NOTE ON SCHRÖDER'S FUNCTIONAL EQUATION

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In the present paper we are concerned with Schröder's functional equation

$$(1) \qquad \varphi[f(x)] = s\varphi(x),$$

where $\varphi(x)$ is the unknown function and s is a number between 0 and 1:

$$(2) \qquad 0 < s < 1.$$

We shall prove a theorem which generalizes some earlier results on convex solutions of the Schröder equation [4], [5]. The basic idea of the proof is the same as in those papers and consists in reducing the original problem to the problem of monotonic solutions of the equation

$$(3) \qquad \rho[f(x)] - \rho(x) = h(x)$$

(with unknown $\rho(x)$). The present treatment, however, is more elementary and avoids the use of Lebesgue measure and integral. Let us note that paper [3] which is fundamental for our present considerations (as well as for those in [4] and [5]) is entirely elementary and makes no use of measure theory.

We consider real functions of a real variable on an interval $\langle 0, a)$, $a \leqq \infty$. The function $f(x)$ is supposed to fulfil the following conditions:

(4) $f(x)$ is continuous and strictly increasing in $\langle 0, a)$.

(5) $f(0) = 0$ and $0 < f(x) < x$ for $x \in (0, a)$.

$$(6) \qquad \lim_{x \to 0+0} \frac{f(x)}{x} = s.$$

In the sequel $f^n(x)$ will denote the n-th iterate of $f(x)$:

$$f^0(x) = x, \; f^{n+1}(x) = f(f^n(x)), \quad n = 0, 1, 2, \cdots.$$

According to (5) the functions $f^n(x)$ are defined in $\langle 0, a)$.

We are going to prove the following

THEOREM. *If $f(x)$ fulfils conditions* (4)—(6), *then equation* (1) *possesses* *at most a one-parameter family of solutions $\varphi(x)$ such that $\varphi(x)/x$ is monotonic in $(0, a)$. These solutions, if they exist, are given by the formula*

$$\varphi(x) = c \lim_{n\to\infty} \frac{f^n(x)}{f^n(d)}, \tag{7}$$

where c is an arbitrary constant and d is a point from $(0, a)$, arbitrarily fixed. If, moreover, the function $f(x)/x$ is monotonic in $(0, a)$, then (7) actually defines a one-parameter family of solutions of (1) such that $\varphi(x)/x$ is monotonic in $(0, a)$.

PROOF. We may disregard the trivial solution $\varphi(x) \equiv 0$ (which is also contained in formula (7) with $c = 0$). So let $\varphi(x)$ be a non-trivial solution of equation (1) such that the function

$$\psi(x) = \frac{\varphi(x)}{x}, \quad x \in (0, a), \tag{8}$$

is monotonic in $(0, a)$. According to (1) the function $\psi(x)$ satisfies the functional equation

$$\psi[f(x)] = \frac{sx}{f(x)}\psi(x), \quad x \in (0, a). \tag{9}$$

From (9) and the monotonity of $\psi(x)$ it follows that $\psi(x)$ has a constant sign in $(0, a)$. In particular $\psi(x) \neq 0$ in $(0, a)$. In fact, if $\psi(x)$ vanished at a point $x_0 \in (0, a)$, then, by (9), it would vanish at all the points $f^n(x_0)$, $n = 0, 1, 2, \cdots$, and thus, being monotonic, it would vanish identically in $(0, a)$. Then by (8) $\varphi(x)$ would vanish identically in $(0, a)$, contrary to our assumption.

Thus we may put $\varepsilon = \operatorname{sgn}\psi(x)$ and

$$\rho(x) = \log \varepsilon\psi(x), \; h(x) = \log\frac{sx}{f(x)}, \quad x \in (0, a). \tag{10}$$

According to (9) the function $\rho(x)$ satisfies the functional equation (3). Now, $\rho(x)$ is monotonic in $(0, a)$, just like $\psi(x)$, and we have in view of (6)

$$\lim_{x\to 0+0} h(x) = 0.$$

On account of a theorem proved [1] in [3] (cf. also [2]) we have

[1] In [3] the inequality $f(x) > x$ was assumed, but the theorem remains true under the conditions of the present paper. Let us note that the case where $f(x) > x$, $s > 1$, can be reduced to that considered here by passing to the equation $\varphi[f^{-1}(x)] = s^{-1}\varphi(x)$.

$$\rho(x) = \rho_0 - \sum_{n=0}^{\infty} \{h[f^n(x)] - h[f^n(d)]\}, \tag{11}$$

where d is an arbitrarily fixed point from $(0, a)$ and ρ_0 is a constant. Hence, taking (10) into account, we get

$$\varphi(x) = \varepsilon \exp \rho_0 \prod_{n=0}^{\infty} \frac{f^{n+1}(x) f^n(d)}{f^{n+1}(d) f^n(x)} = \varepsilon \exp \rho_0 \frac{d}{x} \lim_{n\to\infty} \frac{f^n(x)}{f^n(d)},$$

whence, after putting $c = \varepsilon d \exp \rho_0$, formula (7) follows.

On the other hand, if $f(x)/x$ is monotonic, then so also is the function $h(x)$, and formula (11) actually defines a one-parameter family of monotonic solutions of equation (3) (cf. [3]). Thus in this case formula (7) actually defines a one-parameter family of solutions of equation (1) for which the ratio $\varphi(x)/x$ is a monotonic function. This completes the proof.

It follows from (1), (5) and (2) that every solution of equation (1) fulfils the condition $\varphi(0) = 0$. Thus for every convex or concave solution $\varphi(x)$ of the Schröder equation the ratio $\varphi(x)/x$ is a monotonic function. Since, on the other hand, for convex or concave $f(x)$ the functions defined by (7) are evidently also convex or concave, our previous, analogous results [4], [5] concerning convex solutions of equation (1) are an immediate consequence of the present theorem.

Finally let us note that the Schröder equation is closely connected with the problem of continuous iteration (cf. e.g. [7]). In this connection some results of a similar character to that of the present paper have recently been obtained by A. Lundberg [6] (cf. also [1]).

References

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