浙江大学爱丁堡大学联合学院

ZJU-UoE Institute

Bayesian Statistics and Bayesian Inference

ADS 2, Lecture 22

Zhaoyuan Fang - zhaoyuanfang@intl.zju.edu.cn

Semester 2, 2022/23

Pre-lecture version

This lecture contains a lot of questions that I will ask you to think about in class. Providing the answers beforehand would defeat that purpose. Therefore, the version of the slides available to you before the lecture will not contain all of the information that is presented in the lecture.

A complete version will be uploaded to Learn after the lecture. In the meantime, here is a picture of an adorable baby fox.

Gore Lamar, U.S. Fish and Wildlife Service, Public domain, via Wikimedia Commons

All hypothesis tests are the same . . .

- Formulate the Null Hypothesis, Alternative Hypothesis.
- Design your experiment and collect data
- Summarise and describe your data.
- Think about what you would expect if H0 was true.
- Could your data be explained by the Null Hypothesis? Determine the probability of your data given H0.
- Interpret your p value and make a decision.
- Be aware that hypothesis tests are not perfect.

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Or are they?

Has this ever felt weird to you?

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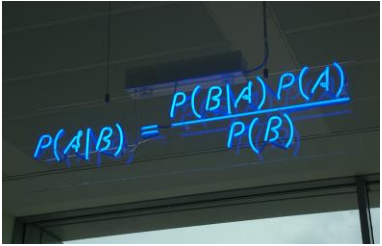
Yes, there is!

Bayesian inference

Key ideas:

- Instead of making a yes/no decision, quantify the strength of our belief.
- Beliefs exist even before we have data, and we take those into account.
- Beliefs are updated as we collect more data.



Learning Objectives

After this lecture, you should be able to do the following:

- Explain what “belief” means to a Bayesian statistician
- Define the terms “prior” and “posterior”
- Specify prior probabilities and distributions
- Use Bayes’ theorem to determine posterior probabilities
- Describe the difference between Bayesian and Frequentist statistics

Outline

We all hold beliefs

- Our beliefs are subjective and influenced by opinion, e.g. do you believe in ghosts?


"If you don't believe in yourself, who will?"

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- Belief determines scientific hypotheses, interesting experiments, direction of research…

"If you don't believe in yourself, who will?"

It’s not just ghosts!

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Example

Dr. Li is studying the effects of childhood exposure to toxic chemicals on growth. For this she measures the height of two groups of adults: Those who have grown up near a chemical plant and those who haven’t.
What beliefs may Dr. Li have that are relevant to this study?

What do I mean by belief?

1. An acceptance that something exists, or is true, especially one without proof.
2. Trust, faith, or confidence in someone or something.

…but how does this fit into my objective view of statistics?

Outline

What is a prior?

The probability (or probability distribution) which describes our belief about the world before we collect any specific data.

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Examples:

- Adult human height is bounded between $50\,cm$ and $3\,m$.
- Adult human height is normally distributed with a mean around $170\,cm$
- An enzymatic reaction is unlikely to proceed with a k_{cat} faster than $10^6\,s^{-1}$ (the k_{cat}) of carbonic anhydrase
- A coin has a $50\,\%$ chance of landing on "heads".
- For a specific somatic gene, if two heterozygous mice (each carrying one mutated and one wildtype allele) mate, $25\,\%$ of their offspring will be homozygous for the mutated allele.
- ...

Priors: Example

Example:

- Hypothesis: Rob's date wants to see Avengers: Endgame.

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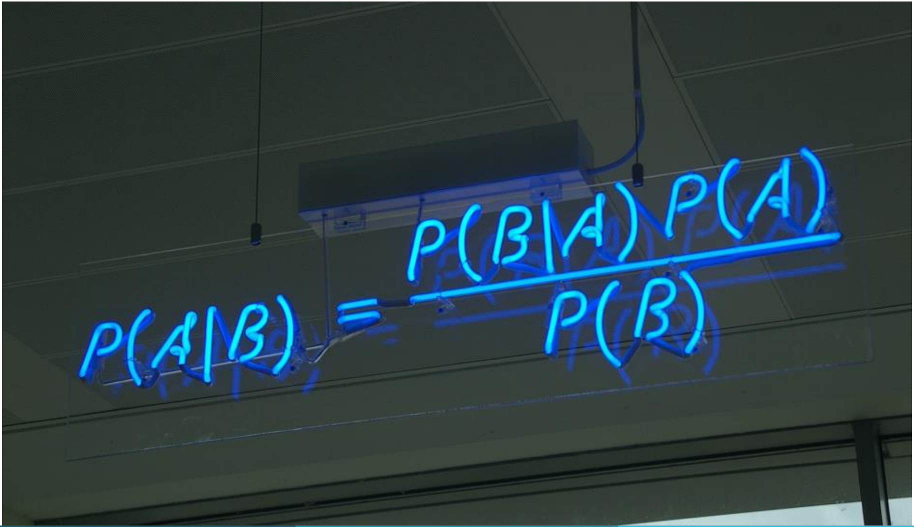
- Hypothesis: Rob's date wants to see Avengers: Endgame.
- What is the prior probability for this hypothesis?
- Rob knows (from talking to his friends) that about 60% of them want to watch Endgame
- Since we haven't collected any data on Rob's date yet, we are going to use this as our prior for the probability of the hypothesis that she wants to watch Endgame:

$$P(H) = 0.6$$

Key idea of Bayesian statistics

We state our beliefs before we do the experiment (prior). We then collect data and use that to update our beliefs (posterior).



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- Now, let's collect some data. Rob does this by asking her if she has seen Avengers: Infinity War. She says yes.
- *How does this change our belief in the hypothesis?*

Posteriors: Example

- Hypothesis: Rob's date wants to see Avengers: Endgame.
  Data: She has seen Avengers: Infinity war.

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- This is where Bayes comes in!

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

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  $P(D)$ ... proportion of people who have seen Infinty War.
  $P(D|H)$ ... proportion of people wanting to see Avengers: Endgame who have seen Infinity War.

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- $P(H)$ ... prior
  $P(D)$ ... proportion of people who have seen Infinty War.
  $P(D|H)$ ... proportion of people wanting to see Avengers: Endgame who have seen Infinity War.
- Assume $P(D) = 0.6$ and $P(D|H) = 0.8$. What is $P(H|D)$?

Posteriors: Example

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)} =$$

$$= \frac{0.8 \times 0.6}{0.6} =$$

$$= 0.8$$

- Our posterior is therefore 0.8.
- We have therefore updated our belief that Rob's date wants to see Avengers: Endgame from a probability of 0.6 to a probability of 0.8.

Posteriors: Example

- OK, so our posterior is 0.8. But what if we collect additional new data?
- For instance: Rob's date has seen every single film starring Robert Downey Jr.
- This is where the magic happens: We can go through the updating process again, this time using the posterior we found previously as the new prior!

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$$= \frac{\frac{P(D|H_1)P(H_1)}{P(D)}}{\frac{P(D|H_2)P(H_2)}{P(D)}} =$$

$$= \frac{P(D|H_1)\, P(H_1)}{P(D|H_2)\, P(H_2)}$$

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We call $\frac{P(D|H_1)}{P(D|H_2)}$ the "Bayes Factor".
(How much more likely are we to see this data if H_1 is true than if H_2 is true?)

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Therefore:
Ratio of posteriors = Bayes factor × Ratio of Priors

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$$\frac{P(H_1|D)}{P(H_2|D)} = 25\frac{0.2}{0.8} =$$

$$= 6.25$$

Some things you may worry about ...

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Some things you may worry about . . .

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  That's OK. The priors are just a starting point. The whole point is that we are updating our beliefs when we get new information.

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- How do I get $P(D|H)$?
  Often this can be done using simulation. You have done this before!
- How do I know $P(D)$?
  If you are comparing two hypotheses, the nice thing is you don't need to know it!

Outline

Are you a Bayesian?

- Up to now in ADS2, we have talked about frequentist statistics, e.g. t-tests, ANOVA, linear regression.
  - Parameters are fixed but unknown, e.g. true average population height, relationship between height and weight.
  - Our major aim was to test and/or falsify various hypotheses and make a decision.

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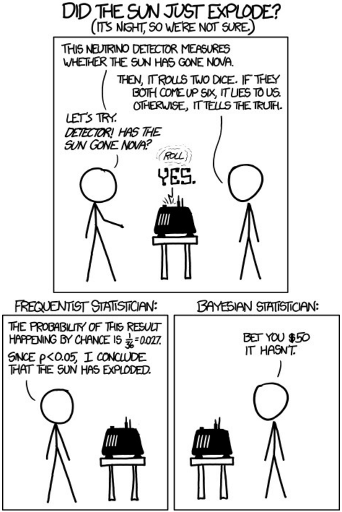
    - Parameters are fixed but unknown, e.g. true average population height, relationship between height and weight.
    - Our major aim was to test and/or falsify various hypotheses and make a decision.
- Bayesian statistics is quite different.
    - Include probabilistic methods and reasoning to the parameters: want to estimate the chance something is true *and* the extent of our belief in the estimate.
    - Can be updated with new knowledge, just like scientists operate normally.
    - Truly explicit/scrutable - did you notice the lack of assumptions in today's lecture?

Are you a Bayesian?

Are you a Frequentist?


DID THE SUN JUST EXPLODE?
(IT'S NIGHT, SO WE'RE NOT SURE.)
THIS NEUTRINO DETECTOR MEASURES WHETHER THE SUN HAS GONE NOVA.
THEN, IT ROLLS TWO DICE. IF THEY BOTH COME UP SIX, IT LIES TO US. OTHERWISE, IT TELLS THE TRUTH.
LET'S TRY. DETECTOR! HAS THE SUN GONE NOVA?
ROLL
YES.
FREQUENTIST STATISTICIAN:
THE PROBABILITY OF THIS RESULT HAPPENING BY CHANCE IS $\frac{1}{36}$=0.027. SINCE p<0.05, I CONCLUDE THAT THE SUN HAS EXPLODED.
BAYESIAN STATISTICIAN:
BET YOU $50 IT HASN'T.

A lot of frequentists may secretly be Bayesians

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Examples of how non-significant results have been described in the literature (for a full list, see: https://mchankins.wordpress.com/2013/04/21/still-not-significant-2/)

barely outside the range of significance ($p = 0.06$)
below (but verging on) the statistical significant level ($p > 0.05$)
bordered on being significant ($p > 0.07$)
borderline level of statistical significance ($p = 0.053$)
close to a marginally significant level ($p = 0.06$)
close to being significant ($p = 0.06$)
close to being statistically significant ($p = 0.055$)
close to borderline significance ($p = 0.072$)

near-to-significance ($p = 0.093$)
non-significant in the statistical sense ($p > 0.05$)
not absolutely significant but very probably so ($p > 0.05$)
not clearly significant ($p = 0.08$)
not completely significant ($p = 0.07$)
not conventionally significant ($p = 0.089$), but ...
not especially significant ($p > 0.05$)
not exactly significant ($p = 0.052$)
not formally significant ($p = 0.06$)
not insignificant ($p = 0.056$)
not markedly significant ($p = 0.06$)

Now, you should be able to do the following:

- Explain what “belief” means to a Bayesian statistician
- Define the terms “prior” and “posterior”
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- Describe the difference between Bayesian and Frequentist statistics

Acknowledgments and Image Credits

This lecture uses materials from an ADS2 lecture from previous years by Rob Young (who is currently at the cinema watching Avengers). Where not otherwise indicated, images are also from that lecture.

Image credits:

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