RiordanPoisson-Stirling

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In [2]:
```
from sympy import *
from IPython.display import *
init_printing()
var('a:z')
var('A:Z');
```

In [3]:
```
V=exp(z)-1
Z=solve(V-v,z)[0]
Z
```

Out[3]:

$$\log(v+1)$$

In [4]:
```
N=9
p=[]
f=series(exp(x*Z),v,0,N)
for i in range(N):
    p.append(factorial(i)*f.coeff(v,i))
p
```

Out[4]:

$$\left[1, \quad x, \quad x^2-x, \quad x^3-3x^2+2x, \quad x^4-6x^3+11x^2-6x, \quad x^5-10x^4+35x^3-50x^2+24x, \quad x^6-15x^5+85x^4-\right.$$

In [5]:
```
N=9
q=[]
f=series(exp(y*V),z,0,N)
for i in range(N):
    q.append(factorial(i)*f.coeff(z,i))
q
```

Out[5]:

$$\left[1, \quad y, \quad y^2+y, \quad y^3+3y^2+y, \quad y^4+6y^3+7y^2+y, \quad y^5+10y^4+25y^3+15y^2+y, \quad y^6+15y^5+65y^4+90y^3+\right.$$

```
In [6]: PCF=Matrix(N,N,lambda n,k: p[n].coeff(x,k))
        QCF=Matrix(N,N,lambda n,k: q[n].coeff(y,k))
        PCF,QCF,simplify(PCF*QCF)
```

Out[6]:

$$\left(\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0\\0 & -6 & 11 & -6 & 1 & 0 & 0 & 0 & 0\\0 & 24 & -50 & 35 & -10 & 1 & 0 & 0 & 0\\0 & -120 & 274 & -225 & 85 & -15 & 1 & 0 & 0\\0 & 720 & -1764 & 1624 & -735 & 175 & -21 & 1 & 0\\0 & -5040 & 13068 & -13132 & 6769 & -1960 & 322 & -28 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 7 & 6 & 1 & 0 & 0 & 0 & 0\\0 & 1 & 15 & 25 & 10 & 1 & 0 & 0 & 0\\0 & 1 & 31 & 90 & 65 & 15 & 1 & 0 & 0\\0 & 1 & 63 & 301 & 350 & 140 & 21 & 1 & 0\\0 & 1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1\end{bmatrix}\right.$$

```
In [7]: qa=[]
        for n in range(N):
            qa.append(bool(sum(p[n-k]*p[k].subs(x,y)*binomial(n,k) for k in range(n+1)).expand
        qa
```

```
Out[7]: [True, True, True, True, True, True, True, True, True]
```

```
In [8]: W=[]
        WW=[]
        for n in range(N):
            W.append(zeros(N,N))
            WW.append(zeros(N,N))
            for k in range(floor(N/2)):
                for l in range(floor(N/2)):
                    W[n][k,l]=sum(binomial(n,j)*PCF[n-j,k]*PCF[j,l] for j in range(n+1))
                    WW[n][k,l]=PCF[n,k+l]*binomial(k+l,l)
        [(W[a]-WW[a]).is_zero for a in range(N)]
```

```
Out[8]: [True, True, True, True, True, True, True, True, True]
```

```
In [9]: for i in range(N):
            display([p[i],q[i]])
```

$$[1, \quad 1]$$

$$[x, \quad y]$$

$$\left[x^2 - x, \quad y^2 + y\right]$$

$$\left[x^3 - 3x^2 + 2x, \quad y^3 + 3y^2 + y\right]$$

$$\left[x^4 - 6x^3 + 11x^2 - 6x, \quad y^4 + 6y^3 + 7y^2 + y \right]$$

$$\left[x^5 - 10x^4 + 35x^3 - 50x^2 + 24x, \quad y^5 + 10y^4 + 25y^3 + 15y^2 + y \right]$$

$$\left[x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x, \quad y^6 + 15y^5 + 65y^4 + 90y^3 + 31y^2 + y \right]$$

$$\left[x^7 - 21x^6 + 175x^5 - 735x^4 + 1624x^3 - 1764x^2 + 720x, \quad y^7 + 21y^6 + 140y^5 + 350y^4 + 301y^3 + 63y^2 + y \right]$$

$$\left[x^8 - 28x^7 + 322x^6 - 1960x^5 + 6769x^4 - 13132x^3 + 13068x^2 - 5040x, \quad y^8 + 28y^7 + 266y^6 + 1050y^5 + 1701y^4 + 96 \right.$$

In [10]:
```
yy=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+p[m].coeff(x,i)*q[i]
    yy.append(simplify(g))

display(yy)

xx=[]
for m in range(N):
    g=0
    for i in range(m+1):
       g=g+q[m].coeff(y,i)*p[i]
    xx.append(simplify(g))

display(xx)
```

$$\left[1, \quad y, \quad y^2, \quad y^3, \quad y^4, \quad y^5, \quad y^6, \quad y^7, \quad y^8 \right]$$

$$\left[1, \quad x, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \quad x^6, \quad x^7, \quad x^8 \right]$$

In [11]:
```
from sympy.functions.combinatorial.numbers import stirling
display([simplify(prod(x-j for j in range(n))-p[n]) for n in range(1,N)])
[(sum(stirling(n,k)*y**k for k in range(0,n+1))-q[n]) for n in range(1,N)]
```

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

Out[11]:

$$[0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

The P-polynomials are falling factorials:

$$p_n(x) = x^{(n)} = x(x-1)\cdots(x-(n-1)) = \sum_{k=0}^{n} s(n,k)x^k$$

where $s(n,k)$ are Stirling numbers of the first kind.

And

$$q_n(y) = \sum_{k=0}^{n} S(n,k)y^k = \mathcal{T}_n(y)$$

the *Touchard* polynomials, with $S(n,k)$ the Stirling numbers of the second kind.

```
In [12]: T=q
         T
```

Out[12]:

$$\left[1, \quad y, \quad y^2 + y, \quad y^3 + 3y^2 + y, \quad y^4 + 6y^3 + 7y^2 + y, \quad y^5 + 10y^4 + 25y^3 + 15y^2 + y, \quad y^6 + 15y^5 + 65y^4 + 90y^3 +\right.$$

```
In [13]: %store T
```

```
Stored 'T' (list)
```