Projekti – OR 2025/2026
pri predmetu Finančni praktikum*

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15. oktober 2025

1 Osnovno o projektu

Osnove:

- Pričakovana obremenitev posameznega študenta za projekt je 60 ur.
- Vsak projekt izdela skupina. Skupina ima lahko dva ali tri študente. Skupine so naključne in jih določi učitelj.
- Študenti v isti skupini lahko dobijo različne ocene.
- Vsaka skupina dobi smernico. Pri določanju točne vsebine imate nekaj **fleksibilnosti**.
- Projekt mora vsebovati programiranje in **eksperimente**. Lahko si izberete programsko okolje, v katerem boste delali. Lahko najdete podatke na internetu, lahko jih pa tudi generirate sami.
- Na koncu projekta bo zagovor.
- Asistent T. Hrga in prof. R. Škrekovski sva na voljo na konzultacije, pomoč itd.

Za programiranje in poročila predlagamo, da uporabljate **GitHub**, ni pa nujno. Priporočamo tudi, da za programiranje uporabljate okolje **Sage** (SageMath) – gre za okolje, osnovano na programskem jeziku Python, z dodano podporo za matematiko (posebej bo prišla v upoštev podpora za grafe in celoštevilsko linearno programiranje).

Na koncu bomo imeli sestanek oziroma zagovor projekta s predavateljem in asistentom, kjer bo določena ocena projekta. Za zagovor, ki bo trajal do pol ure, lahko pripravite kratko predstavitev (max 20 min). Na zagovoru pričakujemo tudi argumentacijo smiselnosti vašega dela (zakaj se ste tako lotili) ter seveda razumevanje kode programa in eksperimentov.

Naloge:

- Razumevanje problema.
- Izbira natančnega problema (ali svojo varianto problema) iz reference.
- Izdelava kratkega (do 2 strani) opisa problema in načrta za nadaljnje delo.
- Programiranje rešitev ali rešitve izbranega problema.

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- Generiranje naključnih ali iskanje realnih podatkov za problem.
- Eksperimentiranje z napisanim programom in generiranimi podatki.
- Poročilo.
- Za naprogramirano kodo je potrebno pripraviti navodili za uporabo (lahko tudi kot komentar v kodi). Koda programa mora biti lepo komentirana.
- Zagovor, kjer predstavite izdelek. Pomembno je, da razumete in lahko zagovarjate, kaj ste delali in uporabljali (tudi kodo programa).

Poročilo:

- Približno 5 strani (fleksibilno).
- Vsebuje jasen opis problema.
- Vsebuje glavne ideje programa ali psevdokodo.
- Vsebuje jasen opis generiranja ali izbiranja podatkov.
- Prikaže grafe, tabele ali drugačen opis eksperimentov (npr. kako se poveča čas izvajanja z velikostjo problema).

2 Koledar

Letos bomo imeli dva termina za projekt. Vsak študent izbere, v katerem terminu bo delal projekt. Termina ne smete spremeniti. *Rok za izbiro termina je 22. oktober 2025*, izbira pa bo potekala preko spletne učilnice.

Z **zvezdico** so označene težje naloge. Če bi se kdo rad lotil težje naloge, naj prosim obvesti asistenta **pred rokom za izbiro termina** (povejte, kdo bo v skupini in katere naloge bi se radi lotili). Ostali boste naključno razdeljeni v skupine, tem pa bo naključno dodeljena ena izmed nalog brez zvezdice.

Prvi termin:

- Dodelitev skupin in tem: 24. oktober 2025.
- Rok za kratek opis: 14. november 2025.
- Rok za poročilo in program: 12. december 2025.
- Zagovori: 15.–19. december 2024. Datumi bodo določeni kasneje.

Drugi termin:

- Dodelitev skupin in tem: 5. december 2025.
- Rok za kratek opis: 2. januar 2025.
- Rok za poročilo in program: 30. januar 2025.
- Zagovori: 2.–6. februar 2026. Datumi bodo določeni kasneje.

3 Projekti

3.1 Metric Dimension

We say that a vertex s *resolves* a pair of vertices x, y in a graph G if $d(s,x) \neq d(s,y)$. A set of vertices S is a *resolving* set of G if every pair of vertices x, y of G is resolved by some vertex s from S. The *(vertex) metric dimension* of a connected graph G, denoted by $\dim(G)$, is defined as the size of a smallest set $S \subseteq V(G)$ which distinguishes all pairs of vertices in G.

We can similarly define the *edge metric dimension* of a connected graph G, denoted by $\mathrm{edim}(G)$, as the size of a smallest set $S \subseteq V(G)$ which distinguishes all pairs of edges, i.e., for each pair of edges e, f of G, there exists a vertex $s \in S$ such that $d(s,e) \neq d(s,f)$. Here, for an edge $e = uv$, we have $d(s,e) = \min\{d(s,u), d(s,v)\}$.

Another invariant which has been studied is the *mixed metric dimension*, where we resolve both edges and vertices in a graph, meaning that we want to distinguish every pair of vertices, every pair of edges, and every vertex from every edge. Denoted by $\mathrm{mdim}(G)$, it is the size of a smallest set $S \subseteq V(G)$ which distinguishes all pairs of vertices and edges.

This part involves various new and old metric dimensions. See the online survey [5] that gives an introduction and the state of the art of the topic.

Group 1: Doubly-Resolving Metric Dimension (2 students, round 2)

A set $S \subseteq V(G)$ is a *doubly-resolving* set if for every pair of vertices $x, y \in V(G)$ there exists a pair of vertices $u, v \in S$ such that $d(x,u) - d(x,v) \neq d(y,u) - d(y,v)$. The *doubly resolving metric dimension* of a graph G is the cardinality of a smallest doubly-resolving set and it is denoted by $\mathrm{dbdim}(G)$. Implement an ILP for computing the doubly-resolving metric dimension according to [4] and answer the following questions.

- It is known that every doubly-resolving set is also a resolving set, hence $\mathrm{dbdim}(G) \leq \dim(G)$. Find graphs (resp. trees) for which equality holds. Maybe among cubic graphs.
- Determine the doubly-resolving metric dimension of trees – possibly it could be a formula.
- Similar strategy apply to cactus graphs.

For small graphs, apply a systematic search; for larger ones, apply a random search. Report your results.

3.2 Metric dimension of directed graphs

In a directed graph, if there is a directed path from a vertex u to a vertex v, then we take for the distance from u to v to be the length of a shortest such path. If such a path exists, we say that v is reachable from u. If there is no directed path from u to v, the we usually have $d(u,v)=\infty$ and consider that v is not reachable from u. If a directed graph is strongly connected then every vertex is reachable from every other vertex.

We say that a vertex s *resolves* a pair of vertices x and y if both x and y are reachable from s but on distinct distances, i.e., $d(s,x)\neq d(s,y)$. A set of vertices S resolves G if every pair of vertices of G is *resolved* by some vertex form S. The smallest size of such a set S is the *metric dimension* of the directed graph G.

Here we will deal with directed circulant graphs $C(n,d)$. If we want to emphasize the list of generatorse we use notaton $C_n(1,2,\ldots,d)$. Note that they are strongly connected, so all distances are finite and consequently metric dimension is finites.

For additonal insides consider the report/project (ask for it, or get it from github) for the last year project:

- Maša Popovič, *Metrične dimenzije usmerjenih grafov*, Poročilo pri predmetu FP, 2024.

There you can find ILP of this problem. So, for many small values of n and d, and we can determine the metric dimension of $C(n,d)$.

Group 2: Metric dimension of directed graphs 1 (2 students, round 1)

We know that $\dim(C_n(1,2,\ldots,k))=k$ for $n>2(k-1)^2$. However, we wonder if the lower bound can be reduced, so your *first task* is to investigate this problem experimentally:

Problem 1. *For $k\geq 3$, find a function $f(k)<2(k-1)^2$ such that $\dim(C_n(1,2,\ldots,k))=k$ for every $n>f(k)$.*

We also know that

$$\begin{aligned}
\dim(C_n(1,2,\ldots,n-1)) &\geq n-1, \quad \text{for } n\geq 3;\\
\dim(C_n(1,2,\ldots,n-2)) &\geq \left\lfloor \tfrac{n}{2} \right\rfloor, \quad \text{for } n\geq 4;\\
\dim(C_n(1,2,\ldots,n-3)) &\geq \left\lfloor \tfrac{n}{2} \right\rfloor, \quad \text{for } n\geq 5.
\end{aligned}$$

Your *second task* is to test the following conjecture for values of n as large as possible:

Conjecture 2. *For $n\geq 7$, we have* $\dim(C_n(1,2,\ldots,n-4))=\left\lceil \frac{2n}{5} \right\rceil$.

Your *final task* is to experimentally explore similar results for $\dim(C_n(1,2,\ldots,n-k))$ where $k\geq 5$.

Group 3: Metric dimension of directed graphs 2 (2 students, round 2)

Building on the results from the 1st round project and on last year's project by Maša Popovič, your task is more open-ended: through experiments, you are invited to search for similar results for $\dim(C_n(a_1,a_2,\ldots,a_k))$. For example:

- Investigate cases where generators are even with step 2, i.e., non-consecutive, such as $2,4,\ldots,$ or $1,2,4,\ldots$ if additionally we want to have 1. Maybe bigger step.

- Consider when the first t odd numbers are included; for instance, for $t=3$ we may consider $\dim(C_n(1,3,5,a_4,\ldots,a_k))$.

- Explore generator lists of the form $(1, 2, 3, 4, \dots, t-1, t, y)$, where y is much larger than t.
- Propose and test additional sequences of your own.

3.3 Odd independent sets

An odd independent set S in a graph $G = (V, E)$ is an independent set of vertices such that, for every vertex $v \in V \setminus S$, either $N(v) \cap S = \emptyset$ or $|N(v) \cap S| \equiv 1 \pmod 2$, where $N(v)$ denotes the open neighborhood of v. The largest cardinality of an odd independent set in a graph G, denoted $\alpha_{\text{od}}(G)$, is called the α_{od} of G.

This new parameter is a natural companion to the recently introduced strong odd chromatic number. A proper vertex coloring of a graph G is called a strong odd coloring if, for every vertex $v \in V(G)$, each color used in the neighborhood of v appears an odd number of times in $N(v)$. The minimum number of colors in a strong odd coloring of G is denoted by $\chi_{\text{so}}(G)$.

A simple relation involving these two parameters and the order $|G|$ of G is

$$\alpha_{\text{od}}(G) \cdot \chi_{\text{so}}(G) \geq |G|,$$

which is analogous to the classical inequality relating the chromatic number and the independence number. See the papers arXiv:2509.20763 and arXiv:2510.01897 for more details.

An ILP formulation for the maximum odd independent set can be stated as follows:

$$\begin{aligned}
\max \quad & \sum_{u \in V} x_u & & \\
\text{s.t.} \quad & 0 \leq x_u \leq 1, & & x_u \in \mathbb{Z}, \quad \forall u \in V, \\
& 0 \leq y_u \leq 1, & & y_u \in \mathbb{Z}, \quad \forall u \in V, \\
& z_u \in \mathbb{Z}, & & \forall u \in V, \\
& x_u + x_v \leq 1, & & \forall uv \in E, \\
& \sum_{uv \in E} x_v \leq n y_u, & & \forall u \in V, \\
& y_u + \sum_{uv \in E} x_v = 2 z_u, & & \forall u \in V.
\end{aligned}$$

Thus, x_u indicates whether u is in the independent set, y_u indicates whether u has a neighbor in the independent set, and z_u is a counter for the vertex u.

Group 4: Graphs with odd independent set of size 1 (2 students, round 1)

We would like to consider the following problem.

Problem 3. *Characterize graphs G with $\alpha_{\text{od}}(G) = 1$.*

Clearly, a necessary condition is that G has diameter at most 2. In addition, the following related observations can be made:

(a) If $\alpha_{\text{od}}(G) = 1$, then also $\chi_{\text{so}}(G + K_r) = 1$.

(b) If G is claw-free, then $\alpha_{\text{od}}(G) = 1$ if and only if G has diameter at most 2.

Your task is to find as many such graphs as possible, and to try to identify their common properties beyond conditions (a), (b), and diameter 2. First, perform a systematic search for small graphs (up to 9 vertices). For larger graphs, you may use stochastic constructions of graphs of diameter 2 and test the hypotheses on them. You may also need to implement a function that checks whether a given graph is claw-free.

Group 5: Odd independent number vs. usual independent number (2 students, round 1)

We know that $\alpha(G) \geq \alpha_{\mathrm{od}}(G) \geq \alpha(G^2)$. Trivially, for a connected graph G, we have $\alpha(G) = \alpha(G^2)$ if and only if $G = K_n$, $n \geq 1$.

We would like to consider the following problem (Actually you have to consider two separate problems).

Problem 4. (i) *Characterize the graphs G for which $\alpha_{\mathrm{od}}(G) = \alpha(G)$ holds.*

(ii) *Characterize the graphs for which $\alpha_{\mathrm{od}}(G) = \alpha(G^2)$ holds.*

Do this:

(a) As a warm-up, try investigating candidates among families such as Kneser graphs, Cartesian products of complete graphs, and (trivially) odd regular bipartite graphs. You may also explore other graph classes that could provide examples.

(b) Test systematically on smaller graphs.

(c) For larger graphs, use some metaheuristic approaches, and if possible incorporate here the findings from (a) and (b) to narrow the search space.

Group 6: Triangle-free r-regular graphs (2 students, round 2)

We want to solve (at least partially) the following three problems:

Problem 5. *Let r be odd. Characterize the connected triangle-free r-regular graphs for which*

(i) *$N(v)$ is a maximum strong odd independent set for every vertex v.*

(ii) *$N(v)$ is a maximum strong odd independent set for some vertex v.*

(iii) $\alpha_{\mathrm{od}}(G) = r$.

Mind that you have three different problem. Note that if a graph satisfies condition (i), then it also satisfies condition (ii), and if a graph satisfies condition (ii), then it also satisfies condition (iii). Two known examples of graphs satisfying (i) are the Petersen graph and $K_{r,r}$, where r is odd.

We know the following:

(a) A necessary condition for (i) and (ii) is that the diameter is at most 3.

(b) Concerning (iii) with $r \geq 3$ (not necessarily odd), since there is no requirement on any $N(v)$, we must have $|G| \leq r(r^2 - 1)$.

Your tasks, taking (a) and (b) into account, are the following:

1. Perform a systematic search for graphs satisfying (i), (ii), and (iii) of small order.
2. Apply random walks among larger graphs of diameter at most 3.
3. Observe that the two mentioned examples belong to the class of strongly regular graphs. Therefore, one may also try other strongly regular graphs $\mathrm{srg}(n, r, \lambda = 0, \mu)$.

Group 7: Outerplanar graphs (2 students, round 2)

We would like to consider the following problem.

Problem 6. *Let G be any outerplanar graph. Is it true that $\alpha_{\mathrm{od}}(G) \geq n/7$ for every outerplanar G?*

Please carry out the following tasks:

- Implement a function that tests whether a given graph is outerplanar. Then, systematically check small outerplanar graphs to verify whether they all satisfy the bound $\alpha_{\mathrm{od}}(G) \geq n/7$.
- Implement a function that generates random outerplanar graphs and check whether the bound $\alpha_{\mathrm{od}}(G) \geq n/7$ holds One can generate a 2-connected outerplanar graph by taking a cycle of arbitrary length and introducing non-crossing diagonals to it. Next by gluing such graphs into a tree-structure, one can obtain outerplanar graphs of connectivity 1.

3.4 Edge coloring

To obtain an edge coloring of a graph G of maximum degree Δ, we assign colors (actually numbers $1, 2, \ldots$) to edges so that every pair of adjacent edges (i.e., sharing a common vertex) has different colors. For such a coloring, we need at least Δ colors.

Group 8: Rich-neighbor edge-colorings (2 students)*

In an edge coloring, an edge e is called *rich* if all edges adjacent to e have different colors. An edge coloring is called a *rich-neighbor edge coloring* if every edge is adjacent to some rich edge. The smallest number of colors for which there exists such a coloring is denoted by $\chi'_{\mathrm{rn}}(G)$. For example, $\chi'_{\mathrm{rn}}(K_4) = 6$.

We want to verify the following conjecture:

Conjecture 7. *For every graph G of maximum degree Δ, $\chi'_{\mathrm{rn}}(G) \leq 2\Delta - 1$ holds.*

Test the above conjecture for

- non-regular graphs;
- regular multigraphs;
- non-regular multigraphs.

In order to do so, use the code from
`https://github.com/anejrozman/Rich-neighbour-edge-coloring`
or implement it yourself. Try to test the conjecture on smaller graphs as systematically as possible. For larger graphs of maximum degree $\Delta \geq 4$, apply a random search. Report your results.

Group 9: $\mathbb{Z}_2^3$-connectivity (2 students)**

First, we define the $(\mathbb{Z}_2^3, +_2)$ *group*. It consists of the set of all eight binary 3-vectors

$$\mathbb{Z}_2^3 = \{(0,0,0), (0,0,1), (0,1,0), \ldots, (1,1,1)\}$$

with operation $+_2$, which sums two elements binary and componentwise, e.g., $(0,0,1) + (1,0,1) = (1,0,0)$.

Now we define the concept of $\mathbb{Z}_2^3$-*connectivity*. Let G be a graph. A function $\delta : V(G) \to \mathbb{Z}_2^3$ is *zero-sum* if $\sum_{v \in V(G)} \delta(v) = (0,0,0)$. We say that a graph G is $\mathbb{Z}_2^3$-*connected* if for every zero-sum function δ of G, there exists a function $f : E(G) \to \mathbb{Z}_2^3 \setminus (0,0,0)$ such that for every vertex v of G and its incident edges $e_1, e_2, \ldots, \delta(v) = f(e_1) + f(e_2) + \cdots$ holds.

Problem 8. *Find a bridgeless graph G that is not $\mathbb{Z}_2^3$-connected with the smallest possible number of 2-edge-cuts.*

Note that the Petersen graph with every edge subdivided is not $\mathbb{Z}_2^3$-connected and it has 15 2-edge-cuts. We also know that graphs with less than four 2-edge-cuts are $\mathbb{Z}_2^3$-connected. So the answer to above problem will be a graph with the number of 2-edge-cuts between 4 and 15. We want to know the right number of 2-edge-cuts or at least to reduce the gap between 4 and 15. Apply a systematic search and report your results.

3.5 Graph indices

Group 10: Interval index (2 students, round 1)

We want to investigate the extremal values of the following index

$$\mathrm{Int}(G) = \sum_{\{u,v\} \subset V} \big(|I(u,v)| - 1 \big),$$

where $I(u,v) = \{w \in V : d(u,w) + d(w,v) = d(u,v)\}$ is the set of vertices lying on the shortest paths between u and v; also known as the *interval* between vertices u and v.

Your tasks are the following:

- Test if the path P_n is the graph with maximum interval index value among graphs on n vertices.
- Find which graphs has minimum and which has maximum among cubic graphs on n vertices. Try to determine some properties of these graphs (maybe small/large diameter, ...)

In order to do so test systematically for smaller graphs. For larger graph use some metahevristic.

Group 11: The Complementary Second Zagreb Index (2 students, round 1)

The *Complementary Second Zagreb Index* (also CSZ index) of a graph G is defined as

$$\mathrm{cM}_2(G) = \sum_{uv \in E(G)} \left| (d_G(u))^2 - (d_G(v))^2 \right|,$$

where $d_G(u)$ denotes the degree of a vertex u in G and $E(G)$ represents the set of edges of G. Note that $+$ is the operation join of two graphs in the following conjecture.

Conjecture 9. *If G^* is a graph having the maximum value of cM_2 among all connected graphs of order n, then G^* is isomorphic to $K_k + \overline{K}_{n-k}$ for some k satisfying $k < \lceil n/2 \rceil$, where $n \geq 5$ and the graph $K_k + \overline{K}_{n-k}$.*

Your tasks are the following:

- Test the above conjecture.
- Assuming it is true, determin k as a function of n.
- Find which graphs has minimum and which has maximum among graphs on n vertices with cyclomatic number k.

In order to do so test systematically for smaller graphs. For larger graph use some metahevristic.

Group 12: General Sombor Index (2 students, round 1)

The *general Sombor index* of a graph G, denoted by $SO_\alpha(G)$, is recently defined as:

$$\mathrm{SO}_\alpha(G) = \sum_{v_i v_j \in E(G)} \left(d_G(v_i)^2 + d_G(v_j)^2 \right)^\alpha ,$$

where $d_G(v_i)$ represents the degree of vertex v_i, and α is an arbitrary real number.

Your task is to identify trees with maximum value for the general Sombor index within the class of n-vertex trees with maximum degree Δ and $\alpha \in (0,1)$. In order to do so:

1. For small graphs, apply a systematic search; for larger ones, apply some stochastic search.
2. Consider different values $\Delta = 3, 4, 5, \ldots$ separately.
3. Take many different values for $c \in (0,1)$. Let some of them be very close to 0, some of them are closing to $\frac{1}{2}$ from both sides, and some very close to 1.

We recommend to have a look in the paper *The General Sombor Index of Extremal Trees with a Given Maximum Degree*.

3.6 Subpath number

For a connected graph G, we define the subpath number, denoted by $\mathrm{pn}(G)$, as the number of paths in the graph, including trivial paths of length 0. Here, we define a path of G of length ℓ as a sequence of vertices $(v_0, v_1, \ldots, v_\ell)$ of G without repetitions (i.e., $v_i \neq v_j$ for all $0 \leq i < j \leq \ell$) such that every pair of consecutive vertices is connected by an edge of G (i.e., $v_{i-1}v_i$ is an edge of G for all $1 \leq i \leq n$).

For more details, you can have look in our online papers, and also in the last year project under the name Pingdingshan number.

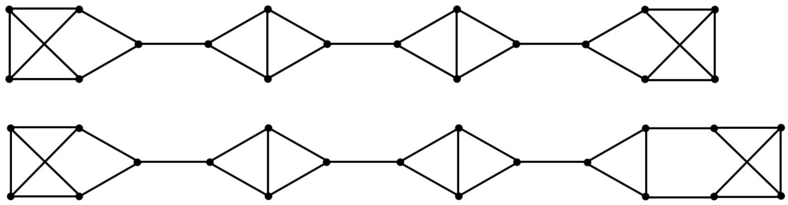
Group 13: Subpath number: minimal cubic graphs (2 students, round 1)

Graphs L_n are illustrated by Figure 1 and defined for even $n \geq 10$ as follows. If $n = 4q - 2$ for some integer q, then the graph L_n is obtained from $(n-10)/4$ copies of $K_4 - e$, and two such copies are connected by an edge which connects a vertices of degree 2 from each copy. Thus, we obtain a path like structrure which is then ended on both sides by a pendant block, each on 5 vertices. Otherwise, if $n = 4q$, then L_n is obtained from $(n-12)/4$ copies of $K_4 - e$ connected into a path like structure the same way as before, but now the structure is ended with two pendant blocks, one on 5 vertices and the other on 7 vertices. We posed the following conjecture.

Conjecture 10. *In the class of cubic graphs on n vertices, where n is even, the graph L_n is the only graph which minimizes the subpath number.*

We have been told that above conjecture is false for large enough n, and so your task is to find a counterexample with n as small as possible. For this:

- Verify the above conjecture for small values of n by systematically considering all cubic graphs for as large as possible n.
- For larger n apply some searching metaheuristic, say simulated annealing, hoping you can find a counterexample.



Slika 1: L_n grafi for $n = 18$ and 20

- After experienced with above two steps, you can start constructing graphs using the above gadgets (from the definition of L_n) as blocks and connecting them in various way (maybe in a tree structure), and apply some search metaheuristic to find such a graph with as small as possible subpath number.

Group 14: Subpath number: maximal cubic vs. small diameter (2 students, round 2)

Cubic graphs with a large subpath number tend to have a rather small diameter—we would like to test to what extent this holds. For suitable values of n, cage graphs such as the Petersen graph and the Heawood graph seem like good candidates. We propose the following working conjecture to test.

Conjecture 11. *In the class of cubic graphs on n vertices, where n is even, the graphs with the smallest possible diameter maximize the subpath number.*

Your tasks are:

- Verify the above conjecture for small values of n by systematically considering all cubic graphs, for n as large as possible.
- For larger n, apply a search metaheuristic (e.g., random walk by rewiring) with the aim of finding a counterexample.

Group 15: Inverse subpath problem (2 students, round 2)

Notice that when the order n is fixed, the graphs with the smallest $\mathrm{spn}(G)$ values are trees, while the one with the largest value is the complete graph K_n.

Inverse Subpath Problem (ISP) asks: *for a given integer k, find a graph G such that* $\mathrm{spn}(G) = k$.

This may not be possible for some small values of k, but as k grows we expect such cases to become increasingly rare. We propose the following working conjecture:

Conjecture 12. *There exists k_0 such that for every sufficiently large $k \geq k_0$, the* ISP *has a solution.*

Your first task is to determine k_0 and the set B of all "bad" integers (i.e., integers for which no graph G satisfies $\mathrm{spn}(G) = k$).

Try to find particular classes of graphs whose subpath numbers will cover the set $\mathbb{N} \setminus B$. For example, this could be graphs consisting of triangles connected together with pending vertices in a tree-like structure. A promising strategy could be to consider separately the intervals

$$\left[\binom{n+1}{2}, \ldots, \binom{n+2}{2}\right]$$

and try to cover all integers from them. Afterall observe that when the order n is fixed, the graphs with the smallest $\mathrm{spn}(G)$ are trees and the one with the largest $\mathrm{spn}(G)$ is the complete graph K_n, with values

$$\binom{n+1}{2} \quad \text{and} \quad \binom{n+2}{2},$$

respectively. Hopefully for each integer from that interval will exist a graph on n vertices that realize it.

3.7 Geodesic subpath number

In a similar way, we define the *geodesic subpath number* $\mathrm{gpn}(G)$ by counting only geodesic paths. Thus, for a connected graph G, we define $\mathrm{gpn}(G)$ as the number of all shortest paths in the graph, including trivial paths of length 0.

This invariant is defined for connected graphs. Note that it attains its minimum for trees. In particular, for each tree T on n vertices we have

$$\mathrm{gpn}(T) = \binom{n}{2}.$$

Group 16: Geodesic subpath number: extremal graphs (2 students, round 1)

Your first task is to implement a function $\mathrm{gpn}(G)$ that (efficiently) calculates gpn for a given graph G.

Next, for each of the following classes of connected graphs on n vertices — all graphs, bipartite graphs, triangle-free graphs, and cubic graphs — we aim to identify which graphs attain the largest possible value of the geodesic subpath number. In order to do so, proceed as follows:

1. For small values of n, say up to 10, perform an exhaustive search through all graphs in the given class on n vertices.
2. Based on the results from the previous step, make a conjecture about the optimal graphs.
3. For larger values of n, test your hypothesis stochastically. We recommend simulated annealing.

Group 17: Geodesic subpath number: cacti (3 students, round 2)

A graph G is called a *cactus graph* if every edge of G belongs to at most one simple cycle. Equivalently, any two cycles in a cactus graph have at most one vertex in common. Thus:

- Every tree is a cactus graph (since it has no cycles).
- Every unicyclic graph (a connected graph with exactly one cycle) is also a cactus graph.
- Cactus graphs can be seen as a natural generalization of trees, where cycles are allowed but in a restricted way.

Implement the following:

1. A function that tests whether a given graph is a cactus graph.
2. A function that stochastically generates a random cactus graph.
3. A function that stochastically performs local modifications of a cactus graph.

Next, using the above functions, try to find *extremal* cactus graphs for the path number / geodesic subpath number among cacti with given n and c. Proceed as follows:

1. For small values of n, perform an exhaustive search within the class on n vertices.
2. Based on the results from the previous step, formulate a conjecture about the optimal graphs.
3. For larger values of n, test your hypothesis with simulated annealing searches.

Group 18: Subpath number: prescribed cyclomatic number (2 students, round 1)

Let $G = (V, E)$ be a connected graph. The *cyclomatic number* (also known as the *circuit rank*) of G is defined as

$$\mu(G) = |E| - |V| + 1.$$

Intuitively, $\mu(G)$ represents the minimum number of edges that must be removed from G to obtain a forest (i.e., an acyclic graph).

Graphs with a prescribed cyclomatic number k form a natural generalization of trees ($\mu = 0$) and unicyclic graphs ($\mu = 1$). They arise in various structural and extremal problems in graph theory, particularly in the study of sparse or nearly acyclic graphs, and have applications in network design, electrical circuits, and software engineering.

In this project, we aim to analyze the subpath number/geodesic subpath number of graphs satisfying $\mu(G) = k$. Your goal is to determine the graphs with the minimum and maximum subpath numbers among all graphs with n vertices and cyclomatic number k. Specifically:

- Perform a systematic analysis for small graphs;
- Apply metaheuristic algorithms (e.g., genetic algorithms, simulated annealing) for larger graphs.

Note that extremal graphs are already known for the subclass of cactus graphs with k cycles (a special case of graphs with $\mu(G) = k$). These may serve as good starting points. See Theorems 10 and 14 in the following paper for details:

[KSSY] M. Knor, J. Sedlar, R. Škrekovski, Y. Yang, *The Subpath Number of Cactus Graphs*, manuscript, 14 pages.

Group 19: Inverse geodesic subpath problem (1 or 2 students, round 2)

Your task is to consider the geodesic variation of the similar project from the previous subsection.

3.8 Sigma irregularities

Irregularity measures quantify to what degree a certain graph is irregular. Thus, they give the value 0 for regular graphs, and a larger value means the graph should be more irregular. Amongst many such measures are also the following two.

The *sigma irregularity* index $\sigma(G)$ is defined as

$$\sigma(G) = \sum_{uv \in E(G)} (d_G(u) - d_G(v))^2.$$

Similarly, the following variant of σ-irregularity, called the *sigma total irregularity*, has been introduced:

$$\sigma_t(G) = \sum_{\{u,v\} \subseteq V(G)} (d_G(u) - d_G(v))^2 .$$

Thus, the sum in the first measure runs through all edges, and in the second it runs through all pairs of vertices.

Group 20: σ-Irregularity for Trees (2 students, round 2)

We aim to determine the optimal trees on n vertices with respect to σ-irregularity, for trees of prescribed maximum degree Δ. The case $\Delta = 5$ has already been studied (see the related work below):

- D. Dimitrov, Ž. Kovijanić Vukićević, G. Popivoda, J. Sedlar, R. Škrekovski, and S. Vujošević, *The μ-Irregularity of Trees with Maximum Degree 5*, manuscript.

We now wish to extend this to larger values of $\Delta \geq 6$. As a starting point, focus on $\Delta = 6$, following the results obtained for $\Delta = 5$ (ask for the paper). In particular, study properties (P1)–(P4) and Theorem 17 from the above paper. See also Fig. 1 for some examples of optimal trees.

Your task is to attempt to obtain a similar characterization for the case $\Delta = 6$. To identify the optimal trees, proceed as follows:

- Perform a systematic search for small n.
- Apply metaheuristic search methods (e.g., simulated annealing) for larger n.

If this approach works for $\Delta = 6$, then repeat it for larger Δ (e.g., $\Delta = 7$). For reference, the case $\Delta = 4$ is discussed here: https://doi.org/10.46793/match.91-1.267K.

3.9 Fullerenes

Fullerenes are a unique class of carbon molecules composed entirely of carbon atoms arranged in a hollow, spherical, ellipsoidal, or cylindrical structure. In 1996, Sir Harold Kroto, Robert Curl, and Richard Smalley were awarded the Nobel Prize in Chemistry for their discovery of buckminsterfullerene (C_{60}), a new form of carbon with a unique spherical structure.

From a graph-theoretical perspective, a fullerene is a 3-regular, planar graph where each face is either a pentagon or a hexagon. This means the molecular structure of fullerenes can be modeled as a graph in which each carbon atom corresponds to a vertex, and each bond between atoms corresponds to an edge. The most famous fullerene, C_{60}, is modeled as a truncated icosahedron, a polyhedral graph with 12 pentagonal and 20 hexagonal faces. These graph-theoretical representations help in understanding fullerenes' structural stability and chemical reactivity, making them useful in various applications such as nanotechnology, drug delivery, energy storage, and molecular electronics.

Group 21: Diameter of icosahedral fullerenes (2 students)**

The construction of icosahedral fullerenes $F_{i,j}$ is based on a hexagonal lattice, where two integers i and j define the steps in two lattice directions. The fullerene is formed by arranging hexagons and introducing 12 pentagons to close the structure, giving it a spherical shape. The order of such a fullerene is given by $n = 20(i^2 + ij + j^2)$. The parameters i and j determine the fullerene's size and symmetry, and by varying them, a family of icosahedral fullerenes can be systematically generated. For a detailed explanation of this construction method, please refer to the paper Fullerene Graphs and Some Relevant Graph Invariants for more details.

Construct the icosahedral fullerene $F_{i,j}$ of type (i, j). Note that its order is given by $n = 20(i^2 + ij + j^2)$. After constructing the fullerene, compare the ratio of the graph's diameter to $\sqrt{n}$, specifically $\frac{\text{diam}(G)}{\sqrt{n}}$. Your task is to find the optimal ratio between i and j that minimizes this value. Additionally, explore the analogous problem of finding the ratio between i and j that maximizes this value.

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