The Operational Calculus.

Von

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Introduction.

The operational calculus[1]) owes its inception to Leibniz, who was struck by the resemblance between the formula for the n-fold differentiation of a product and the formula for the n^{th} power of a sum. Lagrange carried this analogy further, and set up a regular algorithm in which the differential operator plays the part of a sort of hypercomplex unit. It is to him, for example, that we owe the symbolic formula

$$f(x+h) = \left(e^{h\frac{d}{dx}}\right)(f(x)).$$

This operational calculus was further developed by many mathematicians, including Laplace, and more especially Boole, who employed it effectively

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[1]) For a thorough historical account of the earlier phases of this subject, see S. Pincherle, Funktionaloperationen und -gleichungen, Enc. Math. Wiss. II A 11, or Équations et opérations fonctionnelles, Encyclopédie des sciences mathématiques pures et appliquées, tome II, vol. 5, fascicule 1.

in the well-known theory of the linear differential equation with constant coefficients. Other names which deserve particular mention in connection with the history of the theory are those of Arbogast and Oltramare.

The earlier work on the subject is of a purely formal character. The developments obtained are often of the nature of series, but there is little or no attempt to investigate their convergence. The fundamental notion of operator is but ill defined. For example, in the equation of Lagrange just cited, the operation on the left-hand side is applicable to any well-defined function whatever, while that on the right-hand side demands that $f(x)$ be analytic over a region bounded by two parallels to the axis of reals.

In recent years, the operational calculus has experienced a growth in two somewhat antithetical directions. On the one hand, we have the work of the rigorists, who start with a clear definition of an operator and of equations between operators. Chief among these are Pincherle[2]), who considers his operators as transformations which change power series into power series, and Volterra[3]), whose operators are first, such integral transformations as

$$\int_0^x f(\xi)\, F(\xi - x)\, d\xi,$$

and secondarily, such transformations as may be obtained by compounding these transformations with the inverse of a transformation such as

$$\frac{1}{(m-1)!}\int_0^x f(\xi)\,(\xi - x)^{m-1}\, d\xi.$$

On the other hand, we have the practical application of operational methods by O. Heaviside[4]) to the solution of the partial differential equations of the electric circuit. These two developments have not met. The brilliant work of Heaviside is purely heuristic, devoid even of the pretense to mathematical rigor. Its operators apply to electric voltages and currents, which may be discontinuous and certainly need not be analytic. For example, the favorite *corpus vile* on which he tries out his operators is a function which vanishes to the left of the origin and is 1 to the right. This excludes any direct application of the methods of Pincherle. On the other hand, he makes much use of expansions of operators in in-

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[2]) S. Pincherle, Atti R. Accad. Lincei Rendic. (5) **4**, 1 (1895), p. 142; Math. Annalen **49** (1897), p. 349.

[3]) V. Volterra, Leçons sur les fonctions de lignes, Paris, Gauthier-Villars (1913); Functions of Composition, The Rice Institute Pamphlet 7, N° 4, October 1920.

[4]) O. Heaviside, On Operators in Physical Mathematics, Roy. Soc. Proc. 52 (1893), p. 504; **54** (1893), p. 105.

finite series of powers of d/dt, leading to asymptotic series of much value for computation. Such series are not treated by Volterra, whose method is indeed at its best in treating operators of what we may call the integral type.

Although Heaviside's developments have not been justified by the present state of the purely mathematical theory of operators, there is a great deal of what we may call experimental evidence of their validity, and they are very valuable to the electrical engineer. There are cases, however, where they lead to ambiguous or contradictory results. It has hence become important to put them on a sound mathematical basis or failing that, to establish heuristically criteria for the avoidance of contradiction. V. Bush[5]) has given a set of formal criteria of this type, while Carson[6]) has developed a theory of the Volterra type from the standpoint of the engineer, and has amplified it by a somewhat indirect justification of Heaviside's asymptotic series. It is here necessary to refer also to the work of Fry[7]), Berg[8]), and Bromwich[9]).

Nevertheless, a thoroughly satisfactory foundation for the Heaviside theory does not as yet exist. Carson[10]) himself says that his deduction of the asymptotic expansions 'entirely lacks the extraordinary simplicity and directness of the Heaviside solution'. Moreover, he establishes his asymptotic solution separately in each individual case, and does not make it the central feature of a general theory. The problem of obtaining a rigorous interpretation of Heaviside's theory which shall modify the form of his developments as little as possible is therefore still open[11]).

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5) V. Bush, Note on Operational Calculus, Journ. Math. Phys. Massachusetts Institute of Technology **3** (1924), pp. 95—107.

6) J. R. Carson, Theory of the Transient Oscillations of Electrical Networks and Transmission Systems, Proc. Amer. Inst. Elec. Engineers 1919, pp. 345—427; The Heaviside Operational Calculus, The Bell System Technical Journal, November, 1922, pp. 1—13.

7) T. C. Fry, The Application of Modern Theories of Integration to the Solution of Differential Equations, Annals of Mathematics 22 (1920/21), pp. 182—211.

8) E. G. Berg, Heaviside's Operators in Engineering and Physics, Journ. Franklin Institute **198** (1924), pp. 647—702.

9) T. J. I'A. Bromwich, Normal Coordinates in Dynamical Systems, Proc. Lond. Math. Soc. **15** (1916), pp. 401—445.

10) Loc. cit. p. 409.

11) A justification of the method of Carson's second paper which is at once general in standpoint and rigorous is contained in a paper by Gronwall and Carson to appear shortly in the Bull. Amer. Math. Soc. The precise conditions of validity of the methods there developed are in some respects more general, in others less general, than those of the present paper. Cf. also J. R. Carson, Electric Circuit Theory and the Operational Calculus, The Bell System Technical Journal, October, 1925, pp. 685—763.

There are several paths which seem to lead to this goal. Besides the Volterra theory of permutable kernels and the Pincherle theory of transformations of power series, the Laplace transformation[12]) and the Fourier integral appear to be promising tools. Like the theory of Pincherle, however, the theory of the Laplace transformation is directly applicable only to analytic functions. The Fourier integral, which may be regarded as derived from a complex form of the Laplace transformation, is not open to this objection. On the other hand, the functions to which the classical form of the Fourier integral applies are subject to very severe restrictions as to their behavior at infinity. These restrictions are far more severe than those to which Heaviside subjects the functions to which he applies his methods. We must therefore generalize the Fourier integral. This has been done in part by Fry in the paper already cited and in part by the author[13]). It is done to a still wider extent in the first section of the present paper.

When applied to the function e^{nit}, the operator $f(d/dt)$ is equivalent to the multiplier $f(ni)$. The result of applying a given operator to a given Fourier integral may naturally be conceived, then, as the multiplication of each e^{nit} term in the integral by a multiplier depending only on n. That is, the operator d/dt has no particular location in the complex plane, but ranges up and down the entire axis of imaginaries. It is thus too much to expect in general that any particular series expansion or other analytic representation of f should make $f(d/dt)$ converge when applied to a purely arbitrary function. In the present paper, the device is adopted of dissecting a function into a finite or infinite number of ranges of frequency, and of applying to each range the particular expansion of an operator yielding results convergent over that range.

It is by this method of dissection that Heaviside's asymptotic series are justified. If we apply an operator of a certain specified type to a function f of a certain specified type, we obtain a function g with a behavior at infinity essentially the same as that of the function g_λ obtained by applying our operator to the function f_λ which consists, in essence, of the components of f with frequencies within a certain restricted range. Thus if the operator is representable in a convergent series expansion, when applied to functions containing only frequencies withen the given range, we get an asymptotic expansion of $g_\lambda(x)$.

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[12]) Cf. Probleme aus der Theorie der Wärmeleitung. Erste Mitteilung. F. Bernstein u. G. Doetsch. Zweite Mitteilung. G. Doetsch. Math. Zeitschr. 22 (1925), pp. 284–306.

[13]) N. Wiener, On the Representation of Functions by Trigonometrical Integrals, to appear in the Math. Zeitschrift; The Solution of a Difference Equation by Trigonometrical Integrals, Journ. Math. Phys. Massachusetts Institute of Technology, 4 (1925), pp. 153–164.

It may be said in passing that many of the apparent contradictions found by the electrical engineer in the work of Heaviside are due to a misunderstanding of the nature of an asymptotic expansion. Thus the asymptotic expansion of 0 for positive values of t is also an asymptotic expansion for e^{-t}. There is, however, another point where the theory of Heaviside falls into real contradictions. Both the Volterra theory of permutable kernels and the operational calculus here developed are in essence methods of determining the arbitrary constants in the solution of a linear differential equation of constant coefficients. In the operational calculus to which the Volterra theory leads, these are so determined that the result of applying an operator to a function vanishing to the left of a given point itself vanishes to the left of that point. In my theory, on the other hand, the result of applying an operator to a function behaving algebraically at $\pm\infty$ must itself behave algebraically — or at least, must not become exponentially infinite — at $\pm\infty$. These requirements may be in conflict. For example, the general solution of the differential equation

$$\frac{du}{dt} - u = \varphi(t),$$

where

$$\varphi(t) = 0 \qquad [t < 0];$$

$$\varphi(t) = 1 \qquad [t > 0]$$

is

$$u = \varphi(t)\,[e^t - 1] + c\,e^t.$$

The solution corresponding to an operational calculus of the Volterra-Carson type is

$$\varphi(t)\,[e^t - 1],$$

whereas the solution corresponding to an operational calculus of my type is

$$\varphi(t)\,[e^t - 1] - e^t.$$

Now, Heaviside's expansions in negative powers of d/dx correspond to an operational calculus of the Volterra-Carson type, while his expansions in positive powers of d/dx correspond to a theory such as that developed in the present paper. It is thus possible by following out his methods to make very real errors.

There is a large class of cases, however, where the determination of the constants of integration of a differential equation in accordance with the Volterra theory and that here developed are in complete harmony. In these cases, the operators of my theory depend only on the past of the functions to which they apply. I call such operators *retrospective*. In an electrical problem, all operators arising from a system with only positive resistances and leakages will be retrospective. All other operators correspond to electrical systems where there is at some point an unlimited input of energy and hence correspond only to such exceptional conditions as thermionic valves.

Our operators $f(d/dt)$ need special treatment in the case where they possess poles or branch-points on the axis of imaginaries, as well as when their behavior at infinity is too complicated. We shall discuss this, and shall investigate the degree of indeterminateness caused in the problem of inverting an operator by such singularities. We shall here confine ourselves to the simplest cases.

We shall finally investigate the application of our operators to the solution of partial differential equations.

1. The Trigonometric Expansion of a Function of Finite Order at Infinity.

Let $f(x)$ be a function which is summable and of summable square over every finite range, and let there be numbers A and k such that

$$f(x) \leqq A x^k$$

for every x of sufficient positive or negative magnitude. In general, $f(x)$ will not possess a Fourier integral expansion. There is nevertheless a type of extended Fourier integral by which $f(x)$ may be represented.

To establish this, we shall have need of a certain set of auxiliary functions. Let $\Phi(u)$ be a function which is defined over the interval $(0, 1)$, which has at every point derivatives of all orders, which vanishes at 1 and equals 1 at 0, and which has its derivatives of every order vanish at 0 and at 1. We may, for example, put $\Phi(u) = \int\limits_u^1 e^{\frac{1}{\xi^2 - \xi}} d\xi \Big/ \int\limits_0^1 e^{\frac{1}{\xi^2 - \xi}} d\xi$. Let $\Psi(\xi, \lambda)$ be defined by the equation

$$\Psi(\xi, \lambda) = \frac{\sin \lambda \xi}{\pi \xi} + \frac{1}{\pi} \int\limits_0^1 \Phi(u) \cos(u + \lambda) \xi \, du.$$

I then wish to prove that

$$\int\limits_{-\infty}^{\infty} f(\xi)\, \Psi(\xi - x, \lambda)\, d\xi$$

exists, and converges in the mean to $f(x)$ over any finite range as $\lambda \to +\infty$.

To begin with,

$$|\Psi(\xi,\lambda)| = \left| \frac{\sin\lambda\xi}{\pi\xi} + \frac{\Phi(1)\sin(\lambda+1)\xi - \Phi(0)\sin\lambda\xi}{\pi\xi} \right.$$

$$\left. - \frac{1}{\pi\xi}\int_0^1 \Phi'(u)\sin(u+\lambda)\xi\,du \right|$$

$$= \left| \frac{1}{\pi\xi}\int_0^1 \Phi'(u)\sin(u+\lambda)\xi\,du \right|$$

$$= \left| \frac{1}{\pi\xi^2}\int_0^1 \Phi''(u)\cos(u+\lambda)\xi\,du \right|$$

$$= \left| \frac{1}{\pi\xi^m}\int_0^1 \Phi^{(m)}(u)\begin{matrix}\cos\\ \sin\end{matrix}(u+\lambda)\xi\,du \right|$$

$$\leqq \frac{\int_0^1 |\Phi^{(m)}(u)|\,du}{\pi|\xi^m|}.$$

Hence there is a ϑ such that $|\vartheta| \leqq 1$, and

$$\int_a^b dx \left\{ \int_{-\infty}^{\infty} f(\xi)\Psi(\xi - x,\lambda)\,d\xi - f(x) \right\}^2$$

$$= \int_a^b dx \left\{ \int_{-A}^{A} f(\xi)\,\Psi(\xi - x,\lambda)\,d\xi - f(x) \right.$$

$$\left. + \frac{\vartheta}{\pi}\int_0^1 |\Phi^{(k+2)}(u)|\,du \left\{ \left[\int_{-\infty}^{-A} + \int_A^{\infty} \right] \frac{f(\xi)\,d\xi}{(\xi - x)^{k+2}} \right\} \right\}^2$$

$$= \int_a^b dx \left\{ \int_{-A}^{A} f(\xi)\,\Psi(\xi - x,\lambda)\,d\xi - f(x) + \varepsilon(A) \right\}^2$$

$$\leqq \left[\sqrt{\int_a^b dx \left\{ \int_{-A}^{A} f(\xi)\,\Psi(\xi - x,\lambda)\,d\xi - f(x) \right\}^2} + \varepsilon(A)\sqrt{b-a} \right]^2.$$

Here $\varepsilon(A)$ is some quantity depending on A alone, and vanishing as A increases without limit.

We shall therefore have proved the desired convergence in the mean if we can show that for all sufficiently large A,

$$\lim_{\lambda\to\infty} \int_a^b dx \left\{ \int_{-A}^{A} f(\xi)\,\Psi(\xi - x,\lambda)\,d\xi - f(x) \right\}^2 = 0.$$

That is, it is enough to prove that if $g(x)$ vanishes for sufficiently large values of x,

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} dx \left\{ \int_{-\infty}^{\infty} g(\xi)\, \Psi(\xi - x, \lambda)\, d\xi - g(x) \right\}^2 = 0.$$

By hypothesis, we may write

$$g(x) = \underset{B\to\infty}{\mathrm{l.i.m.}} \int_{-B}^{B} \varphi(\xi)\, e^{i\xi x}\, d\xi.$$

in accordance with a theorem of Plancherel[14]). We shall then have

$$\int_{-\infty}^{\infty} g(\eta)\, \Psi(\eta - x, \lambda)\, d\eta = \int_{-\infty}^{\infty} g(x-\eta)\, \Psi(\eta, \lambda)\, d\eta$$

$$= \int_{-\infty}^{\infty} \left[\underset{B\to\infty}{\mathrm{l.\,m.}} \int_{-B}^{B} \varphi(\xi)\, e^{i\xi(x-\eta)}\, d\xi \right] \left[\frac{\sin\lambda\eta}{\eta} + \frac{1}{\pi} \int_0^1 \Phi(u) \cos(u+\lambda)\eta\, du \right] d\eta$$

$$= \int_{-\infty}^{\infty} \varphi(\xi)\, e^{i\xi x}\, d\xi \int_{-\infty}^{\infty} \frac{e^{-i\eta\xi}}{\pi} \left[\frac{\sin\lambda\eta}{\eta} + \int_0^1 \Phi(u) \cos(u+\lambda)\eta\, du \right] d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi)\, e^{i\xi x}\, d\xi \int_{-\infty}^{\infty} e^{-i\eta\xi} \left[\int_{-\infty}^{\infty} F_\lambda(u)\, e^{iu\eta}\, du \right] d\eta,$$

where

$$\begin{aligned}
F_\lambda(u) &= 1; & [-\lambda \leqq u \leqq \lambda]&.\\
F_\lambda(u) &= \Phi(u-\lambda); & [\lambda \leqq u \leqq \lambda + 1]&.\\
F_\lambda(u) &= \Phi(\lambda - u); & [-\lambda - 1 \leqq u \leqq -\lambda]&.\\
F_\lambda(u) &= 0; & [|u| \geqq \lambda + 1]&.
\end{aligned}$$

Therefore, by the theory of the Fourier integral,

$$\int_{-\infty}^{\infty} g(\eta)\, \Psi(\eta - x, \lambda)\, d\eta = \int_{-\infty}^{\infty} \varphi(\xi)\, F_\lambda(\xi)\, e^{i\xi x}\, d\xi,$$

and

$$\int_{-\infty}^{\infty} dx \left\{ \int_{-\infty}^{\infty} g(\xi)\, \Psi(\xi - x, \lambda)\, d\xi - g(x) \right\}^2$$

$$= \int_{-\infty}^{\infty} dx \left[\underset{B\to\infty}{\mathrm{l.\,m.}} \int_{-B}^{B} \varphi(\xi) [1 - F_\lambda(\xi)]\, e^{i\xi x}\, d\xi \right]^2$$

$$= 2\pi \int_{-\infty}^{\infty} [\varphi(\xi)(1 - F_\lambda(\xi))]^2\, d\xi.$$

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[14]) M. Plancherel, Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies, Rend. Palermo **30** (2nd sem. 1910) pp. 289–335.

From this it is easy to conclude that

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} dx \left\{ \int_{-\infty}^{\infty} g(\xi) \Psi(\xi - x, \lambda) \, d\xi - g(x) \right\}^2 = 0,$$

and hence that

$$f_\lambda(x) = \int_{-\infty}^{\infty} f(\xi) \Psi(\xi - x, \lambda) \, d\xi$$

converges in the mean to $f(x)$ as λ becomes infinite.

It will be noticed that $f_\lambda(x)$ may be regarded as consisting of all the components of $f(x)$ with period not exceeding $2\pi\lambda$, together with a portion of the components of $f(x)$ with period exceeding $2\pi\lambda$ but not $2\pi(\lambda+1)$. We have here a method of dividing $f(x)$ into the sum of a number of functions, each of wich contains only harmonics restricted to a given range. This is the key-note of the theory which follows.

2. The Definition of a Non-Singular Operator.

Now let $H(u)$ be a function defined for all pure imaginary values of u, and assuming, in general, complex values. It will be natural to interpret $H(d/dx)\Psi(\xi - x, \lambda)$ as

$$G(\xi - x, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\lambda(u) H(-iu) e^{iu(\xi - x)} \, du.$$

Accordingly, the natural interpretation of

$$H(d/dx) \int_{-\infty}^{\infty} f(\xi) \Psi(\xi - x, \lambda) \, d\xi$$

will be

$$\int_{-\infty}^{\infty} f(\xi) G(\xi - x, \lambda) \, d\xi$$

In case H possesses at every point of the closed interval $(-\lambda i - i, \lambda i + i)$ derivatives of all orders, we may show as in the last paragraph that $G(\xi - x, \lambda)$ is $o(x^{-k})$ at infinity for any positive k, and that

$$\int_{-\infty}^{\infty} f(\xi) G(\xi - x, \lambda) \, d\xi$$

will exist for such functions f as we have been considering. We shall have

$$\int_{-\infty}^{\infty} f(\xi)[G(\xi-x,\lambda_1)-G(\xi-x,\lambda)]\,d\xi \qquad [\lambda_1 \geqq \lambda+1 \geqq 1]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi\left\{\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right][F_{\lambda_1}(u)-F_{\lambda}(u)]\,H(-iu)\,e^{iu(\xi-x)}\,du\right\}$$

$$=\frac{1}{2\pi}\int_{x-1}^{x+1} f(\xi)\,d\xi\left\{\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right][F_{\lambda_1}(u)-F_{\lambda}(u)]\,H(-iu)\,e^{iu(\xi-x)}\,du\right\}$$

$$+\frac{i^n}{2\pi}\left[\int_{-\infty}^{x-1}+\int_{x+1}^{\infty}\right]\frac{f(\xi)}{(\xi-x)^n}\,d\xi\left\{\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]\frac{d^n}{du^n}[(F_{\lambda_1}(u)-F_{\lambda}(u))\,H(-iu)]\,e^{iu(\xi-x)}\,du\right\}$$

$$=\frac{1}{2\pi}\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]du\left\{[F_{\lambda_1}(u)-F_{\lambda}(u)]\,H(-iu)\int_{-1}^{1} f(x+\eta)\,e^{iu\eta}\,d\eta\right.$$

$$\left.+\frac{i^n\,d^n}{du^n}[(F_{\lambda_1}(u)-F_{\lambda}(u))\,H(-iu)]\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)}{\eta^n}\,e^{iu\eta}\,d\eta\right\}.$$

Therefore, by a ready extension of Plancherel's theorem on the Fourier transform, combined with the Schwarz inequality,

$$\left|\int_{-\infty}^{\infty} f(\xi)[G(\xi-x,\lambda_1)-G(\xi-x,\lambda)]\,d\xi\right|^2$$

$$\leqq \frac{1}{2\pi^2}\left\{\left|\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]du\Big[F_{\lambda_1}(u)-F_{\lambda}(u)\Big]H(-iu)\int_{-1}^{1} f(x+\eta)\,e^{iu\eta}\,d\eta\right|^2\right.$$

$$\left.+\left|\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]du\left\{\frac{d^n}{du^n}\Big[\Big(F_{\lambda_1}(u)-F_{\lambda}(u)\Big)H(-iu)\Big]\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)}{\eta^n}\,e^{iu\eta}\,d\eta\right\}\right|^2\right\}$$

$$\leqq \frac{1}{\pi^2}\left\{\operatorname*{Max}_{|v|\geqq\lambda}|H(v)|^2\left|\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]du\int_{-1}^{1} f(x+\eta)\,e^{iu\eta}\,d\eta\right|^2\right.$$

$$+\operatorname*{Max}_{|v|\geqq\lambda}\left|H^{(n)}(v)\right|^2\left|\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]du\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)}{\eta^n}\,e^{iu\eta}\,d\eta\right|^2$$

$$+\left|\left[\int_{-\lambda_1-1}^{-\lambda_1}+\int_{-\lambda-1}^{-\lambda}+\int_{\lambda}^{\lambda+1}+\int_{\lambda_1}^{\lambda_1+1}\right]du\sum_{h=1}^{n}\frac{d^h}{du^h}[F_{\lambda_1}(u)-F_{\lambda}(u)]\frac{d^{n-h}}{du^{n-h}}H(-iu)\frac{n!}{h!\,(n-h)!}\right.$$

$$\left.\left.\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)}{\eta^n}\,e^{iu\eta}\,d\eta\right|^2\right\}.$$

It is now easy to show that if for all k, $\lim\limits_{v\to\pm\infty} H^{(k)}(v)=0$, then given any ε, λ can be chosen so large that for any λ_1, we shall have

$$\int_{-a}^{a}\left|\int_{-\infty}^{\infty} f(\xi)\,[G(\xi-x,\lambda_1)-G(\xi-x,\lambda)]\,d\xi\right|^2 dx$$

$$\leqq \frac{1}{\pi^2}\operatorname*{Max}_{|v|\geqq\lambda}|H(v)|^2\int_{-a}^{a}\left|\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1-1}\right]du\int_{-1}^{1}f(x+\eta)\,e^{iu\eta}\,d\eta\right|^2 dx$$

$$+\frac{1}{\pi^2}\operatorname*{Max}_{|v|\geqq\lambda}|H^{(n)}(v)|^2\int_{-a}^{a}\left|\left[\int_{-\lambda_1-1}^{-\lambda}+\int_{\lambda}^{\lambda_1+1}\right]du\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)}{\eta^n}e^{iu\eta}d\eta\right|dx+\varepsilon$$

$$\leqq \frac{1}{\pi^2}\operatorname*{Max}_{|v|\geqq\lambda}|H(v)|^2\int_{-a}^{a}\left\{\left\{\int_{\lambda}^{\lambda_1+1}du\int_{-1}^{1}(f(x+\eta)+f(x-\eta))\cos u\eta\,d\eta\right\}^2\right.$$

$$\left.+\left\{\int_{\lambda}^{\lambda_1+1}du\int_{-1}^{1}(f(x+\eta)-f(x-\eta))\sin u\eta\,d\eta\right\}^2\right\}dx$$

$$+\frac{1}{\pi^2}\operatorname*{Max}_{|v|\geqq\lambda}|H^{(n)}(v)|^2\int_{-a}^{a}\left\{\left\{\int_{\lambda}^{\lambda_1+1}du\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)+f(x-\eta)}{\eta^n}\cos u\eta\,d\eta\right\}^2\right.$$

$$\left.+\left\{\int_{\lambda}^{\lambda_1+1}du\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{f(x+\eta)-f(x-\eta)}{\eta^n}\sin u\eta\,d\eta\right\}^2\right\}dx+\varepsilon.$$

Here n is taken to be even.

We shall consider n to be chosen so large that

$$f(x)=o(x^{n+2})$$

at infinity. I then say that all the integrals in this expression will tend to zero with increasing λ. For exemple, we shall have

$$I=\int_{-a}^{a}\left\{\int_{\lambda}^{\lambda_1+1}du\int_{-1}^{1}(f(x+\eta)-f(x-\eta))\cos u\eta\,d\eta\right\}^2dx$$

$$=\int_{\lambda}^{\lambda_1+1}du\int_{\lambda}^{\lambda_1+1}dv\int_{-1}^{1}\cos u\xi\,d\xi\int_{-1}^{1}\cos v\eta\,d\eta\int_{-a}^{a}(f(x+\xi)+f(x-\xi))(f(x+\eta)+f(x-\eta))\,dx$$

Now, if $\psi(x,\eta)$ is any summable function of summable square, it is easy to prove by expanding it in a double Fourier series that

$$\int_{-1}^{1}\cos u\xi\,d\xi\int_{-1}^{1}\psi(\xi,\eta)\cos v\eta\,d\eta$$

is a function of summable square over an infinite range of both variables.

From this we may conclude that I vanishes with increasing λ, independently of λ_1.

The proof goes through in a similar manner for the other cases, and the factor η^n in the denominator guarantees that all the infinite integrals we obtain from our second integral shall be convergent. Hence if $H(v)$ is bounded at infinity and if all its derivatives vanish at infinity, we have

$$\lim_{\lambda\to\infty} \int_{-a}^{a} \left| \int_{-\infty}^{\infty} f(\xi)[G(\xi - x, \lambda_1) - G(\xi - x, \lambda)]\, d\xi \right|^2 dx = 0,$$

and

$$\underset{\lambda\to\infty}{\mathrm{l.\,m.}} \int_{-\infty}^{\infty} f(\xi) G(\xi - x, \lambda)\, d\xi$$

exists. We shall term this $H(d/dx)f(x)$.

We shall have

$$\int_{-\infty}^{\infty} G_1(\xi - x; \lambda_1)\, d\xi\, \mathrm{l.\,m.} \int_{-\infty}^{\infty} f(\eta) G_2(\eta + \xi, \lambda_2)\, d\eta$$

$$= \lim_{\lambda_2\to\infty} \int_{-\infty}^{\infty} f(\eta)\, d\eta \int_{-\infty}^{\infty} G_1(\xi - x, \lambda_1) G_2(\eta - \xi, \lambda_2)\, d\xi$$

$$= \lim_{\lambda_2\to\infty} \int_{-\infty}^{\infty} f(\eta)\, d\eta \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} F_{\lambda_1}(u) H_1(-iu) e^{iu(\xi - x)}\, du \int_{-\infty}^{\infty} F_{\lambda_2}(v) H_2(-iv) e^{iv(\eta-\xi)}\, dv$$

$$= \lim_{\lambda_2\to\infty} \int_{-\infty}^{\infty} f(\eta)\, d\eta \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\lambda_1}(u) F_{\lambda_2}(u) H_1(-iu) H_2(-iu) e^{iu(\eta - x)}\, du$$

$$= \int_{-\infty}^{\infty} f(\eta)\, d\eta \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\lambda_1}(u) H_1(-iu) H_2(-iu) e^{iu(\eta - x)}\, du,$$

where H_1 and H_2 are functions everywhere infinitely differentiable, finite at infinity, and with all their derivatives finite at infinity, and G_1 and G_2 bear to them the same relation which G bears to H. If H satisfies this condition as to its differentiability and its order at infinity, we shall call $H(d/dx)$ a *non-singular operator*. It is clear that if $H_1(d/dx)$ and $H_2(d/dx)$ are non-singular, $H_1(d/dx) H_2(d/dx)$ will also be non-singular. We have just seen that the result of applying to $f(x)$ first $H_2(d/dx)$ and then $H_1(d/dx)$ is $H_1(d/dx) H_2(d/dx)$, so that the operational product of two non-singular operators is always a non-singular operator.

A particularly interesting non-singular operator is $F_\lambda(i\,d/dx)$. By multiplying this with an arbitrary operator, we obtain that operator as applied to a given frequency range. The operator thus obtained may be non-singular even when the operator multiplied with $F_\lambda(i\,d/dx)$ is singular. This is the case, for instance, with the operator d/dx itself.

In addition to the non-singular operators, there are other operators of the form $H(d/dx)$ which apply to all summable functions bounded over every finite region which are of finite order at infinity. For example,

$$e^{a\,d/dx} f_\lambda(x) = f_\lambda(x+a).$$

Hence if we define $a^{a\,d/dx} f(x)$ as the limit in the mean of $e^{a\,d/dx} f_\lambda(x)$ as λ becomes infinite, we have

$$e^{a\,d/dx} f(x) = f(x+a).$$

We can similarly be sure of the existence of

$$H(d/dx) f(x) = \underset{\lambda\to\infty}{\text{l. m.}}\, H(d/dx) f_\lambda(x),$$

where

$$H(d/dx) = e^{a\,d/dx} J(d/dx),$$

while $J(d/dx)$ is non-singular. An example in point is

$$e^{\sqrt{(d/dx+a)\,(d/dx+b)}}.$$

We shall call such operators *non-singular in the generalized sense.*

3. The Analytic Expansion of a Non-Singular Operator.

Let $H(u)$ be a function analytic within a circle of radius $r > 1$ about the origin. Let $\lambda < r-1$. We shall have

$$H\left(\frac{d}{dx}\right)\Psi(\xi - x, \lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\lambda(u) H(-iu)\, e^{iu(\xi-x)}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\lambda(u)\Big\{ H(0) - iu\,H'(0) + \ldots + \frac{(-iu)^{n-1}}{(n-1)!} H^{(n-1)}(0)$$

$$+ \frac{(-i)^n}{(n-1)!}\int_0^u (u-\eta)^{n-1} H^{(n)}(-i\eta)\,d\eta\Big\} e^{iu(\xi-x)}\, du$$

$$= H(0)\,\Psi(\xi-x,\lambda) + H'(0)\frac{d\Psi(\xi-x,\lambda)}{dx} + \ldots$$

$$+ \frac{H^{(n-1)}(0)}{(n-1)!}\frac{d^{n-1}\Psi(\xi-x,\lambda)}{dx^{n-1}} + \frac{\vartheta(\lambda+1)^{n+1}}{\pi(n-1)!}\max_{|\eta|\leqq\lambda+1}\left|H^{(n)}(-i\eta)\right| \qquad [|\vartheta|\leqq 1].$$

We may conclude at once from this that

$$\sum \frac{H^{(n)}(0)}{n}\frac{d^n\Psi}{dx^n}(\xi - x, \lambda)$$

converges uniformly to $H(d/dx)\,\Psi(\xi-x,\lambda)$ over any range, finite or infinite.

Again,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_\lambda(u)\frac{(-i)^n}{(n-1)!}\int_0^u (u-\eta)^{n-1} H^{(n)}(-i\eta)\,d\eta\, e^{iu(\xi-x)}\,du$$

$$=\frac{1}{2\pi(\xi-x)^k}\int_{-\infty}^{\infty} e^{iu(\xi-x)}\frac{d^k}{du^k}\left[F_\lambda'(u)\frac{(-i)^{n-k}}{(n-1)!}\int_0^u (u-\eta)^{n-1} H^{(n)}(-i\eta)\,d\eta\right]du$$

$$=\frac{1}{2\pi(\xi-x)^k}\int_{-\infty}^{\infty} e^{iu(\xi-x)}\left[\sum_{h=0}^{k}\frac{(-i)^{n-k}\,k!}{(n-1)!\,h!\,(k-h)!}F_\lambda^{(h)}(u)\right.$$

$$\left.\cdot\frac{d^{k-h}}{du^{k-h}}\int_0^u (u-\eta)^{n-1} H^{(n)}(-i\eta)\,d\eta\right]du$$

$$=\frac{1}{2\pi(\xi-x)^k}\int_{-\infty}^{\infty} e^{iu(\xi-x)}\left[\sum_{h=0}^{k}\frac{(-i)^{n-h}\,k!}{(n-k+h-1)!\,h!\,(k-h)!}F_\lambda^{(h)}(u)\right.$$

$$\left.\int_0^u (u-\eta)^{n-k+h-1} H^{(n-k+h)}(-i\eta)\,d\eta\right]du$$

$$=\frac{\theta_1}{\pi(\xi-x)^k}\sum_{h=0}^{k}(\lambda+1)^{n-k+h+1}\frac{k!\,F_\lambda^{(h)}(u)}{(n-k+h-1)!\,h!\,(k-h)!}\operatorname*{Max}_{|\eta|\leqq\lambda+1} H^{(n-k+h)}(-i\eta).$$

It follows that

$$\lim_{N\to\infty}(\xi-x)^k\left[\sum_0^N\frac{H^{(n)}(0)}{n!}\frac{d^n}{dx^n}\Psi(\xi-x,\lambda)-H\left(\frac{d}{dx}\right)\Psi(\xi-x,\lambda)\right]=0$$

uniformly in $\xi-x$, whatever k may be.

It resulte that in case f satisfies the conditions which we have laid down for it,

$$\sum_0^\infty\frac{H^{(n)}(0)}{n!}\frac{d^n}{dx^n}f_\lambda(x)=H\left(\frac{d}{dx}\right)f_\lambda(x)$$

uniformly over any finite range whatever. That is, in case f contains only components of sufficiently low frequency, if we apply to f an operator $H(d/dx)$, we may expand this operator in a Maclaurin series. In order to show this, it is enough to prove that

$$\frac{d}{dx}f_\lambda(x)=\int_{-\infty}^{\infty} f(\xi)\frac{d}{dx}\Psi(\xi-x,\lambda)\,d\xi,$$

with a similar theorem for the derivatives of higher order. Now,

$$\int_{-\infty}^{\infty} f(\xi)\frac{d}{dx}\Psi(\xi-x,\lambda)\,d\xi=\left[\int_{-A}^{A}+\int_{-\infty}^{-A}+\int_A^\infty\right]f(\xi)\frac{d}{dx}\Psi(\xi_1-x,\lambda)\,d\xi$$

$$=\lim_{A\to\infty}\frac{d}{dx}\int_{-A}^{A} f(\xi)\,\Psi(\xi-x,\lambda)\,d\xi$$

uniformly in x over any finite range of x. Now, it is easy to show that if $\varphi_k(x)$ converges uniformly to $\varphi(x)$ over some closed range of x, while $\varphi_k'(x)$ converges uniformly to $\psi(x)$, then within this range,

$$\psi(x) = \varphi'(x).$$

Thus

$$\int_{-\infty}^{\infty} f(\xi) \frac{d}{dx} \Psi(\xi - x, \lambda)\, d\xi = f_\lambda'(x).$$

In the case where $H(u)$ possesses a Taylor expansion valid within a circle of radius $r > 1$ about the point ia, a precisely similar argument will show that if $\lambda < r - 1$ and

$$g_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \left[\int_{a-\lambda-1}^{a-\lambda} \Phi(\lambda - a - u)\, e^{iu(\xi - x)}\, du + \int_{a-\lambda}^{a+\lambda} e^{iu(\xi-x)}\, du \right.$$

$$\left. + \int_{a+\lambda}^{a+\lambda+1} \Phi(u - a - \lambda)\, e^{iu(\xi-x)}\, du \right]$$

we shall have

$$H\left(\frac{d}{dx}\right) g_\lambda(x) = \sum_0^\infty \frac{H^{(n)}(ia)}{n!} \left(\frac{d}{dx} - ia\right)^n g_\lambda(x)$$

uniformly over any finite range of x.

We shall now consider Taylor expansions about the point at infinity. Let $H(u)$ possess such an expansion. We can then put

$$H(u) = H(\infty) + \left(\frac{1}{u}\right) K\left(\frac{1}{u}\right),$$

where K is analytic in some circle about the origin of radius R^{-1}. Let λ be any number greater than R. We shall have

$$H\left(\frac{d}{dx}\right)[f(x) - f_\lambda(x)] = \frac{1}{2\pi} \operatorname*{l.m.}_{\mu\to\infty} \int_{-\infty}^{\infty} [f(\xi) - f_\lambda(\xi)]\, d\xi \int_{-\infty}^{\infty} F_\mu(u)\, H(-iu)\, e^{iu(\xi-x)}\, d$$

$$= \frac{1}{2\pi} \operatorname*{l.m.}_{\mu\to\infty} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} F_\mu(u)\, H(-iu)\, e^{iu(\xi-x)}\, du$$

$$- \frac{1}{4\pi^2} \operatorname*{l.m.}_{\mu\to\infty} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(\eta)\, d\eta \int_{-\infty}^{\infty} F_\lambda(v)\, e^{iv(\eta-\xi)}\, dv \int_{-\infty}^{\infty} F_\mu(u)\, H(-iu)\, e^{iu(\xi-x)}\, du$$

$$= \frac{1}{2\pi} \operatorname*{l.m.}_{\mu\to\infty} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} F_\mu(u)\, H(-iu)\, e^{iu(\xi-x)}\, du$$

$$- \frac{1}{2\pi} \operatorname*{l.m.}_{\mu\to\infty} \int_{-\infty}^{\infty} f(\eta)\, d\eta \int_{-\infty}^{\infty} F_\mu(u)\, F_\lambda(u)\, H(-iu)\, e^{iu(\eta-x)}\, du$$

$$= \frac{1}{2\pi} \underset{\mu\to\infty}{\mathrm{l.m.}} \int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{\infty} [F_\mu(u) - F_\lambda(u)]\, H(-iu)\, e^{iu(\xi-x)}\,du$$

$$= H(\infty)[f(x) - f_\lambda(x)] + \frac{1}{2\pi} \underset{\mu\to\infty}{\mathrm{l.m.}} \int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{\infty} \frac{F_\mu(u) - F_\lambda(u)}{-iu} K\left(\frac{i}{u}\right) e^{iu(\xi-x)}\,du$$

$$= H(\infty)[f(x) - f_\lambda(x)] + \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{\infty} \frac{i[1 - F_\lambda(u)]}{u} K\left(\frac{i}{u}\right) e^{iu(\xi-x)}\,du$$

$$= H(\infty)[f(x) - f_\lambda(x)]$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{\infty} \frac{i[1 - F_\lambda(u)]}{u} \Big[K(0) + \frac{i}{u} K'(0) + \dots$$

$$+ \frac{i^{n-1}}{u^{n-1}(n-1)!} K^{n-1}(0) + \frac{i^n}{(n-1)!} \int_u^{\infty} \left(\frac{1}{u} - \frac{1}{\eta}\right)^{n-1} K^{(n)}\left(\frac{i}{\eta}\right) \frac{d\eta}{\eta^2}\Big] e^{iu(\xi-x)}\,du$$

$$= H(\infty)[f(x) - f_\lambda(x)] + K(0) \frac{1}{d/dx} [f(x) - f_\lambda(x)] + \dots$$

$$+ \frac{K^{(n-1)}(0)}{(d/dx)^n (n-1)} [f(x) - f_\lambda(x)] + \int_{-\infty}^{\infty} f(\xi)\, Q(\xi - x)\, d\xi,$$

where $\frac{1}{(d/dx)^n}[f(x) - f_\lambda(x)]$ signifies that solution of the differential equation

$$g^{(n)}(x) = f(x) - f_\lambda(x)$$

which has the property that $g_\mu(x)$ vanishes for sufficiently small values of μ, while

$$Q(w) = \frac{i^{n+1-k}}{2(n-1)!\,\pi w^k} \int_{-\infty}^{\infty} e^{iuw} \frac{d^k}{du^k} \left[\frac{1 - F_\lambda(u)}{u} \int_u^{\infty} \left(\frac{1}{u} - \frac{1}{\eta}\right)^{n-1} K^{(n)}\left(\frac{i}{\eta}\right) \frac{d\eta}{\eta^2}\right] du.$$

It is easy to establish that the analytic character of K makes

$$\lim_{n\to\infty} [w^k Q(w)] = 0$$

for all positive values of k. We thus establish the analytic expansion of an operator about the point at infinity.

4. Singular Operators with Poles.

Let $H(d/dx)$ be a non-singular operator without finite or infinite zeros on the axis of imaginaries. Then $H(d/dx)$ will be another such operator, which will be the inverse of the first. The question arises, has H any other inverses?

It has not. Suppose $H(d/dx)f(x) = H(d/dx)g(x)$, where f and g differ at more than a set of points of measure zero. Then $H(d/dx)(f(x) - g(x))$ is identically zero, except on such a set. Hence

$$f(x) - g(x) = \frac{1}{H(d/dx)} H\left(\frac{d}{dx}\right)(f(x) - g(x)) = \frac{1}{H(d/dx)} 0 = 0,$$

except at the most at a set of points of zero measure. This is however a contradiction.

We now turn to the case where $H(d/dx)$ has a finite number of zeros of finite orders. Let us suppose, for purposes of simplicity, that there is a zero of the n^{th} order at the origin, and that no other zero is as near as $\lambda + 1$. We have

$$\begin{aligned} H\left(\frac{d}{dx}\right) f(x) &= H\left(\frac{d}{dx}\right) f_\lambda(x) + H\left(\frac{d}{dx}\right)(f(x) - f_\lambda(x)) \\ &= H_1\left(\frac{d}{dx}\right) f_\lambda(x) + H_2\left(\frac{d}{dx}\right)(f(x) - f_\lambda(x)), \end{aligned}$$

where H_1 equals H within a circle of radius exceeding $\lambda + 1$ about the origin, and defines a non-singular operator with no zeros on the imaginary axis outside this circle, except for a zero of order n at infinity, while H_2 equals H outside a circle of radius less than λ about the origin, and defines a non-singular operator with no zeros on the imaginary axis within this circle. It is easy to show that the equation

$$H\left(\frac{d}{dx}\right) f(x) = g(x)$$

reduces to the two simultaneous equations

$$H_1\left(\frac{d}{dx}\right) f_\lambda(x) = g_\lambda(x)$$

and

$$H_2\left(\frac{d}{dx}\right)(f(x) - f_\lambda(x)) = g(x) - g_\lambda(x).$$

We may without difficulty transform the first into the form

$$\frac{H_2(d/dx)}{(d/dx)^n}\left(\frac{d}{dx}\right)^n f_\lambda(x) = g_\lambda(x).$$

The solution of this may be achieved by the inversion of a non-singular operator without zeros, followed by an n-fold integration, which will in general introduce n constants of integration. The second equation will now involve the inversion of an operator with one less zero than H. Thus by a repetition of the same process, we ultimately reduce the inversion of our operator to the inversion of a non-singular operator without zeros in the finite portion of the plane. The inversion of an operator with an n-fold zero at infinity may similarly be reduced to an n-fold differentiation.

We shall represent the inverse of the non-singular operator $H(d/dx)$ by $1/H(d/dx)$. This will manifestly contain one arbitrary constant for every finite zero of H, counted with its multiplicity, when we apply it to an arbitrary function. The effect of a zero at infinity will be to restrict the class of functions to which $1/H(d/dx)$ can apply. We shall consider an operator containing both poles and zeros on the axis of imaginaries as the product of two operators, one with poles alone and the other with zeros alone. We can investigate the problem of inverting such an operator by our now-familiar method of dissecting each function to which we apply it into frequency ranges, each containing only a single singularity or zero, and replacing the operator over this range by one without other zeros or singularities.

We shall later investigate the question of removing the arbitrary character of the constants in the definition of singular operators.

5. Branch-Points and Fractional Differentiation and Integration.

Let us now consider operators such as $(d/dx)^{1/2}$, with branch points. We can limit ourselves to begin with to the case of an operator whose only notable point on the axis of imaginaries is a single branch-point of order n, where the sheets of the Riemann surface interchange in circular order. Within a certain distance of this point, our operator $H(d/dx)$ is determined by an analytic function, and every where except at the branch-point this is continuous, and has continuous derivatives of all orders. Our function $H(-iu)$ is bounded at infinity, whele all its derivatives vanish there. For purposes of simplicity, we shall only consider the case where the origin is the branch-point.

Let it be noted that our operator is singular, and accordingly has not yet received any precise definition. As before, we shall write

$$G(\xi - x, \lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\lambda(u) H(-iu) e^{iu(\xi - x)} du.$$

Since H is analytic within some circle about the origin, except at the center,

$$\frac{1}{2\pi}\int_{-a}^{a} F_\lambda(u) H(-iu) e^{iu(\xi - x)} du$$

for sufficiently small values of a may be regarded as an integral along some path in the complex plane other than the axis of imaginaries. The choice of this path depends on the choice we make of the sheets of the Riemann surface in which our function is taken to the right and the left of the origin. If the junction between these sheets is made trough positive real arguments of $H(u)$, our path of integration will pass above the origin; in the contrary case, below.

Let us now assume that our path of integration from $-\infty$ to ∞, as modified by our detour at the origin, passes above the origin, and let us call it C. We shall then have

$$G(\xi - x, \lambda) = \frac{1}{2\pi}\int_C F_\lambda(u) H(-iu)\, e^{iu(\xi-x)}\, du$$
$$= \frac{i k}{2\pi(\xi - x)^k}\int_C \frac{d^k}{du^k}[F_\lambda(u) H(-iu)]\, e^{iu(\xi-x)}\, du.$$

Hence if ε is any positive quantity,

$$|G(\xi - x, \lambda)| \leqq \frac{1}{2\pi(\xi - x)^k} \max_C \frac{d^k}{du^k}[F_\lambda(u) H(-iu)]$$

if $\xi - x \leqq 0$, and

$$|G(\xi - x, \lambda)| \leqq \frac{e^{\varepsilon(\xi - x)}}{2\pi(\xi - x)^k} \max_C \frac{d^k}{du^k}[F_\lambda(u) H(-iu)]$$

otherwise. Therefore, if $f(x)$ is any function which is $O(e^{-qx})$ to the right of the origin, q being greater than 0, and is $O(x^m)$ to the left,

$$\int_{-\infty}^{\infty} f(\xi)\, G(\xi - x, \lambda)\, d\xi,$$

exists. We shall term this

$$H\left(\frac{d}{dx}\right) f_\lambda(x).$$

This function will become only algebraically infinite at infinity. We may use the same methods as previously to show that

$$H\left(\frac{d}{dx}\right) f(x) = \underset{\lambda \to \infty}{\mathrm{l.\,m.}}\, H\left(\frac{d}{dx}\right) f_\lambda(x)$$

exists. We shall call this, for obvious reasons, the backward value of $H(d/dx) f(x)$. The forward value of this symbol will correspond to a path of integration passing below the origin. It will apply to functions $f(x)$ which only become algebraically infinite for negative large values of x, and which become exponentially small for positive large values of x. In the applications of the operational calculus, x generally stands for the time, and $f(x)$ represents an impressed voltage or current. It is hence the backward values of our operators which have physical significance. We shall discuss this in more detail when we come to the theory of prospective and retrospective operators. Unless the contrary is stated, then, our operators with branch-points will be taken as backward.

As far as what we have said about the existence of $H(d/dx)f_\lambda(x)$, the properties of H for arguments greater than $\lambda+1$ in modulus are irrelevant. Thus if $f(x)$ is determined to have the value $(1+\operatorname{sgn} x)/2$, $(d/dx)^{1/2} f_\lambda(x)$ exists. We have:

$$\left(\frac{d}{dx}\right)^{1/2} f_\lambda(x) = \frac{1}{2\pi}\int_0^\infty d\xi \int_{-\lambda-1}^{\lambda+1} F_\lambda(u)\sqrt{-iu}\, e^{iu(\xi-x)}\,du$$

$$= \frac{1}{2\pi}\int_0^\infty d\xi \left[\int_0^{\lambda+1} F_\lambda(u)\frac{1-i}{\sqrt{2}}\sqrt{u}\, e^{iu(\xi-x)}\,du\right.$$

$$\left. + \int_0^{\lambda+1} F_\lambda(u)\frac{1+i}{\sqrt{2}}\sqrt{u}\, e^{iu(\xi-x)}\,du\right]$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^\infty d\xi \int_0^{\lambda+1} F_\lambda(u)\left[\sin u(\xi-x)+\cos u(\xi-x)\right]\sqrt{u}\,du$$

$$= \lim_{H\to\infty}\frac{\sqrt{2}}{2\pi}\int_0^{\lambda+1} F_\lambda(u)\sqrt{u}\,\frac{-\cos u(H-x)+\cos ux+\sin u(H-x)+\sin ux}{u}\,du$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{\lambda+1} F_\lambda(u)\,\frac{\sin ux+\cos ux}{\sqrt{u}}\,du,$$

since $F_\lambda(u)/du$ is summable from 0 to $\lambda+1$, so that

$$\lim_{H\to\infty}\frac{\sqrt{2}}{2\pi}\int_0^{\lambda+1}\frac{F_\lambda(u)}{\sqrt{u}}\cos u(H-x)\,du = \lim_{H\to\infty}\frac{\sqrt{2}}{2\pi}\int_0^{\lambda+1}\frac{F_\lambda(u)}{\sqrt{u}}\sin u(H-x)\,du = 0.$$

Thus

$$\left(\frac{d}{dx}\right)^{1/2} f_\lambda(x) = \frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{\sin ux+\cos ux}{\sqrt{u}}\,du - \frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{[1-F_\lambda(u)]\,[\sin ux+\cos ux]}{\sqrt{u}}\,du$$

$$\begin{cases} = \dfrac{1}{\sqrt{\pi x}} - \dfrac{\sqrt{2}}{2\pi}\displaystyle\int_0^\infty \frac{[1-F_\lambda(u)]\,[\sin ux+\cos ux]}{\sqrt{u}}\,du; & [x>0] \\[2ex] = -\dfrac{\sqrt{2}}{2\pi}\displaystyle\int_0^\infty \frac{[1-F_\lambda(u)]\,[\sin ux+\cos ux]}{\sqrt{u}}\,du. & [x<0] \end{cases}$$

Now,

$$\frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{[1-F_\lambda(u)]\,[\sin ux+\cos ux]}{\sqrt{u}}\,du$$

$$=\pm\frac{\sqrt{2}}{2\pi x^n}\int_0^\infty (\sin ux \pm \cos ux)\frac{d^n}{du^n}\left(\frac{1-F_\lambda(u)}{\sqrt{u}}\right)du$$

$$=\frac{1}{x^n}\int_0^\infty \frac{(-iu)}{u^{n-1}}(\sin ux \pm \cos ux)\,du$$

where $G(u)$ is a continuous function which is $o(1/u)$ at infinity. We may readily conclude that

$$\frac{d^n}{dx^n}\frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{[1-F_\lambda(u)]\,[\sin ux+\cos ux]}{\sqrt{u}}\,du$$

exists, and is $o(1/u^k)$ at infinity for every value of k. Thus we get *the* fundamental theorem that

$$\left(\frac{d}{dx}\right)^{n+1/2} f_\lambda(x)=\frac{1}{\sqrt{\pi}}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-n+\frac{3}{2}\right)x^{-n+1/2}+o(x^{-k})$$

for all values of k, as x becomes positively infinite, and

$$\left(\frac{d}{dx}\right)^{n+1/2} f_\lambda(x)=o(x^{-k})$$

for all values of k, as x becomes negatively infinite. Here $(d/dx)^{n+1/2}f_\lambda(x)$ is taken as the n^{th} derivative of $(d/dx)^{1/2}f_\lambda(x)$. It is easy to show that this agrees with the direct definition of the derivative of order $n+1/2$.

In the same way, if $H(d/dx)$ is an operator which is non-singular except at the origin, and analytic about the origin, except for a finite branch-point at the origin, we may show that

$$H\left(\frac{d}{dx}\right)f(x)-H\left(\frac{d}{dx}\right)f_\lambda(x)=o(x^{-k})$$

at $\pm\infty$, $f(x)$ being given as above the interpretation $(1+\operatorname{sgn} x)/2$.

We now wish to discuss the definition of an operator with a branch point of finite order which is at the same time an infinity. As in the case of a pole, we can first isolate this singularity and then reduce our operator to a non-singular operator combined with an operator of the form $(d/dx-ai)^{-n}$. We shall only take up in detail the case of the operator $(d/dx)^{-1/2}$, but our methods will be of general applicability.

Let us consider a function $f(x)$ which is $O(e^{kx})$ at $-\infty$, k being positive, and which is at most algebraically infinite at $+\infty$. It will be natural to make the definition

$$\left(\frac{d}{dx}\right)^{-1/2} f(x) = \underset{\lambda\to\infty}{\mathrm{l.\,m.}} \int_{-\infty}^{x} \left(\frac{d}{dx}\right)^{1/2} f_\lambda(x)\,dx.$$

We have, however,

$$\int_{-\infty}^{x} \left(\frac{d}{dx}\right)^{1/2} f_\lambda(x)\,dx = \frac{1}{2\pi}\int_{-x}^{x} dx \int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{\infty} F_\lambda(u)\sqrt{-iu}\,e^{iu(\xi-x)}\,du$$

$$= \frac{1}{2\pi}\lim_{A\to\infty}\int_{-\infty}^{x} dx \int_{-A}^{A} f(\xi)\,d\xi \int_{-\infty}^{\infty} F_\lambda(u)\sqrt{-iu}\,e^{iu(\xi-x)}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{x} dx \int_{-\infty}^{\infty} F_\lambda(u)\sqrt{-iu}\,e^{iu(\xi-x)}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \left[\int_{-\infty}^{\infty} \frac{F_\lambda(u)}{\sqrt{-iu}}\,e^{iu(\xi-\eta)}\,du\right]_{\eta=-\infty}^{\eta=x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{-\infty}^{\infty} \frac{F_\lambda(u)}{\sqrt{-iu}}\,e^{iu(\xi-x)}\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} \frac{f(\xi)}{\sqrt{x-\xi}}\,d\xi - \frac{\sqrt{2}}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \int_{0}^{\infty} \frac{1-F_\lambda(u)}{\sqrt{u}}\left[\sin u(x-\xi) - \cos u(x-\xi)\right]du.$$

It may then be established without much difficulty that the second term in this sum converges in the mean to 0 over any finite range as λ increases without limit. Thus

$$\left(\frac{d}{dx}\right)^{-1/2} f(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} \frac{f(\xi)}{\sqrt{x-\xi}}\,d\xi.$$

As far as functions of the form $f_\lambda(x)$ are concerned, the operator $(d/dx)^{-1/2}$ here defined will be a true inverse to the operator $(d/dx)^{1/2}$. It is only for such functions that we have defined the derivative of order $1/2$. In the more general case, we shall define it as the inverse of $(d/dx)^{-1/2}$. The theory of the latter is well known, and it is known that it can have only one inverse on the range running from $-\infty$ up.

It is important to note that the operator $(d/dx)^{-1/2}$, and if it apply to f, the operator $(d/dx)^{1/2}$, are not affected by values of f for arguments greater than the point where we examine the function generated from f by the operator in question.

The operator $(d/dx)^{n+1/2}$ is to be conceived as the result of applying first $(d/dx)^{1/2}$ and then $(d/dx)^n$. Similar definitions are to be made for other improper fractional powers of the differentiator, negative as well as positive. If we confine our attention to functions f which are $O(e^{kx})$ at $-\infty$ and algebraically infinite or finite at $+\infty$ and let all the integrations in the definitions of our inverse operators be made from $-\infty$, we shall find that the laws of indices will always be obeyed.

6. Asymptotic Series.

Let $f(x)$ be $(1+\operatorname{sgn} x)/2$. Let $H(d/dx)$ be defined by a function $H(u)$ determined everywhere on the axis of imaginaries, non-singular, and such that

$$M(u) = \frac{H(u)}{\sqrt{u}}$$

is analytic within a circle of radius $r > \lambda + 1$ about the origin. Then

$$\begin{aligned}
H\left(\frac{d}{dx}\right) f(x) &= H\left(\frac{d}{dx}\right) f_\lambda(x) + o\left(\frac{1}{x^k}\right) \\
&= M\left(\frac{d}{dx}\right)\left[\left(\frac{d}{dx}\right)^{1/2} f_\lambda(x)\right] + o\left(\frac{1}{x^k}\right)
\end{aligned}$$

$$\begin{aligned}
&= M(0)\left[\left(\frac{d}{dx}\right)^{1/2} f_\lambda(x)\right] + M'(0)\left[\left(\frac{d}{dx}\right)^{3/2} f_\lambda(x)\right] + M''(0)\left[\left(\frac{d}{dx}\right)^{5/2} f_\lambda(x)\right] + \dots \\
&\qquad + \frac{M^{(k-1)}(0)}{(k-1)}\left[\left(\frac{d}{dx}\right)^{k-1/2} f_\lambda(x)\right] + N\left(\frac{d}{dx}\right)\left[\left(\frac{d}{dx}\right)^{k+1/2} f_\lambda(x)\right] + o\left(\frac{1}{x^k}\right) \\
&= \frac{1}{\sqrt{\pi x}}\left[M(0) - \frac{1}{2}\frac{M'(0)}{x} + \frac{1}{2}\cdot\frac{3}{2}\frac{M''(0)}{x^2} - \dots\right. \\
&\qquad \left. + (-1)^{k-1}\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-3}{2}\right)\frac{M^{(k-1)}(0)}{x^{k-1}}\right] + o\left(\frac{1}{x^k}\right) \\
&\qquad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{d}{dx}\right)^{k+1/2} f_\lambda(\xi)\,d\xi\int_{-\infty}^{\infty} F_{\lambda+2}(u)\,N(-iu)\,e^{iu(\xi-x)}\,du \\
&= \frac{1}{\sqrt{\pi x}}\left[M(0) - \dots + (-1)^{k-1}\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-3}{2}\right)\frac{M^{(k-1)}(0)}{x^{k-1}}\right] + o\left(\frac{1}{x^k}\right) \\
&\qquad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(d/d\xi)^{1/2} f_\lambda(\xi)}{(x-\xi)^k}\int_{-\infty}^{\infty} F_{\lambda+2}(u)\,N(-iu)\,e^{iu(\xi-x)}\,du \\
&= \frac{1}{\sqrt{\pi x}}\left[M(0) - \dots + (-1)^{k-1}\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-3}{2}\right)\frac{M^{(k-1)}(0)}{x^{k-1}}\right] + O\left(\frac{1}{x^k}\right).
\end{aligned}$$

Here $N(u)$ is a function without singularities on the imaginary axis between $-\lambda-2$ and $\lambda+2$. We have proved, that is, that

$$\frac{1}{\sqrt{\pi x}}\left[M(0)-\ldots+(-1)^{k-1}\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-3}{2}\right)\frac{M^{k-1}(0)}{x^{k-1}}+\ldots\right]$$

is an asymptotic series for $H(d/dx)f(x)$.

7. Prospective and Retrospective Operators.

Let us now consider a non-singular operator $H(d/dx)$, where $H(u)$ is analytic in the entire right half-plane, and continuous in the right half-plane together with the axis of imaginaries. Furthermore, let $u^2 H(u)$ be bounded in the right half-plane. Then if $\xi > x$.

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} H(-iu)\, e^{iu(\xi-x)}\, du\right| = \left|\lim_{r\to\infty}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r H(re^{i\vartheta})\, e^{re^{i\vartheta}(x-\xi)}\, d\vartheta\right|$$

$$\leqq \frac{\max|uH(u)|}{2\pi}\lim_{r\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{r\cos\vartheta(x-\xi)}\, d\vartheta = 0.$$

That is, $\frac{1}{2\pi}\int_{-\infty}^{\infty} H(-iu)\, e^{iyu}\, du$ vanishes for positive values of y. We then have, as we may easily show,

$$H(d/dx)f(x) = \frac{1}{2\pi}\int_{-\infty}^{x} f(\xi)\, d\xi \int_{-\infty}^{\infty} H(-iu)\, e^{iu(\xi-x)}\, du.$$

Consequently, if $f(x)$ vanishes for negative values of x, we have

$$H(d/dx)f(x) = \frac{1}{2\pi}\int_{0}^{x} f(\xi)\, d\xi \int_{-\infty}^{\infty} H(-iu)\, e^{iu(\xi-x)}\, du.$$

This is however an operator of the type discussed by Carson and Volterra, and when applied to a function $f(x)$ yields a function $g(x)$ which only depends on the values of f for arguments in the neighborhood of x and smaller arguments. Such an operator we call *retrospective*. Clearly, when applied to doubly differentiable functions, $(d/dx)^2 H(d/dx)$ is likewise retrospective. By hypothesis, however, $u^2 H(u)$ is bounded on the axis of imaginaries, so that the operator $(d/dx)^2 H(d/dx)$ is continuous, if we consider as the *écart* between two functions the square root of the integral between $-\infty$ and $+\infty$ of the square of their difference. From this we may conclude at once that $(d/dx)^2 H(d/dx)$ is retrospective in its most general application. In other words, an operator $H_1(d/dx)$ is retrospective if $H_1(u)$ is continuous and bounded in the right half-plane together with the axis of imaginaries, is analytic to the right of the axis, vanishes like u^2 at the origin, and is such that $H_1(d/dx)/(d/dx)^2$ is non-singular. The last two conditions are readily shown to be superfluous if H_1 is non-singular in the generalized sense. The presence of a finite set of branch-points or poles on the axis of imaginaries does not interfere with this retrospective character of $H_1(d/dx)$, provided that in all the integrals which arise from carrying out our operator, we let the lower bound of integration be $-\infty$. The sum of two retrospective operators is retrospective, as is the inverse of a retrospective operator, if it be properly chosen. There is an analogues theory of prospective operators, involving only the present and future of a function, not its past. Here the rôles of the right and the left half-planes are interchanged.

If an operator has singularities in the right half-plane, it will not in general be retrospective, as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H(-iu)\, e^{iu(\xi-x)}\, du - \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r H(re^{i\vartheta})\, e^{re^{i\vartheta}(x-\xi)}\, d\vartheta$$

will now represent the sum of the residues of

$$H(u)\, e^{u(x-\xi)}$$

within the region bounded by the axis of imaginaries and the semi-circle of radius r in the right half-plane with the origin as center. In such a case $H(d/dx)$ may represent an operator both in the Volterra-Carson theory and in that here developed, but it will not represent the same operator. Thus the present paper demands that

$$\frac{1}{d/dx-1}[1+\operatorname{sgn} x] = (e^x-1)(1+\operatorname{sgn} x) - 2e^x,$$

while the Volterra-Carson theory requires that

$$\frac{1}{d/dx-1}[1+\operatorname{sgn} x] = (e^x-1)(1+\operatorname{sgn} x).$$

In other words, the asymptotic expansions of Heaviside and his expansions in positive powers of t can only be depended on to yield the same results when we confine ourselves to operators analytic in the right half-plane. As a matter of fact, such operators correspond to problems arising from positive resistances only.

8. The Operational Solution of Partial Differential Equations.

Before we enter on this topic in detail, it is important to consider the nature of the solution of a partial differential equation. Let us consider the linear equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

where for simplicity's sake, we shall suppose that the coefficients have as many derivatives as we shall need in the work which follows. If u satisfies this equation, it must manifestly possess the various derivatives indicated in the equation. As is familiar, however, in the case of the equation of the vibrating string, there are cases where u must be regarded as a solution of our differential equation in a general sense without possessing all the orders of derivatives indicated in the equation, and indeed without being differentiable at all. It is a matter of some interest, therefore, to render precise the manner in which a non-differentiable function may satisfy in a generalized sense a differential equation.

Let $G(x, y)$ be a function positive and infinitely differentiable within a certain bounded polygonal region R of the XY plane, vanishing with its derivatives of all orders on the periphery of R, and zero outside R. Then there is a function $G_1(x, y)$ such that

$$\iint_R (Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu)\,G(x, y)\,dx\,dy$$
$$=\iint_R u(x, y)\,G_1(x, y)\,dx\,dy$$

for all u with bounded summable derivatives of the first two orders, as we may show by integration by parts. Thus a necessary and sufficient condition for u to satisfy our differential equation almost everywhere is that

$$\iint_R u(x, y)\,G_1(x, y)\,dx\,dy=0$$

for every possible G (as the G's form a complete set over any region), and that u possess the requisite derivatives. We can thus regard a function orthogonal to every G_1 as satisfying our differential equation in a generalized sense. This sense is more general than that developed by Bôcher [15]), as not even the existence of first order derivatives is postulated. If a sequence of generalized solutions of our differential equation converges in the mean over a given area to a function $\Phi(x, y)$, it is now manifest

[15]) M. Bôcher, On harmonic functions in two dimensions, Proc. Amer. Acad. Sc. **41** (1905–1906); G. C. Evans, On the reduction of integro-differential equations, Trans. Amer. Math. Soc. **15** (1914), pp. 477–496.

that Φ is itself a generalized solution of this differential equation over this area.

Now let us discuss

$$e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f_\lambda(t)$$

where a and b are positive, where f vanishes exponentially at $-\infty$, and is algebraically infinite at $+\infty$, and where x is positive.

$$e^{-x\sqrt{(d/dt+a)(d/dt+b)}}$$

is then a non-singular operator (in the generalized sense), and is retrospective. Let $|\lambda+1|<a$, $|\lambda+1|<b$. Then

$$e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f_\lambda(t) = \sum_h \sum_j A_{hj} x^h f_\lambda^{(j)}(t),$$

where the methods of section 3 enable us to show that this double series is uniformly and absolutely convergent over any finite range. The coefficients are the same as those obtained in the expansion of

$$e^{-x\sqrt{(y+a)(y+b)}}$$

in a double series in x and y. Similarly, we have uniformly and absolutely

$$(d/dt+a)(d/dt+b)\, e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f_\lambda(t) = \sum_h \sum_j B_{hj} x^h f_\lambda^{(j)}(t),$$

where

$$\begin{aligned}\sum_h \sum_j B_{hj} x^h y^j = (y+a)(y+b)\, e^{-x\sqrt{(y+a)(y+b)}} &= \frac{\partial^2}{\partial x^2} \sum_h \sum_j A_{hj} x^h y^j \\ &= \sum_h \sum_j h(h-1) A_{hj} x^{h-2} y^j.\end{aligned}$$

Thus

$$B_{hj} = h(h-1) A_{hj},$$

from which we may conclude that

$$\begin{aligned}(d/dt+a)(d/dt+b)\, e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f_\lambda(t) \\ = \frac{\partial^2}{\partial x^2} e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f_\lambda(t).\end{aligned}$$

By using in addition to expansions about the origin, expansions about other points on the axis of imaginaries, we may prove in a similar manner that for any value of λ, this last equation holds. We know, however, that for any given value of x,

$$e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f(t) = \underset{\lambda\to\infty}{\text{l. m.}}\; e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f_\lambda(t)$$

and there is no difficulty in showing that this holds uniformly over any finite range of x. Thus

$$u = e^{-x\sqrt{(d/dt+a)(d/dt+b)}} f(t)$$

satisfies in the generalized sense the differential equation

$$u_{tt} + (a + b) u_t + abu = u_{xx}.$$

If then it possesses the necessary derivatives, it satisfies this equation in the ordinary sense. When $x = 0$, it reduces to $f(t)$, and it depends retrospectively on $f(t)$. It may be shown, moreover, that if $f(t)$ is everywhere continuous, so is $u(x, t)$. We thus have completely solved the telegrapher's equation for a semi-infinite line with a given impressed voltage at the end.

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