Lattice regularization of non-relativistic interacting fermions in 1+1 dimensions

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- The continuum Hamiltonian of a $(1+1)$ system[†] reads:

$$H = \sum_{\sigma=1}^{2} -\psi_\sigma^\dagger \frac{\partial_x^2}{2m_a}\psi_\sigma + \lambda\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1 + c\left(\psi_1^\dagger i\overleftrightarrow{\partial_x}\psi_2^\dagger\right)\left(\psi_2 i\overleftrightarrow{\partial_x}\psi_1\right). \quad (1)$$

- ψ_1 and ψ_2 are two species of non-relativistic fermion particles.
- λ is a parameter for an on-site interaction.
- c is a parameter for a pairwise local velocity-dependent interaction.

[†] S.Y Yong and D. T. Son (2018)

Lattice Regularization

- Regulating a system on a lattice, we want to study few-body systems non-perturbatively.

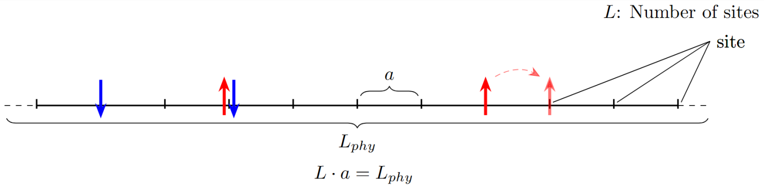


Figure 1: A graphical illustration of lattice theory.

- We are interested in whether we can recover the continuum theory at the continuum limit ($a \to 0$ or $L \to \infty$).

System with Contact Interaction ($c = 0$)

- With only contact interaction, (also known as the Hubbard Model) the Hamiltonian is:

$$H_1 = H_0 + H_{\mathrm{int}}^{(\lambda)}. \tag{2}$$

- $H_0 = \sum_{x,\sigma} \epsilon(a)(\psi^\dagger_{\sigma,x}\psi_{\sigma,x+\alpha} - 2\psi^\dagger_{\sigma,x}\psi_{\sigma,x} + \psi^\dagger_{\sigma,x+\alpha}\psi_{\sigma,x})$
- $H_{\mathrm{int}}^{(\lambda)} = \epsilon(a)\lambda(a)\psi^\dagger_{1,x}\psi_{1,x}\psi^\dagger_{2,x}\psi_{2,x}$
- In the continuum limit, $a\lambda(a)\epsilon(a)$ is constant.

System with Contact Interaction ($c = 0$)

- We can show that, using exact diagonalization, $\lim_{a\to 0} a\lambda(a)\epsilon(a) = -0.140966$.

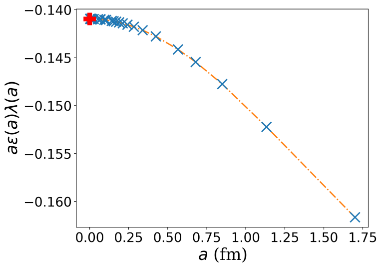


Figure 2: Interaction coefficient $a\lambda(a)\epsilon(a)$ as a function of lattice spacing a.

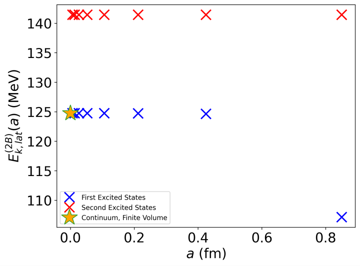


Figure 3: Comparison of energy levels generated by our method and the finite-volume calculation†.

† C. Körber, E. Berkowitz, and T. Luu (2019).

System with Contact Interaction ($c = 0$)

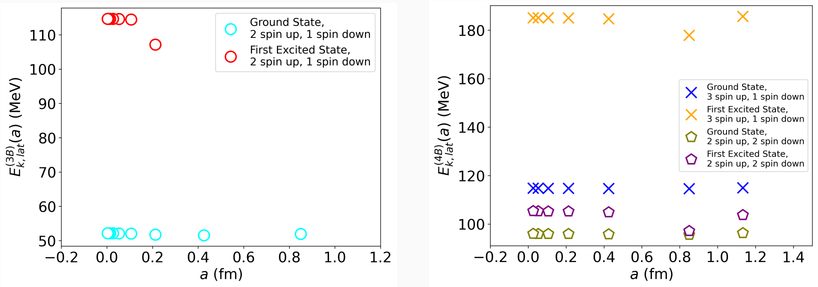


Figure 4: Three- and four-body ground state and first excited state energy as functions of lattice spacing. They have well-defined continuum limits.

System with Contact Interaction ($c = 0$)

- We present an infinite volume extrapolation method, fixing the lattice spacing $a \leq 0.1$fm, and increase site number, L.
- The interaction strength, $U = -0.140966$ is used for delta potential calculation.

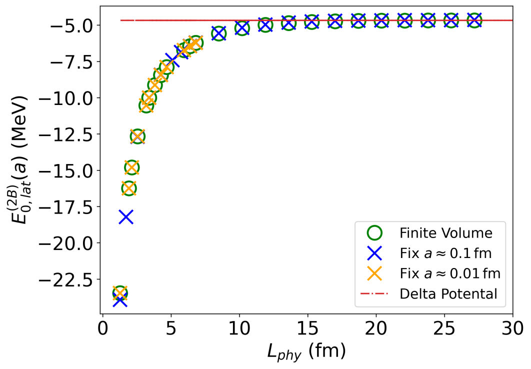


Figure 5: Comparison of different lattice spacing a.

System with Momentum-Dependent Interaction ($\lambda = 0$)

- With only the momentum-dependent interaction, the Hamiltonian reads

$$H_2 = H_0 + H^{(c)}_{\text{int}}. \tag{3}$$

- $H^{(c)}_{\text{int}} = \epsilon c_2 \sum_x \left(\psi_{2,x+\alpha}\psi_{1,x} - \psi_{2,x}\psi_{1,x+\alpha}\right)^\dagger \left(\psi_{2,x+\alpha}\psi_{1,x} - \psi_{2,x}\psi_{1,x+\alpha}\right)$

- At the continuum limit, $\lim_{a\to 0} c_2 = -1$.

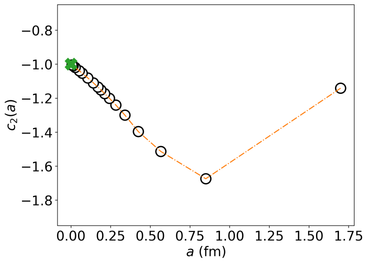
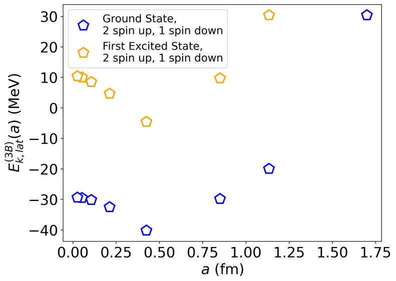


Figure 6: Plot of interaction strength c as a function of lattice spacing, a.

System with Momentum-Dependent Interaction ($\lambda = 0$)

- Energy for three- and four-body systems can support further study of three-body force.



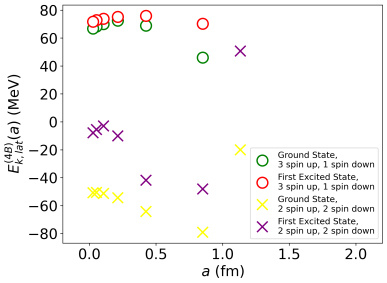


Figure 7: Three- and four-body ground state and first excited state energy as functions of lattice spacing.

System with Momentum-Dependent Interaction ($\lambda = 0$)

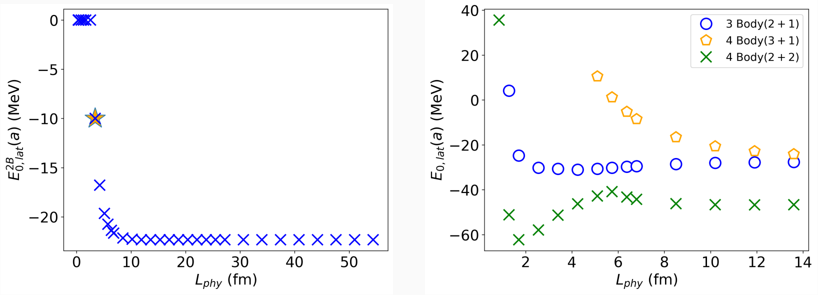


Figure 8: Infinite volume extrapolation for two- (Left), three-, and four-body (Right) systems. at $a \approx 0.1$ fm. $E_{3B}/E_{2B} \approx 1.23$.

- In the continuum effective theory calculation, $E_{3B}/E_{2B} = 1.375$.

Summary and Outlook

- In this work, we study a lattice regularization of a continuum theory, and use exact diagonalization method to verify the consistency of two approaches at the continuum and infinite length limit.
- Further study the full Hamiltonian, including the renormalization of a three-body force and extend this method to many-body systems, using quantum Monte Carlo simulations.
- This method can be applied on more general mass-spin imbalanced system.

Thank you!

Fixing Energy Scales

- The discretized version of a free Hamiltonian would look like

$$H_0 = -\frac{1}{2m_a\epsilon^2}(\psi^\dagger_{\sigma,x}\psi_{\sigma,x+\alpha} - 2\psi^\dagger_{\sigma,x}\psi_{\sigma,x} + \psi^\dagger_{\sigma,x+\alpha}\psi_{\sigma,x}) \tag{4}$$

- Notice that m_a here is not equal to the physical mass, m_{phy}, and depends on the lattice spacing a.
- Energy of a free Hamiltonian can be solved using exact diagonalization method. The energy scale $\epsilon(a) = -\frac{1}{2m_a\epsilon^2}$ is fixed by comparing to the ground state energy of a one-dimensional infinite square well problem.

$$2\epsilon(a)\left[1 - \cos\left(\frac{2\pi}{L}\right)\right] = \frac{2\pi^2}{m_{\mathrm{phy}}L^2_{\mathrm{phy}}} \tag{5}$$

System with Momentum-Dependent Interaction ($\lambda = 0$)

- With only contact interaction, the Hamiltonian reads

$$H_2 = H_0 + H_{\rm int}^{(c)} \tag{6}$$

- $H_0 = \sum_{x,\sigma} \epsilon(a)(\psi^\dagger_{\sigma,x}\psi_{\sigma,x+\alpha} - 2\psi^\dagger_{\sigma,x}\psi_{\sigma,x} + \psi^\dagger_{\sigma,x+\alpha}\psi_{\sigma,x})$
- $H_{\rm int}^{(c)} = \epsilon(a) c \sum_x \left(\psi_{2,x+\alpha}\psi_{1,x} - \psi_{2,x}\psi_{1,x+\alpha}\right)^\dagger \left(\psi_{2,x+\alpha}\psi_{1,x} - \psi_{2,x}\psi_{1,x+\alpha}\right)$
- We introduce a relative momentum q, and relative position, δ to reduce to a one-body system.

$$|q, \delta\rangle = \frac{1}{\sqrt{L}} \sum_x \psi^\dagger_{2,x}\psi^\dagger_{1,x+\delta} e^{i(2\pi/L)qx} |0\rangle \tag{7}$$

- Define $I_{lat}^{(c)} = 2\epsilon(a)\delta = 0(E - H_0)^{-1}\delta = 0$ that appears in the denominator of the first-order expansion of Green function.
- At pole,

$$\lim_{a\to 0} c_2(a) = 2/I_0^{(c)} = -2/\int_0^{2\pi} \frac{dp}{2\pi} \frac{(1-\cos 2p)}{1-\cos p)} = -1. \tag{8}$$