Finite Element Analysis for Electrostatic and Transmission Line Problems

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***Abstract*—The finite element method (FEM) is a numerical technique for obtaining approximate solutions to boundary-value-problems of mathmatical physics, such like Partial Differential Equation (PDE). Compared with finite difference method, FEM is very competitive when solving two-dimensional and three-dimensional PDEs with complicated geometry and mesh. Hence it plays a very important role in electromagnetic analysis. In this paper, I first formulate the finite element solution for a general one-dimensional and a general two-dimensional boundary-value problem (Poisson's Equation) using Ritz's method and linear elements. Then I study the higher order element, such as quadratic and cubic interpolation. Finally I illustrate the application of FEM on Electrostatic and Transmission Line problems in two-dimensional case.**

***Index Terms*—Finite Element Method, Ritz Method, Interpolation, Microstrip, Transmission Line.**

I. Introduction

Electromagnetic analysis has been an indispensable part of many engineering and scientific studies since J.C.Maxwell published his electromagnetic theory in 1873. The problem of electromagnetic analysis is basically solving a set of Maxwell's equations subject to boundary conditions[1]. In the past 200 years people developed a lot of methods to solve boundary-value Maxwell's equations analytically, such as using Green's fuction with vector potential. However, the mathematical models of most physical problems are so complicated that an analytical or closed-form solution is often not available, especially for nonlinear problem in optics and photonics area. For this reason, numerical methods are introduced to help solve these linear or nonlinear PDEs. The well-known numerical methods consist of difference method like finite difference (FD) and FEM, integral method like method of moment (MoM) and others like spectrum method. For electromagnetic problems, there are more methods like finite-difference time-domain method (FDTD), Beam tracing method and transmission line model method (TLM). Today, the finite element method has been very well developed for two-dimensional and three-dimensional problems and widely used in electromagnetic reseach, since many famous commercial softwares use FEM as their main algorithm, such like COMSOL, ANSYS, CST and so forth. Instead of having intuitive scheme like finite difference method, FEM is based on more complex mathematical theorem. As is discussed, for solving the boundary-value electromagnetic problems it is hard to find the analytical solution in most of cases. To overcome this difficulty, various of approximate methods have been developed, among which Ritz and Galerkin's methods have been used most widely. The basic idea of these two methods is selecting trial functions (or test functions) defined over the entire solution domain that can represent the true solution, at least approxiamately, and satisfy the proper boundary conditions. However, for many problems it is still very difficult to find the appropriate test functions, if not impossible, especially for two- and three- dimensional problems. To alleviate this diffuculty, we can divide the entire domain into several small subdomains and use trial functions defined over each subdomain. In this case such trial functions are usually simpler than entire-domain and we can use interpolation method to form these trial functions on each subregion. Hence, a finite element analysis of a boundary-value problem should include the following steps:
1. Discretization the entire domain into subdomains;
2. Selection of the interpolation functions (trial functions);
3. Formulation of the system of equations via Ritz or Galerkin's method;
4. Solution of the system of equations.

I will discuss the above steps in the following sections in great detail. In secion II, I introduce Ritz's method of formulation; in section III, I discuss the discretizing method and interpolation scheme; in section IV, I will show how to formulate the system of equations created from discretization and how to deal with three kinds of boundary conditions. In section V, there are the numerical solutions to PDEs solved by FEM. Those problems include basic 1D and 2D poisson's equations with Dirichlet or Neumann boundary condtions, electrostatic problem, as well as the analysis of microstrip and quasistatic transmission line. Note that transmission line plays a significant role in communication engineering, thus using numerical method to study the electromagnetic properties of it helps us with better understanding.

II. The variational theorem and Ritz Method

A. Formulation of the Differential Equation

We start with a differential equation of the following form: Under the domain Ω with boundary Γ:

$$\mathcal{L}\phi = f \tag{1}$$

From equation (1), $\mathcal{L}$ denotes the differential equation, ϕ is the variable, and f is the excitation source. For electromagnetic problems, the form of governing differential equation ranges from simple Poisson equations, like electrostatic problem, to complicated scalar and vector wave equations, like antenna problem.

To formulate the system matrix with the Ritz method, we must first define the inner product for functions inside the solution space:

$$\langle \phi, \psi \rangle = \int_{\Omega} \phi \psi^* d\Omega \tag{2}$$

where the asterisk denotes the complex conjugate. With the definition of inner product it can be shown that if the operator $\mathcal{L}$ is self-adjoint, the solution of (1) can be obtained by minimizing the functional

$$F(\widetilde{\phi}) = \frac{1}{2}\langle \mathcal{L}\widetilde{\phi}, \widetilde{\phi} \rangle - \frac{1}{2}\langle \widetilde{\phi}, f \rangle - \frac{1}{2}\langle f, \widetilde{\phi} \rangle \tag{3}$$

Here we deploy $\widetilde{\phi}$ instead of ϕ - where $\widetilde{\phi}$ is the trial (test) function. The test function we use in FEM represent a weak solution - a guess to the real solution. The $\widetilde{\phi}$ can be discretized and approximated by the expansion:

$$\widetilde{\phi} \approx \sum_{i=1}^{N} c_i v_i = C^T V = V^T C \tag{4}$$

In the expansion above, the v_i are chosen basis function over the entire domain, and c_i are its coefficients. So C and V are column vectors. Combining equation (3) and (4), we have

$$F(\widetilde{\phi}) = \frac{1}{2} C^T \int_{\Omega} V \mathcal{L} V^T d\Omega C - C^T \int_{\Omega} V f d\Omega \tag{5}$$

In order to minimize the functionals, we just need to set all of the partial derivatives of f to be zero:

$$\begin{aligned} \frac{\partial F}{\partial C_i} &= \frac{1}{2}\int_{\Omega} V L V^T d\Omega C + \frac{1}{2} C^T \int_{\Omega} V L V_i d\Omega - \int_{\Omega} V_i f d\Omega \\ &= \frac{1}{2}\sum_{j=1}^{N} C_j \int_{\Omega} (V_i \mathcal{L} V_j + V_j \mathcal{L} V_i) d\Omega - \int_{\Omega} V_i f d\Omega \\ &= 0 \qquad \forall i \in [0, \ldots, N] \end{aligned} \tag{6}$$

which can be written as the matrix format:

$$MC = b \tag{7}$$

The matrix M can be composed as follow:

$$M_{ij} = \frac{1}{2}\int_{\Omega} (V_i \mathcal{L} V_j + V_j \mathcal{L} V_i) d\Omega \tag{8}$$

The column vector b can be composed as follow:

$$b_i = \int_{\Omega} V_i f d\Omega \tag{9}$$

III. Discretizing and Inerpolation method

A. *Discretization of the entire domain*

As is discussed in I, finding the appropriate trial function for the hole domain is hard, hence we need to discretize the entire domain for simplification. Actually, the discretization is perhaps the most important step in any finite element analysis because the manner in which the domian is discretized will affected the computer storage requirement, CPU time, and especially the accuracy of numerical result. In this step, the entire domain Ω is divided into M subdomains Ω^e which is always referred to as *element*. For a one-dimensional domain, the elements are always short line segmentations to form (at least approximately) the original line; for a two-dimensional domain, the elements are often triangles (for irregular region) and rectangles (for regular region). In a three-dimensional problem, the domain can be discretized into tetrahedra, triangular prisms, or rectangular bricks. Figure 3.1 shows these discretization schemes.

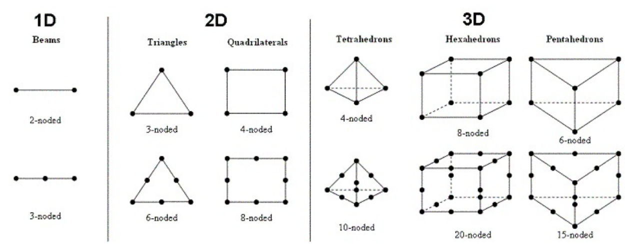

fig 3.1 Basic finite elements.

The domain discretization is always considered as the preprocessing task because it can be completely seperated from the other steps. Therefore, many well-developed finite element programs are created and open sourced to us. In this project I used gmsh as the mesh generator to discretize the entire domain for 2D cases.

B. *Selection of Interpolation Functions*

The second step of finite element analysis is the selection of an interpolation function that provides an approxiamation of the unknown solution within an element. The interpolation can be a polynomial of first (linear), second (quadratic), third (cubic) or higher order. The higher the order is, the more accurate the interpolation can be, while the interpolation formula will be more complicated. Once the order of the polynomial is selected, we can derive an expression for the unknown solution in an element in the following form:

$$\widetilde{\phi} = \sum_{i=1}^{N} N_j^e \phi_j^e = \{N^e\}^T \{\phi^e\} \tag{10}$$

where n is the number of nodes in each element; ϕ_j^e is the value of ϕ at node j in the element; and N_j^e is the interpolation function (or basis function) for node j which has the polynomial form. The most important feature of functions N_j^e is that they are nonzero only within the element e, and outside this element they must vanish, due to the feature of Hilbert space. Another thing we need to note is that the order of the interpolation functions N_j^e is the same as the order of element. In this paper I used Linear, quadratic and cubic interpolation for 1D problem and only linear interpolation for 2d cases.

IV. Formulation and Boundary condition

A. Formulation via Ritz Method

The third step in a finite element analysis is to formulate the system of equations. Reconsider the equation (1) and for simplicity assume that the problem is real valued. When the entire domain is discretized into subdomains, The *functional* function F($\widetilde{\phi}$) defined in (3) can be expressed as

$$F(\widetilde{\phi}) = \sum_{e=1}^{M} F^e(\widetilde{\phi}^e) \tag{11}$$

where M is the number of elements making up the entire domian and

$$F^e(\widetilde{\phi}^e) = \frac{1}{2}\int_{\Omega^e} \widetilde{\phi}^e \mathcal{L}\widetilde{\phi}^e d\Omega - \int_{\Omega^e} f\widetilde{\phi}^e d\Omega \tag{12}$$

where Ω^e is the domain of each element. Substituting the interpolation scheme (10) into (12), we obtain

$$F^e = \{\phi^e\}^T\{\frac{1}{2}\int_{\Omega^e}\{N^e\}\mathcal{L}\{N^e\}^T d\Omega\{\phi^e\} - \int_{\Omega^e} f\{N^e\}d\Omega\} \tag{13}$$

which can be written in matrix form as

$$F^e = \frac{1}{2}\{\phi^e\}^T[K^e]\{\Phi^e\} - \{\Phi^e\}^T\{b^e\} \tag{14}$$

where $[K^e]$ is an n × n matrix while b^e an n × 1 vector, n is the number of nodes in an element. The elements in $[K^e]$ and b^e are

$$K_{ij}^e = \int_{\Omega^e} N_i^e \mathcal{L} N_j^e d\Omega \tag{15}$$

$$b_i^e = \int_{\Omega^e} f N_i^e d\Omega \tag{16}$$

Note that since $\mathcal{L}$ is self-adjoint, matrix [K^e] is symmetric. Substituting (14) to (11), we can obtain the formula of F in the entire domain as

$$F(\widetilde{\phi}) = \sum_{e=1}^{M}(\frac{1}{2}\{\phi^e\}^T[K^e]\{\phi^e\} - \{\phi^e\}^T\{b^e\}) \tag{17}$$

By performing the summation and replacing the local node numbers with the global node numbers, (17) can be written as

$$F = \frac{1}{2}\{\phi\}^T[K]\{\phi\} - \{\phi\}^T\{b\} \tag{18}$$

where [K] is an N × N matrix, with N being the total number of nodes in the entire domain; $\{\phi\}$ is an N × 1 vector referring to the unknown value on these nodes; $\{b\}$ is the N × 1 vector standing for source term. According to variational theorem, the solution to the system of equations can be obtained by imposing the stationarity requirement δ F = 0. Here we can equivalently set the partial derivative of F with respect to ϕ_i to be zero:

$$\frac{\partial F}{\partial \phi_i} = \sum_{j=1}^{N} K_{ij}\phi_j - b_i = 0 \quad i = 1, 2, ..., N \tag{19}$$

or in a matrix form,

$$[K]\{\phi\} = \{b\} \tag{20}$$

With equation (20), we are able to solve the problem via LU decomposition and backward/forward substitution.

B. Application of Boundary Condition

Before we solve the system of equations, we need to apply the required boundary conditions. There are three kinds of boundary conditions that are often encountered: Dirichlet, Neumann and Robin's boundary condition. The Dirichlet condition is an *essential* boundary condition that must be imposed explicitely; in contrast, the Neumann condition is always satisfied implicitly and automatically in the solution process, hence Neumann condition is also called the *natural boundary condition*. For Robin condition, since it is the linear combination of Dirichlet and Neumann condition, emplicitely implementing is also needed.

V. Numerical Solution via Finite Element Analysis

In the foregoing sections I introduced the steps of solving one boundary-valued problem via FEM analytically and numerically. Hence in this section, I will apply the Finite Element Analysis to one-dimensional and two-dimensional problems with both Dirichlet and Neumann boundary conditions, including electrostatic and transmission line cases.

A. One-dimensional Poisson Equation

At the first beginning I implement the Finite Element analysis to 1D poisson equation

$$-\frac{d^2\phi}{dt^2} = \pi^2 sin(\pi x) \quad 0 < x < L \tag{21}$$

Here L = 1, x ∈[0,L] is the entire domain, with Dirichlet boundary condition

$$\phi|_{x=0} = \phi|_{x=L} = 0 \tag{22}$$

From the boundary condition we can easily find the analytical solution is

$$\phi = sin(\pi x) \quad 0 < x < L \tag{23}$$

Firstly I choose linear element and interpolation, which is

$$\phi^e(x) = \sum_{j=1}^{2} N_j^e(x)\phi_j^e \tag{24}$$

where N_j^e denotes the interpolation function given by

$$N_1^e(x) = \frac{x_2^e - x}{l^e} \quad and \quad N_2^e(x) = \frac{x - x_1^e}{l^e}$$

where l^e is the length of element. Then, formulating via the Ritz method. By intergrating over the element I calculate K_{ij}^e and b_j^e from

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \frac{dN_i^e}{dx}\frac{dN_j^e}{dx} dx \tag{25}$$

$$b_j^e = \int_{x_1^e}^{x_2^e} N_i^e f dx \tag{26}$$

where f = π^2sin(πx) here, as the source term. The next step is assemblying to form the system of equations by a summation over all elements and then an imposition of the stationarity requirement:

$$\sum_{e=1}^{M}([\overline{K}^e]\{\overline{\phi}^e\} - \{\overline{b}^e\}) = \{0\} \tag{27}$$

Here I discretized the entire domain into M = 18 elements, with 19 nodes. Having obtained [K] and b, I applied the Dirichlet boundary condition by setting the value of ϕ at the boundary points to be zero, hence the size of [K] and b reduced by 2. Then I solved the vector ϕ by LU decomposition. I also used finite difference menthod to solve the same problem with the same discretization, which is shown in figure 5.1

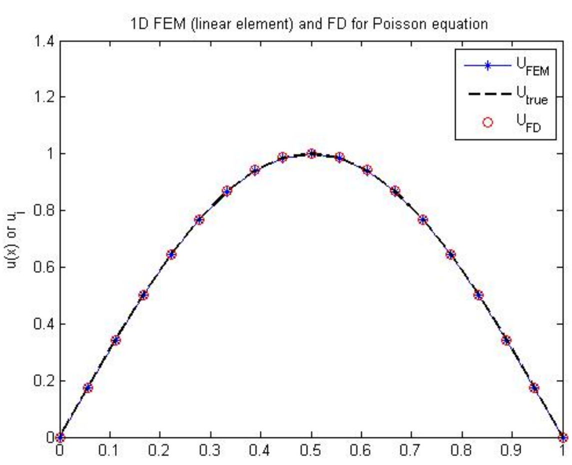

fig 5.1 FEM and FD solution to one-dimensional Poisson's equation

To exam the accuracy of FEM and FD, I calculated the L_∞ norm of it. Note that FEM's solution is extremely accurate on each node, hence I choose h_e = h/N as the size of interval to sample the interpolared solution on the whole region. The error distribution (compared with analytical solution with the same sampling frequency) is shown in fig 5.2

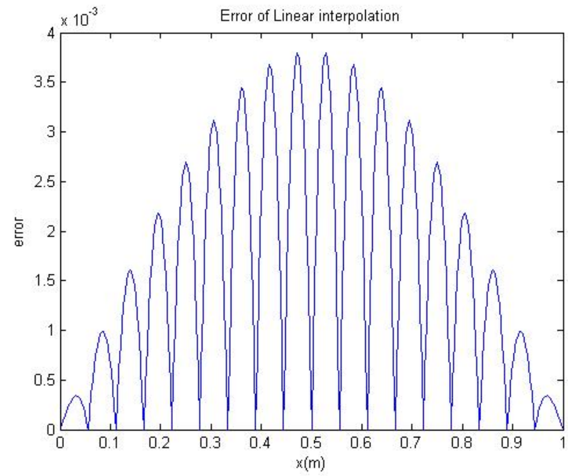

fig 5.2 Error distribution from linear interpolation

From figure 5.2, the L_∞ norm is 0.0038, while the L_∞ norm of finite difference method is 0.0025. The error of these methods shares the same order of magnitudes. Hence linear interpolation is not with a very high accuracy. To study how the order of interpolation affects the accuracy, I tried quadratic and cubic interpolation as

$$\phi_q^e(x) = \sum_{j=1}^{3} N_j^e(x)\phi_j^e \;\; and \;\; \phi_c^e(x) = \sum_{j=1}^{4} N_j^e(x)\phi_j^e \tag{28}$$

where N_j^e for quadratic interpolation is given by

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)}$$
$$N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}$$
$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)}$$

while N_j^e for cubic interpolation is given by

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)(x - x_4^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)(x_1^e - x_4^e)}$$
$$N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)(x - x_4^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)(x_2^e - x_4^e)}$$
$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)(x - x_4^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)(x_3^e - x_4^e)}$$
$$N_4^e(x) = \frac{(x - x_1^e)(x - x_2^e)(x - x_3^e)}{(x_4^e - x_1^e)(x_4^e - x_2^e)(x_4^e - x_3^e)}$$

Using the same size of discretization as N = 18, the $L\infty$ norm of these two interpolation are 3.350e-4 and 3.769e-5, respectively, which is much smaller than both FEM with linear interpolation and finite difference method. The error distribution of the two kinds of interpolation is shown in figure 5.3 and 5.4

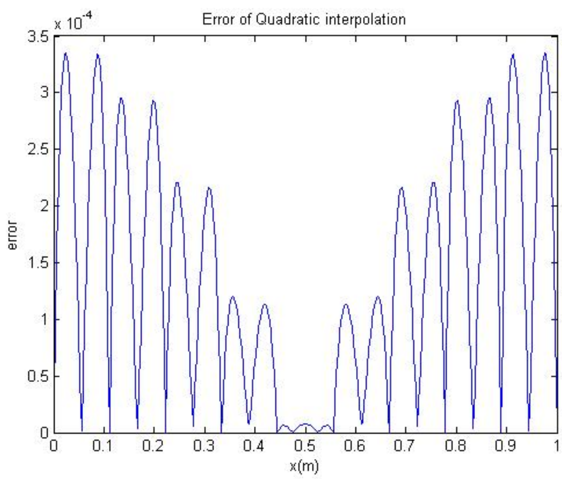

fig 5.3 Error distribution from quadratic interpolation

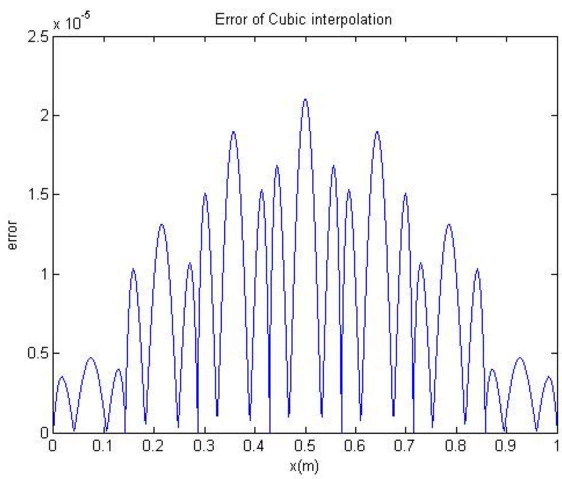

fig 5.4 Error distribution from cubic interpolation

I then test the scheme for convergence by doing seven runs. I start with N = 18 (with 19 nodes) and find the L_∞ norm of finite differece method, FEM with linear interpolation, quadratic interpolation as well as cubic interpolation. Then I halve the size of element and repeat the same procedure for 7 runs. The results is shown in Table 5.1 to 5.4

Num	$\|err\|_{l^\infty}$	l^∞ ratio	Num	$\|err\|_{l^\infty}$	l^∞ ratio
18	0.0025	4.0046	288	9.916e-6	4.0000
36	6.349e-4	4.0011	576	2.479e-6	4.0000
72	1.587e-4	4.0003	1152	6.198e-7	4.0000
144	3.966e-5	4.0001	2304	1.549e-7	

Table 5.1 Error of L^∞ norm of finite difference method

Num	$\|err\|_{l^\infty}$	l^∞ ratio	Num	$\|err\|_{l^\infty}$	l^∞ ratio
18	0.0038	3.9867	288	1.487e-5	4.0000
36	9.508e-4	3.9967	576	3.719e-6	4.0000
72	2.379e-4	3.9992	1152	9.296e-7	4.0000
144	5.949e-5	3.9998	2304	2.324e-7	

Table 5.2 Error of L^∞ norm of FEM with linear interpolation

Num	$\|err\|_{l^\infty}$	l^∞ ratio	Num	$\|err\|_{l^\infty}$	l^∞ ratio
18	3.350e-4	7.8897	288	8.326e-8	7.9996
36	4.246e-5	7.9768	576	1.041e-8	8.0000
72	5.323e-6	7.9930	1152	1.301e-9	
144	6.660e-7	7.9985			

Table 5.3 Error of L^∞ norm of FEM with quadratic interpolation

Number of element	$\|err\|_{l^\infty}$	l^∞ ratio
18	3.768e-5	15.5501
36	2.424e-6	15.8874
72	1.526e-7	15.9695
144	9.553e-9	

Table 5.4 Error of L^∞ norm of FEM with cubic interpolation

From the convergence analysis we can see that both finite difference method and FEM with linear interpolation are of 2^{nd} order accuracy, quadratic is of 3^{rd} order accuracy while cubic is of 4^{th} order accuracy. Note that for quadratic and cubic interpolation I didn't try as much as 7 runs, that's because the error decreases really fast when I halve the size of element, when it reaches to the order of round-off error as 1e-10, the result will be contaminated.

B. Two-dimensional Poisson Equation

With all the effort on one-dimensional case, now we can move to two-dimensional problems. For electromagnetic problems, we always solve it in frequence domain by assuming the EM wave is time-harmonic. In general case, the 2D anisotropic Helmholtz equation can be written as

$$-\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\alpha_y\frac{\partial\phi}{\partial y}\right)+\beta\phi=f \qquad (29)$$

Where ϕ can be electric field, magnetic field, potential or vector potential which satisfies Lorentz condition. At the first beginning I solve the equation with the simplest case given as following:

$$-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}=2\pi^2 sin(\pi x)cos(\pi y) \qquad (30)$$

where x, y defined on [0,1]×[0,1], with the boundary condition

$$\phi|_{x=0}=\phi|_{x=1}=0 \;\; and \;\; \frac{\partial\phi}{\partial n}|_{y=0}=\frac{\partial\phi}{\partial n}|_{y=1}=0$$

Therefore the analytical solution is ϕ = sin(πx)cos(πy). Solved with linear interpolation, result is shown in figure 5.5, while the analytical solution is shown in figure 5.6

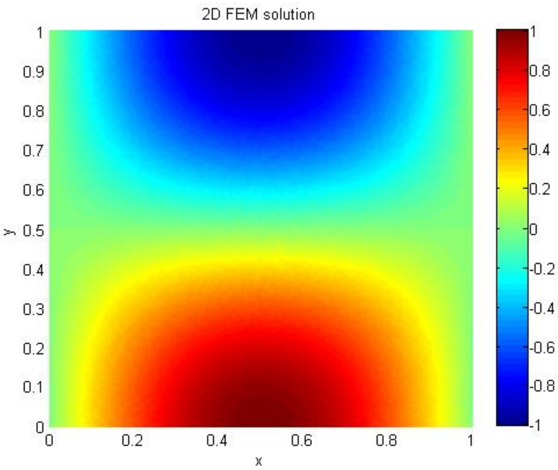

fig 5.5 FEM solution to 2D Poisson equation

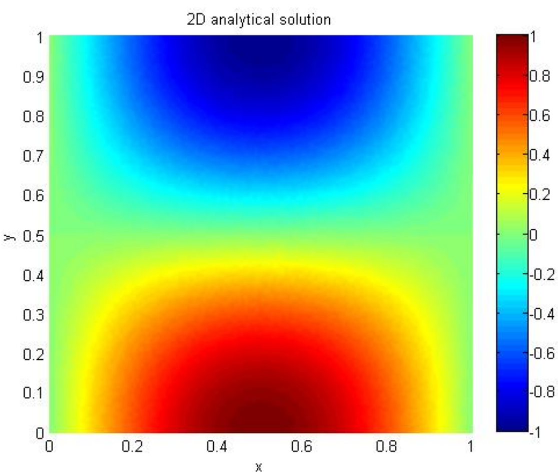

fig 5.6 Analytical solution to 2D Poisson equation

C. Electrostatic Problem: Electric Potential distribution on a Microstrip Line

Microstrip Line is widely used in communication system. It consists of a substrate made of dielectric material (such as FR-4) with a metal strip on the top surface of it. The size of it is always in micro-meter, hence is called ”microstrip line”. It can be a type of electrical transmission line implemented on printed board circuit technology or multi-chip modules (MCM). Transmission line acts as an interconnect between various devices on platform. When we excite one terminal of it, the wave will propagate through it to other terminals. Figure 5.7 shows the transection of the basic model of a microstrip line with E and H field distribution.

In this section, I will solve the electric potential distribution on a microstrip line.

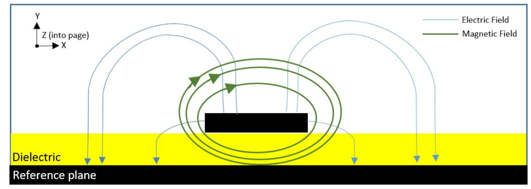

fig 5.7 E and H field distribution on a microstrip line

The two-dimensional electrostatic field problems are governed by the two-dimensional Poisson equation

$$-\frac{\partial}{\partial x}(\epsilon_r\frac{\partial \phi}{\partial x}) - \frac{\partial}{\partial y}(\epsilon_r\frac{\partial \phi}{\partial y}) = \frac{\varrho}{\epsilon_0} \tag{31}$$

where ϱ denotes the charge density, and ϵ_r denotes the relative permittivity. The geometry and mesh of the microstrip model is shown in fig 5.8 and fig 5.9

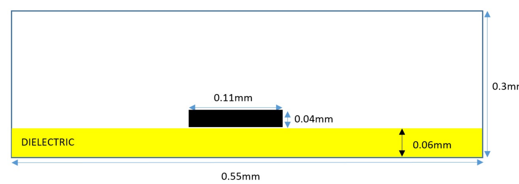

fig 5.8 Geometry information of the microstrip line

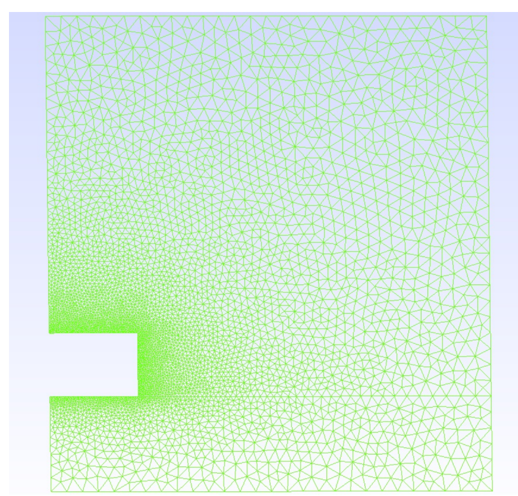

fig 5.9 mesh of the microstrip line

Note that since the model, and governing equations are symmetric, hence I can reduce the solution domain by half by applying the homogeneous Neumann boundary condition at the plane of symmetry.

From figure 5.8, the width of the metal is 0.11mm, with thickness be 0.04mm. I used copper as the metal whose conductivity if 4.0×10^7 S/m. The height of substrate is 0.06mm with the material whose relative permittivity is 3.

As a simplification we can assume that both electric and magnetic fields are orthogonal to z-direction, in which the wave propagates in transverse electro-magnetic mode (TEM). Therefore we can decompose the transmission line into discrete segments along the z-direction and analyze the distribution on the cross section.

The boundary condition of this model requires classified discussion. First is the boundary on the surface of metal. For a electrostatic problem, since I am interested in electric potential distribution, I can set the potential on the surface to be 1 V because it is equipotential on the surface of good conductors, and the potential to be 0 on the ground surface. On the interface of substrate and air, as potentil jump can lead to infinite electric field intensity, it must follow the Neumann boundary as $\frac{\partial \phi}{\partial n} = 0$. The most difficult one is the outer boundary. For most of the cases microstrip line is within an relatively open or unbounded domain, which means field must vanish at boundaries and no reflection is allowed. Hence, a perfect matching layer will be the optimal choice, which is often used in radiation and antenna problems. For simplicity here I let the outer box to be large enough and use Dirichlet boundary contion. The result of electric potential distribution is shown in figure 5.10.

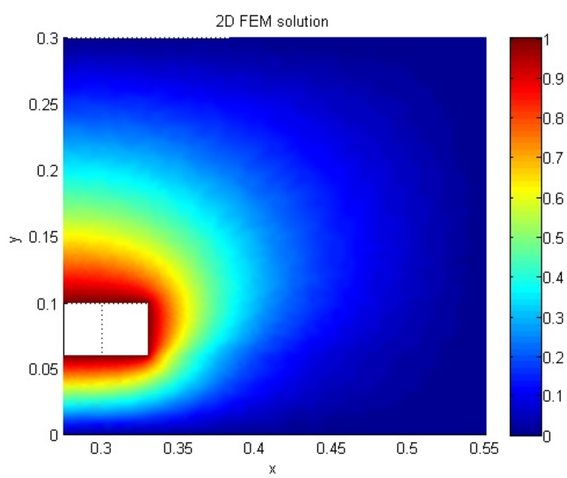

fig 5.10 FEM solution of electric potention distribution on a microstrip line

This time I have no way to obtain the analytical solution. For checking the correctness of FEM solution I solve the same problem via COMSOL. The result from COMSOL is shown in figure 5.11. We can have a intuitive impression that the result from my solver is fairly close to the result solved by COMSOL. Note that we don't care about the potential distribution inside the metal while COMSOL still calculate them out.

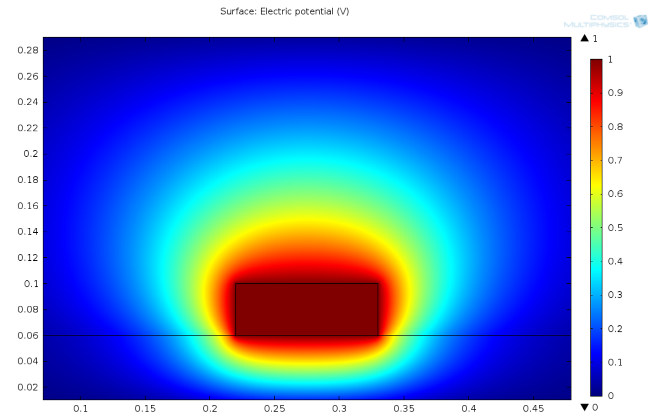

fig 5.11 electric potention distribution on a microstrip line from COMSOL.

D. FEM to Quasistatic Problem: Analysis of Multiconductor Transmission Lines

Analysis of multiconductor transmission lines is an important problem for the design of high-frequency integrated circuits. For transmission lines made of lossy conductions, we can employ a quasistatic analysis that keeps the conduction current and ignores the displacement current introduced by electromagnetic induction. With this approximation, $\mathbf{E} \approx$ -j$\omega\mathbf{A}$, where $\mathbf{E}$ is the electric field, ω is the angular frequency and $\mathbf{A}$ is the vector potential defined by $\mathbf{B} = \bigtriangledown\times\mathbf{A}$. From Maxwell's equations, we can obtain

$$\bigtriangledown \times (\frac{1}{\mu_r} \bigtriangledown \times A_z \hat{z}) + j\omega\mu_0\sigma A_z \hat{z} = \mu_0 \bar{J}_{imp}\hat{z} \tag{32}$$

where σ denotes the conductivity of the lossy conductors and $\hat{J}_{imp}$ is the impressed current along the longitudinal direction (for this problem is z direction). Conductors are made of aluminum-oxide with $\sigma = 3.6\times10^7$ and $\mu_r = 1$. With relation $\mathbf{E} \approx$ -j$\omega\mathbf{A}$, (32) can be written as

$$-\frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + j\omega\mu_0\sigma A_z = \mu_0\sigma E_{imp} \tag{33}$$

The information of geometry and mesh of the multicondutors transmission line model are shown in figure 5.12 and 5.13

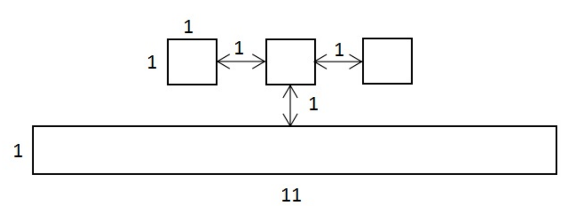

fig 5.12 Geometry information of the multiconductor transmission line model

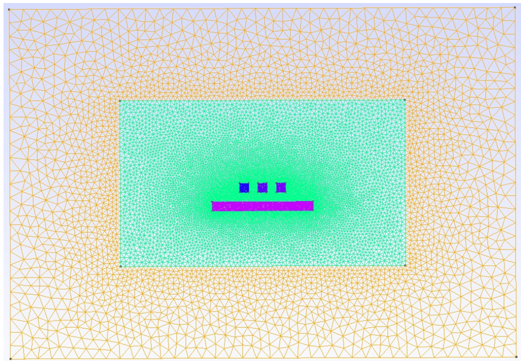

fig 5.13 Mesh information of the multiconductor transmission line model

Note that I used two region of mesh, the outer one is of coarse mesh size while the inner one is of finer mesh size. The boundary condition here for any interface can be set as Neumann boundary condition. With the system of elements, I excited the conductors one by one with E_{imp} = 1 and ω = 10 MHz to obtain the electric field distribution among the region. As usual, I compare my numerical solution with the result from COMSOL, as is shown in figure 5.14 to 5.17 (Here I only show the result when excit)

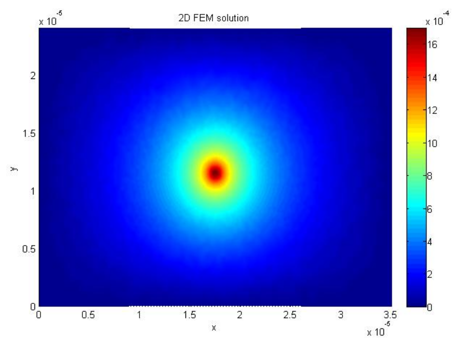

fig 5.14 E field distribution from my solution when excite conductor 2

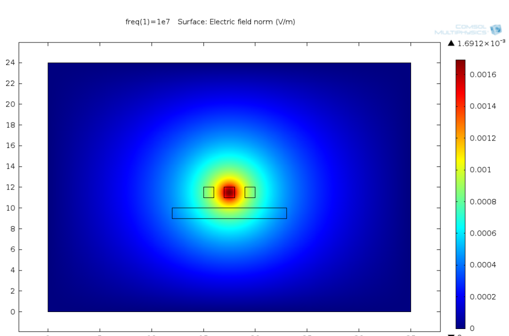

fig 5.15 E field distribution from COMSOL when excite conductor 2

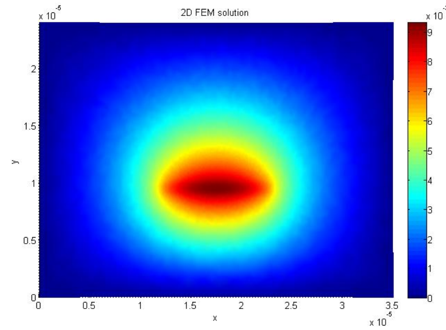

fig 5.16 E field distribution from my solution when excite conductor 4

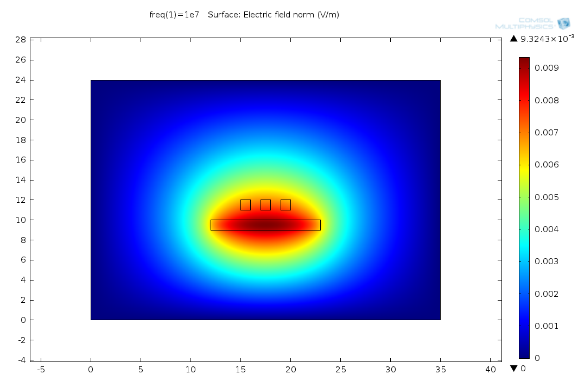

fig 5.17 E field distribution from COMSOL when excite conductor 4

Compare the solutions from my FEM solver and COMSOL, they are fairly close to each other. Hence My algorithm is relatively reliable like commercial softwares.

VI. Conclusion

Through this paper, I implemented finite element method for solving the following problems:

- 1-dimensional Poisson equation: I solved it with FEM and finite difference method. The convergence analysis shows that with linear interpolation and center-difference, both FEM and FD have a 2^{nd} order accuracy; with quadratic interpolation FEM has a 3^{rd} order accuracy; while with cubic interpolation FEM has a 4^{th} order accuracy.
- 2-dimensional Poisson equation: I solved it with FEM and compared with analytical solution.
- 2-dimensional Microstrip Line: I solved it with FEM and compared with result from COMSOL.
- 2-dimensional Transmission Line: I solved it with FEM and compared with result from COMSOL.

The comparison shows my FEM algorithm is reliable for the correct solution which is close to the commercial software. However, it also took much longer time than COMSOL when solving the two-dimensional cases. Hence more modification is needed for improving the performance.

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