Newton's Method and its Ramifications

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and information about how English mathematicians made use of infinite series in their research.

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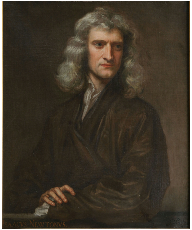

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Similarly, he approximates that $r \approx -0.00004852$. So, $y \approx 2 + (0.1 + (-0.0054 + -0.00004852)) = 2.09455148$.

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Indeed, Newton has the following diagram in his letter.

		$+2{,}10000000$ $-0{,}00544852$ $2{,}09455148$
$2+p=y$	y^3	$+8+12p+6pp+p^3$
	$-2y$	$-4-2p$
	-5	-5
	summa	$-1+10p+6pp+p^3$
$+0{,}1+q=p$	$+p^3$	$+0{,}001+0{,}03q+0{,}3qq+q^3$
	$+6pp$	$+0{,}06 \quad +1{,}2 \quad +6$
	$+10p$	$+1 \quad +10,$
	-1	-1
	summa	$0{,}061 \quad +11{,}23q+6{,}3qq+q^3$
$-0{,}0054+r=q$	$+q^3$	$-0{,}0000001+0{,}000r$ &c
	$+6{,}3qq$	$+0{,}0001837-0{,}068$
	$+11{,}23q$	$-0{,}060642 \quad +11{,}23$
	$+0{,}061$	$+0{,}061$
	summa	$+0{,}0005416+11{,}162r$
$-0{,}00004852+s=r$		

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There is a similar diagram below the previous diagram, where Newton solves $f(x, y) = y^3 + axy + a^2y - x^3 - 2x^3 = 0$ for y in "almost the same way", meaning that he expresses y as a power series in x.

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		$\left(a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512aa} + \frac{509x^4}{16384a^3}\right.$ &c
$a+p=y$	y^3	$a^3+3aap+3app+p^3$
	$+axy$	$+aax+axp$
	$+aay$	$+a^3+aap$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$-\frac{1}{4}x+q=p$	p^3	$-\frac{1}{64}x^3+\frac{3}{16}xxq$ &c
	$+3app$	$+\frac{3}{16}axx-\frac{3}{2}axq+3aqq$
	$+axp$	$-\frac{1}{4}axx+axq$
	$+4aap$	$-axx+4aaq$
	$+aax$	$+aax$
	$-x^3$	$-x^3$
$+\frac{xx}{64a}+r=q$	$3aqq$	$+\frac{3x^4}{4096a}$ &c
	$+\frac{3}{16}xxq$	$+\frac{3x^4}{1024a}$ &c
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3-\frac{1}{2}axr$
	$+4aaq$	$+\frac{1}{16}axx+4aar$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$
	$-\frac{1}{16}axx$	$-\frac{1}{16}axx$

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Fig. 1

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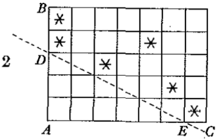
x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
0	y	y^2	y^3	y^4

A C

Fig. 2



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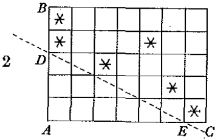

Fig. 2

Newton drew the diagrams above, the *Newton diagram* of $f(x, y)$, and wrote that "I apply the ruler DE to the lower of the places marked in the lefthand column, and rotate it from the lower to the higher to the right till it begins to reach likewise another or perhaps several of the remaining marked places."

Letter 2

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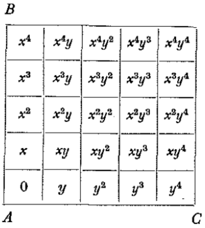

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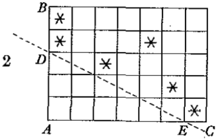

So, a first approximation of a solution to

$$f(x,y) = y^6 - 5xy^5 + (x^3/a)y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$$

is a root of $y^6 - 7a^2x^2y^2 + 6a^3x^3 = 0$. Since the slope is $-1/2$, we want that $y = t(x)^{1/2}$ for some $t \in \mathbb{R}$. Then,

$$x^3(t^6 - 7a^2t^2 + 6a^3) = 0$$

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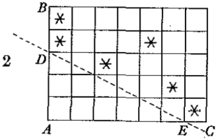

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$$x^3(t^6 - 7a^2t^2 + 6a^3) = 0 \implies t \in \{\pm\sqrt{a}, \pm\sqrt{2a}\} \text{ could work.}$$

We make a choice $\boxed{y = \sqrt{ax} + p}$, substitute it back to $f(x,y)$ to get a new equation $g(\sqrt{x}, p)$, and continue likewise to get q, r, etc.

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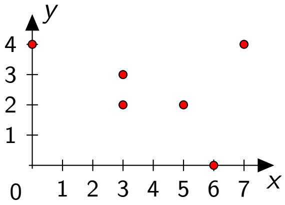

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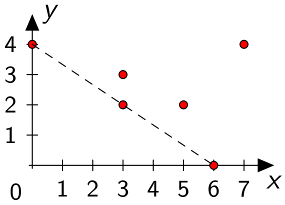

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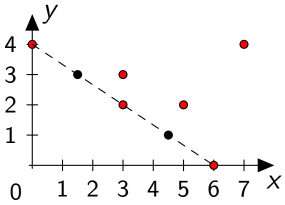

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With a change of variable $x \to x^2$ ($\widetilde{x} := \sqrt{x}$), all points on the dashed line with integer y-coordinates belong to $\mathbb{Z} \times \mathbb{Z}$.

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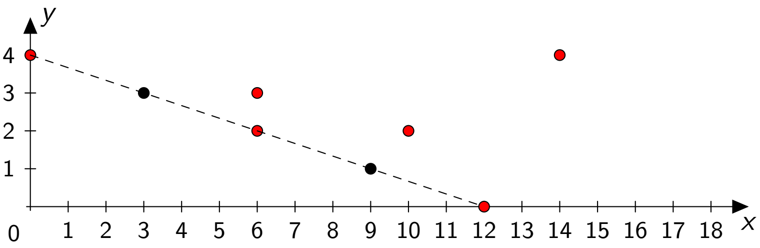

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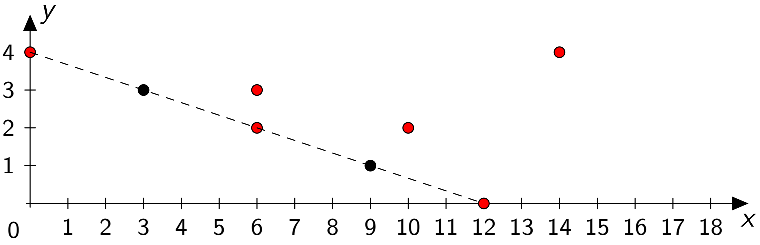

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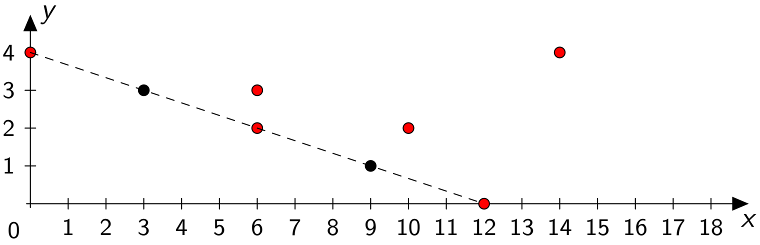

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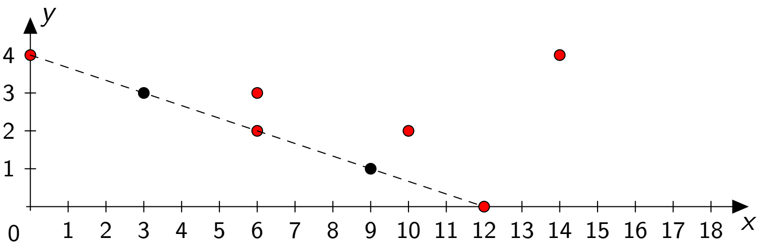

Observe that the dashed line is given by $3y + x = 12$, and other red points (x, y) not on the dashed line must satisfy $3y + x > 12$. (Define *weight* of (x, y) to be $3y + x$). Those terms in $f_0(x, y) = 0$ corresponding to red points of weight 12 form a homogeneous polynomial in y, x^3.

Proof

Since $\mathbb{C}$ is algebraically closed, we can factorize this homogeneous polynomial as follows.

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$$
y \mid y^4 + b_3(x^3)^3 y + b_2(x^3)^2 y^2 + b_1(x^3)y^3 + b_0(x^3)^4
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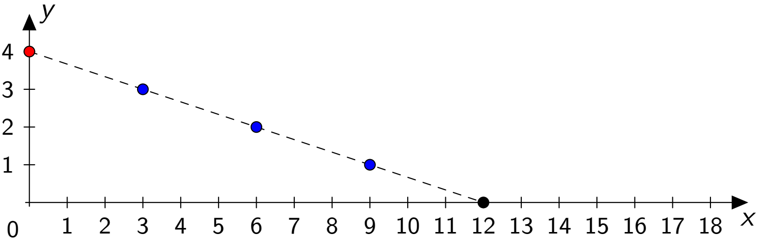
since the change of variable preserves weight. Hence, $b_0 = 0$.

Proof

Geometrically, the vanishing of b_0 means that the lowest point on the dashed line is black.



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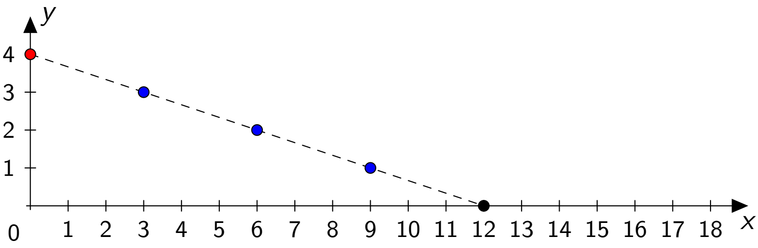

We are not sure whether the intermediate points are red or black, so let them be blue.

Proof

Do a change of variable $y \to x^3 y$ ($\widetilde{y} := y/x^3$).

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Then, $y^4 \to x^{12}y^4, x^3y^3 \to x^{12}y^3, (x^3)^2y^2 \to x^{12}y^2$, and $(x^3)^3y \to x^{12}y$.

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Dividing everything by x^{12}, the Newton diagram is obtained from the previous one by "left-shifting" w.r.t. y-coordinates.

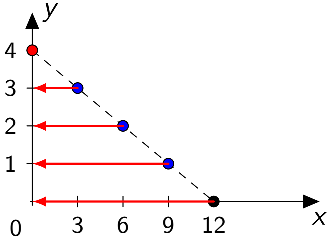

Figure 1: y

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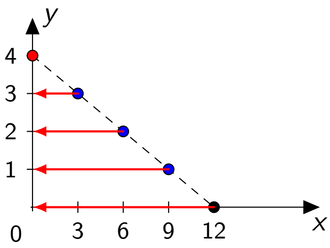

Figure 1: y

$\rightsquigarrow$

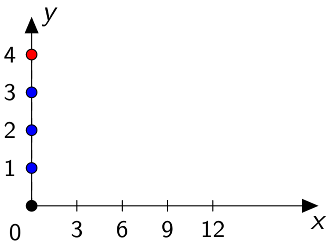

Figure 2: y/x^3 with x^{12} factored out.

Proof

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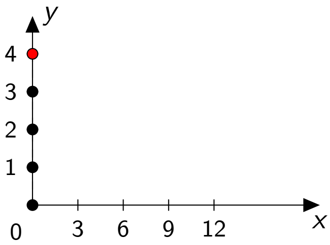

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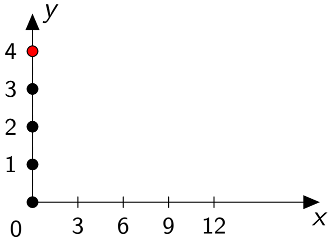

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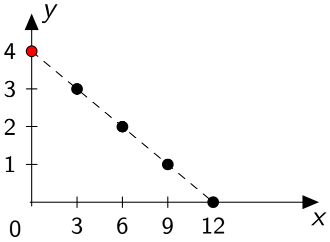

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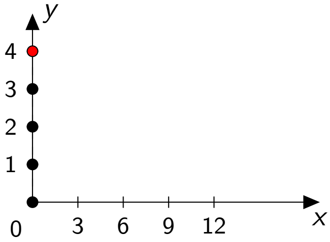

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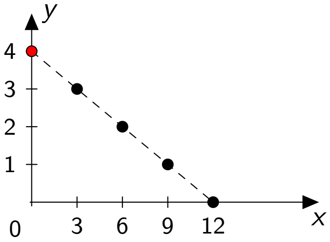
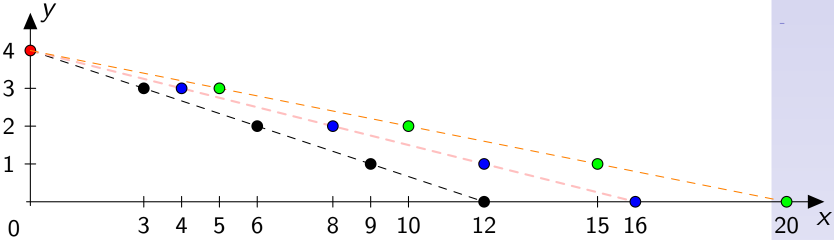


, which means that Newton's ruler should continue to rotate after $y \to y + \alpha_1 x^3$.

Proof

Rotating the ruler and performing changes of variable, we eliminate the coefficients of $y^i x^j$ for all $i < 4$ and all j (up to arbitrarily large constant N).



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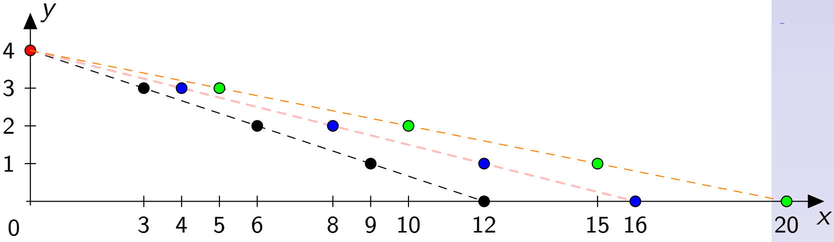

Thus, $y \to y + c_1 x^3 + c_2 x^4 + c_3 x^5 + \dots$ makes

$$y^4 \mid \widetilde{f}\left(x, y + \sum_{n \geq 1} c_n x^{n+2}\right).$$

So, $\widetilde{f}(x, \sum_{n \geq 1} c_n x^{n+2}) = 0$.

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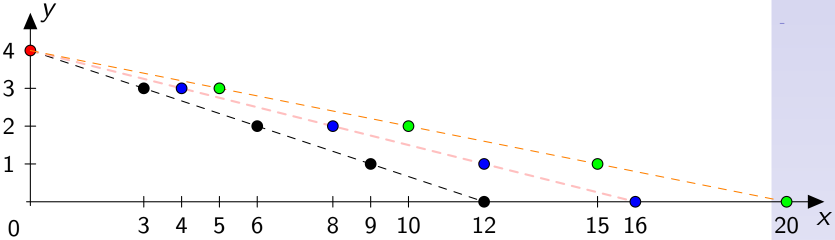

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So, $\widetilde{f}(x, \sum_{n\geq 1} c_n x^{n+2}) = 0$. **Exercise**: Finish the proof.

A Worked-out Example

We take the idea of the proof to solve the following equation

$$
\begin{aligned}
f(x,y) &= 4x^5 - 3x^4 - 4x^2(y-x) + 4(y-x)^2 \\
&= 4x^5 - 3x^4 - 4x^2y + 4x^3 + 4y^2 - 8xy + 4x^2 = 0
\end{aligned}
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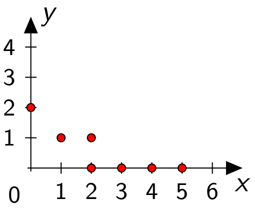

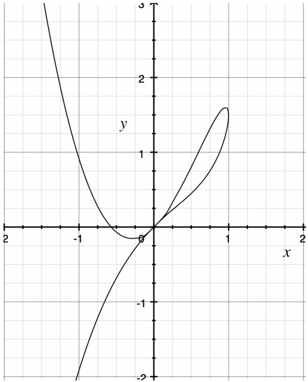

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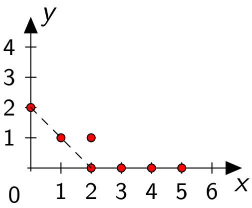
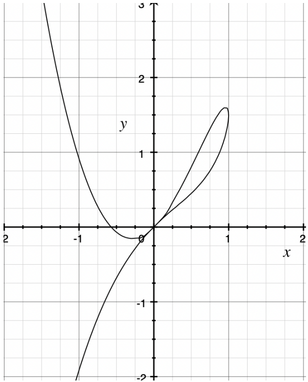
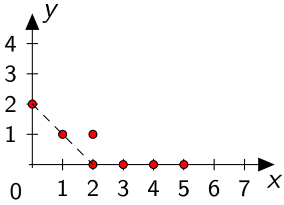




A Worked-out Example

$$f(x,y) = 4x^5 - 3x^4 - 4x^2(y-x) + 4(y-x)^2.$$



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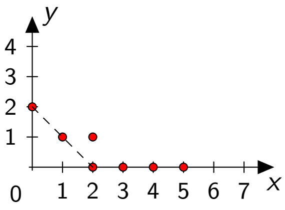

We solve $4y^2 - 8xy - 4x^2 = 4(y - x)^2$ and get $y = x$ as a root.

A Worked-out Example

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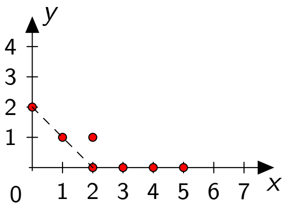


We solve $4y^2 - 8xy - 4x^2 = 4(y-x)^2$ and get $y = x$ as a root. Then, we do $y \to y + x$ ($y_1 = y - x$) to get

$$4x^5 - 3x^4 - 4x^2y + 4y^2 = 0$$

Now, $y \to xy$ ($y_2 = y_1/x$) gives us

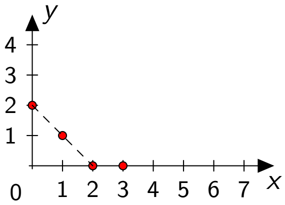
$$4x^5 - 3x^4 - 4x^2(xy) + 4(xy)^2 = x^2 \underbrace{\left(4x^3 - 3x^2 - 4xy + 4y^2\right)}_{g(x,y)}.$$

A Worked-out Example

The Newton diagram of

$$g(x, y) = 4x^3 - 3x^2 - 4xy + 4y^2$$

is



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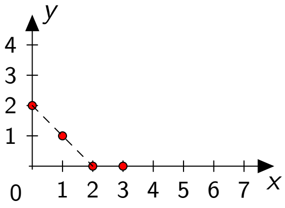

Then, we solve $4y^2 - 4xy - 3x^2 = 0$ to get approximate solutions $y = \frac{3}{2}x$ or $y = -\frac{1}{2}x$.

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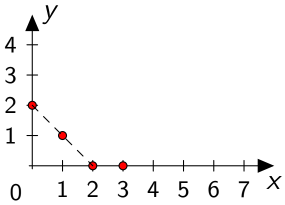

Then, we solve $4y^2 - 4xy - 3x^2 = 0$ to get approximate solutions $y = \frac{3}{2}x$ or $y = -\frac{1}{2}x$. Choosing $y = \frac{3}{2}x$, we do $y \to y + \frac{3}{2}x$ ($y_3 = y_2 - \frac{3}{2}x$).

A Worked-out Example

Substituting $y + \frac{3}{2}x$ into $g(x, y)$, we have

$$g(x, y + \frac{3}{2}x) = 4x^3 - 3x^2 - 4x(y + \frac{3}{2}x) + 4(y + \frac{3}{2}x)^2$$

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With $y \to xy$ $(y_4 = y_3/x)$, we have

$$4x^3 + 8x(xy) + 4(xy)^2 = 4x^2 \underbrace{\left(x + 2y + y^2\right)}_{h(x,y)}.$$

A Worked-out Example

Substituting $y + \frac{3}{2}x$ into $g(x, y)$, we have

$$\begin{aligned} g(x, y + \frac{3}{2}x) &= 4x^3 - 3x^2 - 4x(y + \frac{3}{2}x) + 4(y + \frac{3}{2}x)^2 \\ &= 4x^3 - \cancel{3x^2} - 4xy + \cancel{6x^2} + 4y^2 + 12xy + \cancel{9x^2} \\ &= 4x^3 + 8xy + 4y^2 \end{aligned}$$

With $y \to xy$ $(y_4 = y_3/x)$, we have

$$4x^3 + 8x(xy) + 4(xy)^2 = 4x^2 \underbrace{\left(x + 2y + y^2\right)}_{h(x,y)}.$$

Here, we can just apply the quadratic formula:

$$y_4 = y = \frac{-2 \pm \sqrt{4 - 4x}}{2} = -1 \pm \sqrt{1 - x}.$$

A Worked-out Example

By Newton's generalized binomial theorem:

$$\boxed{(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k, \quad \forall r \in \mathbb{R}}$$

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$$\sqrt{1-x} = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} x^k$$

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So, $y_4 = -\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} + O(x^6)$ or
$y_4 = -2 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{5x^4}{128} + \frac{7x^5}{256} + O(x^6)$.

A Worked-out Example

So, $y = x + (x(\frac{3}{2}x + x(-1 \pm \sqrt{1-x})))$ are two solutions to $f(x,y) = 0$.

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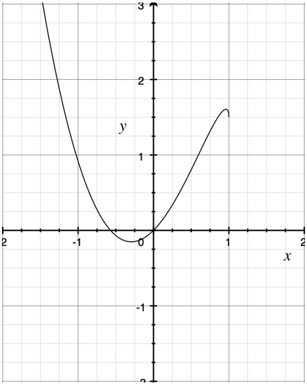
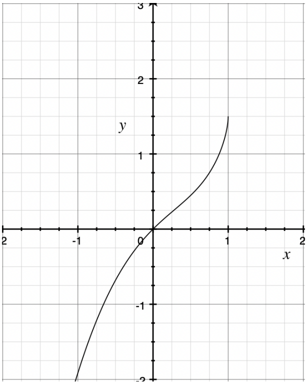
So, $y = x + (x(\frac{3}{2}x + x(-1 \pm \sqrt{1-x})))$ are two solutions to $f(x, y) = 0$. Their graphs are:



$+\sqrt{1-x}$



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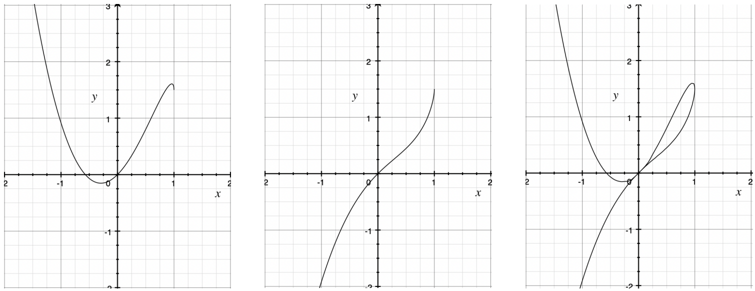
A Worked-out Example

Comparing the following three graphs:



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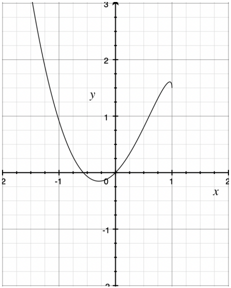

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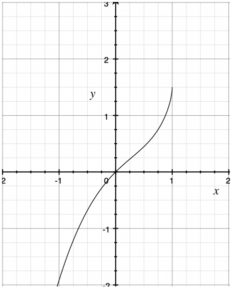

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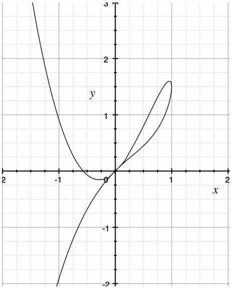

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The End

Thank you for your attention!

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[4, 3, 5, 2, 1]

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