Notes on Riemann-Roch Theorem

My reference is the first chapter of Stichtenoth's "Algebraic Function Fields and Codes".

Definition 0.0.1. An algebraic function field F/K of one variable over K is a finite algebraic extension $F/K(x)$ for some $x \in F$ that is of transcendental degree 1 over K.
The field of constants of F/K is $\widetilde{K} := \{z \in F | z \text{ is algebraic over } K\}$.
A valuation ring of the function field F/K is a ring $\mathcal{O} \subset F$ satisfying:

(1) $K \subsetneq \mathcal{O} \subsetneq F$,
(2) for every $z \in F$, we have $z \in \mathcal{O}$ or $z^{-1} \in \mathcal{O}$.

A place of the function field F/K is the unique maximal ideal P of some valuation ring $\mathcal{O}$ of F/K. We define $\mathbf{P}_F := \{P | P \text{ is a place of } F/K\}$.
For a valuation ring $\mathcal{O}$ with a place P, we have the quotient map $\pi : \mathcal{O} \to \mathcal{O}/P$.
Since $\widetilde{K}$ consists of algebraic elements, $\widetilde{K} \subset \mathcal{O}$ and $\widetilde{K} \cap P = \{0\}$ as P consists of non-units of $\mathcal{O}$. So, $\pi|_{\widetilde{K}}$ is an embedding of $\widetilde{K}$ into the residue field $\mathcal{O}/P$.
The degree of P is $\deg P := [F_P : K]$, where $F_P = \mathcal{O}/P$ and K is the image of K under $\pi|_K$.

Proposition 0.0.2. *For a place P and any nonzero $x \in P$, we have $\deg P \leq [F : K(x)]$. Since x is necessarily transcendental over K, $\deg P < \infty$.*

Proof. If $[z_1], \dots, [z_n] \subset F_P$ are K-LI, $z_1, \dots, z_n \subset \mathcal{O}$ are $K(x)$-LI. (Just apply $\pi : \mathcal{O} \to \mathcal{O}/P$ to $K(x) \subset \mathcal{O}$) □

Corollary 0.0.3. *We have $[\widetilde{K} : K] \leq [F_P : K] = \deg P < \infty$.*

Lemma 0.0.4. *If R is a subring of F containing K, and I is a nonzero proper ideal of R, then there exists a place P such that $P \supset I$ and $\mathcal{O}_P \supset R$.*

Proof. A maximal element $\mathcal{O}$ of the nonempty poset

$$\mathcal{F} := \{S | S \text{ is a subring of } F \text{ with } S \supset R \text{ and } IS \neq S\}$$

is a valuation ring having the desired properties. □

Theorem 0.0.5. *Every transcendental $z \in F$ has at least one zero and one pole. In particular, $\mathbf{P}_F \neq \emptyset$.*

Proof. Let $R = K[z]$ and $I = zK[z]$. Since z is transcendental, R, I satisfy the assumptions of the previous lemma. So, $z \in P$ for some place P by the lemma.

Let $R = K[z^{-1}]$ and $I = z^{-1}K[z^{-1}]$. By the lemma, $z^{-1} \in Q$ for some place Q, and z has a pole at Q. □

Theorem 0.0.6 (Weak Approximation Theorem)**.** *Let $P_1, \dots, P_n$ be distinct places. Then for any $x_1, \dots, x_n \in F$ and $r_1, \dots, r_n \in \mathbb{Z}$, there exists $x \in F$ such that $v_i(x - x_i) = r_i$ for $1 \leq i \leq n$.*

Proposition 0.0.7. *Let $P_1, \dots, P_r$ be any finite subset of zeros of an nonzero element $x \in F$. Then,*

$$\sum_{i=1}^{r} v_{P_i}(x) \deg P_i \leq [F : K(x)].$$

Sketch of Proof. By the weak approximation theorem, for each i we can choose $t_i \in F$ with

$$v_j(t_i) = \delta_{ij}, \quad \forall 1 \leq j \leq r.$$

For any K-basis of F_{P_i}, say $[s_{i1}], \dots, [s_{id_i}]$ with $s_{ij} \in \mathcal{O}_{P_i}$ for $1 \leq j \leq d_i = \deg P_i$. By the weak approximation theorem, we can modify them by some elements in P_i to get $\{z_{ij}\}_{j=1}^{d_i}$ such that

$$\left\{ t_i^{k_i} z_{ij_i} \middle| 1 \leq i \leq r, 0 \leq k_i \leq v_{P_i}(x) - 1, 1 \leq j_i \leq \deg P_i \right\}$$

is $K(x)$-linearly independent. □

Corollary 0.0.8. *Every nonzero element $0 \neq x \in F$ has finitely many zeros (and poles).*

Proof. if $x \in \widetilde{K}$, x has neither zeros nor poles.
If x is transcendental over K, by the inequality (proposition) above, its number of zeros is bounded from above by $[F : K(x)] < \infty$.
By considering x^{-1}, we see that x has finitely many poles. □

Since $[\widetilde{K} : K] < \infty$, $F/\widetilde{K}$ is also a function field. From now on, we assume that $K = \widetilde{K}$.

Definition 0.0.9. The divisor group of F/K is the free abelian group generated by $\mathbf{P}_F$. For a divisor, $D = \sum_{P\in\mathbf{P}_F} n_P P =: \sum_{P\in\mathbf{P}_F} v_P(D)P$, its degree $\deg D$ is $\sum_P v_P(D)\deg P$.
For $0 \neq x \in F$, define

$$(x)_0 = \sum_{P \text{ zero}} v_P(x)P, \quad (x)_\infty = \sum_{P \text{ pole}} (-v_P(x))P, \quad (x) := (x)_0 - (x)_\infty.$$

By the "algebraically closed" assumption, since (transcendental $\Rightarrow$ at least one zero and one pole), we have that for $x \in F \setminus \{0\}$, x is algebraic (i.e. $x \in K$) $\iff$ $(x) = 0$.

Just as before, we define the principal divisors $\mathrm{Princ}(F)$ as $\{(x) | 0 \neq x \in F\}$, the divisor class group $\mathrm{Cl}(F)$ as the quotient of divisor group by the principal divisors, and any two divisors sharing the same divisor class are equivalent.

The Riemann-Roch space of a divisor A is again the K-vector space

$$\mathscr{L}(A) = \{0 \neq x \in F | (x) \geq -A\} \cup \{0\} = \{x \in F | v_P(x) \geq -v_P(A), \forall P\}$$

with $\dim(\mathscr{L}(A)) =: \ell(A)$, which is nonzero if and only if $A \sim A'$ for some effective A'. Again, equivalent divisors have isomorphic Riemann-Roch spaces ($\in \underline{\mathrm{Vect}_K}$):

$$A' \sim A \implies \mathscr{L}(A) \cong \mathscr{L}(A').$$

Lemma 0.0.10. *(a)* $\mathscr{L}(0) = K$*,*
(b) If $A < 0$ *then* $\mathscr{L}(A) = \{0\}$*.*

Proof. Any transcendental element has at least one zero and at least one pole. □

Next Goal: $\ell(A)$ is finite for every divisor A.

Lemma 0.0.11. *For divisors* $A \leq B$*, we have* $\mathscr{L}(A) \subset \mathscr{L}(B)$ *and*

$$\deg A - \ell(A) \leq \deg B - \ell(B)$$

Proof. Clearly we can prove it by induction: so we may let $B = A + P$ for some place P. Choose $t \in F$ such that $v_P(t) = v_P(B)$. Then, $T : \mathscr{L}(B) \to \mathcal{O}_P \to \mathcal{O}_P/P = F_P, x \mapsto [(xt)]$ is K-linear s.t. $\ker T = \{0\} \cup \{x | (xt) \in P \text{ (i.e. } v_P(x) \geq -v_P(t) + 1 = -v_P(A))\} = \mathscr{L}(A)$. □

Proposition 0.0.12. *If* $A = A_+ - A_-$ *for effecitve* A_+ *and* A_-*, then* $\ell(A) \leq \deg A_+ + 1$*.*

Proof. It is clear that $\mathscr{L}(A) \subset \mathscr{L}(A_+)$ because $A \leq A_+$. Applying the previous lemma to $0 \leq A_+$ yields $\deg A_+ - \ell(A_+) \geq \deg 0 - \ell(0) \implies \ell(A_+) \leq \deg A_+ + 1$. □

Corollary 0.0.13. *For every* A *with* $\deg A > 0$*,* $\ell(A) \leq 1 + \deg A$*.*

Proof. The previous proposition addresses the case of effective A.
We may assume that $\ell(A) > 0$. Then $A \sim A'$ for an effective A'. Thus, $\ell(A) = \ell(A'), \deg(A) = \deg(A')$ here I'm assuming that $\deg((x)) = 0, \forall 0 \neq x \in F$, and we are back to the effective case. □

Proposition 0.0.14 (For the red part of the proof above)**.** *For* $x \in F \setminus K$*,* $\deg((x)_\infty) = [F : K(x)]$*. Then,* $\deg((x)_0) = \deg((x^{-1})_\infty) = [F : K(x^{-1})] = [F : K(x)]$.

Proof. Let $P_1, \ldots, P_r$ be all the poles of x. Then $\deg(x_\infty) = \sum_{i=1}^r v_i(x^{-1}) \deg P_i \le [F : K(x)]$ by a previous result.

We claim that $n := [F : K(x)] \le \deg(x_\infty)$. Let $e_1, \ldots, e_n$ be a $K(x)$-basis of F. Choose an effective $D \in Div(F)$ such that $D \ge -(e_i), \forall i$. Then, for all $m \ge 0$,

$$\left\{x^i e_j \middle| 0 \le i \le m, 1 \le j \le n\right\} \text{ is a } K\text{-linearly independent subset of } \mathscr{L}(m(x_\infty) + D)$$

, because $e_1, \ldots, e_n$ are $K(x)$-LI. Consequently, $(m+1)n \le \ell(m(x_\infty)+D) \le \deg(m(x_\infty) + D) + 1$ since $m(x_\infty) + D$ is effective. Thus,

$$m(\deg(x_\infty) - n) \ge n - \deg D - 1 \implies \deg(x_\infty) - n \ge 0. \quad \square$$

Proposition 0.0.15. *There is $\gamma \in \mathbb{Z}$ such that $\deg A - \ell(A) \le \gamma$ for every divisor A.*

Proof. For a fixed $x \in F \setminus K$, $\exists D$ s.t. $\forall m \ge 0$, $\ell(m(x_\infty) + D) \ge (m+1)n = (m+1)\deg(x_\infty)$. We have already proven that $\deg A - \ell(A)$ is an increasing function in A, so $\ell(m(x_\infty)+D) \le \deg D + \ell(m(x_\infty))$. Hence,

$$(m+1)\deg(x_\infty) \le \deg D + \ell(m(x_\infty)) \implies \boxed{\deg(m(x_\infty)) - \ell(m(x_\infty)) \le \deg D - n =: \gamma.}$$

Claim. *For a given A, we can find A', C and $m \ge 0$ such that $A \le A' \sim C \le m(x_\infty)$, from which it is clear that $\deg A - \ell(A) \le \deg(m(x_\infty)) - \ell(m(x_\infty)) \le \gamma$.*

We choose A' to be any effective divisor that is $\ge A$, and compute the dimension

$$\ell(m(x_\infty) - A') \ge \ell(m(x_\infty)) - \deg A' \ge (\deg(m(x_\infty)) - \gamma) - \deg A' > 0 \text{ for large } m.$$

Thus, we choose non-zero $z \in \mathscr{L}(m(x_\infty) - A')$ and put $C := A' - (z) \le m(x_\infty)$. $\square$

Definition 0.0.16. The genus of F/K is $g := \max\{\deg(A) - \ell(A) + 1 | A \in \mathrm{Div}(F)\}$. With $A = 0$, we see that $g \ge \deg(0) - \ell(0) + 1 = 0$, so the genus is non-negative. For every divisor A, $\ell(A) \ge \deg(A) + 1 - g$.

Theorem 0.0.17 (Riemann's Theorem). *There exists an integer c, depending only on F/K, such that if $\deg A \ge c$, then $\ell(A) = \deg A + 1 - g$.*

Proof. Let A_0 be a divisor such that $\deg A_0 - \ell(A_0) + 1 = g$ and put $c = \deg A_0 + g$.

If $\deg(A) \ge c$ then

$$\ell(A - A_0) \ge \deg(A - A_0) + 1 - g \ge c - \deg A_0 + 1 - g = 1.$$

So, we can pick $0 \ne z \in \mathscr{L}(A - A_0)$ and set $A' := A + (z) \ge A_0$; then

$$\deg A - \ell(A) = \deg A' - \ell(A') \ge \deg A_0 - \ell(A_0) = g - 1.$$

By the definition of g, we have $\ell(A) = \deg A + 1 - g$. $\square$

Definition 0.0.18. For $A \in \mathrm{Div}(F)$ the integer

$$i(A) = \ell(A) - \deg A + g - 1$$

is called the index of specialty of A.

Riemann's Theorem stats that $i(A) \geq 0$ and $i(A) = 0$ if $\deg A$ is sufficiently large.

Definition 0.0.19. An adele of F/K is a mapping

$$\alpha : \mathbf{P}_F \to F, P \mapsto \alpha_P,$$

such that $\alpha_P \in \mathcal{O}_P$ for almost every $P \in \mathbf{P}_F$.
Considering a mapping as an element in the direct product, we write $\alpha = (\alpha_P)_P$.
The set $\mathcal{A}_F := \{\alpha | \alpha \text{ is an adele of } F/K\}$ is the adele space of F/K, which is an K-algebra.
The principal adele of $x \in F$ is the adele whose components are all equal to x (since x has finitely many poles). Then $F \hookrightarrow \mathcal{A}_F$ is an embedding.
For each place P and adele $\alpha = (\alpha_P)$, we define a valuation $v_P(\alpha) := v_P(\alpha_P)$. By the definition of adele space, $v_P(\alpha) \geq 0$ for almost all P.
For each divisor A, its Riemann-Roch space of adeles is a K-subspace of $\mathcal{A}_F$ defined as

$$\mathcal{A}_F(A) = \{\alpha \in \mathcal{A}_F | v_P(\alpha) \geq -v_P(A), \forall P \in \mathbf{P}_F\} \text{ “} \supset \mathscr{L}(A)\text{” via } F \hookrightarrow \mathcal{A}_F.$$

Theorem 0.0.20. *For every A, $i(A) = \dim(\mathcal{A}_F/(\mathcal{A}_F(A) + F))$.*

Proof. We proceed as follows

(1) If $A_1 \leq A_2$, then $\deg A_2 - \deg A_1 = \dim \mathcal{A}_F(A_2) - \dim \mathcal{A}_F(A_1) = \dim((\mathcal{A}_F(A_2))/(\mathcal{A}_F(A_1)))$.

Proof. By induction, let $A_2 = A_1 + P$. Let $t \in F$ such that $v_P(t) = v_P(A_2) = v_P(A_1) + 1$ Consider the residue map $\mathcal{A}_F(A_2) \to F_P, \alpha \mapsto [t\alpha_P]$. We still have kernel $= \mathcal{A}_F(A_1)$, but this time the map is surjective. ☐

(2) We have a short exact sequence

$$0 \to \frac{\mathscr{L}(A_2)}{\mathscr{L}(A_1)} \to \frac{\mathcal{A}_F(A_2)}{\mathcal{A}_F(A_1)} \to \frac{\mathcal{A}_F(A_2) + F}{\mathcal{A}_F(A_1) + F} \to 0.$$

Thus,

$$\begin{aligned}\dim(\mathcal{A}_F(A_2) + F) - \dim(\mathcal{A}_F(A_1) + F) &= x(\dim \mathcal{A}_F(A_2) - \dim \mathcal{A}_F(A_1)) - (\ell(A_2) - \ell(A_1)) \\ &= (\deg A_2 - \deg A_1) - (\ell(A_2) - \ell(A_1)) \qquad \text{by (1)}\end{aligned}$$

(3) If B is such that $\ell(B) = \deg B + 1 - g$, then $\mathcal{A}_F = \mathcal{A}_F(B)$.

Proof. Since $\deg B - \ell(B)$ is increasing in B, for any $B' \geq B$, we have $\ell(B) \leq \deg B' + 1 - g \leq \ell(B)$. So $\deg B' - \ell(B') = g - 1$ for all $B' \geq B$. For any $\alpha \in \mathcal{A}_F$, we find $B' \geq B$ such that $\alpha \in \mathcal{A}_F(B')$. Apply (2) to $B' \geq B$ to conclude $\alpha \in \mathcal{A}_F(B') + F = \mathcal{A}_F(B) + F$. ☐

For an arbitrary A, there is $B \geq A$ such that $\ell(B) = \deg B + 1 - g$ by Riemann's theorem. Thus $\dim(\mathcal{A}_F) = \dim(\mathcal{A}_F(B) + F)$ by (3), and

$$\begin{aligned}\dim(\mathcal{A}_F/(\mathcal{A}_F(A) + F)) &= \dim(\mathcal{A}_F(B) + F) - \dim(\mathcal{A}_F(A) + F) \\ &= (\deg B - \deg A) - (\ell(B) - \ell(A)) \qquad \text{by (2)} \\ &= (g-1) - \deg A + \ell(A) = i(A).\end{aligned}$$

☐

We have shown that $\boxed{\ell(A) - \dim(\mathcal{A}_F/(\mathcal{A}_F(A) + F)) = \deg A + 1 - g}$.
Goal: show that $\dim(\mathcal{A}_F/(\mathcal{A}_F(A)+F)) = \dim \mathscr{L}((\omega)-A)$ for any differential $\omega \neq 0$. What? Observe that since the dimension is finite, $\mathcal{A}_F/(\mathcal{A}_F(A) + F)$ is isomorphic to its dual space, which can be thought of as the subspace of $\mathcal{A}_F^*$ that vanish on $\mathcal{A}_F(A)+F$, denoted by $\Omega_F(A)$. Therefore, $i(A) = \dim(\mathcal{A}_F/(\mathcal{A}_F(A)+F)) = \dim(\Omega_F(A))$: $\boxed{\ell(A) - \dim(\Omega_F(A))) = \deg A + 1 - g}$.
So, it suffices to show that $\Omega_F(A) \cong \mathscr{L}((\omega) - A)$.

Definition 0.0.21. The space of Weil differentials of F/K is

$$\Omega_F := \{\omega \in \mathcal{A}_F^* | \exists A \in \mathrm{Div}(F) \text{ s.t. } (\mathcal{A}_F(A) + F) \subset \ker \omega\}.$$

It is clear that $\Omega_F(A) \subset \Omega_F$ for every $A \in \mathrm{Div}(F)$.
If $\deg A \leq -2$, we have $\dim(\Omega_F(A)) = i(A) = \ell(A) - \deg A + g - 1 \geq 0 + 2 + 0 - 1 = 1$, hence $0 \neq \Omega_F(A) \subset \Omega_F$.

Definition 0.0.22. For a non-zero Weil differential $0 \neq \omega$, we define its divisor (ω) to be

$$\max\{A \in \mathrm{Div}(F) | \omega \text{ vanishes on } \mathcal{A}_F(A)\}.$$

Justification. By Riemann, if $\deg A \geq c$ for some c, then $i(A) = 0$ and $\mathcal{A}_F = \mathcal{A}_F(A) + F$. If $\omega \neq 0$, then $\deg A < c$. Take (ω) to be a divisor of highest degree. □

And $v_P(\omega) := v_P((\omega))$.
It follows from the definitions that $\Omega_F(A) = \{\omega | \omega = 0 \text{ or } (\omega) \geq A\}$. In particular, $\Omega_F(0) = \{\omega | \omega = 0 \text{ or holomorphic}\}$, and $g = i(0) = \dim(\Omega(0))$.

Proposition 0.0.23. *$x \in F$ acts on $\omega \in \Omega_F$ by $(x\omega : (\alpha_P)_{P \in \mathbf{P}_F} \mapsto \omega((x\alpha_P)_{P\in\mathbf{P}_F}) \in \Omega_F$. We have Ω is 1-dimensional F-vector space.*

Proof. Since $i(A) = \dim(\Omega_F(A))$, we know that $\Omega_F \neq 0$.
Let ω_1, ω_2 be non-zero differentials vanishing on $\mathcal{A}_F(A_1)$ and $\mathcal{A}_F(A_2)$ respectively.
For $x \in \mathscr{L}(A_i + B)$, $(x) \geq -A_i - B$, so $(x\omega_i) \geq -A_i - B + A_i = B$ and $x\omega_i \in \Omega_F(-B)$.
We consider the injective K-linear maps $\varphi_i : \mathscr{L}(A_i + B) \to \Omega_F(-B)$. It suffices to show that

$$\mathrm{Im}(\varphi_1) \cap \mathrm{Im}(\varphi_2) \neq \{0\}$$

because then $x_1\omega_1 = x_2\omega_2$ for non-zero $x_1, x_2 \in F$.
But this is a simple consequence of the dimension formula

$$\dim \mathrm{Im}(\varphi_1) \cap \mathrm{Im}(\varphi_2) \geq \dim(\mathrm{Im}(\varphi_1)) + \dim(\mathrm{Im}(\varphi_2)) - \dim(\Omega_F(-B)).$$

Indeed, $\dim(\Omega_F(-B)) = i(-B) = \dim(-B) - \deg(-B) + g - 1 = \deg B - 1 + g$; and by taking large enough $\deg B$, by Riemann's theorem

$$\dim(\mathrm{Im}(\varphi_i)) = \ell(A_i + B) = \deg A_i + \deg B + 1 - g.$$

Thus, $\dim \mathrm{Im}(\varphi_1) \cap \mathrm{Im}(\varphi_2) \geq \deg B + (\deg A_1 + \deg A_2 + 3(1-g))[> 0 \text{ if } \deg B \text{ is large.}]$ □

Proposition 0.0.24. *For non-zero $x \in F$ and non-zero $\omega \in \Omega_F$, we have $(xw) = (x) + (w)$. Thus any two non-zero differentials have equivalent (canonical) divisors, as $\dim_F(\Omega) = 1$.*

Proof. If ω vanishes on $\mathcal{A}_F(A)$, then xw vanishes on $\mathcal{A}_F(A + (x))$.
Taking $A = (\omega)$, we have $(x\omega) \geq (\omega) + (x)$. Similarly, $(\omega) \geq (x\omega) + (x^{-1}) = (x\omega) - (x)$. □

Theorem 0.0.25 (Duality Theorem)**.** *Let $A \in \mathrm{Div}(F)$, and $W = (\omega)$ be a canonical divisor. Then,*

$$\mu : \mathscr{L}(W - A) = \mathscr{L}((\omega) - A) \to \Omega_F(A), x \mapsto x\omega$$

is a K-linear isomorphism.

Proof. If $x \neq 0$ and $(x) \geq A - (\omega)$, then $(x\omega) = (x) + (\omega) \geq A$, hence $x\omega \in \Omega_F(A)$.
It is clear that μ is well-defined and injective.
To show that μ is surjective, let $\omega' \in \Omega_F(A) \setminus \{0\}$. Then there exists $x \in F$ s.t. $\omega' = x\omega$ for a unique non-zero $x \in F$. Since $(x\omega) \geq A$, we have $(x) \geq A - (\omega)$ and $x \in \mathscr{L}((\omega) - A)$. $\square$