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In [1]:
```
from sympy import *
from IPython.display import *
init_printing()
var('a:z')
var('A:Z');
```

In [2]:
```
V=tanh(z)
Z=solve(V-v,z)[0]
Z=atanh(v)
Z
```

Out[2]:

$$\operatorname{atanh}(v)$$

In [44]:
```
N=9
p=[]
f=series(exp(x*Z),v,0,N)
for i in range(N):
    p.append(factorial(i)*f.coeff(v,i))
p
```

Out[44]:

$$\left[1, \quad x, \quad x^{2}, \quad x^{3}+2x, \quad x^{4}+8x^{2}, \quad x^{5}+20x^{3}+24x, \quad x^{6}+40x^{4}+184x^{2}, \quad x^{7}+70x^{5}+784x^{3}+720x, \quad x^{8}+11\right.$$

In [4]:
```
N=9
q=[]
f=series(exp(y*V),z,0,N)
for i in range(N):
    q.append(factorial(i)*f.coeff(z,i))
q
```

Out[4]:

$$\left[1, \quad y, \quad y^{2}, \quad y^{3}-2y, \quad y^{4}-8y^{2}, \quad y^{5}-20y^{3}+16y, \quad y^{6}-40y^{4}+136y^{2}, \quad y^{7}-70y^{5}+616y^{3}-272y, \quad y^{8}-112\right.$$

In [5]:
```
PCF=Matrix(N,N,lambda n,k: p[n].coeff(x,k))
QCF=Matrix(N,N,lambda n,k: q[n].coeff(y,k))
PCF,QCF,simplify(PCF*QCF)
```

Out[5]:

$$\left(\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&2&0&1&0&0&0&0&0\\0&0&8&0&1&0&0&0&0\\0&24&0&20&0&1&0&0&0\\0&0&184&0&40&0&1&0&0\\0&720&0&784&0&70&0&1&0\\0&0&8448&0&2464&0&112&0&1\end{bmatrix},\ \begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&-2&0&1&0&0&0&0&0\\0&0&-8&0&1&0&0&0&0\\0&16&0&-20&0&1&0&0&0\\0&0&136&0&-40&0&1&0&0\\0&-272&0&616&0&-70&0&1&0\\0&0&-3968&0&2016&0&-112&0&1\end{bmatrix},\ \begin{bmatrix}1&0\\0&1\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix}\right.$$

In [6]:
```
qa=[]
for n in range(N):
    qa.append(bool(sum(p[n-k]*p[k].subs(x,y)*binomial(n,k) for k in range(n+1)).expand
qa
```

Out[6]: [True, True, True, True, True, True, True, True, True]

In [7]:
```
W=[]
WW=[]
for n in range(N):
    W.append(zeros(N,N))
    WW.append(zeros(N,N))
    for k in range(floor(N/2)):
        for l in range(floor(N/2)):
            W[n][k,l]=sum(binomial(n,j)*PCF[n-j,k]*PCF[j,l] for j in range(n+1))
            WW[n][k,l]=PCF[n,k+l]*binomial(k+l,l)
[(W[a]-WW[a]).is_zero for a in range(N)]
```

Out[7]: [True, True, True, True, True, True, True, True, True]

In [12]:
```
for i in range(N):
     display([p[i],q[i]])
```

$$[1,\quad 1]$$

$$[x,\quad y]$$

$$[x^2,\quad y^2]$$

$$[x^3+2x,\quad y^3-2y]$$

$$\left[x^4 + 8x^2, \quad y^4 - 8y^2\right]$$

$$\left[x^5 + 20x^3 + 24x, \quad y^5 - 20y^3 + 16y\right]$$

$$\left[x^6 + 40x^4 + 184x^2, \quad y^6 - 40y^4 + 136y^2\right]$$

$$\left[x^7 + 70x^5 + 784x^3 + 720x, \quad y^7 - 70y^5 + 616y^3 - 272y\right]$$

$$\left[x^8 + 112x^6 + 2464x^4 + 8448x^2, \quad y^8 - 112y^6 + 2016y^4 - 3968y^2\right]$$

In [9]:
```
yy=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+p[m].coeff(x,i)*q[i]
    yy.append(simplify(g))

display(yy)

xx=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+q[m].coeff(y,i)*p[i]
    xx.append(simplify(g))

display(xx)
```

$$\left[1, \quad y, \quad y^2, \quad y^3, \quad y^4, \quad y^5, \quad y^6, \quad y^7, \quad y^8\right]$$

$$\left[1, \quad x, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \quad x^6, \quad x^7, \quad x^8\right]$$

In [10]:
```
display([expand(sum(binomial(n,k)*(-1)**k*ff(x/2,n-k)*ff(-x/2,k) for k in range(n+1))
from sympy.functions.combinatorial.numbers import stirling
%store -r L
display([simplify(sum(stirling(n,m)*2**(n-m)*(L[m].subs(x,y)) for m in range(1,n+1))-
```

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

"Time-zero" Krawtchouk polynomials:

$$p_n(x) = K_n(x/2; 1/2, 0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{x}{2}\right)^{(n-k)} \left(-\frac{x}{2}\right)^{(k)}$$

referring to DLMF, 18.23.3 .

For the tanh-polynomials, we have

$$q_n(y) = \sum_{m=1}^{n} S(n,m) 2^{n-m} p_m^{\text{Laguerre}}(y)$$

In [52]:
```
for n in range(0,N-1,2):
        display(expand(y*q[n]-sum(binomial(n,2*j)*2**(2*j-1)*q[n+1-2*j] for j in range(1,
```

$$0$$

$$0$$

$$0$$

$$0$$

In [53]:
```
for n in range(1,N,2):
          display(expand(y*q[n]-sum(binomial(n,2*j)*2**(2*j-1)*q[n+1-2*j] for j in range
```

$$0$$

$$0$$

$$0$$

$$0$$

In [ ]: