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In [1]:
```
from sympy import *
from IPython.display import *
init_printing()
var('a:z')
var('A:Z');
```

In [2]:
```
V=asinh(z)
Z=solve(V-v,z)[0]
Z
```

Out[2]:

$$\sinh\left(v\right)$$

In [3]:
```
N=9
p=[]
f=series(exp(x*Z),v,0,N)
for i in range(N):
    p.append(factorial(i)*f.coeff(v,i))
p
```

Out[3]:

$$\left[1, \quad x, \quad x^2, \quad x^3 + x, \quad x^4 + 4x^2, \quad x^5 + 10x^3 + x, \quad x^6 + 20x^4 + 16x^2, \quad x^7 + 35x^5 + 91x^3 + x, \quad x^8 + 56x^6 + 336\right.$$

In [4]:
```
#N=8
q=[]
f=series(exp(y*V),z,0,N)
for i in range(N):
    q.append(factorial(i)*f.coeff(z,i))
q
```

Out[4]:

$$\left[1, \quad y, \quad y^2, \quad y^3 - y, \quad y^4 - 4y^2, \quad y^5 - 10y^3 + 9y, \quad y^6 - 20y^4 + 64y^2, \quad y^7 - 35y^5 + 259y^3 - 225y, \quad y^8 - 56y^6 +\right.$$

In [5]:
```
PCF=Matrix(N,N,lambda n,k: p[n].coeff(x,k))
QCF=Matrix(N,N,lambda n,k: q[n].coeff(y,k))
PCF,QCF,simplify(PCF*QCF)
```

Out[5]:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 20 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 91 & 0 & 35 & 0 & 1 & 0 \\ 0 & 0 & 64 & 0 & 336 & 0 & 56 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & -10 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 64 & 0 & -20 & 0 & 1 & 0 & 0 \\ 0 & -225 & 0 & 259 & 0 & -35 & 0 & 1 & 0 \\ 0 & 0 & -2304 & 0 & 784 & 0 & -56 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In [6]:
```
qa=[]
for n in range(N):
    qa.append(bool(sum(p[n-k]*p[k].subs(x,y)*binomial(n,k) for k in range(n+1)).expand
qa
```

Out[6]: [True, True, True, True, True, True, True, True, True]

In [7]:
```
W=[]
WW=[]
for n in range(N):
    W.append(zeros(N,N))
    WW.append(zeros(N,N))
    for k in range(floor(N/2)):
        for l in range(floor(N/2)):
            W[n][k,l]=sum(binomial(n,j)*PCF[n-j,k]*PCF[j,l] for j in range(n+1))
            WW[n][k,l]=PCF[n,k+l]*binomial(k+l,l)
[(W[a]-WW[a]).is_zero for a in range(N)]
```

Out[7]: [True, True, True, True, True, True, True, True, True]

In [12]:
```
for i in range(N):
    display([p[i].factor(),q[i].factor()])
```

$$[1, \quad 1]$$

$$[x, \quad y]$$

$$[x^2, \quad y^2]$$

$$\left[x\left(x^2+1\right), \quad y(y-1)(y+1)\right]$$

$$\left[x^2\left(x^2+4\right),\quad y^2\left(y-2\right)\left(y+2\right)\right]$$

$$\left[x\left(x^4+10x^2+1\right),\quad y\left(y-3\right)\left(y-1\right)\left(y+1\right)\left(y+3\right)\right]$$

$$\left[x^2\left(x^4+20x^2+16\right),\quad y^2\left(y-4\right)\left(y-2\right)\left(y+2\right)\left(y+4\right)\right]$$

$$\left[x\left(x^6+35x^4+91x^2+1\right),\quad y\left(y-5\right)\left(y-3\right)\left(y-1\right)\left(y+1\right)\left(y+3\right)\left(y+5\right)\right]$$

$$\left[x^2\left(x^6+56x^4+336x^2+64\right),\quad y^2\left(y-6\right)\left(y-4\right)\left(y-2\right)\left(y+2\right)\left(y+4\right)\left(y+6\right)\right]$$

In [13]:
```
yy=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+p[m].coeff(x,i)*q[i]
    yy.append(g)

display(yy)

xx=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+q[m].coeff(y,i)*p[i]
    xx.append(g)

display(xx)
```

$$\left[1,\quad y,\quad y^2,\quad y^3,\quad y^4,\quad y^5,\quad y^6,\quad y^7,\quad y^8\right]$$

$$\left[1,\quad x,\quad x^2,\quad x^3,\quad x^4,\quad x^5,\quad x^6,\quad x^7,\quad x^8\right]$$

In [10]:
```
%store -r T
[simplify(sum(binomial(n,k)*(-1)**k*T[n-k].subs(y,x/2)*T[k].subs(y,-x/2) for k in rang
```

Out[10]:

$$\left[0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0\right]$$

In [11]:
```
display([simplify(y*prod(y**2-(2*k+1)**2 for k in range(1+Rational((n-3)/2)))-q[n]) fo
[simplify(prod(y**2-(2*k)**2 for k in range(1+Rational((n-2)/2)))-q[n]) for n in range
```