Math 223: Homework 7

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Problem 1

Find the first two terms of the asymptotic expansion of

$$\int_5^9 \frac{e^{-x t}}{t} \, dt, \; x \to +\infty.$$

Solution:

Let us apply integration by parts with $u = f(t) = \frac{1}{t}$ and $dv = e^{-x t} dt$. This yields,

$$\begin{aligned} I(x) &= -\frac{1}{x t} e^{-x t} \Big|_5^9 + \frac{1}{x} \int_5^9 \left(\frac{-1}{t^2}\right) e^{-x t} \, dt \\ &= \frac{1}{5x} e^{-5x} - \frac{1}{9x} e^{-9x} - \frac{1}{x} \int_5^9 \frac{e^{-x t}}{t^2} \, dt \\ &= \sum_{n=0}^{m-1} \frac{e^{-5x}}{(5x)^{n+1}} (-1)^n - \sum_{n=0}^{m-1} \frac{e^{-9x}}{(9x)^{n+1}} (-1)^n + \frac{1}{x^m} \int_5^9 (-1)^m t^{-m-1} e^{-x t} \, dt \end{aligned}$$

Since $e^{-9x} << e^{-5x}$ as $x \to +\infty$, we can neglect the second sum and rewrite it as

$$I(x) \sim \sum_{n=0}^{m-1} \frac{e^{-5x}}{(5x)^{n+1}} (-1)^n + \frac{1}{x^m} \int_5^9 (-1)^m t^{-m-1} e^{-x t} \, dt.$$

$f^{(m)}(t)$ is continuous on [5, 9] for all positive integer m. To get the first two terms, we can write it as

$$I(x) \sim \sum_{n=0}^{1} \frac{e^{-5x}}{(5x)^{n+1}} (-1)^n + \frac{1}{x^2} \int_5^9 t^{-3} e^{-x t} \, dt = \frac{e^{-5x}}{5x} - \frac{e^{-5x}}{25x^2} + O\left(x^{-3}\right) \text{ as } x \to \infty$$

Therefore, the first two terms of the asymptotic expansion is $\frac{e^{-5x}}{5x} - \frac{e^{-5x}}{25x^2}$**,** $x \to +\infty$**.**

In[∘]:=
```
AsymptoticIntegrate[e^(-x t)/t, {t, 5, 9}, {x, ∞, 2}]
```

Out[∘]=

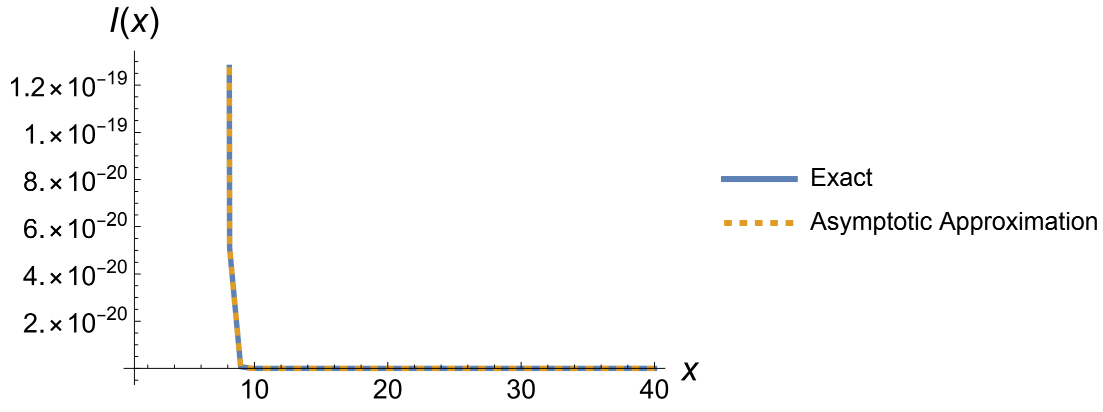
$$\frac{2 e^{-5x}}{125 x^3} - \frac{e^{-5x}}{25 x^2} + \frac{e^{-5x}}{5x}$$

In[∘]:=
```
ExactIntegral1[x_] = Integrate[e^(-x t)/t, {t, 5, 9}, Assumptions → x > 0]
```

Out[∘]=
```
ExpIntegralEi[-9 x] - ExpIntegralEi[-5 x]
```

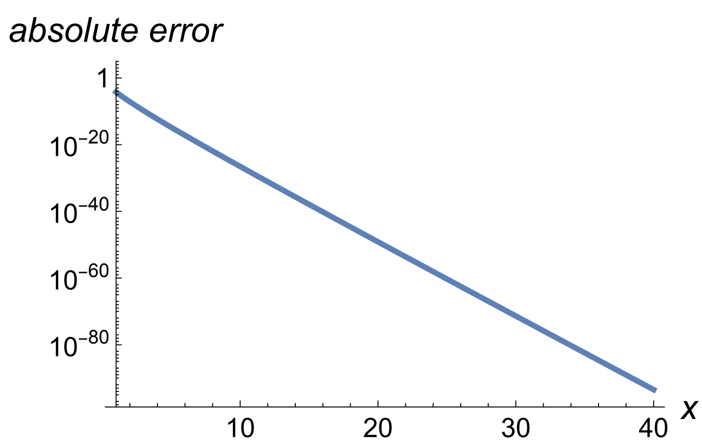
In[ ]:=
```
Plot[{ExactIntegral1[x], E^(-5 x)/(5 x) - E^(-5 x)/(25 x^2)}, {x, 1, 40},
 PlotStyle →
  {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Exact", "Asymptotic Approximation"}]
```

Out[ ]=



In[ ]:=
```
LogPlot[Abs[ExactIntegral1[x] - (E^(-5 x)/(5 x) - E^(-5 x)/(25 x^2))], {x, 1, 40},
 PlotStyle → Directive[Solid, Thickness[0.01]],
 AxesLabel → {Style["x", Italic, 18], Style["absolute error", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

Out[ ]=



Problem 2

Use Watson’s lemma to determine the asymptotic expansion of

$$I(x) = \int_0^{\pi} e^{-x\,t}\, t^{-1/3}\, dt,\ x \to +\infty.$$

Solution:

$f(t) = t^{-1/3}$ is integrable on $[0, \pi]$. Consider a parameter $0 < R < \pi$, and write $I(x) = \int_0^R t^{-1/3}\, e^{-x\,t}\, dt + \int_R^{\pi} t^{-1/3}\, e^{-x\,t}\, dt$.

Let's consider the second integral. Since the upper bound is π, we can bound $f(t)$ by a constant, $\left| t^{-1/3} \right| \le R^{-1/3}$, $t \in [R, \pi]$, and then find

$\left|\int_R^{\pi} t^{-1/3}\, e^{-x\,t}\, dt\right| \le R^{-1/3} \int_R^{\pi} e^{-x\,t}\, dt = \frac{R^{-1/3}}{x}\left(e^{-R\,t} - e^{-\pi\,t}\right) \sim \frac{e^{-R\,t}}{x},\ x \to +\infty.$

To compute the first integral, $\int_0^R t^{-1/3}\, e^{-x\,t}\, dt = \int_0^{\infty} t^{-1/3}\, e^{-x\,t}\, dt - \int_R^{\infty} t^{-1/3}\, e^{-x\,t}\, dt$.

According to the definition of the Gamma function, $\Gamma(z) = \int_0^{\infty} x^{z-1}\, e^{-x}\, dx$, we can find $\int_0^{\infty} t^{-1/3}\, e^{-x\,t}\, dt = \frac{\Gamma(-1/3+1)}{x^{-1/3+1}} = \frac{\Gamma\left(\frac{2}{3}\right)}{x^{2/3}}$.

From the analysis we have used for $\int_R^{\pi} t^{-1/3}\, e^{-x\,t}\, dt$, we can find $\int_R^{\infty} t^{-1/3}\, e^{-x\,t}\, dt \sim \frac{e^{-R\,t}}{x},\ x \to +\infty$.

$I(x) = \int_0^R t^{-1/3}\, e^{-x\,t}\, dt + \int_R^{\pi} t^{-1/3}\, e^{-x\,t}\, dt \sim \frac{\Gamma\left(\frac{2}{3}\right)}{x^{2/3}} - \frac{e^{-R\,t}}{x} + \frac{e^{-R\,t}}{x} = \frac{\Gamma\left(\frac{2}{3}\right)}{x^{2/3}},\ x \to +\infty.$

Thus, we have the asymptotic expansion is $\frac{\Gamma\left(\frac{2}{3}\right)}{x^{2/3}}$, $x \to +\infty$.

In[ ]:=
```
AsymptoticIntegrate[e^(-x t) t^(-1/3), {t, 0, π}, {x, ∞, 1}]
```

Out[ ]=

$$\frac{\text{Gamma}\left[\frac{2}{3}\right]}{x^{2/3}}$$

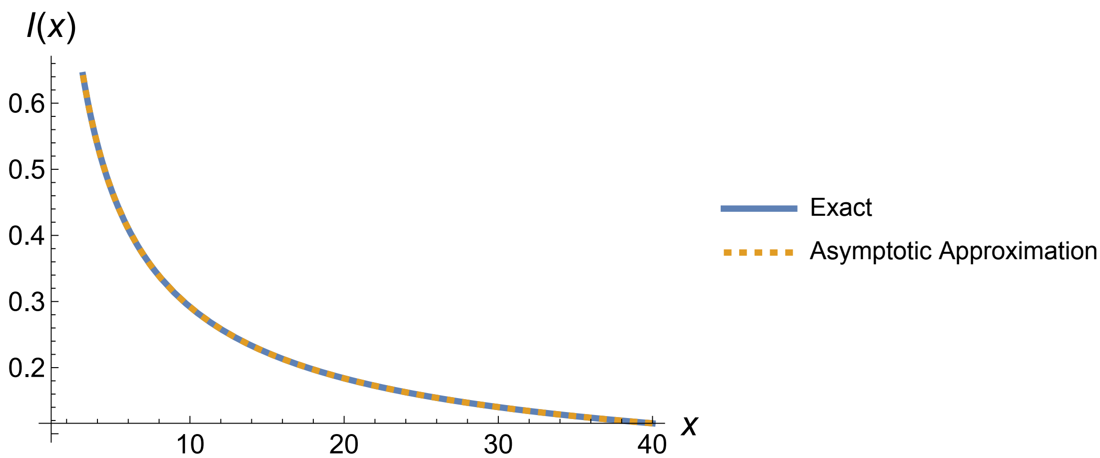
In[ ]:=
```
ExactIntegral2[x_] = Integrate[e^(-x t) t^(-1/3), {t, 0, π}, Assumptions → x > 0]
```

Out[ ]=

$$\frac{\text{Gamma}\left[\frac{2}{3}\right] - \text{Gamma}\left[\frac{2}{3}, \pi\, x\right]}{x^{2/3}}$$

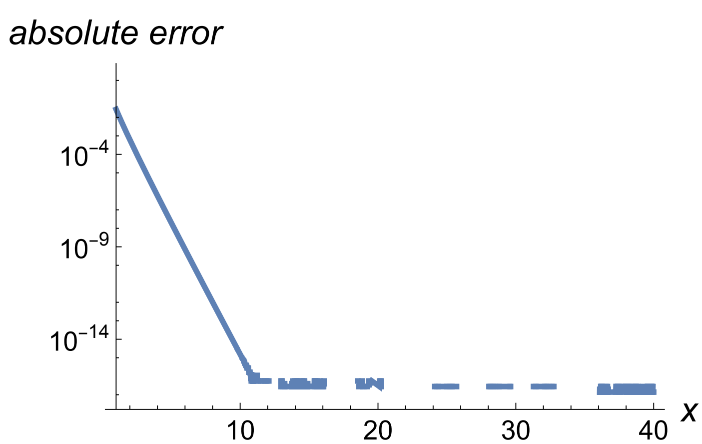
In[ ]:=
```
Plot[{ExactIntegral2[x], Gamma[2/3]/x^(2/3)}, {x, 1, 40},
 PlotStyle →
  {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Exact", "Asymptotic Approximation"}]
```

Out[ ]=



In[ ]:=
```
LogPlot[Abs[ExactIntegral2[x] - Gamma[2/3]/x^(2/3)], {x, 1, 40},
 PlotStyle → Directive[Solid, Thickness[0.01]],
 AxesLabel → {Style["x", Italic, 18], Style["absolute error", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

Out[ ]=



Problem 3

Use Laplace's method to determine the leading behavior of

$$I(x) = \int_{-1/2}^{1/2} e^{-x \sin^4 t}\, dt, \; x \to +\infty.$$

Solution:

Consider the general form of a Laplace integral, $I(x) = \int_a^b f(t)\, e^{-x\,\phi(t)}\, dt, \quad x \to +\infty$. We have $f(t) = 1$ and $\phi(t) = \text{Sin}^4[t]$.

The function $\phi(t)$ has a *local minimum* at an interior point $c = 0$ satisfying $-1/2 < c < 1/2$, so that $\phi'(c) = 0$ and $\phi''(c) = 0$.

We won't be able to directly use the result of Laplace's method, but we can follow the method by expanding about $t = 0$, we find

In[ ]:=
```
ϕ = Series[Sin[t]^4, {t, 0, 4}]
```

Out[ ]=

$t^4 + O[t]^5$

Thus, we find that

$$I(x) \sim \int_{-\infty}^{\infty} e^{-x\,t^4}\, dt\,, \quad x \to +\infty.$$

We can evaluate $\int_{-\infty}^{\infty} e^{-x\,t^4}\, dt$ using Mathematica and find

In[ ]:=
```
Integrate[e^(-x t^4), {t, -∞, ∞}, Assumptions → x > 0]
```

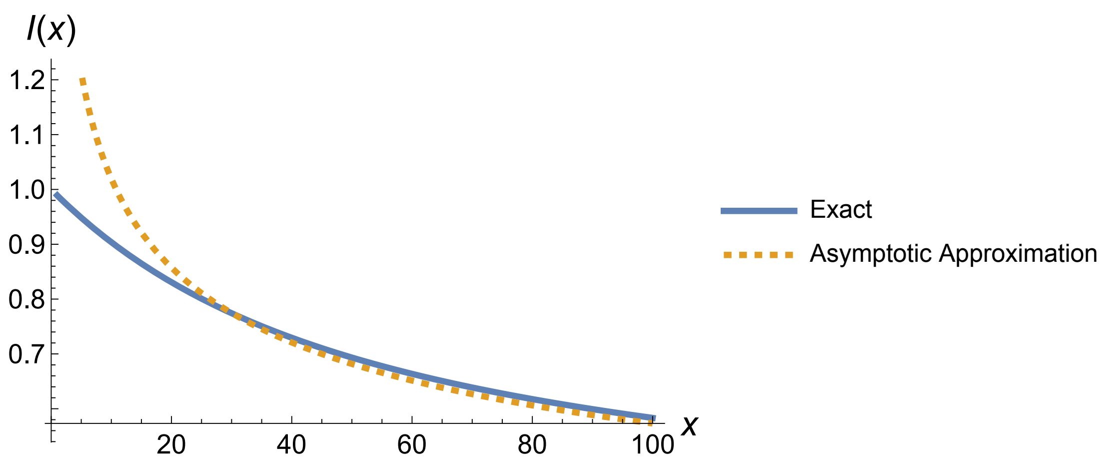
Out[ ]=

$$\frac{2\,\text{Gamma}\left[\frac{5}{4}\right]}{x^{1/4}}$$

Thus, the leading behavior is $\mathbf{I(x) \sim \dfrac{2\,Gamma\left[\frac{5}{4}\right]}{x^{1/4}}, \; x \to +\infty.}$

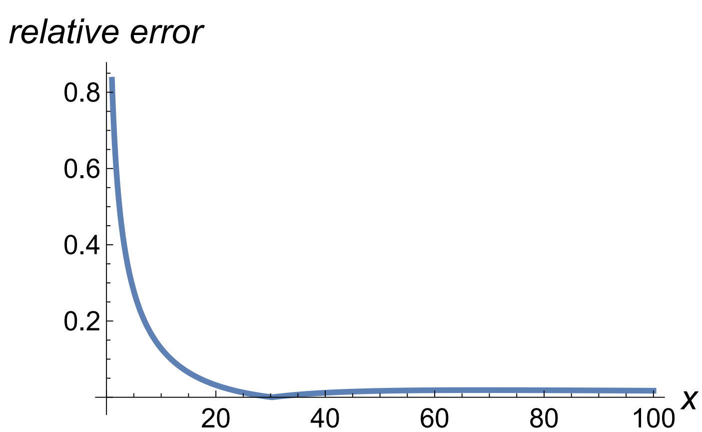
In[∘]:=
```
Plot[{NIntegrate[e^(-x Sin[t]^4), {t, -1/2, 1/2}], (2 Gamma[5/4])/x^(1/4)}, {x, 1, 100},
 PlotStyle →
  {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Exact", "Asymptotic Approximation"}]
```

Out[∘]=



In[∘]:=
```
Plot[Abs[NIntegrate[e^(-x Sin[t]^4), {t, -1/2, 1/2}] - (2 Gamma[5/4])/x^(1/4)] / Abs[NIntegrate[e^(-x Sin[t]^4), {t, -1/2, 1/2}]], {x, 1, 100},
 PlotRange → All, PlotStyle → Directive[Solid, Thickness[0.01]],
 AxesLabel → {Style["x", Italic, 18], Style["relative error", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

Out[∘]=



Problem 4

The L_p norm of a function g is given by $\| g \|_p = (I(p))^{1/p}$ where

$$I(p) = \int_a^b | g(t) |^p \, dt.$$

Assuming that $|g(t)| \in C^4$ and that it attains its unique maximum on $t = c$ inside $[a, b]$ with $g(c) \neq 0$, use Laplace's method to show that the L_p norm converges to the "maximum" norm as $p \to \infty$.

Solution:

$(\| g \|_p)^p = \int_a^b | g(t) |^p \, dt = \int_a^b e^{p \log(|g(t)|)} \, dt$

$\phi(t) = -\log(g(t))$, $\phi'(t) = \frac{-g'(t)}{g(t)}$ and $\phi''(t) = \frac{-g(t) g''(t) + g'(t)^2}{g(t)^2}$

The function $\phi(t)$ has a *local minimum* at an interior point c satisfying $a < c < b$, so that $\phi'(c) = 0$ and $\phi''(c) = \frac{-g''(c)}{g(c)}$.

Using Laplace's method, we have

$$(\| g \|_p)^p \sim \sqrt{\frac{2\pi}{p\,\phi''(c)}}\, e^{-p\,\phi(c)} =$$

$$\sqrt{\frac{2\pi}{p\,\phi''(c)}}\, e^{-p\,\phi(c)} = \sqrt{\frac{-2\pi\, g(c)}{p\, g''(c)}}\, g(c)^p, \quad p \to +\infty$$

Taking the power $1/p$, we can get:

$$\| g \|_p \sim \left(\frac{-2\pi\, g(c)}{p\, g''(c)}\right)^{1/2p} g(c), \quad p \to +\infty$$

where $\left(\frac{-2\pi\, g(c)}{p\, g''(c)}\right)^{1/2p} = \exp\left(\frac{1}{2p}\left[\log\left(\frac{-2\pi\, g(c)}{g''(c)}\right) - \log(p)\right]\right)$

We can follow the method by expanding exponential function about $u = 0$, we find

In[ ]:=
```
Series[E^u, {u, 0, 1}]
```

Out[ ]=

$1 + u + O[u]^2$

Consider $u = \frac{1}{2p}\left[\log\left(\frac{-2\pi\, g(c)}{g''(c)}\right) - \log(p)\right]$, we can get

$$\left(\frac{-2\pi\, g(c)}{p\, g''(c)}\right)^{1/2p} \sim$$

$$1 + \frac{1}{2p}\left[\log\left(\frac{-2\pi\, g(c)}{g''(c)}\right) - \log(p)\right] + O\left(\frac{1}{p^2}\right), \quad p \to +\infty$$

$$\| g \|_p \sim g(c) \left(1 + \frac{1}{2p} \left[\log\left(\frac{-2\pi g(c)}{g''(c)}\right) - \log(p)\right]\right), \quad p \to +\infty$$

In[ ]:=

```
Limit[(Log[-2 π g(c)/g''(c)] - Log[p])/(2 p), p → ∞]
```

Out[ ]=

```
0
```

Therefore, $\| g \|_p \sim g(c)$ as $p \to \infty$. Since $\phi(t) = -\log(g(t))$ has a local minimum at an interior point c satisfying $a < c < b$, we must have $g(t)$ has a local maximum at c. In other words, $g(c)$ is the maximum norm of function g.

Hence, the L_p norm $\| g \|_p \sim g(c)$, where $g(c)$ is the maximum norm of function g on $[a, b]$, as $p \to \infty$.

Problem 5

Show that

$$\int_0^\infty \log\left(\frac{u}{1-e^{-u}}\right) \frac{e^{-ku}}{u} \, du \sim \frac{1}{2k}, \quad k \to \infty.$$

Solution:

We can write the integral as following:

$$\int_0^\infty \log\left(\frac{u}{1-e^{-u}}\right) \frac{e^{-ku}}{u} \, du = \int_0^\infty \frac{[\log(u) - \log(1-e^{-u})]}{u} e^{-ku} \, du$$

We can apply Watson's lemma where $f(u) = \frac{[\log(u) - \log(1-e^{-u})]}{u}$. $f(u)$ is integrable on $[0, \infty)$.

Consider a parameter $0 < R < \pi$, and write $\int_0^\infty \log\left(\frac{u}{1-e^{-u}}\right) \frac{e^{-ku}}{u} \, du = \int_0^R f(u) e^{-ku} \, du + \int_R^\infty f(u) e^{-ku} \, du$.

To evaluate the second integral $\int_R^\infty f(u) e^{-ku} \, du$, we need to find the upper bound of $f(u)$. Let's take its first derivative

In[ ]:=

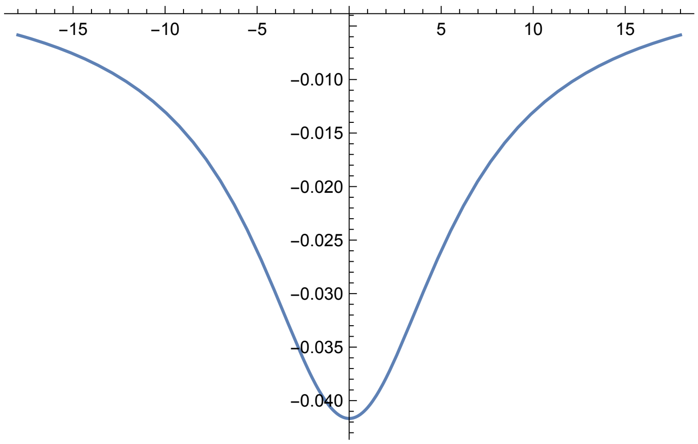
```
D[(Log[u] - Log[1 - e^-u])/u, u]
```

Out[ ]=

$$\frac{-\frac{e^{-u}}{1-e^{-u}} + \frac{1}{u}}{u} - \frac{-\text{Log}[1 - e^{-u}] + \text{Log}[u]}{u^2}$$

In[∘]:=
$$\text{Plot}\left[\frac{-\frac{e^{-u}}{1-e^{-u}}+\frac{1}{u}}{u}-\frac{-\text{Log}[1-e^{-u}]+\text{Log}[u]}{u^2}, \{u, -18., 18.\}\right]$$

Out[∘]=



In[∘]:=
$$\text{Limit}\left[\frac{-\frac{e^{-u}}{1-e^{-u}}+\frac{1}{u}}{u}-\frac{-\text{Log}[1-e^{-u}]+\text{Log}[u]}{u^2}, u \to \infty\right]$$

Out[∘]=

0

We can observe that $f'(u) < 0,\ \forall\ u \geq 0$, so $f(u)$ is a monotonically decreasing function.

We can find its upper bound by taking limit:

In[∘]:=
$$\text{Limit}\left[\frac{\text{Log}[u]-\text{Log}[1-e^{-u}]}{u}, u \to 0, \text{Direction} \to \text{"FromAbove"}\right]$$

Out[∘]=

$$\frac{1}{2}$$

We can conclude that $|f(u)| \leq \frac{1}{2}$, and then we find that

$$\left|\int_R^\infty f(u)\, e^{-k u}\, du\right| \leq \int_R^\infty \frac{1}{2}\, e^{-k u}\, du = \frac{1}{2}\,\frac{e^{-k R}}{k} \sim \frac{e^{-k R}}{k}, \quad k \to +\infty.$$

To compute the first integral $\int_0^R f(u)\, e^{-k u}\, du$. We assume $f(u)$ has the following asymptotic expansion,

In[∘]:=
$$\text{Series}\left[\frac{\text{Log}[u]-\text{Log}[1-e^{-u}]}{u}, \{u, 0, 0\}\right]$$

Out[∘]=

$$\frac{1}{2}+O[u]^1$$

By substituting this asymptotic expansion into the integral, we find

$$\int_0^R f(u)\, e^{-k u}\, dt \sim \int_0^R \left(\frac{1}{2}\right) e^{-k u}\, du, \quad k \to +\infty.$$

Next, we write

$$\int_0^R \left(\frac{1}{2}\right) e^{-k u}\, du = \int_0^\infty \frac{1}{2}\, e^{-k u}\, du - \int_R^\infty \frac{1}{2}\, e^{-k u}\, du.$$

We can get $\int_0^\infty \frac{1}{2}\, e^{-k u}\, du = \frac{1}{2k}$.

In[ ]:=

```
Integrate[1/2 e^(-k u), {u, 0, ∞}, Assumptions → k > 0]
```

Out[ ]=

$$\frac{1}{2\,k}$$

From the analysis we have used above, we can find that

$$\int_R^\infty \frac{1}{2}\, e^{-k u}\, du \sim \frac{e^{-k R}}{k}, \quad k \to +\infty.$$

$\int_0^\infty \log\left(\frac{u}{1-e^{-u}}\right) \frac{e^{-k u}}{u}\, du \sim \frac{1}{2k} - \frac{e^{-k R}}{k} + \frac{e^{-k R}}{k} = \frac{1}{2k}, k \to +\infty.$

Hence, we can get $\int_0^\infty \log\left(\frac{u}{1-e^{-u}}\right) \frac{e^{-k u}}{u}\, du \sim \frac{1}{2k}, k \to +\infty.$