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March 5, 2020

In [5]:
```
from sympy import *
from IPython.display import *
init_printing()
var('a:z')
var('A:Z');
```

In [6]:
```
alpha=Symbol('alpha',positive=True)
V=-log(1-2*alpha*z+z**2)
Z=solve(V-v,z)[0]
Z
```

Out[6]:

$$\alpha - \sqrt{\alpha^2 - 1 + e^{-v}}$$

In [11]:
```
N=8
p=[]
f=series(exp(x*Z),v,0,N)
for i in range(N):
    p.append(factorial(i)*f.coeff(v,i).collect(x))
p
```

Out[11]:

$$\left[1, \quad \frac{x}{2\alpha}, \quad 2x\left(-\frac{1}{4\alpha}+\frac{1}{8\alpha^3}\right)+\frac{x^2}{4\alpha^2}, \quad 6x^2\left(-\frac{1}{8\alpha^2}+\frac{1}{16\alpha^4}\right)+6x\left(\frac{1}{12\alpha}-\frac{1}{8\alpha^3}+\frac{1}{16\alpha^5}\right)+\frac{x^3}{8\alpha^3}, \quad 24x^3\left(-\frac{1}{32\alpha^3}+\right.\right.$$

In [12]:
```
N=8
q=[]
f=series(exp(y*V),z,0,N)
for i in range(N):
    q.append(factorial(i)*f.coeff(z,i).collect(y))
q[:5]
```

Out[12]:

$$\left[1, \quad 2\alpha y, \quad 4\alpha^2y^2+2y\left(2\alpha^2-1\right), \quad 8\alpha^3y^3+6y^2\left(4\alpha^3-2\alpha\right)+6y\left(\frac{8\alpha^3}{3}-2\alpha\right), \quad 16\alpha^4y^4+24y^3\left(4\alpha^4-2\alpha^2\right)+24y\right.$$

```
In [13]: PCF=Matrix(N,N,lambda n,k: p[n].coeff(x,k))
         QCF=Matrix(N,N,lambda n,k: q[n].coeff(y,k))
         PCF[:5,:5],QCF[:5,:5],simplify(PCF*QCF)
```

Out[13]:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2\alpha} & 0 & 0 & 0 \\ 0 & -\frac{1}{2\alpha}+\frac{1}{4\alpha^3} & \frac{1}{4\alpha^2} & 0 & 0 \\ 0 & \frac{1}{2\alpha}-\frac{3}{4\alpha^3}+\frac{3}{8\alpha^5} & -\frac{3}{4\alpha^2}+\frac{3}{8\alpha^4} & \frac{1}{8\alpha^3} & 0 \\ 0 & -\frac{1}{2\alpha}+\frac{7}{4\alpha^3}-\frac{9}{4\alpha^5}+\frac{15}{16\alpha^7} & \frac{7}{4\alpha^2}-\frac{9}{4\alpha^4}+\frac{15}{16\alpha^6} & -\frac{3}{4\alpha^3}+\frac{3}{8\alpha^5} & \frac{1}{16\alpha^4} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2\alpha & 0 \\ 0 & 4\alpha^2-2 & 4\alpha^2 \\ 0 & 16\alpha^3-12\alpha & 24\alpha^3-12\alpha \\ 0 & 96\alpha^4-96\alpha^2+12 & 176\alpha^4-144\alpha^2+12 \end{bmatrix} \right.$$

```
In [14]: qa=[]
         for n in range(N):
             qa.append(bool(sum(p[n-k]*p[k].subs(x,y)*binomial(n,k) for k in range(n+1)).expand
         qa
```

Out[14]: [True, True, True, True, True, True, True, True]

```
In [15]: m=PCF.shape[0]
         W=[]
         WW=[]
         for n in range(m):
             W.append(zeros(m,m))
             WW.append(zeros(m,m))
             for k in range(floor(m/2)):
                 for l in range(floor(m/2)):
                     W[n][k,l]=simplify(sum(binomial(n,j)*PCF[n-j,k]*PCF[j,l] for j in range(n
                     WW[n][k,l]=PCF[n,k+l]*binomial(k+l,l)
         [(simplify(W[a]-WW[a])).is_zero for a in range(m)]
```

Out[15]: [True, True, True, True, True, True, True, True]

```
In [16]: for i in range(5):
             display([p[i],q[i]])
```

$$[1, \quad 1]$$

$$\left[\frac{x}{2\alpha}, \quad 2\alpha y\right]$$

$$\left[2x\left(-\frac{1}{4\alpha}+\frac{1}{8\alpha^3}\right)+\frac{x^2}{4\alpha^2}, \quad 4\alpha^2y^2+2y\left(2\alpha^2-1\right)\right]$$

$$\left[6x^2\left(-\frac{1}{8\alpha^2}+\frac{1}{16\alpha^4}\right)+6x\left(\frac{1}{12\alpha}-\frac{1}{8\alpha^3}+\frac{1}{16\alpha^5}\right)+\frac{x^3}{8\alpha^3}, \quad 8\alpha^3y^3+6y^2\left(4\alpha^3-2\alpha\right)+6y\left(\frac{8\alpha^3}{3}-2\alpha\right)\right]$$

$$\left[24x^3\left(-\frac{1}{32\alpha^3}+\frac{1}{64\alpha^5}\right)+24x^2\left(\frac{7}{96\alpha^2}-\frac{3}{32\alpha^4}+\frac{5}{128\alpha^6}\right)+24x\left(-\frac{1}{48\alpha}+\frac{7}{96\alpha^3}-\frac{3}{32\alpha^5}+\frac{5}{128\alpha^7}\right)+\frac{x^4}{16\alpha^4}, \quad 16\alpha^4\right.$$

In [17]:
```
yy=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+p[m].coeff(x,i)*q[i]
    yy.append(simplify(g))

display(yy)

xx=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+q[m].coeff(y,i)*p[i]
    xx.append(simplify(g))

display(xx)
```

$$\left[1, \quad y, \quad y^2, \quad y^3, \quad y^4, \quad y^5, \quad y^6, \quad y^7\right]$$

$$\left[1, \quad x, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \quad x^6, \quad x^7\right]$$

In [18]:
```
from sympy.functions.combinatorial.numbers import stirling
%store -r B
display([simplify(sum(stirling(n,k)*(-1)**(n-k)*(2*alpha**2)**(-k)*B[k].subs(x,alpha*
[simplify(gegenbauer(n,y,alpha)*factorial(n)-q[n]) for n in range(0,N)]
```

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

Out[18]:

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

In [19]:
```
%store -r B
display(B)
```

$$\left[1, \quad x, \quad x^2+x, \quad x^3+3x^2+3x, \quad x^4+6x^3+15x^2+15x, \quad x^5+10x^4+45x^3+105x^2+105x, \quad x^6+15x^5+105\right.$$

P-polynomials:

$$p_n(x)=\sum_{k=0}^{n} S(n,k)(-1)^{n-k}(2\alpha^2)^{-k}p_k^{\text{Bessel}}(\alpha x)$$

Q-polynomials are Gegenbauer polynomials:

$$q_n(y) = n! C_n^{(y)}(\alpha) = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^{2k}\, k!} (-1)^k (y)_{n-k} (\alpha)^{n-2k}$$

```
In [35]: for n in range(0,N,2):
             display(simplify(q[n]-(-2)**n*sum(binomial(n,2*k)*ff(-y,n-k)*factorial(2*k)/facto
```

0

0

0

0

```
In [36]: for n in range(1,N,2):
             display(simplify(q[n]-(-2)**n*sum(binomial(n,2*k)*ff(-y,n-k)*factorial(2*k)/facto
```

0

0

0

0

```
In [65]: for n in range(N-1):
             display(simplify((expand(2*alpha*(y+n)*q[n]-n*(2*y+n-1)*q[n-1])).collect(y)-q[n+1]
```

0

0

0

0

0

0

0

```
In [ ]:
```