RiordanHermite-Bessel

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In [10]:
```
from sympy import *
from IPython.display import *
init_printing()
var('a:z')
var('A:Z');
```

In [11]:
```
V=z-z**2/2
Z=solve(V-v,z)[0]
Z
```

Out[11]:

$$-\sqrt{-2v+1}+1$$

In [12]:
```
N=8
p=[]
f=series(exp(x*Z),v,0,N)
for i in range(N):
    p.append(factorial(i)*f.coeff(v,i))
p
```

Out[12]:

$$\left[1, \quad x, \quad x^2+x, \quad x^3+3x^2+3x, \quad x^4+6x^3+15x^2+15x, \quad x^5+10x^4+45x^3+105x^2+105x, \quad x^6+15x^5+105$$

In [13]:
```
N=8
q=[]
f=series(exp(y*V),z,0,N)
for i in range(N):
    q.append(factorial(i)*f.coeff(z,i))
q
```

Out[13]:

$$\left[1, \quad y, \quad y^2-y, \quad y^3-3y^2, \quad y^4-6y^3+3y^2, \quad y^5-10y^4+15y^3, \quad y^6-15y^5+45y^4-15y^3, \quad y^7-21y^6+105y^5$$

```
In [14]: PCF=Matrix(N,N,lambda n,k: p[n].coeff(x,k))
         QCF=Matrix(N,N,lambda n,k: q[n].coeff(y,k))
         PCF,QCF,simplify(PCF*QCF)
```

Out[14]:

$$\left(\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&3&3&1&0&0&0&0\\0&15&15&6&1&0&0&0\\0&105&105&45&10&1&0&0\\0&945&945&420&105&15&1&0\\0&10395&10395&4725&1260&210&21&1\end{bmatrix}, \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&-1&1&0&0&0&0&0\\0&0&-3&1&0&0&0&0\\0&0&3&-6&1&0&0&0\\0&0&0&15&-10&1&0&0\\0&0&0&-15&45&-15&1&0\\0&0&0&0&-105&105&-21&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right.$$

```
In [15]: qa=[]
         for n in range(N):
             qa.append(bool(sum(p[n-k]*p[k].subs(x,y)*binomial(n,k) for k in range(n+1)).expand
         qa
```

```
Out[15]: [True, True, True, True, True, True, True, True]
```

```
In [16]: W=[]
         WW=[]
         for n in range(N):
             W.append(zeros(N,N))
             WW.append(zeros(N,N))
             for k in range(floor(N/2)):
                 for l in range(floor(N/2)):
                     W[n][k,l]=sum(binomial(n,j)*PCF[n-j,k]*PCF[j,l] for j in range(n+1))
                     WW[n][k,l]=PCF[n,k+l]*binomial(k+l,l)
         [(W[a]-WW[a]).is_zero for a in range(N)]
```

```
Out[16]: [True, True, True, True, True, True, True, True]
```

```
In [17]: for i in range(N):
             display([p[i],q[i]])
```

$$[1, \quad 1]$$

$$[x, \quad y]$$

$$\left[x^2 + x, \quad y^2 - y\right]$$

$$\left[x^3 + 3x^2 + 3x, \quad y^3 - 3y^2\right]$$

$$\left[x^{4}+6x^{3}+15x^{2}+15x,\quad y^{4}-6y^{3}+3y^{2}\right]$$

$$\left[x^{5}+10x^{4}+45x^{3}+105x^{2}+105x,\quad y^{5}-10y^{4}+15y^{3}\right]$$

$$\left[x^{6}+15x^{5}+105x^{4}+420x^{3}+945x^{2}+945x,\quad y^{6}-15y^{5}+45y^{4}-15y^{3}\right]$$

$$\left[x^{7}+21x^{6}+210x^{5}+1260x^{4}+4725x^{3}+10395x^{2}+10395x,\quad y^{7}-21y^{6}+105y^{5}-105y^{4}\right]$$

```
In [18]: yy=[]
         for m in range(N):
             g=0
             for i in range(m+1):
                g=g+p[m].coeff(x,i)*q[i]
             yy.append(simplify(g))

         display(yy)

         xx=[]
         for m in range(N):
             g=0
             for i in range(m+1):
                g=g+q[m].coeff(y,i)*p[i]
             xx.append(simplify(g))

         display(xx)
```

$$\left[1,\quad y,\quad y^{2},\quad y^{3},\quad y^{4},\quad y^{5},\quad y^{6},\quad y^{7}\right]$$

$$\left[1,\quad x,\quad x^{2},\quad x^{3},\quad x^{4},\quad x^{5},\quad x^{6},\quad x^{7}\right]$$

```
In [39]: display([(sum(gamma(n+k)/gamma(n-k)/2**k/factorial(k)*x**(n-k) for k in range(n+1))-p
         [simplify(sum(binomial(n,2*j)*factorial(2*j)/2**j/factorial(j)*(-1)**j*y**(n-j) for j
```

$$[0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0]$$

Out[39]:

$$[0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0]$$

Bessel polynomials

$$x\theta_{n-1}(x)=\sum_{k=0}^{n-1}\frac{\Gamma(n+k)}{\Gamma(n-k)\,k!}\,\frac{x^{n-k}}{2^{k}}$$

Hermite polynomials

$$He_n(x, xt) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k\, k!} (-1)^k x^{n-k} t^k$$

In [56]:
```
B=p
%store B
```

```
Stored 'B' (list)
```

In [ ]: