# POWER SUM SYMMETRIC FUNCTIONS

## 1. Powersum family

A powersum symmetric function with a single integer index, $n$, is the sum of all $n^{\text{th}}$ powers of the variables $\{x_1 \ldots, x_d\}$:

$$\mathrm{p}_n = x_1^n + x_2^n + \cdots + x_d^n$$

In terms of monomial symmetric functions,

$$\mathrm{p}_n = \mathrm{m}_{(n)}$$

The powersum symmetric function indexed by $\lambda$ is the product of the corresponding single-indexed functions:

$$\mathrm{p}_\lambda = \mathrm{p}_{\lambda_1}\mathrm{p}_{\lambda_2}\cdots\mathrm{p}_{\lambda_L} = \mathrm{p}_1^{\rho_1}\mathrm{p}_2^{\rho_2}\cdots\mathrm{p}_n^{\rho_n} = \mathrm{p}^\rho$$

in multi-index notation. Taken together $\{\mathrm{p}_\lambda\}_\lambda$ form a basis for the symmetric functions under consideration.

The homogeneous and elementary symmetric functions have expansions in terms of the $\mathrm{p}_\lambda$'s.

**Definition** Given $\rho = (1^{\rho_1}2^{\rho_2}\cdots n^{\rho_n})$, define

$$z_\rho = 1^{\rho_1}2^{\rho_2}\cdots n^{\rho_n}\,\rho_1!\,\rho_2!\cdots\rho_n!$$

with $z_\lambda$ defined accordingly for $\rho = \rho(\lambda)$.

**Proposition 1.1.** *We have the expansions*

$$\mathrm{h}_n = \sum_{\rho\vdash n} \frac{1}{z_\rho}\,\mathrm{p}^\rho$$

$$\mathrm{e}_n = \sum_{\rho\vdash n} \frac{(-1)^{n-\sum_j \rho_j}}{z_\rho}\,\mathrm{p}^\rho$$

**Example.** With the partitions of 2, $\{[2], [11]\}$ and of 3, $\{[3], [21], [111]\}$, we have values of $z_\rho$ for 2: (2,1) and for 3: (3,2,6) with

$$\begin{aligned} \mathrm{h}_2 &= \mathrm{p}_2/2 + \mathrm{p}_1^2/2 \\ \mathrm{h}_3 &= \mathrm{p}_3/3 + \mathrm{p}_2\mathrm{p}_1/2 + \mathrm{p}_1^3/6 \\ \mathrm{e}_2 &= -\mathrm{p}_2/2 + \mathrm{p}_1^2/2 \\ \mathrm{e}_3 &= \mathrm{p}_3/3 - \mathrm{p}_2\mathrm{p}_1/2 + \mathrm{p}_1^3/6 \end{aligned}$$

etc.

1.1. **Diagrams.** The powersum symmetric functions correspond to a sum of single-rowed SSYT's with the same entry in each box. E.g., $\mathrm{p}_4$ is the sum of SSYT's with shape

$$\boxed{\phantom{1}}\boxed{\phantom{1}}\boxed{\phantom{1}}\boxed{\phantom{1}}$$

For $d = 3$, we have

$$\mathrm{p}_4 = \boxed{1}\boxed{1}\boxed{1}\boxed{1} + \boxed{2}\boxed{2}\boxed{2}\boxed{2} + \boxed{3}\boxed{3}\boxed{3}\boxed{3}$$

the diagrams indicating the corresponding monomials. The number of terms is exactly $d$.