I,\mathbf{k}}_n$$

that is defined as the composite of elements from $\{q_{i,\sigma} | 1 \le i \le n, \sigma \in \Sigma_K\}$ in any order in which $q_{i,\sigma}$ appears with multiplicity $k_{i,\sigma}$.

Proof. (i) Passing to a finite admissible cover $\{\operatorname{Sp}(A_l)\}_{l=1}^r$ of $\operatorname{Sp}(A)$, we assume the $\mathcal{L}_i$ are trivial. By the definition of U_i, we have that for all $x \in \operatorname{Sp}(A)$,

$$\{\mathrm{wt}_\sigma(\delta_{1,x}), \dots, \mathrm{wt}_\sigma(\delta_{n-i,x})\} \cap \{\mathrm{wt}_\sigma(\delta_{n-i+1,x}), \dots, \mathrm{wt}_\sigma(\delta_{n,x})\} = \emptyset.$$

inside the residue field $k(x)$ for all $x \in \mathrm{Sp}(A)$. Thus, for $Q_{i,\sigma}(T) := \prod_{j=n-i+1}^{n}(T - \mathrm{wt}_\sigma(\delta_j))$ and $S_{i,\sigma}(T) := \prod_{j=1}^{n-i}(T - \mathrm{wt}_\sigma(\delta_j))$, we have $(Q_{i,\sigma}(T), S_{i,\sigma}(T)) = (1)$ and a factorization

$$P_{\mathrm{Sen},\sigma}(T) = Q_{i,\sigma}(T)S_{i,\sigma}(T) \in A[T],$$

which is independent of the choice of $\mathrm{Fil}^\bullet(D)$: indeed, if $\{\delta'_m\}_{m=1}^n$ is the parameters attached to another triangulation of D over $\mathcal{R}_{K,A}$, then we get another co-maximal factorization

$$P_{\mathrm{Sen},\sigma}(T) = Q'_{i,\sigma}(T)S'_{i,\sigma}(T) \in A[T]$$

where we similarly put $Q'_{i,\sigma}(T) := \prod_{j=n-i+1}^{n}(T - \mathrm{wt}_\sigma(\delta'_j))$ and $S'_{i,\sigma}(T) := \prod_{j=1}^{n-i}(T - \mathrm{wt}_\sigma(\delta'_j)))$. Moreover, we have $(Q'_{i,\sigma}(T), S_{i,\sigma}(T)) = (Q_{i,\sigma}(T), S'_{i,\sigma}(T)) = 1$ by Lemma 4.9(iii). We conclude that $Q_{i,\sigma}(T) = Q'_{i,\sigma}(T)$ and $S_{i,\sigma}(T) = S'_{i,\sigma}(T)$ by Lemma 4.9(i).

Let $\{e_1, \dots, e_{n-i}, e_{n-i+1}, \dots, e_n\}$ be a basis of $D^+_{\mathrm{dif}}(D)$ as in the proof of Proposition 4.11 with

$$I := \{k | n - i + 1 \le k \le n\} \subset \{1, \dots, n\}$$

such that for the splitting of $D_{\mathrm{Sen},\sigma}(D)$ as

$$\ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \cong D_{\mathrm{Sen},\sigma}(D/\,\mathrm{Fil}^{n-i}(D)) \oplus D_{\mathrm{Sen},\sigma}(\mathrm{Fil}^{n-i}(D))$$

according to the derivative Θ_σ of the Γ_K-action, the mod-t_σ reduction $\{\overline{e_1}, \dots, \overline{e_{n-i}}\}$ is a basis of $\ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$ and $\{\overline{e_{n-i+1}}, \dots, \overline{e_n}\}$ is a basis of $\ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$.

From the proof of Proposition 4.11, we see that the submodule N of basis

$$\{e_1, \dots, e_{n-i}, t_\sigma e_{n-i+1}, \dots, t_\sigma e_n\}$$

in $D^+_{\mathrm{dif}}(D)$ corresponds to the module M given by Wu in Theorem 4.10(i). Thus $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))$ corresponds to M since the localization of $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))$ clearly has the same basis

$$\{e_1, \dots, e_{n-i}, t_\sigma e_{n-i+1}, \dots, t_\sigma e_n\}$$

by the definition of pullback, and (i) is proven.

(ii) Keep the notations in the proof of (i), in particular the basis $e_1, \dots, e_n$. By (i), $q_{i,\sigma}(D)$ can be computed using any triangulation $\mathrm{Fil}^\bullet(D)$ on D with respect to which our basis is chosen, because all triangulation induce the same factorization of σ-Sen polynomial. We know that $q_{i,\sigma}(D)$ is trianguline by Proposition 4.11. Let $h_1, \dots, h_n$ be the ordered σ-Sen weights of D with respect to any triangulation $\mathrm{Fil}^\bullet(D)$. Then,

$$h'_m = \begin{cases} h_m & \text{if } 1 \le m \le n - i, \\ h_m + 1 & \text{if } n - i + 1 \le m \le n. \end{cases}$$

are the ordered σ-Sen weights of an induced triangulation on the pullback $D' := p_{i,\sigma}(D)$. Since we assume $D \in \mathfrak{X}_n^{\sigma\text{-wu},i,1}(A) = \mathfrak{X}_n^{\sigma\text{-wu},i,[0,1]}(A)$, we have

$$h'_j \not\equiv h'_k \pmod{\mathfrak{m}_x}, \quad h'_j \not\equiv h'_k - 1 \pmod{\mathfrak{m}_x}$$

for all $x \in \mathrm{Sp}(A)$ with the corresponding maximal ideal $\mathfrak{m}_x$ and for all $1 \le j \le n - i < k \le n$. In particular, we can apply $q_{i,\sigma}$ to D'. It remains to show that D' is σ-weight-uniform. Assume, for the sake of contradiction, that there exists another triangulation on D' such that at some point $x \in \mathrm{Sp}(A)$, the induced ordering on σ-Sen weights is

$$(h'_{w(1),x}, \dots, h'_{w(n),x})$$

for some $w \in S_n$ such that for some $1 \le m_0 \le n$, $h'_{w(m_0),x} \neq h'_{m_0,x} \in k(x)$. By Proposition 4.11, this other triangulation on D' induces on $q_{i,\sigma}(D') = q_{i,\sigma}(p_{i,\sigma}(D)) = D$ a triangulation whose ordered σ-Sen weights are $(h''_1, \dots, h''_n)$ with

$$h''_m = \begin{cases} h'_{w(m)} - 1 = h_{w(m)} & \text{if } w(m) \in I := \{k | n-i+1 \le k \le n\}, \\ h'_{w(m)} = h_{w(m)} & \text{if } w(m) \notin I. \end{cases}$$

<u>Case (a)</u>. If $\{m_0, w(m_0)\}$ is contained in either I or I^c, then $h'_{w(m_0),x} \neq h'_{m_0,x} \in k(x)$ implies

$$h''_{m_0,x} = h_{w(m_0),x} \neq h_{m_0,x} \in k(x)$$

contrary to the assumption that D is σ-weight-uniform.
<u>Case (b)</u>. Otherwise, either $m_0 \in I$ but $w(m_0) \notin I$, or $m_0 \notin I$ but $w(m_0) \in I$. Then we have

$$h''_{m_0,x} = h_{w(m_0),x} \neq h_{m_0,x} \in k(x)$$

because $D \in \mathfrak{X}_n^{\sigma\text{-wu},i}(A)$, contrary to the assumption that D is σ-weight-uniform.

Hence, we conclude that $p_{i,\sigma} : \mathfrak{X}_n^{\sigma\text{-wu},i,1} \to \mathfrak{X}_n^{\sigma\text{-wu},i,-1}$ preserves σ-weight-uniformity.

(iii) By Lemma 4.2, the independence (i) of triangulations for the pullback interpretation of $p_{i,\sigma}$ under the given assumption on weights, and (ii) that $p_{i,\sigma}$ preserves weight-uniformity under the given assumption on weights, it follows that any ways of composing $p_{i,\sigma}$ with multiplicity $k_{i,\sigma}$ as i varies in $\{1, \dots n\}$ and σ varies in Σ_K produce the same result, which is the morphism $p_{\mathbf{k}}$.

Since $q_{i,\sigma}$ is the inverse of $p_{i,\sigma}$ by Theorem 4.10(iii), it follows that we can define $q_{\mathbf{k}}$ by composing $q_{i,\sigma}$'s with multiplicity $k_{i,\sigma}$ in any order. It is clear that $p_{\mathbf{k}}$ and $q_{\mathbf{k}}$ are mutual inverses. □

4.2.1. To summarize, we have $p_{i,\sigma} : \mathfrak{X}_B \to \mathfrak{X}_B$, which is often invertible by Theorem 3.1. By Theorem 4.12, this induces a "change of weights" morphism of stacks

$$p_{i,\sigma} : \mathfrak{X}_n^{\sigma\text{-wu},i} \to \mathfrak{X}_n$$

which, if the "regularity" of the Sen weights is unchanged after the weight change, is invertible:

$$p_{i,\sigma} : \mathfrak{X}_n^{\sigma\text{-wu},i,1} \xrightarrow{\sim} \mathfrak{X}_n^{\sigma\text{-wu},i,-1}$$

by Theorem 4.12(iii). Moreover, we know by Lemma 4.2 that for any $1 \le i, j \le n$ and $\tau, \sigma \in \Sigma_K$,

$$(p_{j,\tau} \circ p_{i,\sigma})(D) = (p_{i,\sigma} \circ p_{j,\tau})(D)$$

whenever the pullbacks are independent of the choice of triangulation.

$$p_{\mathbf{k}} : \mathfrak{X}_n^{\mathrm{wu},\mathbf{k}} \xrightarrow{\sim} \mathfrak{X}_n^{\mathrm{wu},-\mathbf{k}}$$

as long as the "regularity" of the Sen weights is unchanged after the change. See [Wu, Example 3.18] for the general (non-invertible) case of changing the Sen weights on $\mathfrak{X}_n$ in a similar way.

One family of "nice" D are those non-critical crystabelline (φ, Γ_K)-modules $D \in \mathfrak{X}_n^{\mathrm{wu}}(A)$ of regular Sen weights. When $A = E$ is a finite extension of $\mathbb{Q}_p$, we verify in the last section that $p_{i,\sigma}$ on D corresponds to the translation functor on $\pi_{\mathrm{fs}}(D)$, for Ding's construction $\pi_{\mathrm{fs}}(D)$.

4.3. **Étaleness.** According to Jean-Marc Fontaine and others, there is an equivalence of categories D_{rig} between continuous E-linear $\mathrm{Gal}(\overline{K}|K)$-representations and étale (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$. We study in this subsection how the pullback operations interplay with étaleness on certain very generic or non-critical crystabelline (φ, Γ_K)-modules.

In this subsection, we write $\mathcal{R} := \mathcal{R}_{K,E}$; by [BC09, Lemma 2.2.5], this discussion also applies to A-linear Galois representations and (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$ for $A \in \mathcal{C}_E$ any commutative local Artinian E-algebra with residue field E.

Let us recall Kedlaya's theory of slopes. For a (φ, Γ_K)-module D of rank n over $\mathcal{R}$, we define its **degree** by $\deg(D) := \deg(\wedge^n D)$, which is the p-adic valuation of a "φ-eigenvalue" on $\wedge^n D$. Then, the **slope** of D is defined by $\mu(D) := \deg(D)/\mathrm{rank}(D)$. We say that D is **semistable** if for any finite free φ-submodule M of D satisfying $\varphi^* M \cong M$, one has $\mu(M) \geq \mu(D)$. Then, D is called **étale** if it is semistable of slope zero, cf. [Liu08, p.8].

Remark 4.13. The following properties clearly hold for (φ, Γ)-modules.

(i) If $0 \to D' \to D \to D'' \to 0$ is exact, then $\deg(D) = \deg(D') + \deg(D'')$.
(ii) One has $\mu(D_1 \otimes D_2) = \mu(D_1) + \mu(D_2)$.
(iii) We have $\deg(D^\vee) = -\deg(D)$ and $\mu(D^\vee) = -\mu(D)$.

Example 4.14. By [KPX14, Construction 6.2.4], for any continuous character $\delta : K^\times \to E^\times$, if we factorize $\delta = \delta^{\mathrm{unr}} \delta^{\mathrm{wt}}$ such that $\delta^{\mathrm{unr}}(\pi_K) = \delta(\pi_K)$ and $\delta^{\mathrm{wt}}|_{\mathcal{O}_K^\times} = \delta|_{\mathcal{O}_K^\times}$, then

$$\mathcal{R}(\delta) = \mathcal{R}(\delta^{\mathrm{unr}}) \otimes \mathcal{R}(\delta^{\mathrm{wt}}).$$

Since $\varphi^f = \delta(\pi_K)$ on $\mathcal{R}(\delta^{\mathrm{unr}}) = D_{f,\delta(\pi_K)} \otimes_{K_0 \otimes E} \mathcal{R}$, and $\mathcal{R}(\delta^{\mathrm{wt}})$ is étale, we get by Remark 4.13(ii)

$$\mu(\mathcal{R}(\delta)) = \deg(\mathcal{R}(\delta)) = \frac{1}{f} v_p(\delta(\pi_K)).$$

The following is the slope filtration theorem by Kedlaya.

Theorem 4.15 ([Liu08, Theorem 2.4])**.** *Every (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$ admits a unique filtration $0 = D_0 \subset D_1 \subset \cdots \subset D_\ell = D$ by saturated sub-(φ, Γ)-modules whose successive quotients are semistable with increasing slopes $\mu(M_1/M_0) < \cdots < \mu(M_{\ell-1}/M_\ell)$.*

In view of Theorem 4.15 and Remark 4.13(i), we deduce

Corollary 4.16. *A (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$ is étale if and only if $\mu(D) = 0$ and it does not contain any saturated (φ, Γ_K)-submodule of strictly negative slope.*

4.3.1. Suppose $(D, \mathrm{Fil}^\bullet(D))$ is strongly non-split of parameters in $\mathcal{T}_\circ^n(E)$. By Corollaries 3.5 and 4.16, D is étale if and only if $\mathrm{Fil}^i(D)$ has non-negative slopes and D has slope zero. So,

Proposition 4.17. *(i) If $(D, \mathrm{Fil}^\bullet(D))$ is strongly non-split of parameter $(\delta_1, \ldots, \delta_n) \in \mathcal{T}_\circ^n(E)$, then D is étale if and only if $\sum_{i=1}^m v_p(\delta_i(\pi_K)) \geq 0$ for $1 \leq m < n$ and $\sum_{i=1}^n v_p(\delta_i(\pi_K)) = 0$.*
(ii) If $(D, \mathrm{Fil}^\bullet(D))$ is strongly non-split of parameter $(\delta_1, \ldots, \delta_n) \in \mathcal{T}_\circ^n(E)$ and étale, then $p_{j,\sigma}(D)$ is étale up to twist if and only if

$$\begin{cases} \displaystyle\sum_{i=1}^m v_p(\delta_i(\pi_K)) \geq \frac{mj}{ne} & 1 \leq m \leq n-j; \\ \displaystyle\sum_{i=1}^m v_p(\delta_i(\pi_K)) + \frac{m-(n-j)}{e} \geq \frac{mj}{ne} & n-j < m < n. \end{cases}$$

Proof. (i) By Remark 4.13(i) and the definition of slope,

$$\mu(\mathrm{Fil}^i(D)) = \deg(\mathrm{Fil}^i(D))/\mathrm{rank}(\mathrm{Fil}^i(D)) = \frac{1}{i}\sum_{j=1}^{i}\deg(\mathcal{R}(\delta_j))$$

By Example 4.14, we see that $\mu(\mathrm{Fil}^i(D)) = \dfrac{1}{fi}\displaystyle\sum_{j=1}^{i} v_p(\delta_j(\pi_K))$, for all i.

(ii) The new parameter of $p_{j,\sigma}(D)$ is $(\delta_1, \ldots, \delta_{n-j}, x_\sigma\delta_{n-j+1}, x_\sigma\delta_{n-j+2}, \ldots, x_\sigma\delta_n)$, and

$$v_p(x_\sigma(\pi_K)) = v_p(\sigma(\pi_K)) = v_p(\pi_K) = 1/e.$$

Let $\chi : K^\times \to E^\times$ be any continuous character. The twist $p_{j,\sigma}(D)(\chi)$ is again non-split and very generic of parameter $(\delta_1\chi, \ldots, \delta_{n-j}\chi, \ldots x_\sigma\delta_{n-j+1}\chi, x_\sigma\delta_{n-j+2}\chi, \ldots, x_\sigma\delta_n\chi)$, which is étale if and only if we have

$$\begin{cases} \sum_{i=1}^{m} v_p(\delta_i(\pi_K)) + m v_p(\chi(\pi_K)) \geq 0 & \text{if } 1 \leq m \leq n-j \\ \sum_{i=1}^{j} v_p(\delta_i(\pi_K)) + (m-(n-j))v_p(\pi_K) + m v_p(\chi(\pi_K)) \geq 0 & \text{if } n-j < m \leq n-1 \\ \sum_{i=1}^{n} v_p(\delta_i(\pi_K)) + j v_p(\pi_K) + n v_p(\chi(\pi_K)) = 0 & \text{if } m = n \end{cases}$$

by (i) above. Since D is étale, $\cancel{\sum_{i=1}^{n} v_p(\delta_i(\pi_K))} + j v_p(\pi_K) + n v_p(\chi(\pi_K)) = 0$ implies $v_p(\chi(\pi_K)) = -j/(ne)$, which completes the proof, by taking an unramified χ. ☐

4.3.2. In this paragraph, we consider the étaleness for $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. By Corollary 4.16 and Theorem 3.9, all saturated (φ, Γ_K)-submodules of D are $\mathrm{Fil}^i_w(D) = D_{w,i}$ as in Proposition 3.12 for $0 \leq i \leq n$ and $w \in S_n$. Let $\{\alpha_i := \phi_i(\pi_K)\}_{1\leq i\leq n}$ be the φ^f-eigenvalues on $D^{K_m}_{\mathrm{cris}}(D)$. Then, by Proposition 3.12(ii) and Example 4.14, we have

$$v_p(\delta_{w,i}(\pi_K)) = v_p\left(\prod_\sigma \sigma(\pi_K)^{h_{i,\sigma}}\phi_{w(i)}(\pi_K)\right) = v_p(\alpha_{w(i)}) + \sum_\sigma h_{i,\sigma}\frac{1}{e}. \tag{4.3.1}$$

Proposition 4.18. *Suppose $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. Let $\tau \in S_n$ be a refinement such that*

$$v_p(\alpha_{\tau(1)}) \leq v_p(\alpha_{\tau(2)}) \leq \cdots \leq v_p(\alpha_{\tau(n)}).$$

Then, D is étale if and only if

$$\begin{cases} \displaystyle\sum_{i=1}^{j} v_p(\alpha_{\tau(i)}) \geq -\frac{1}{e}\sum_\sigma\sum_{i=1}^{j} h_{i,\sigma} & \text{if } 1 \leq j < n, \\ \displaystyle\sum_{i=1}^{n} v_p(\alpha_i) = -\frac{1}{e}\sum_{\sigma,i} h_{i,\sigma} & \text{if } j = n. \end{cases}$$

Proof. It follows immediately from Corollary 4.16, our choice of τ, and Equation (4.3.1). ☐

Remark 4.19. Alternatively, to prove Proposition 4.18 we can observe that, by our choice of τ, the n inequalities amount to having that the Hodge filtration on $D_{\mathrm{cris}}(D)$ is weakly admissible in the sense of [Fon94, Definiton 4.4.3], or concretely [BS07, Equations (4) and (5)]. Indeed, by [BM02, Proposition 3.1.1.5] and [Fon94, Proposition 4.4.9], it suffices to check $t_H(D') \geq t_N(D')$ for all E-*filtered* $(\varphi, G(K_m/K))$-submodules over K of $D_{\mathrm{cris}}(D)$ with the induced filtration. But these D' are in bijection with the $n!$ refinements, so their Newton numbers are clear, and their

Hodge numbers are clear by the non-criticalness assumption. A computation similar to that for [BS07, Proposition 3.2, $(i) \Rightarrow (ii)$] gives the n inequalities.

Corollary 4.20. *Suppose $D \in \mathfrak{X}_n(E)$ satisfies either the hypothesis of Proposition 4.17 or that of Proposition 4.18 and is étale of regular Sen weights. Suppose that $p_{i,\sigma}(D)$ again satisfies the hypothesis of Proposition 4.17 or that of Proposition 4.18 and is étale after twist by a character $\chi : K^\times \to E^\times$ (whether this is possible can be checked by Proposition 4.17 or 4.18).*

Then, for any local Artinian E-algebra $A \in \mathcal{C}_E$ of residue field E and for any deformation D_A of D to an element in $\mathfrak{X}_n(A)$, $p_{i,\sigma}(D_A)$ is again étale after twisting by the same character χ.

Proof. First of all, by $p_{i,\sigma}(D_A)$ we mean the result of applying Wu's construction (Theorem 4.10) to D_A, which is applicable and deforms $p_{i,\sigma}(D)$ from $\mathcal{R}_{K,E}$ to $\mathcal{R}_{K,A}$ since at the unique closed point $x \in \mathrm{Sp}(A)$, $D_{A,x} = D$ is assumed to have regular Sen weights.

The étaleness of $p_{i,\sigma}(D_A)(\chi)$ follows from the assumption that $p_{i,\sigma}(D)(\chi)$ is étale and the fact that extensions of pure of slope s φ-modules are pure of slope s by [BC09, Lemma 2.2.5]. ☐

5. Relation with translation functors

For $K = \mathbb{Q}_p$ and $\mathbf{h} = (h_1 > h_2 > \cdots > h_n)$, we verify that those change of weights maps

$$f_{\mathbf{h},\mathbf{h}'} : \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h}) \to \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h}')$$

between regular Sen weights that are realizable by a sequence of pullback operators $p_{\mathbf{k}} = \prod_i p_i^{k_i}$ corresponds to translation functors on $\pi_{\min}$ and π_{fs} constructed by Ding in [Din25, (3.44)]. Let us fix $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ and write $p_{\mathbf{k}}(D) = f_{\mathbf{h},\mathbf{h}'}(D) \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h}')$.

5.1. **Translation of (φ, Γ)-modules.** Recall that the Baer sum "$D_1 + D_2$" of two extensions of (φ, Γ)-modules over $\mathcal{R}_{\mathbb{Q}_p,E}$

$$0 \to A \to D_1 \to B \to 0, \quad 0 \to A \to D_2 \to B \to 0$$

is calculated by taking the direct sum

$$0 \to A \oplus A \to D_1 \oplus D_2 \to B \oplus B \to 0,$$

pushing out via $A \oplus A \xrightarrow{\text{sum}} A$, and then pulling back via the diagonal $B \xrightarrow{\Delta} B \oplus B$:

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\text{sum}"] & D_1 \oplus D_2 \arrow[r] \arrow[d] & B \oplus B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \sqcup_{A \oplus A} (D_1 \oplus D_2) \arrow[r] & B \oplus B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & D_1 + D_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd} \tag{5.1.1}$$

Moreover, if there are further extensions

$$0 \to A' \to A \to A'' \to 0, \quad 0 \to B' \to B \to B'' \to 0, \quad 0 \to D_i' \to D_i \to D_i'' \to 0$$

for $i = 1, 2$ inducing 9-term commutative diagrams with exact columns and rows:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & D_i' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & D_i \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & D_i'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{5.1.2}
$$

then we naturally have a sequence

$$0 \to (D_1' + D_2') \to (D_1 + D_2) \to (D_1'' + D_2'') \to 0 \tag{5.1.3}$$

which is exact since the columns in (5.1.2) are split as $\mathcal{R}$-modules (because each term is free over $\mathcal{R}$), so the pushout and pullback used in defining the Baer sum (5.1.1) preserve exact sequences.

The extension group $\mathrm{Ext}^1_w(D, D)$ is the set of *trianguline deformations* of D with respect to the refinement $w(\phi)$ from E to $E[\varepsilon]/\varepsilon^2$. A deformation $\widetilde{D} \in \mathrm{Ext}^1(D, D)$, viewed as a (φ, Γ)-module over $\mathcal{R}_{\mathbb{Q}_p, E[\varepsilon]/\varepsilon^2}$, belongs to $\mathrm{Ext}^1_w(D, D)$ if and only if it has a filtration $\mathrm{Fil}^\bullet(\widetilde{D})$ such that $\mathrm{Fil}^i(\widetilde{D}) \in \mathrm{Ext}^1(\mathrm{Fil}^i_w(D), \mathrm{Fil}^i_w(D))$ and $\mathrm{Fil}^i(\widetilde{D})/\mathrm{Fil}^{i-1}(\widetilde{D}) \in \mathrm{Ext}^1(\mathcal{R}(\delta_{w,i}), \mathcal{R}(\delta_{w,i}))$ for all i.

The exactness of (5.1.3) implies that we have a group homomorphism

$$\mathrm{Ext}^1_w(D, D) \twoheadrightarrow \prod_{i=1}^n \mathrm{Ext}^1(\mathcal{R}(\delta_{w,i}), \mathcal{R}(\delta_{w,i})), \quad \widetilde{D} \mapsto (\mathrm{Fil}^i(\widetilde{D})/\mathrm{Fil}^{i-1}(\widetilde{D}))_{1 \le i \le n}$$

which is surjective by [BC09, Proposition 2.3.10].

Recall that $\dim_E \mathrm{Ext}^1(\mathcal{R}(\delta_{w,i}), \mathcal{R}(\delta_{w,i})) = 2$. For any continuous character $\psi : \mathbb{Q}_p^\times \to E$, we can define a deformation of $\mathcal{R}(\delta_{w,i})$ as $\widetilde{\mathcal{R}(\delta_{w,i})} := \mathcal{R}(\delta_{w,i}) \oplus \varepsilon\mathcal{R}(\delta_{w,i})$ by

$$\widetilde{\gamma}(x) := \gamma(x) + \psi(\gamma)(\gamma(x))\varepsilon, \quad \widetilde{\varphi}(x) := \varphi(x) + \psi(p)(\varphi(x))\varepsilon, \tag{5.1.4}$$

for all $x \in \mathcal{R}(\delta_{w,i}) \subset \mathcal{R}(\delta_{w,i}) \oplus \varepsilon\mathcal{R}(\delta_{w,i})$. This is just the (φ, Γ)-module $\mathcal{R}_{\mathbb{Q}_p, E[\varepsilon]/\varepsilon^2}(\delta_{w,i}(1 + \psi\varepsilon))$ over $\mathcal{R}_{\mathbb{Q}_p, E[\varepsilon]/\varepsilon^2}$ associated to $\delta_{w,i}(1+\psi\varepsilon) : \mathbb{Q}_p^\times \to (E[\varepsilon]/\varepsilon^2)^\times$. Since $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mu_{p-1} \times \mathbb{Z}_p$, we have $\dim_E \mathrm{Hom}(\mathbb{Q}_p^\times, E) = 2$ and an isomorphism $\prod_{i=1}^n \mathrm{Ext}^1(\mathcal{R}(\delta_{w,i}), \mathcal{R}(\delta_{w,i})) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p), E)$. The bijection $\mathrm{Ext}^1(\mathcal{R}(\delta), \mathcal{R}(\delta)) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Q}_p^\times, E)$ is a group homomorphism with respec to Baer sum of extensions and addition of characters, as one can check using (5.1.1) and (5.1.4), cf. the proof of Lemma 5.1 below. Then, we define κ_w to be the composite homomorphism

$$\kappa_w : \mathrm{Ext}^1_w(D, D) \to \prod_{i=1}^n \mathrm{Ext}^1(\mathcal{R}(\delta_{w,i}), \mathcal{R}(\delta_{w,i})) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p), E).$$

Let $\mathrm{Ext}^1_0(D,D) := \ker(\kappa_w)$, which is independent of the choice of w by [Din25, Lemma 2.11]. We write $\overline{\mathrm{Ext}}^1_w(D,D) := \mathrm{Ext}^1_w(D,D)/\mathrm{Ext}^1_0(D,D)$ for the quotient, so that we have

$$\kappa_w : \overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p),E).$$

We claim that there is a commutative diagram of E-linear maps:

$$\begin{tikzcd} \mathrm{Ext}^1_w(D,D) \arrow[r, "\kappa_w"] \arrow[d, "p_{\mathbf{k}}"] & \mathrm{Hom}(T(\mathbb{Q}_p),E) \arrow[d, equal] \\ \mathrm{Ext}^1_w(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D)) \arrow[r, "\kappa_w"] & \mathrm{Hom}(T(\mathbb{Q}_p),E) \end{tikzcd} \tag{5.1.5}$$

for each $w \in S_n$. For the proof, by induction we may assume $p_{\mathbf{k}} = p_i$ for some i.

Given $\widetilde{D} \in \mathrm{Ext}^1_w(D,D)$ with the corresponding triangulation $\mathrm{Fil}^\bullet(\widetilde{D})$ over $E[\varepsilon]/\varepsilon^2$, we apply p_i to obtain a (φ,Γ)-submodule $p_i(\widetilde{D})$, which again belongs to $p_i(\widetilde{D}) \in \mathrm{Ext}^1_w(p_i(D),p_i(D))$. For $\widetilde{D_1}, \widetilde{D_2} \in \mathrm{Ext}^1_w(D,D)$, the Baer sum $p_i(\widetilde{D_1}) + p_i(\widetilde{D_2})$ is the (φ,Γ)-submodule of the Baer sum $\widetilde{D_1} + \widetilde{D_2}$ containing $\mathrm{Fil}^{n-i}(\widetilde{D_1}) + \mathrm{Fil}^{n-i}(\widetilde{D_2}) \simeq \mathrm{Fil}^{n-i}(\widetilde{D_1} + \widetilde{D_2})$ with quotient being

$$t(\widetilde{D_1}/\mathrm{Fil}^{n-i}(\widetilde{D_1})) + t(\widetilde{D_2}/\mathrm{Fil}^{n-i}(\widetilde{D_2})) \simeq t((\widetilde{D_1}+\widetilde{D_2})/\mathrm{Fil}^{n-i}(\widetilde{D_1}+\widetilde{D_2}))$$

by (5.1.3), hence $p_i(\widetilde{D_1}) + p_i(\widetilde{D_2}) = p_i(\widetilde{D_1}+\widetilde{D_2})$. We deduce that

$$p_i : \mathrm{Ext}^1_w(D,D) \to \mathrm{Ext}^1_w(p_i(D),p_i(D))$$

is a group homomorphism. The commutativity of (5.1.5) follows from the effect of p_i on triangulation parameters. We also obtain a commutative diagram of group homomorphisms:

$$\begin{tikzcd} \overline{\mathrm{Ext}}^1_w(D,D) \arrow[r, "\kappa_w", "\sim"'] \arrow[d, "p_{\mathbf{k}}"] & \mathrm{Hom}(T(\mathbb{Q}_p),E) \arrow[d, equal] \\ \overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D)) \arrow[r, "\kappa_w", "\sim"'] & \mathrm{Hom}(T(\mathbb{Q}_p),E) \end{tikzcd} \tag{5.1.6}$$

where $\overline{\mathrm{Ext}}^1_w(D,D) := \mathrm{Ext}^1_w(D,D)/\ker(\kappa_w)$, and thus $p_{\mathbf{k}}$ induces a bijection on $\overline{\mathrm{Ext}}^1_w$.

For any $\widetilde{D} \in \mathrm{Ext}^1(D,D)$, its Sen polynomial $P_{\mathrm{Sen}}(T) \in (E[\varepsilon]/\varepsilon^2)[T]$ reduces modulo ε to the Sen polynomial of D which is $\prod_{i=1}^n (T-h_i)$. Since $E[\varepsilon]/\varepsilon^2$ is (ε)-adically complete, Hencel's lemma applies and $P_{\mathrm{Sen}}(T)$ splits as $\prod_{i=1}^n (T-(h_i+a_i\varepsilon))$ for $a_i \in E$. By [Wu, Proposition 3.16], there is a unique (φ,Γ)-submodule of $\widetilde{D}$ containing $t\widetilde{D}$ whose Sen polynomial is

$$\prod_{j=1}^{n-i}(T-(h_j+a_j\varepsilon)) \cdot \prod_{k=n-i+1}^{n}(T-(h_k+1+a_k\varepsilon))$$

which we denote by $p_i(\widetilde{D})$. Modulo ε, we get the Sen polynomial of $p_i(D)$, so by uniqueness we have that $p_i(\widetilde{D}) \in \mathrm{Ext}^1(p_i(D),p_i(D))$.

Lemma 5.1. *The pullback map*

$$p_{\mathbf{k}} : \mathrm{Ext}^1(D,D) \to \mathrm{Ext}^1(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D))$$

is a group homomorphism.

Proof. By induction we may assume $p_{\mathbf{k}} = p_i$ for some i. For any $\widetilde{D_1}, \widetilde{D_2} \in \mathrm{Ext}^1(D, D)$, as in the trianguline case we have two inclusions

$$t(\widetilde{D_1} + \widetilde{D_2}) \subset p_i(\widetilde{D_1}) + p_i(\widetilde{D_2}) \subset \widetilde{D_1} + \widetilde{D_2}, \quad t(\widetilde{D_1} + \widetilde{D_2}) \subset p_i(\widetilde{D_1} + \widetilde{D_2}) \subset \widetilde{D_1} + \widetilde{D_2}$$

so in order to show that

$$p_i(\widetilde{D_1}) + p_i(\widetilde{D_2}) = p_i(\widetilde{D_1} + \widetilde{D_2})$$

by [Wu, Proposition 3.16], it suffices to show that they have the same Sen polynomials.

Choose $a_1, \dots, a_n, b_1, \dots, b_n \in E$ such that $\widetilde{D_1}$ has Sen polynomial $\prod_{i=1}^n (T - (h_i + a_i\varepsilon))$ and $\widetilde{D_2}$ has Sen polynomial $\prod_{i=1}^n (T - (h_i + b_i\varepsilon))$.

Since D_{Sen} is an exact functor, we have $D_{\mathrm{Sen}}(\widetilde{D_1} + \widetilde{D_2}) = D_{\mathrm{Sen}}(\widetilde{D_1}) + D_{\mathrm{Sen}}(\widetilde{D_2})$, and

$$0 \to \varepsilon D_{\mathrm{Sen}}(D) \to D_{\mathrm{Sen}}(\widetilde{D_\bullet}) \to D_{\mathrm{Sen}}(D) \to 0$$

for $\bullet \in \{1, 2\}$, which splits as free $(\mathbb{Q}_p)_\infty \otimes_{\mathbb{Q}_p} E$-module $D_{\mathrm{Sen}}(\widetilde{D_\bullet}) = \varepsilon D_{\mathrm{Sen}}(D) \oplus D_{\mathrm{Sen}}(D)$. Let e_i be the Sen-eigenvector of weight $h_i \in D_{\mathrm{Sen}}(D)$. By (5.1.1), $D_{\mathrm{Sen}}(\widetilde{D_1}) + D_{\mathrm{Sen}}(\widetilde{D_2})$ equals

$$\{(\varepsilon d_1, d_2 + \varepsilon d_3, d_2 + \varepsilon d_4) | d_1, d_2, d_3, d_4 \in D_{\mathrm{Sen}}(D)\} \,/\, \{(-\varepsilon(r_1 + r_2), \varepsilon r_1, \varepsilon r_2) | r_1, r_2 \in D_{\mathrm{Sen}}(D)\}$$

which has a $(\mathbb{Q}_p)_\infty \otimes_{\mathbb{Q}_p} E[\varepsilon]/\varepsilon^2$-basis being represented by the classes of $\{(0, e_i, e_i) | 1 \le i \le n\}$. Then, we compute

$$\begin{aligned}
\Theta_\sigma([(0, e_i, e_i)]) &= [(0, (h_i + a_i\varepsilon)e_i, (h_i + b_i\varepsilon)e_i] \\
&= h_i[(0, e_i, e_i)] + [\varepsilon(a_i + b_i)e_i, 0, 0] \\
&= (h_i + (a_i + b_i)\varepsilon)[(0, e_i, e_i)]
\end{aligned}$$

from which we see that the Sen polynomial of $\widetilde{D_1} + \widetilde{D_2}$ is $\prod_{i=1}^n (T - (h_i + (a_i + b_i)\varepsilon))$. Thus, the Sen polynomials of $p_i(\widetilde{D_1}) + p_i(\widetilde{D_2})$ and $p_i(\widetilde{D_1} + \widetilde{D_2})$ are equal, as desired. $\square$

Since $p_{\mathbf{k}}$ on $\mathrm{Ext}^1(D, D)$ extends the pullbacks $p_{\mathbf{k}}$ on $\mathrm{Ext}^1_w(D, D)$ for each w, we have a commutative diagram of E-linear homomorphisms by Lemma 5.1,

(5.1.7)
$$\begin{tikzcd}
\bigoplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(D, D) \arrow[r, two heads] \arrow[d, "p_{\mathbf{k}}", "\simeq"'] & \overline{\mathrm{Ext}}^1(D, D) \arrow[d, "p_{\mathbf{k}}", "\simeq"'] \\
\bigoplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) \arrow[r, two heads] & \overline{\mathrm{Ext}}^1(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D))
\end{tikzcd}$$

with surjective rows [Nak13, Theorem 2.62] and bijective columns: the right vertical column is surjective by commutativity, and is bijective for dimension reason, cf. [Din25, (2.16)]. Alternatively, the bijectivity follows from Theorem 4.10(iii), which is applicable since D and $p_{\mathbf{k}}(D)$ are of regular Sen weights.

5.2. **Translation of locally analytic representations.** Let $\theta = (0, -1, \dots, 1 - n) \in \mathbb{Z}^n$.

5.2.1. We prove the commutativity of

$$\begin{tikzcd}
\mathrm{Hom}(T(\mathbb{Q}_p),E) \arrow[r, "i_{\mathbf{h}}"] \arrow[d, equal] & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\mathrm{PS}(w(\phi),\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h})) \arrow[d, "T_{\mathbf{h}-\theta}^{\mathbf{h}'-\theta}"] \\
\mathrm{Hom}(T(\mathbb{Q}_p),E) \arrow[r, "i_{\mathbf{h}'}"] & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\mathrm{PS}(w(\phi),\mathbf{h}'),\mathrm{PS}(w(\phi),\mathbf{h}'))
\end{tikzcd} \tag{5.2.1}$$

where the horizontal arrows are given by parabolic inductions: for $\psi \in \mathrm{Hom}(T(\mathbb{Q}_p),E)$, we set

$$i_{\mathbf{h}}(\psi) := \mathrm{Ind}_{B^-}^{\mathrm{GL}_n}(w(\phi)\eta z^{\mathbf{h}-\theta}(1+\psi\varepsilon)), \quad i_{\mathbf{h}'}(\psi) := \mathrm{Ind}_{B^-}^{\mathrm{GL}_n}(w(\phi)\eta z^{\mathbf{h}'-\theta}(1+\psi\varepsilon))$$

where $B^- \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is the lower triangular Borel subgroup, and η is the smooth character

$$\eta := 1 \boxtimes |\cdot| \boxtimes \cdots \boxtimes |\cdot|^{n-1} = |\cdot|^{-1} \circ \theta.$$

By [JLS24, (4.2.2)], there is a canonical isomorphism describing the strong dual of $\mathrm{Ind}_{B^-}^G(\tau)$:

$$D(G) \otimes_{D(B^-)} E_{\tau^{-1}} \xrightarrow{\sim} \mathrm{Ind}_{B^-}^G(\tau)'_b \tag{5.2.2}$$

of $D(G)$-modules for any continuous $\tau : T(\mathbb{Q}_p) \to E^\times$. Thus, if $\widetilde{\tau} : T(\mathbb{Q}_p) \to (E[\varepsilon]/\varepsilon^2)^\times$ is any deformation of τ from E to $E[\varepsilon]/\varepsilon^2$, we have a commutative diagram with exact rows
(5.2.3)

$$\begin{tikzcd}
0 \arrow[r] & D(G) \otimes_{D(B^-)} E_{\tau^{-1}} \arrow[r] \arrow[d, "(5.2.2)"', "\simeq"] & D(G) \otimes_{D(B^-)} E_{\widetilde{\tau}^{-1}} \arrow[r] \arrow[d] & D(G) \otimes_{D(B^-)} E_{\tau^{-1}} \arrow[r] \arrow[d, "(5.2.2)"', "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{Ind}_{B^-}^G(\tau)'_b \arrow[r] & \mathrm{Ind}_{B^-}^G(\widetilde{\tau})'_b \arrow[r] & \mathrm{Ind}_{B^-}^G(\tau)'_b \arrow[r] & 0
\end{tikzcd}$$

By the 5-lemma, we conclude that (5.2.2) also holds for continuous $\tau : T(\mathbb{Q}_p) \to (E[\varepsilon]/\varepsilon^2)^\times$.

Set $\lambda := \mathbf{h} - \theta$ and $\lambda' := \mathbf{h}' - \theta$. For any weight ξ, let χ_ξ denote the infinitesimal character of the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ on the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} E_\xi$. Let χ_ξ^- denote the infinitesimal character of $\mathcal{Z}(\mathfrak{g})$ on the opposite Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} E_\xi$.

Lemma 5.2. *For any weight $\xi \in \mathfrak{t}_E^*$, we have $\chi_\xi^- = \chi_{w_0\xi}$.*

Proof. The twisted Harish-Chandra homomorphism has two formulations ([HTT08, Theorem 9.4.3] or [KV95, p. 290]), i.e., the following diagram is commutative

$$\begin{tikzcd}
Z(\mathfrak{g}) \arrow[r, hook] \arrow[d, equal] & U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \arrow[r, "-\rho\circ\mathrm{pr}_1"] & U(\mathfrak{h}) \arrow[d, equal] \\
Z(\mathfrak{g}) \arrow[r, hook] & U(\mathfrak{h}) \oplus (\mathfrak{n}^+ U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-) \arrow[r, "+\rho\circ\mathrm{pr}_1"] & U(\mathfrak{h})
\end{tikzcd}$$

where

$$\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+3}{2}, \frac{-n+1}{2}\right)$$

is the half-sum of the positive roots of GL_n with respect to B, so we have

$$\theta = \rho + \left(\frac{1-n}{2}, \dots, \frac{1-n}{2}\right).$$

The top row gives $\chi_{\xi-\rho}$ and the bottom row gives $\chi^-_{\xi+\rho}$. Hence, $\chi^-_\xi = \chi_{\xi-2\rho} = \chi_{w_0\xi}$, where the last equality follows from

$$w_0 \cdot (w_0\xi) = w_0(w_0\xi + \rho) - \rho = \xi - 2\rho$$

and the fact that χ_ξ depends only on the linkage class of ξ. $\square$

By [Dinb, Lemma 3.9] and (5.2.3), translation functors "commute" with strong duals:

$$\begin{aligned} T_\lambda^{\lambda'}\left(\mathrm{Ind}_{B^-}^{\mathrm{GL}_n}(w(\phi)\eta z^\lambda(1+\psi\varepsilon))\right)'_b &= T_{\lambda^*}^{\lambda'^*}(\mathrm{Ind}_{B^-}^{\mathrm{GL}_n}(w(\phi)\eta z^\lambda(1+\psi\varepsilon))'_b) \\ &= T_{-w_0\lambda}^{-w_0\lambda'}\left(D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{\left(w(\phi)\eta z^\lambda(1+\psi\varepsilon)\right)^{-1}}\right) \end{aligned}$$

where $\lambda^* := -w_0\lambda$ and $\lambda'^* := -w_0\lambda$ are infinitesimal characters of the dual. By Lemma 5.2,

$$T_{-w_0\lambda}^{-w_0\lambda'}\left(D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{\left(w(\phi)\eta z^\lambda(1+\psi\varepsilon)\right)^{-1}}\right) = \overline{T}_{-\lambda}^{-\lambda'}\left(D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{\left(w(\phi)\eta z^\lambda(1+\psi\varepsilon)\right)^{-1}}\right)$$

where the right-hand side uses $\chi^-_{(-\lambda)}$ instead of $\chi_{(-w_0\lambda)}$ as the infinitesimal character, hence the notation $\overline{T}$ instead of T. We write $(\bar{\cdot})$ for the other dot action $w\bar{\cdot}\lambda := w(\lambda-\rho)+\rho$ fixing ρ.

Proposition 5.3. *The translation functor*

$$\overline{T}_{w_0\bar{\cdot}(-\lambda)}^{w_0\bar{\cdot}(-\lambda')} = \overline{T}_{-\lambda}^{-\lambda'} : D(G)\text{-mod}_{|-\lambda|} \to D(G)\text{-mod}_{|-\lambda'|}$$

induces an equivalence of categories.

Proof. Note that $w_0\bar{\cdot}(-\lambda) = -(w_0\cdot\lambda) = -w_0(\mathbf{h}) + \theta$ and $w_0\bar{\cdot}(-\lambda') = -w_0(\mathbf{h}') + \theta$ are anti-dominant with respect to B^- because $\mathbf{h}$ and $\mathbf{h}'$ are regular, the difference of weights

$$w_0\bar{\cdot}(-\lambda') - w_0\bar{\cdot}(-\lambda) = -w_0(-\lambda'+\lambda) = \left(\sum_{i=1}^n k_i, \sum_{i=2}^n k_i, \ldots, k_n\right)$$

lifts to an algebraic character of $T(\mathbb{Q}_p)$, the stabilizers of $-\lambda'$ and $-\lambda$ for the $(\bar{\cdot})$-action of W are trivial, and $(w_0\bar{\cdot}(-\lambda'))^\natural, (w_0\bar{\cdot}(-\lambda))^\natural$ lie in the same open Weyl chamber $\mathscr{C}$ of $\mathscr{E} := \mathbb{R}\otimes_{\mathbb{Z}}\Phi$, i.e., the condition of [JLS24, (4.2.8)] is satisfied. Therefore, [JLS24, Theorem 1] applies. $\square$

By the paragraph preceding Proposition 5.3, to prove the commutativity of diagram (5.2.1), it suffices to prove Lemma 5.4 below, for then we get

$$T_\lambda^{\lambda'}\left(\mathrm{Ind}_{B^-}^{\mathrm{GL}_n}(w(\phi)\eta z^\lambda(1+\psi\varepsilon))\right)'_b = \left(\mathrm{Ind}_{B^-}^{\mathrm{GL}_n}(w(\phi)\eta z^{\lambda'}(1+\psi\varepsilon))\right)'_b$$

and taking the strong dual again proves the commutativity of (5.2.1).

Lemma 5.4. *We have*

$$\overline{T}_{-\lambda}^{-\lambda'}\left(D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{\left(w(\phi)\eta z^\lambda(1+\psi\varepsilon)\right)^{-1}}\right) = D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{\left(w(\phi)\eta z^{\lambda'}(1+\psi\varepsilon)\right)^{-1}}$$

We follow the argument given in [JLS24] except that we change the coefficient ring from E to $E[\varepsilon]/(\varepsilon^2)$. Recall that there is an exact functor

$$\check{\mathcal{F}}_{B^-}^G : \mathcal{O}^{B^-,\infty} \longrightarrow D(G)\text{-mod}, \quad \check{\mathcal{F}}_{B^-}^G(M) := D(G)\otimes_{D(\mathfrak{g},B^-)} M$$

from a subcategory of $D(\mathfrak{g}, B^-)$-modules to the category of $D(G)$-modules, cf. [JLS24, (4.1.2) and (4.1.6)] for precise definition. If M has an $E[\varepsilon]/(\varepsilon^2)$-structure, so does $\check{\mathcal{F}}_{B^-}^G(M)$.

Lemma 5.5. *For any locally $\mathbb{Q}_p$-analytic character $\tau : B^- \to (E[\varepsilon]/\varepsilon^2)^\times$, one has*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\tau} \cong D(\mathfrak{g}, B^-) \otimes_{D(B^-)} (E[\varepsilon]/\varepsilon^2)_\tau$$

as $D(\mathfrak{g}, B^-)$-modules, and therefore

$$\check{\mathcal{F}}^G_{B^-}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\tau}) = D(G) \otimes_{D(B^-)} (E[\varepsilon]/\varepsilon^2)_\tau.$$

Proof. When τ takes values in $E^\times$, this is [JLS24, Examples 4.1.4 and 4.1.7]. The proof given there respects the $E[\varepsilon]/\varepsilon^2$-structure, and hence the proof still holds. □

Lemma 5.6. *For any $M \in \mathcal{O}^{B^-,\infty}$ with an $E[\varepsilon]/\varepsilon^2$-structure and any compatible weights $\mu, \lambda \in \mathfrak{t}^*_E$ (that is, $\lambda - \mu$ is algebraic), we have*

$$\overline{T}^\mu_\lambda(\check{\mathcal{F}}^G_{B^-}(M)) = \check{\mathcal{F}}^G_{B^-}(\overline{T}^\mu_\lambda(M))$$

as $D(G)$-modules with $E[\varepsilon]/\varepsilon^2$-structures.

Proof. Without the $E[\varepsilon]/\varepsilon^2$-structure, this is [JLS24, Theorem 4.1.12]. The proof given there respects the $E[\varepsilon]/\varepsilon^2$-structure, and hence the proof still holds. □

Proof of Lemma 5.4. It suffices to show that under our assumption, there is an isomorphism (5.2.4)

$$\overline{T}^{w_0\cdot(-\lambda')}_{w_0\cdot(-\lambda)}\left(U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\left(w(\phi)\eta z^\lambda(1+\psi\varepsilon))\right)^{-1}}\right) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\left(w(\phi)\eta z^{\lambda'}(1+\psi\varepsilon))\right)^{-1}}$$

as $D(\mathfrak{g}, B^-)$-modules. As $U(\mathfrak{g})$-module, this is immediate by [Hum08, Lemma 7.5 & Theorem 7.6], which are applicable by the proof of Proposition 5.3. Via Lemma 5.5, [Hum08, Theorem 7.6] admits an analogue for $D(\mathfrak{g}, B^-)$-modules as in [JLS24, Proposition 4.2.10], and the proof respects $E[\varepsilon]/\varepsilon^2$-deformations of the weight character, because self-extension of $U(\mathfrak{g})$-module does not change the generalized infinitesimal character.

For $\tilde{\lambda} := w(\phi)\eta z^\lambda(1 + \psi\varepsilon))$ and $\tilde{\lambda}' := w(\phi)\eta z^{\lambda'}(1 + \psi\varepsilon))$, we have

$$\begin{aligned}
\overline{T}^{-\lambda'}_{-\lambda}\left(D(G) \otimes_{D(B^-)} (E[\varepsilon]/\varepsilon^2)_{\tilde{\lambda}^{-1}}\right) &= \overline{T}^{-\lambda'}_{-\lambda}(\check{\mathcal{F}}^G_{B^-}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\tilde{\lambda}^{-1}})) && \text{Lemma 5.5}\\
&= \check{\mathcal{F}}^G_{B^-}(\overline{T}^{-\lambda'}_{-\lambda}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\tilde{\lambda}^{-1}})) && \text{Lemma 5.6}\\
&= \check{\mathcal{F}}^G_{B^-}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} (E[\varepsilon]/\varepsilon^2)_{d\tilde{\lambda}'^{-1}}) && (5.2.4)\\
&= D(G) \otimes_{D(B^-)} (E[\varepsilon]/\varepsilon^2)_{\tilde{\lambda}'^{-1}} && \text{Lemma 5.5}
\end{aligned}$$

as desired. □

5.2.2. Recall that $\zeta_w : \mathrm{Hom}(T(\mathbb{Q}_p), E) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h}))$ in [Din25, (3.11)] is defined to be the following composite,

$$\begin{aligned}
\zeta_w : \mathrm{Hom}(T(\mathbb{Q}_p), E) &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\mathrm{PS}(w(\phi), \mathbf{h}), \mathrm{PS}(w(\phi), \mathbf{h}))\\
&\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}(w(\phi), \mathbf{h}))\\
&\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h}))
\end{aligned}$$

where the first map is the parabolic induction $i_{\mathbf{h}}$, the second map is the pullback by [Din25, (3.2)], and the third map is the pushforward by [Din25, (3.3)] but with $\pi_1(\phi, \mathbf{h})$ replaced by the larger representation $\pi(\phi, \mathbf{h})$. As noted in the paragraph preceding [Din25, Theorem 1.3], a spectral sequence argument shows that ζ_w is bijective.

By Proposition 5.3 and Lemma 5.4, $T_\lambda^{\lambda'}$ is an equivalence of categories with $T_\lambda^{\lambda'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h})) = \pi_{\mathrm{alg}}(\phi,\mathbf{h}')$ and $T_\lambda^{\lambda'}(\mathrm{PS}(w(\phi),\mathbf{h})) = \mathrm{PS}(w(\phi),\mathbf{h}')$. Since $\pi(\phi,\mathbf{h})$ is the unique quotient of the pushout $\bigoplus^{w\in S_n}_{\pi_{\mathrm{alg}}(\phi,\mathbf{h})} \mathrm{PS}(w(\phi),\mathbf{h})$ whose socle is $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by [Din25, Remark 3.9], it follows that

$$T_\lambda^{\lambda'}(\pi(\phi,\mathbf{h})) = \pi(\phi,\mathbf{h}').$$

We thus obtain a commutative diagram
(5.2.5)

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathrm{PS}(w(\phi),\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h})) \arrow[r, "\mathrm{pull}"] \arrow[d, "T_\lambda^{\lambda'}"] & \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h})) \arrow[r, "\mathrm{push}"] \arrow[d, "T_\lambda^{\lambda'}"] & \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})) \arrow[d, "T_\lambda^{\lambda'}"] \\
\mathrm{Ext}^1(\mathrm{PS}(w(\phi),\mathbf{h}'),\mathrm{PS}(w(\phi),\mathbf{h}')) \arrow[r, "\mathrm{pull}"] & \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\phi,\mathbf{h}'),\mathrm{PS}(w(\phi),\mathbf{h}')) \arrow[r, "\mathrm{push}"] & \mathrm{Ext}^1(\pi_{\mathrm{alg}}(\phi,\mathbf{h}'),\pi(\phi,\mathbf{h}'))
\end{tikzcd}$$

5.3. **Intertwining of the two translations.**

5.3.1. For each $w\in S_n$, we can now deduce that

$$\begin{tikzcd}
\overline{\mathrm{Ext}}^1_w(D,D) \arrow[r, hook, "\zeta_w\circ\kappa_w"] \arrow[d, "p_{\mathbf{k}}", "\simeq"'] & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})) \arrow[d, "T_{\mathbf{h}-\theta}^{\mathbf{h}'-\theta}", "\simeq"'] \\
\overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D)) \arrow[r, hook, "\zeta_w\circ\kappa_w"] & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}'),\pi(\phi,\mathbf{h}'))
\end{tikzcd} \tag{5.3.1}$$

is a commutative diagram. Indeed, putting diagrams (5.1.6), (5.2.1) and (5.2.5) together and unwinding the definitions, we see the commutativity of (5.3.1).

By the commutative diagrams (5.1.7) and (5.3.1), if we define the subspaces

$$\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})) := \mathrm{Im}\left(\overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow{\zeta_w\circ\kappa_w} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h}))\right),$$

following [Din25, before (3.17)], then we have two commutative diagrams

$$\begin{tikzcd}
\bigoplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D) \arrow[r, hook, "\zeta_w\circ\kappa_w"] \arrow[d, "p_{\mathbf{k}}", "\simeq"'] & \bigoplus_{w\in S_n}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})) \arrow[d, "T_{\mathbf{h}-\theta}^{\mathbf{h}'-\theta}", "\simeq"'] \\
\bigoplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D)) \arrow[r, hook, "\zeta_w\circ\kappa_w"] & \bigoplus_{w\in S_n}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}'),\pi(\phi,\mathbf{h}'))
\end{tikzcd} \tag{5.3.2}$$

and

$$\begin{tikzcd}
\bigoplus_{w\in S_n}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})) \arrow[r, "T_{\mathbf{h}-\theta}^{\mathbf{h}'-\theta}", "\sim"'] \arrow[d, two heads, "\mathrm{sum}"] & \bigoplus_{w\in S_n}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}'),\pi(\phi,\mathbf{h}')) \arrow[d, two heads, "\mathrm{sum}"] \\
\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})) \arrow[r, "T_{\mathbf{h}-\theta}^{\mathbf{h}'-\theta}", "\sim"'] & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}'),\pi(\phi,\mathbf{h}'))
\end{tikzcd} \tag{5.3.3}$$

where the vertical amalgamation maps are surjective by [Din25, (3.13), (3.16)].

5.3.2. *Universal extension.* We discuss the notion of universal representation in general. Let E be a field, and let $\mathcal{D}$ be an E-linear abelian category with objects A and B. Then, to any finite dimensional E-linear subspace W of $\mathrm{Ext}^1_{\mathcal{D}}(A,B)$, one can attach a "universal extension" $\mathcal{E}^{W\text{-univ}}$ of $A\otimes_E W$ ($\cong A^{\dim_E W}$) by B, satisfying the property that for any extension

$$0\to B\to \mathcal{E}_e\to A\to 0$$

with corresponding class $e\in W\subset \mathrm{Ext}^1_{\mathcal{D}}(A,B)$, the map $\alpha_e: A\to A\otimes_E W, a\mapsto a\otimes e$ induces

$$\alpha_e^*:\mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B)\to \mathrm{Ext}^1_{\mathcal{D}}(A,B)$$

such that the pullback $\alpha_e^*(\mathcal{E}^{W\text{-univ}})$ is precisely $\mathcal{E}_e$.

Proposition 5.7. *(i) Abstractly, this extension $\mathcal{E}^{W\text{-univ}}$ is the image of the inclusion map*

$$i_W: W\hookrightarrow \mathrm{Ext}^1_{\mathcal{D}}(A,B)$$

under the canonical isomorphisms

$$i_W\in \mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) \xleftarrow[\text{can}]{\sim} \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E W^\vee \xrightarrow[\text{can}]{\sim} \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B)\ni \mathcal{E}^{W\text{-univ}}$$

where the first map is a canonical isomorphism for finite dimensional W, and the second map is defined as follows: given $\mathcal{E}_e\in \mathrm{Ext}^1_{\mathcal{D}}(A,B)$ and a linear functional $f\in W^\vee$, we pullback $\mathcal{E}_e$ via $\beta_f: A\otimes W\to A, a\otimes w\mapsto f(w)a$ to $\beta_f^*(\mathcal{E}_e)\in \mathrm{Ext}^1_{\mathcal{D}}(A,B)$, which defines

$$\mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E W^\vee \xrightarrow[\text{can}]{\sim} \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B),\quad \mathcal{E}_e\otimes f\mapsto \beta_f^*(\mathcal{E}_e)$$

with its inverse given by

$$\mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B)\xrightarrow[\text{can}]{\sim} \mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)),\quad \mathcal{E}\mapsto (e\mapsto \alpha_e^*(\mathcal{E})).$$

(ii) Concretely, choose any E-basis $\{e_1,\dots,e_d\}$ of W, corresponding to extensions $\mathcal{E}_1,\dots,\mathcal{E}_d\in \mathrm{Ext}^1_{\mathcal{D}}(A,B)$. Form the pushout of $\mathcal{E}_1,\dots,\mathcal{E}_d$ over the common object B, denoted $\bigoplus_B^{1\le i\le d}\mathcal{E}_i$:

$$\left(0\to B\to \bigoplus_B^{1\le i\le d}\mathcal{E}_i\to \bigoplus_{i=1}^d Ae_i\cong A\otimes_E W\to 0\right)\in \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B).$$

Then, $\bigoplus_B^{1\le i\le d}\mathcal{E}_i\cong \mathcal{E}^{W\text{-univ}}$ is the universal extension associated to W.

Proof. For any $e\in W$, the inclusion $i_e: E\hookrightarrow W, 1\mapsto e$ induces a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) \arrow[r, "\sim"] \arrow[d, "i_e^*"] & \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E W^\vee \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\otimes i_e^\vee"] & \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B) \arrow[d, "\alpha_e^*"] \\
\mathrm{Hom}_E(E,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) \arrow[r, "\sim"] & \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E E^\vee \arrow[r, "\sim"] & \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E E,B)
\end{tikzcd}$$

Keeping track of the element $i_W\in \mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B))$ proves (i).

As for the second statement, we have a commutative diagram of E-linear isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) \arrow[r, "\sim"] \arrow[d, "\bigoplus_{i=1}^d i_{e_i}^*", "\simeq"'] & \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E W^\vee \arrow[r, "\sim"] \arrow[d, "\bigoplus_{i=1}^d(\mathrm{id}\otimes i_{e_i}^\vee)", "\simeq"'] & \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B) \arrow[d, "\bigoplus_{i=1}^d \alpha_{e_i}^*", "\simeq"'] \\
\bigoplus_{i=1}^d \mathrm{Hom}_E(E,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) \arrow[r, "\sim"] & \bigoplus_{i=1}^d \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E E^\vee \arrow[r, "\sim"] & \bigoplus_{i=1}^d \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E E,B)
\end{tikzcd}$$

for the chosen basis $\{e_1, \dots, e_d\}$ of W. Recall that the pushout $\bigoplus_B^{1\le i\le d} \mathcal{E}_i$ sitting in

$$0 \to B \to \bigoplus_B^{1\le i\le d} \mathcal{E}_i \to \bigoplus_{i=1}^{d} Ae_i \cong A \otimes_E W \to 0$$

is by construction such that $\alpha_{e_i}^*\left(\bigoplus_B^{1\le i\le d} \mathcal{E}_i\right) = \mathcal{E}_i$ for each $1 \le i \le d$. Hence, it corresponds to $i_W \in \mathrm{Hom}_E(W, \mathrm{Ext}^1_{\mathcal{D}}(A, B))$, and it equals the universal extension attached to W by (i). $\square$

5.3.3. *The proof.* For the representation $\pi_{\mathrm{fs}}(D)$, by [Din25, Theorem 1.3], for $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$, there is a unique surjection $t_D : \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{\varphi,\Gamma}(D, D)$ such that

$$\begin{tikzcd} \bigoplus_{w\in S_n} \overline{\mathrm{Ext}}^1_w(D, D) \arrow[r, "(\zeta_w\circ\kappa_w)", "\sim"'] \arrow[d, two heads, "\mathrm{sum}"] & \bigoplus_{w\in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \arrow[d, two heads, "\mathrm{sum}"] \\ \overline{\mathrm{Ext}}^1_{\varphi,\Gamma}(D, D) & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \arrow[l, two heads, "t_D"] \end{tikzcd} \tag{5.3.4}$$

and $\pi_{\mathrm{fs}}(D)$ is set to be the universal extension $\mathcal{E}^{\ker(t_D)\text{-univ}}$ of $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \otimes_E \ker(t_D)$ by $\pi(\phi, \mathbf{h})$.

We can use the change of weights operators $p_{\mathbf{k}}$ and $T_\lambda^{\lambda'}$ to form a cube:
(5.3.5)

$$\begin{tikzcd} & \bigoplus_{w\in S_n} \overline{\mathrm{Ext}}^1_w(D, D) \arrow[rr, "\zeta_w\circ\kappa_w", "\simeq"'] \arrow[dl, two heads, "\mathrm{sum}"] \arrow[dd, "p_{\mathbf{k}}"] & & \bigoplus_{w\in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \arrow[dl, two heads, "\mathrm{sum}"] \arrow[dd, "T_\lambda^{\lambda'}"] \\ \overline{\mathrm{Ext}}^1_{\varphi,\Gamma}(D, D) \arrow[dd, "p_i"] & & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \arrow[ll, two heads, "t_D"] \arrow[dd, "T_\lambda^{\lambda'}"] & \\ & \bigoplus_{w\in S_n} \overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) \arrow[rr, "\zeta_w\circ\kappa_w", "\simeq"'] \arrow[dl, two heads, "\mathrm{sum}"] & & \bigoplus_{w\in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi(\phi, \mathbf{h}')) \arrow[dl, two heads, "\mathrm{sum}"] \\ \overline{\mathrm{Ext}}^1_{\varphi,\Gamma}(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) & & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi(\phi, \mathbf{h}')) \arrow[ll, two heads, "t_{p_{\mathbf{k}}(D)}"] & \end{tikzcd}$$

where those arrows labeled by "sum", "t_D" or "$t_{p_{\mathbf{k}}(D)}$" are surjective, and all other arrows are bijective. By the commutativity of (5.1.7), (5.3.2), (5.3.3), and the defining diagram (5.3.4) for t_D and $t_{p_{\mathbf{k}}(D)}$, all but the "front" faces of the cube are commutative. By the surjectivity of

$$\bigoplus_{w\in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \xrightarrow{\mathrm{sum}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h}))$$

and a diagram chase, we deduce that the "front face"

$$\begin{tikzcd} \overline{\mathrm{Ext}}^1_{\varphi,\Gamma}(D, D) \arrow[d, "p_{\mathbf{k}}", "\simeq"'] & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi(\phi, \mathbf{h})) \arrow[l, two heads, "t_D"] \arrow[d, "T_\lambda^{\lambda'}", "\simeq"'] \\ \overline{\mathrm{Ext}}^1_{\varphi,\Gamma}(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbb{Q}_p)}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi(\phi, \mathbf{h}')) \arrow[l, two heads, "t_{p_{\mathbf{k}}(D)}"] \end{tikzcd} \tag{5.3.6}$$

also commutes. Hence, $T_\lambda^{\lambda'}(\ker(t_D)) = \ker(t_{p_{\mathbf{k}}(D)})$, by which we conclude

$$T_\lambda^{\lambda'}(\pi_{\mathrm{fs}}(D)) = T_\lambda^{\lambda'}(\mathcal{E}^{\ker(t_D)\text{-univ}}) = \mathcal{E}^{\ker(t_{p_{\mathbf{k}}(D)})\text{-univ}} = \pi_{\mathrm{fs}}(p_{\mathbf{k}}(D)) = \pi_{\mathrm{fs}}(f_{\mathbf{h},\mathbf{h}'}(D)). \qquad \square$$

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