This article is a slightly expanded version of my presentation at the 1994 DMV conference in Duisburg. The aim is to try to convey to the reader some of the fascination that the Bruhat-Tits buildings hold for the author. All graphics are created using a computer program developed by my co-workers Erdmann, Landvogt and Wettig.

I would like to start with an analogy that is familiar to all. The upper half-plane $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ arises by means of the bijection

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}) &\to \mathbb{H} \\ g &\mapsto gi \end{aligned}$$

as a homogeneous space of the Lie group $\mathrm{SL}_2(\mathbb{R})$ associated to the maximal compact subgroup $\mathrm{SO}(2)$. The fundamentally important role that the upper half-plane $\mathbb{H}$ plays in various areas of mathematics does not need to be explained in more detail here. For comparison with what will be said later, however, one aspect should be emphasized: suitable spaces of functions on $\mathbb{H}$ provide explicit models for certain series of (infinite-dimensional) representations of the Lie group $\mathrm{SL}_2(\mathbb{R})$.
The above bijection is a special case of the general principle

$$\begin{matrix}\text{semisimple}\\ \text{real Lie group}\end{matrix} \Big/ \begin{matrix}\text{maximal compact}\\ \text{subgroup}\end{matrix} \quad = \quad \text{symmetric space}$$

The choice of the maximal compact subgroup does not play any role, since they are all conjugate according to Cartan's fixed point theorem.

In number theory, alongside the field $\mathbb{R}$ of real numbers, the local or p-adic number fields $\mathbb{Q}_p$ have equal status for every prime number p. The field $\mathbb{Q}_p$ arises from the field of rational numbers $\mathbb{Q}$ by completion with respect to the p-adic absolute value $|\ |_p$ which is defined as follows: if we write the rational number $x \in \mathbb{Q}$ as $x = p^m \cdot \frac{a}{b}$ with $m \in \mathbb{Z}$ and for $a, b \in \mathbb{Z}$ coprime to p, then $|x|_p := p^{-m}$. In other words, the p-adic absolute value measures how often a given number is divisible by the prime number p. An important feature here is that $|\ |_p$ is a non-archimedean absolute value, i.e. it satisfies the strict triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$. This means that $\mathbb{Q}_p$ contains the discrete valuation ring

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \big| |x|_p \leq 1\}$$

with maximal ideal

$$p\mathbb{Z}_p := \{x \in \mathbb{Q}_p \big| |x| < 1\}$$

The corresponding residue class field $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$ is of course the finite field with p elements. So:

$$\mathbb{Q}_p \xleftarrow{\supseteq} \mathbb{Z}_p \xrightarrow{pr} \mathbb{F}_p$$

Using $|\ |_p$ as a metric, $\mathbb{Q}_p$ is naturally provided with a topology. But this is totally disconnected by the strict triangle inequality. The subset $\mathbb{Z}_p$ is compact and open; hence $\mathbb{Q}_p$ is locally compact. In this context, the analogue of the Lie group $\mathrm{SL}_2(\mathbb{R})$ is obviously the group $G := \mathrm{SL}_2(\mathbb{Q}_p)$, which we want to look at first in more detail. The topology on $\mathbb{Q}_p$ leads to a natural topology on G via the vector space of 2×2 matrices. It is easy to see:

- G is a locally compact totally disconnected topological group.
- $K_0 := \mathrm{SL}_2(\mathbb{Z}_p)$is a maximal compact subgroup of G.

But there are two phenomena that differ significantly from the situation in real Lie groups:

- K_0 is open in G. Consequently, the homogeneous space G/K_0 is discrete and thus topologically uninteresting.
- $K_1 := \left\{ \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_p \right\}$ is also a maximal compact subgroup of G; K_1 and K_0 are not conjugated in G.

The latter fact is due to the fact that the two $\mathbb{Z}_p$-lattices $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and $\mathbb{Z}_p \oplus p\mathbb{Z}_p$ in the vector space $\mathbb{Q}_p \oplus \mathbb{Q}_p$ cannot transformed into one another by a matrix in G.

We thus establish that although the individual sets G/K_i have no topological structure, however, the problem arises of classifying the conjugacy classes of maximal compact subgroups and of understanding the position of such non-conjugated subgroups in relation to one another. It turns out that the answer can be best given in a geometric way. Let

$$I := K_0 \cap K_1$$

be the so-called Iwahori subgroup. Then the following holds:

Fact: The maximal compact subgroups which contain I (i.e. K_0 and K_1) form a set of representatives for the conjugacy classes of all maximal compact subgroups.

We now define a 1-dimensional simplicial complex in that we

- assign a vertex to each maximal compact subgroup, and that we
- connect two vertices represented by K and K' by an edge if and only if $K \cap K'$ is conjugate to I.

This gives the well-known tree X, which looks as in Fig. 1 for $p = 2$:

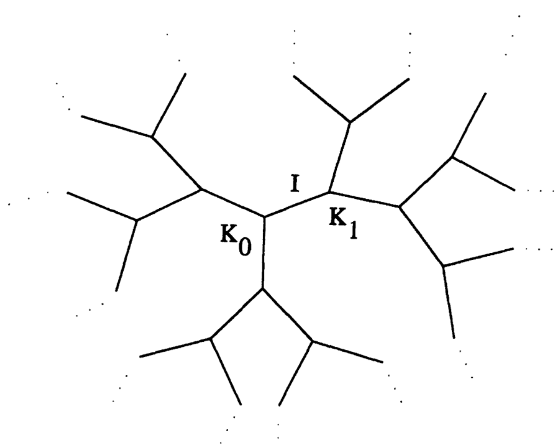

Fig. 1

Obviously, conjugation induces an action from G on X by simplicial automorphisms. Their fundamental property is expressed through the identity

$$\text{stablizer of a vertex} \quad = \quad \begin{array}{c}\text{maximal compact subgroup} \\ \text{representing the vertex}\end{array}$$

Moreover, the size of the intersection $K \cap K'$ of two maximal compact subgroups K and K' is controlled by how far apart the associated vertices are in X. In summary, we may say that the action of G on X describes the position of the maximal compact subgroups in G completely.

This principle of construction can be generalized without much difficulty to the group $G = \mathrm{SL}_n(\mathbb{Q}_p)$, where $n \in \mathbb{N}$ is arbitrary. The associated *Bruhat-Tits building* $X = X(G)$ is now an $(n-1)$-dimensional simplicial complex whose vertices again correspond to the maximal compact subgroups of G. As expected, its global structure is combinatorially very complicated. The advantage lies in the fact that difficult combinatorics are expressed in a geometrically clear way. Good statements about X can be made in two ways:

1) The local structure around a vertex:
   One has the bijection

$$\begin{array}{rcl} \begin{array}{c}\text{vertices adjacent to} \\ \text{the vertex } K_0 = \mathrm{SL}_n(\mathbb{Z}_p)\end{array} & \longleftrightarrow & \begin{array}{c}\text{maximal parabolic} \\ \text{subgroups in } \mathrm{SL}_n(\mathbb{F}_p)\end{array} \\ K & \longmapsto & (K \cap K_0)/1 + p \cdot \mathrm{SL}_n(\mathbb{Z}_p). \end{array}$$

   For instance, K_1 corresponds to the Borel subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{F}_p)$. Therefore, the appearance of the building X in the neighborhood of a vertex can be traced back to the well-understood combinatorics of the parabolic subgroups in the algebraic group SL_n over the finite field $\mathbb{F}_p$.
   Two pictures: In the case of the group $\mathrm{SL}_3(\mathbb{Q}_2)$, each vertex of X is contained in exactly 21 two-simplices. It can no longer be presented without self-intersections. Therefore, Fig. 2 shows only 12 of them, chosen purely for aesthetic reasons. What has been described so far can also be generalized to the effect that one works over a finite extension of $\mathbb{Q}_p$ instead of over $\mathbb{Q}_p$ itself. If one considers the group SL_3 over a quadratic extension of $\mathbb{Q}_2$, whose residue class field is $\mathbb{F}_4$, then there are already 105 two-simplices containing a given vertex of X. Fig. 3 shows 15 of them, namely, those that are adjacent to the simplices in dark gray that are in a common apartment (see below).

2) The so-called apartments:
   Denote by A the subcomplex of X spanned by all vertices tK_0t^{-1}, where $t \in T \subset \mathrm{GL}_n(\mathbb{Q}_p)$ traverses all invertible diagonal matrices. For example, one obtains

   - for $\mathrm{SL}_2(\mathbb{Q}_p)$ a path in the tree that is unbounded in both directions (i.e. a triangulated real line) (Fig. 4),
   - for $\mathrm{SL}_3(\mathbb{Q}_p)$ a real plane triangulated as in Fig. 5.

   The subcomplexes gA of X that result from applying a group element $g \in G$ to A are called the *apartments* of the building. They cover X, i.e.

$$X = \bigcup_{g \in G} gA.$$

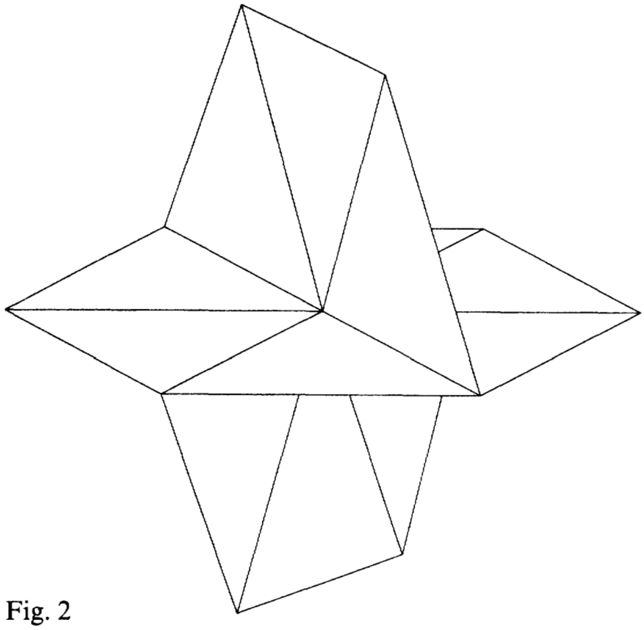

Fig. 2

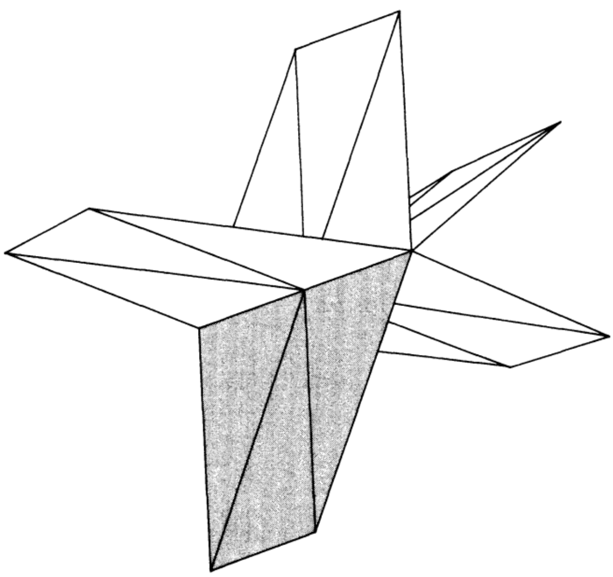

Fig. 3

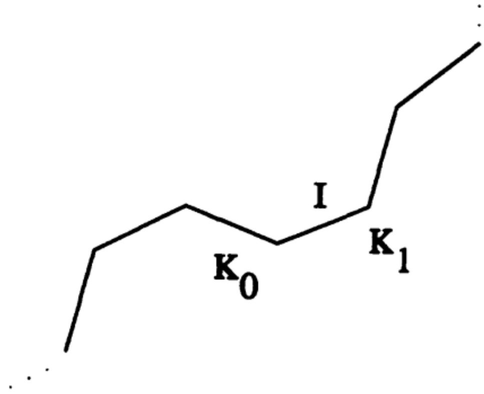

Fig. 4

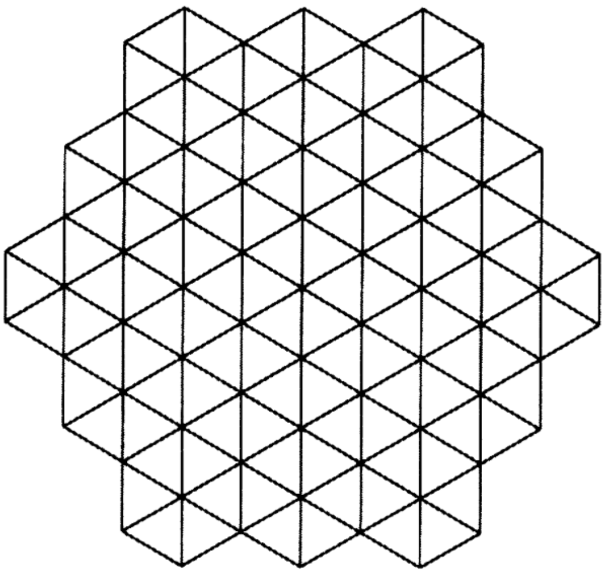

Fig. 5

The general theory begins with a connected reductive group $\mathbb{G}$ over $\mathbb{Q}_p$. The group $G = \mathbb{G}(\mathbb{Q}_p)$ of the $\mathbb{Q}_p$-rational points of $\mathbb{G}$ is in a natural way a locally compact totally disconnected topological group. One sees this most easily by realizing $\mathbb{G}$ as a matrix group. Following similar, yet incomparably more complicated principles, Bruhat and Tits ([BT]) construct the *building* $X = X(G)$ for G. This is a topological space that is naturally provided with a cell structure, a metric and a G-action. The latter respects cell structure and metric. Structure statements 1) and 2) are again special cases of general facts:

Ad 1): For each vertex x of X, Bruhat and Tits construct a "model" of $\mathbb{G}$ over $\mathbb{Z}_p$, i.e. a group scheme $\mathcal{G}_x$ over $\mathbb{Z}_p$, whose generic fiber over $\mathbb{Q}_p$ is just $\mathbb{G}$. Then,

- the group $\mathcal{G}_x(\mathbb{Z}_p)$ essentially coincides with the stabilizer P_x of x in G;
- the structure of X near x is determined by the structure of $\mathcal{G}_x(\mathbb{F}_p)$.

Another picture: In the building for group $\mathrm{Sp}_4(\mathbb{Q}_2)$ there are vertices that are contained in exactly 45 two-simplices. According to our selection criterion, we only show 19 of them, which, however, already forces self-intersections. All simplices filled with light and dark are even in a common apartment (Fig. 6).

Ad 2): The apartments of X are isometric to the Euclidean space $\mathbb{R}^d$ as metric spaces, where d denotes the semi-simple $\mathbb{Q}_p$ rank of the group $\mathbb{G}$. The cell structure of the apartments is essentially determined by the root system in $\mathbb{R}^d$ belonging to the reductive group $\mathbb{G}$, and X is always covered by its apartments.

An apartment for group $\mathrm{Sp}_4(\mathbb{Q}_p)$ looks like in Fig. 7.

Just for fun, in Fig. 8, the combination of an apartment image with a "local" image.

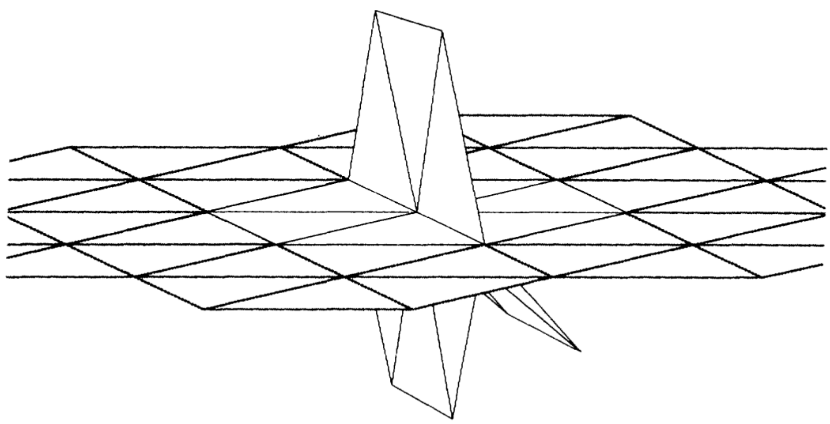

Fig. 8

Summary: The action of G on the Bruhat-Tits building $X(G)$ describes the inner structure of G in a geometric way.

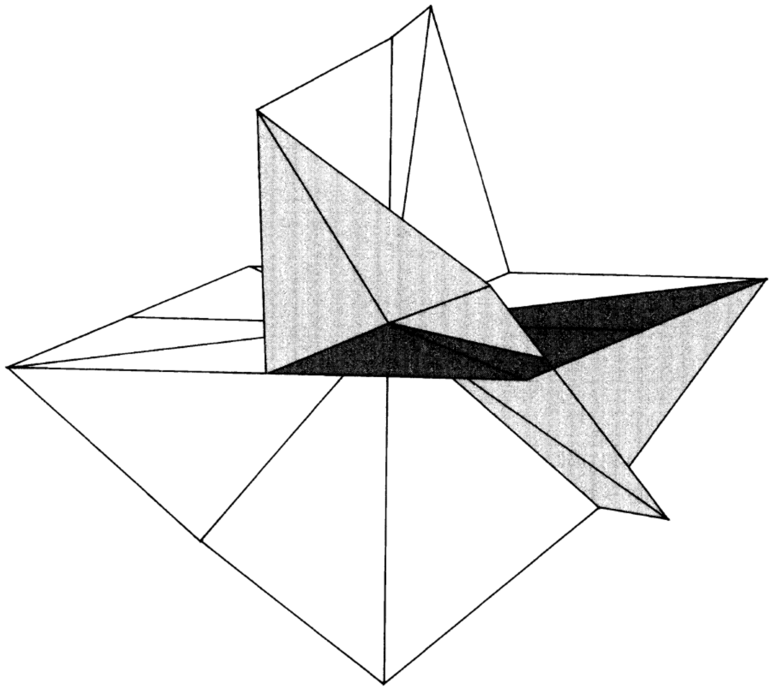

Fig. 6

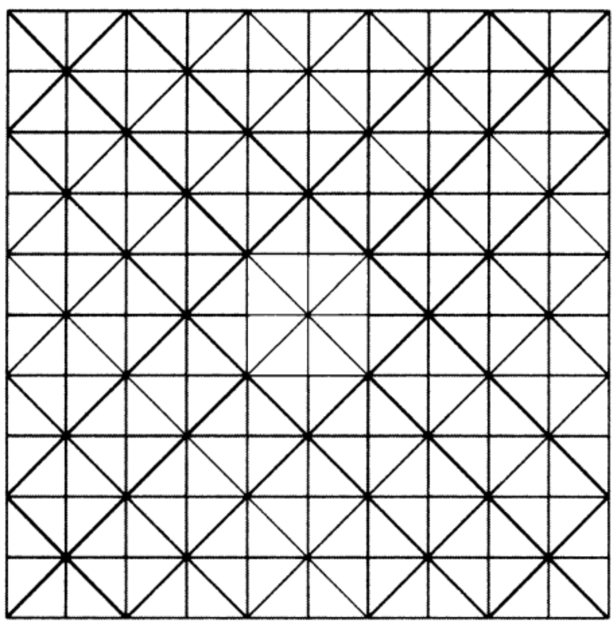

Fig. 7

In the following, I would like to show that the building X is also suitable for describing "external" structures of the group G. We limit to the so-called smooth representation theory of G and report results of the joint work [SS] with U. Stuhler.

Definition: A smooth representation V of G is a $\mathbb{C}$-vector space V equipped with a linear G-action such that for all $v \in V$,

$$\{g \in G | gv = v\} \text{ is open in } G.$$

It is important to realize that the obvious standard action of the group $\mathrm{SL}_n(\mathbb{Q}_p)$ on the n-dimensional $\mathbb{Q}_p$-vector space is not smooth. In fact, apart from the trivial representation, all irreducible smooth representations of $\mathrm{SL}_n(\mathbb{Q}_p)$ are infinite-dimensional! This is a typical phenomenon that is characteristic of this theory. In particular, it is not surprising that the harmonic analysis on the locally compact group G plays a major role. A very good example of an irreducible smooth representation is the so-called Steinberg representation St of $\mathrm{SL}_2(\mathbb{Q}_p)$: for this purpose, V denotes the space of $\mathbb{C}$-valued local constant functions on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over the field $\mathbb{Q}_p$. Let $\mathrm{SL}_2(\mathbb{Q}_p)$ operate by left translation on V, this results in a smooth representation, and the constant functions clearly form an invariant subspace in V. The quotient $\mathrm{St} := V/\mathbb{C}$ is irreducible.

At this point, a brief explanation is appropriate as to why this concept formation, which at first glance appears so little arithmetic, is of the greatest importance for number theory. A fundamental interest of number theory is to understand the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of the algebraic closure $\overline{\mathbb{Q}_p}$ over $\mathbb{Q}_p$, or its finite-dimensional representations. One would like to classify the latter with the help of data that are directly given by the base field $\mathbb{Q}_p$. In local class field theory this problem was solved for 1-dimensional representations, i.e. characters of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. In the so-called local Langlands program, the conjecture stated that the set of parameters for the general classification is essentially just the set of isomorphism classes of irreducible smooth representations of the groups $\mathrm{GL}_n(\mathbb{Q}_p)$ with varying $n \in \mathbb{N}$ (for a precise formulation cf. [Ta]).

Back to smooth representation theory itself. A standard attempt to get a handle on irreducible smooth representations can be roughly described as follows. Let $K \subseteq G$ be a maximal compact subgroup. It is a direct consequence of the smoothness condition that every irreducible smooth K-representation is finite-dimensional and it factors through a finite quotient of K. In other words, the smooth representation theory of the compact group K reduces to the representation theory of finite groups, which we will generously consider "known" in our context.

If V is a smooth G-representation of finite length, it follows (cf. [Ca]) that V as a K-representation is a direct sum

$$V \cong \bigoplus_{\pi \in \widehat{K}} m_V(\pi) \cdot \pi$$

over irreducible smooth K-representations $\pi \in \widehat{K}$ with finite multiplicities $m_V(\pi)$. The hope is now that all the irreducible V are characterized by the choice of K and the multiplicities $m_V(\pi)$. For the groups $\mathrm{GL}_n(\mathbb{Q}_p)$, this strategy has recently been successfully implemented in [BK].

The desire is obvious to bring this approach into a functorial form. Certainly all possible K must be considered simultaneously, which is probably where the building

$X = X(G)$ comes into play. In [SS] we proceed as follows (for the sake of simplicity, the reductive group $\mathbb{G}$ is assumed to be semisimple):

To each cell $F \subseteq X$ denote by $P_F^\dagger \subseteq G$ the stabilizer; this is a compact open subgroup. We construct a natural G-equivariant filtration

$$P_F^\dagger \supseteq U_F^{(0)} \supseteq U_F^{(1)} \supseteq \dots \supseteq U_F^{(e)} \supseteq \dots$$

by compact open normal subgroups $U_F^{(e)} \trianglelefteq P_F^\dagger$. For the following let us fix a "level" $e \geq 0$. Let V be a smooth G-representation of finite length and set

$$V^{U_F^{(e)}} := \left\{ v \in V \middle| gv = v \text{ for all } g \in U_F^{(e)} \right\}.$$

Then one has that

Fact: The space of invariants $V^{U_F^{(e)}}$ is a finite-dimensional representation of the finite group $P_F^\dagger / U_F^{(e)}$.

The pretty simple observation is that these spaces of invariants for varying F (but fixed e) can be combined into a sheaf $\underset{\approx}{V}$ on the building X, so that:

$$\text{the stalk of } \underset{\approx}{V} \text{ at the point } x \text{ is } V^{U_F^{(e)}}, \text{ if } x \in F.$$

More or less by construction we have:

- the sheaf $\underset{\approx}{V}$ is constructible;
- the group G acts on $\underset{\approx}{V}$;
- the assignment $V \mapsto \underset{\approx}{V}$ is an exact functor.

The justification for this formation is provided by a deeper result in [SS]:

- If the level e is chosen big enough (depending on V), then the G-representation V can be recovered from the sheaf $\underset{\approx}{V}$ by passing to a suitable homology group.

Summary: The sheaf $\underset{\approx}{V}$ on building X is a "localization" of the G-rep V.

The methods of algebraic topology are available for the investigation of sheaves. Indeed, in [SS] we obtain statements about the homological algebra of the category of smooth G-representations by computing (co)homology groups. But I would like to end this report by describing an application to the harmonic analysis of G.

If V is infinite-dimensional, then it is meaningless to compute the trace of an element $g \in G$ on V in the sense of linear algebra. But the character of such a representation V as a distribution exists. This means the following. Denote by $\mathcal{H}$ the space of $\mathbb{C}$-valued locally constant functions with compact support on G. This is an associative (usually non-unital) algebra with respect to convolution

$$(\varphi * \psi)(h) := \int_G \varphi(g)\psi(g^{-1}h)dg;$$

where dg is a fixed Haar measure on the locally compact group G. The algebra $\mathcal{H}$ is called the Hecke algebra, to be regarded as the correct version of the group algebra in this context. The smoothness condition means that every smooth G-representation V is automatically an $\mathcal{H}$-module by means of

$$\varphi * v := \int_G \varphi(g) g v dg.$$

If V is of finite length, then the convolution operator $\varphi * . : V \to V$ has finite rank, so that the trace $Tr(\varphi; V)$ is defined ([Ca]). So, we obtain a linear form

$$\mathrm{Tr}(.; V) : \mathcal{H} \to \mathbb{C}.$$

A deep theorem due to Harish-Chandra and Howe from harmonic analysis (cf. [Si]) states that there exists a locally integrable function θ_V on G such that:

$$\mathrm{Tr}(\varphi; V) = \int_G \varphi(g)\theta_V(g) dg \quad \text{for all} \quad \varphi \in \mathcal{H}.$$

This character function θ_V has the usual property of characterizing irreducible V up to isomorphism. But the meaning of the values of the function θ_V remains unclear.

At least for the so-called elliptic elements in G, our localization theory leads to an answer there. An element $g \in G$ is called elliptic, if its centralizer in G is compact. In many problems of harmonic analysis, the study of non-elliptic elements can be reduced to the elliptic case by going to suitable reductive subgroups of $\mathbb{G}$. In some way, the elliptic elements form the "hard kernel" of G. If $g \in G$ is elliptic, then the set of fixed points

$$X^g := \{x \in G | gx = x\}$$

in the building is compact.

Since the sheaf $\underset{\approx}{V}$ is constructible, the cohomology groups $H^*(X^g, \underset{\approx}{V})$ are finite-dimensional. Due to the G-equivariance of $\underset{\approx}{V}$, the group element g also acts on this cohomology. In [SS] we establish the following trace formula of Hopf-Lefschetz type.

Trace formula: For V of finite length, big enough level e, and elliptic $g \in G$, we have that

$$\theta_V(g) = \sum_{i=0}^{d} (-1)^i \cdot \mathrm{Tr}(g; H^i(X^g, \underset{\approx}{V})).$$

Literature

[BK] Bushnell C., Kutzko P.: The admissibledual of GL(N) via compact open subgroups. Ann. Math. Studies **129.** Princeton Univ. Press 1993

[BT] Bruhat F., Tits J.: Groupes réductifs sur un corps local I. Données radicielles valuées. Publ. Math. IHES **41** (1972). II. Schémas en groupes. Existence d'une donnée radicielle valuée. Publ. Math. IHES **60** (1984)

[Ca] Cartier P.: Representations of p-adic groups: a survey. In Automorphic Forms, Representations and L-Functions. Proc. Symp. Pure Math. **33** (1), 111-155. American Math. Soc. 1979

[Si] Silberger A.: Introduction to harmonic analysis on reductive p-adic groups. Princeton Univ. Press 1979

[SS] Schneider P., Stuhler U.: Representation theory and sheaves on the Bruhat-Tits building. Preprint 1995

[Ta] Tate J.: Number theoretic background. In Automorphic Forms, Representations and L-Functions. Proc. Symp. Pure Math. **33** (2), 3-26. American Math. Soc. 1979