

# Schneider-Stuhler complex of locally analytic principal series for $\mathrm{PGL}_n$ .

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# Problem

## Theorem (Schneider-Stuhler, 1993)

Let  $U := U_{[\mathbb{Z}_p^n]}^{(e)} = \left\{ \mathrm{GL}_n(\mathbb{Z}_p) \ni A \equiv I_n \pmod{p^e} \right\}$ . For any  $V$  generated by  $V^U$  as  $G$ -representation, the Schneider-Stuhler complex  $\mathrm{SS}_\bullet := C_c^{or}(X_\bullet, \underline{V})$  is a resolution of  $V$ .

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We may consider the larger category of locally  $\mathbb{Q}_p$ -analytic  $E$ -linear  $G$ -representations, and replace the fixed vectors  $V^{U_F}$  by the analytic vectors  $V^{U_F\text{-an}}$ . Then we again get a chain complex.



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## Question

Suppose  $V$  is generated by  $V^{U\text{-an}}$ . Is it true that its locally analytic Schneider-Stuhler complex gives a resolution of  $V$ ?

## Related results for locally analytic principal series $V$

### Theorem (Lahiri, 2020)

For  $n = 2$ , i.e.,  $G = \mathrm{GL}_2(\mathbb{Q}_p)$ , let  $B$  be the upper-triangular Borel subgroup,  $T$  the diagonal torus, and  $V = \mathrm{Ind}_B^G(\chi)$  the locally analytic induction of any locally  $\mathbb{Q}_p$ -analytic character of  $T$ .

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### Remark

The Schneider-Stuhler complex can be defined in exactly the same way for  $\mathrm{char} \, p$  smooth  $V$ , but it is *not* always a resolution of  $V$  ( $\exists$  counterexample by Ollivier-Schneider, using results of Breuil).

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### Theorem (Kohlhaase-Schraen, 2013)

Let  $I \subset G := \mathrm{PGL}_n(\mathbb{Q}_p)$  be the Iwahori subgroup, i.e., represented by those in  $\mathrm{PGL}_n(\mathbb{Z}_p)$  which are upper-triangular modulo  $p$ .

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$$\left( \bigwedge^{\bullet} E^{\Delta} \right) \otimes_E \mathrm{c}\text{-Ind}_I^G(\mathcal{A}) \rightarrow V = \mathrm{Ind}_B^G(\chi) \rightarrow 0$$

for large enough  $e$ , where  $\Delta$  is the set of simple roots of  $G$ .

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To each  $\alpha \in \Delta$ , the authors define  $y_\alpha \in \mathrm{End}_G(\mathrm{c}\text{-Ind}_I^G(\mathcal{A}))$ . Then, the resolution is the Koszul complex of  $\mathrm{c}\text{-Ind}_I^G(\mathcal{A})$  defined by these endomorphisms.

## Related results for locally analytic principal series $V$

Since the two complexes have the same length  $n$  and the same 0-th homology  $V$ , one may hope to relate them. Indeed, we can show

### Proposition

For  $n = 2$ , there is a  $\mathrm{PGL}_2(\mathbb{Q}_p)$ -equivariant isomorphism from the Kohlhaase-Schraen resolution to the Schneider-Stuhler complex.

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### Remark

For  $n \geq 3$ , if there is a morphism of complexes  $\mathrm{KS}_\bullet \rightarrow \mathrm{SS}_\bullet$ , it seems that it is not an isomorphism degree-wise.

# Consequences and the next step

## Application

There is a work in progress by Shishir Agrawal and Matthias Strauch, which, assuming the exactness of the Schneider-Stuhler complex, uses the theory of solid locally analytic representations and a spectral sequence argument to show that if  $V$  and  $W$  are admissible, then the Schneider-Stuhler resolution of  $V$  can be used to compute  $\mathrm{Ext}_G(V, W)$ .

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## Future

The next step is to see if the arguments of Ollivier can be adapted to locally analytic principal series.