Thesis on Stochastic Integral Course

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May 23, 2023

Abstract

This paper provides definitions for the solutions and uniqueness of stochastic differential equations, and discusses the existence and uniqueness of local solutions under certain given conditions for the stochastic differential equation

$$\begin{aligned} dX_t &= f(X_t)dt + \int_{-1}^{0} g(r)X_{t+r}drdt + \sigma(t, X_t)dB_t \\ X_t &= k(t), t \in [-1, 0] \end{aligned} \tag{1}$$

as well as the existence of global solutions.

Definition 0.1 (Definition of Global Solution). *Given a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_*, P)$ satisfying the usual conditions, we define X_t as a solution to equation (1) if it satisfies:*
(1) $\{X_t : t \in [0, T]\}$ is continuous P-a.s.
(2) $\{X_t : t \in [0, T]\}$ is $\mathcal{F}_*$-adapted.
(3) $\int_0^T (|f(X_s)| + \int_{-1}^0 |g(r)||X_{r+s}|dr)ds + \int_0^T |\sigma(s, X_s)|^2 ds < \infty$ P-a.s.
(4) $dX_t = f(X_t)dt + \int_{-1}^0 g(r)X_{t+r}drdt + \sigma(t, X_t)dB_t$

Definition 0.2 (Definition of Uniqueness of Global Solution). *We say the solution is unique if X_t and Y_t are two solutions such that for any $t \in [0, T]$, we have $P(X_t = Y_t) = 1$.*

Definition 0.3 (Definition of Local Solution). *Given a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_*, P)$ satisfying the usual conditions, we define (X_t, τ) as a local solution to equation (1) if it satisfies:*
(1) τ is an $\mathcal{F}_*$-stopping time.
(2) $X^\tau := X(\bullet \wedge \tau) \in C([0, \tau), \mathbb{R})$ P-a.s.
(3) X is $\mathcal{F}_*$-adapted.
(4) $\forall t \geq 0$ P-a.s., $X(t \wedge \tau)$ satisfies equation (1).

Definition 0.4 (Definition of Maximal Local Solution). *(X, τ) is called a maximal local solution if:*
(1) There exists a sequence of stopping times τ_n such that $\tau_n \nearrow \tau$.
(2) (X, τ_n) is a local solution.
(3) On $\{\tau < \infty\}$, P-a.s., $\lim_{t \nearrow \tau} \sup_{0 \leq s \leq t} |X(t, \omega)| = \infty$.

Since f is polynomial and locally Lipschitz, we first prove the existence and uniqueness of global solutions under the conditions that f is globally Lipschitz, $g \in L^1([-1, 0], \mathbb{R})$, $j(t)$ is bounded, $\sigma(t, x)$ has linear growth, and $k \in C([-1, 0])$.

Theorem 1. *When f, g, j, k, σ satisfy the above conditions, equation (1) has a unique global solution.*

Proof. First, we prove uniqueness: Let X_t and Y_t be two solutions. By Itô's formula, we have

$$\begin{aligned} (X_t - Y_t)^2 = 2 &\int_0^t (X_s - Y_s)(f(X_s) - f(Y_s))ds \\ &+ 2 \int_0^t (X_s - Y_s) \int_{-1}^0 g(r)(X_{s+r} - Y_{s+r})drds \\ &+ 2 \int_0^t (X_s - Y_s)(\sigma(s, X_s) - \sigma(s, Y_s))dB_s \\ &+ \int_0^t (\sigma(s, X_s) - \sigma(s, Y_s))^2 ds \end{aligned} \tag{2}$$

Let n be sufficiently large (greater than $k(0)$), and let the stopping time $\tau_n = \inf\{t \in [0,T] : |X_t| + |Y_t| \geq n\}$. Replacing t with $t \wedge \tau_n$ still holds. For convenience, let $X_t^n = X(t \wedge \tau_n)$. Next, estimate $\mathbb{E}[\sup_{0\leq s\leq t} |X_s^n - Y_s^n|^2]$. Denote the above integrals as $I_1(s), I_2(s), I_3(s), I_4(s)$, taking the supremum on both sides, we get

$$\mathbb{E} \sup_{0\leq s\leq t} |X_s^n - Y_s^n|^2 \leq \sum_{k=1}^{4} \mathbb{E} \sup_{0\leq s\leq t} |I_k(s \wedge \tau_n)|. \tag{3}$$

Estimate I_k respectively:

$$\sup_{0\leq s\leq t} |I_1(s \wedge \tau_n)| \leq 2K \int_0^t \sup_{0\leq l\leq s} |X_l^n - Y_l^n|^2 ds. \tag{4}$$

Note that $\sup_{-1\leq s\leq t} |X_s^n - Y_s^n| = \sup_{0\leq s\leq t} |X_s^n - Y_s^n|$ because X and Y are equal on $[-1, 0]$ and the stopping time is greater than 0, thus

$$\sup_{0\leq s\leq t} |I_2(s \wedge \tau_n)| \leq ||g||_{L^1} \int_0^t \sup_{0\leq l\leq s} |X_l^n - Y_l^n|^2 ds. \tag{5}$$

Using the BDG inequality and the mean value inequality, we obtain

$$\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq t} |I_3| &\leq \mathbb{E}\left(\int_0^t |j(s)|^2 |X_s^n - Y_s^n|^4 ds\right)^{1/2} \\
&\leq ||j||_{L^\infty} \mathbb{E}[\sup_{0\leq s\leq t} |X_s^n - Y_s^n| (\int_0^t \sup_{0\leq l\leq s} |X_l^n - Y_l^n|^2 ds)^{1/2}] \\
&\leq \frac{1}{2}\mathbb{E}[\sup_{0\leq s\leq t} |X_s^n - Y_s^n|^2] + C_j \int_0^t \mathbb{E}[\sup_{0\leq l\leq s} |X_l^n - Y_l^n|^2] ds.
\end{aligned} \tag{6}$$

The handling of I_4 is similar to that of I_1:

$$\sup_{0\leq s\leq t} |I_4| \leq ||j||_{L^\infty}^2 \int_0^t \sup_{0\leq l\leq s} |X_l^n - Y_l^n|^2 ds \tag{7}$$

Thus, from equations (3) to (7), we have

$$\mathbb{E} \sup_{0\leq s\leq t} |X_s^n - Y_s^n|^2 \leq 2(2K + ||g||_{L^1} + C_j + ||j||_{L^\infty}^2) \int_0^t \mathbb{E}[\sup_{0\leq l\leq s} |X_l^n - Y_l^n|^2] ds \tag{8}$$

By Gronwall's inequality, we obtain

$$\mathbb{E} \sup_{0\leq s\leq t} |X_s^n - Y_s^n|^2 \leq 0$$

Letting n approach ∞, and applying Fatou's lemma, uniqueness is proven.

Next, we prove existence using the Picard iteration method:
Let

$$X^0(t) = \begin{cases} k(0) & t \geq 0 \\ k(t) & -1 \leq t \leq 0. \end{cases} \tag{9}$$

When $t \geq 0$,

$$X_t^{n+1} = k(0) + \int_0^t f(X_s^n) ds + \int_0^t \int_{-1}^0 g(r) X_{r+s}^n dr ds + \int_0^t \sigma(s, X_s) dB_s$$

When $t \leq 0$, $X_t^n = k(t)$.

First, we prove that $E[\sup_{0\leq t\leq T} |X_t^n|^2] < \infty$.

By the Cauchy inequality, we have

$$|X_t^{n+1}|^2 \leq 4\left(|k(0)|^2 + \left(\int_0^t f(X_s^n) ds\right)^2 + \left(\int_0^t \int_{-1}^0 g(r) X_{r+s}^n dr ds\right)^2 + \left(\int_0^t \sigma(s, X_s) dB_s\right)^2\right) \tag{10}$$

Given that f is globally Lipschitz, σ has linear growth, and $g \in L^1$, using similar methods as discussed in class, we have

$$\begin{aligned}
E[\sup_{0\leq t\leq T} |X_t^{n+1}|^2] \leq 4k(0)^2 &+ C_T E[\sup_{0\leq s\leq T} |f(X_s^n)|^2] + ||g||_{L^1}^2 C_T E[\sup_{-1\leq t\leq T} |X_t^n|^2] \\
&+ T E[\sup_{0\leq t\leq T} |\sigma(s, X_s)|^2] < \infty
\end{aligned} \tag{11}$$

The treatment of the martingale part uses the BDG inequality, and we have $\sup_{t\in[-1,T]}|X_t^n|^2 \leq ||k||_\infty^2 + \sup_{t\in[0,T]}|X_t^n|^2$. The last inequality utilizes the linear growth of f and σ along with induction.

Next, we estimate the rate of convergence. First, by the Cauchy inequality,

$$\begin{aligned}E[\sup_{0\leq t\leq T}|X_t^{n+1}-X_t^n|^2] \leq 3\Bigg(&E[\sup_{0\leq t\leq T}\left|\int_0^t f(X_s^n)-f(X_s^{n-1})ds\right|^2] \\ &+E[\sup_{0\leq t\leq T}\left|\int_0^t\int_{-1}^0 g(r)(X_{r+s}^n-X_{r+s}^{n-1})drds\right|^2] \\ &+E[\sup_{0\leq t\leq T}\left|\int_0^t \sigma(s,X_s^n)-\sigma(s,X_s^{n-1})dB_s\right|^2]\Bigg)\end{aligned} \tag{12}$$

Let us denote each of the integrals as I_1, I_2, I_3 and estimate them respectively.
For I_1 and I_3, using similar methods from class, with f and σ being globally Lipschitz and the BDG inequality, we have

$$\begin{aligned}I_1 &\leq K^2T\int_0^T E[\sup_{0\leq s\leq t}|X_s^n-X_s^{n-1}|^2]dt, \\ I_3 &\leq C_2||j||_{L^\infty}\int_0^T E[\sup_{0\leq s\leq t}|X_s^n-X_s^{n-1}|^2]dt.\end{aligned} \tag{13}$$

It is easy to see that the upper limit T in the above integrals can be replaced by any number u that is less than T, which facilitates the use of induction.
Next, we estimate I_2:

$$\begin{aligned}E\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_{-1}^0 g(r)(X_{r+s}^n-X_{r+s}^{n-1})drds\right|^2\right] &\leq TE\left[\sup_{[0,T]}\int_0^t\left(\int_{-1}^0|g(r)||X_{r+s}^n-X_{r+s}^{n-1}|dr\right)^2 ds\right] \\ &\leq TE\left[\sup_{[0,T]}\int_0^t ||g||_{L^1}^2\sup_{[-1,s]}|X_u^n-X_u^{n-1}|^2 ds\right] \\ &\leq T||g||_{L^1}^2 E\left[\int_0^T\sup_{[0,s]}|X_u^n-X_u^{n-1}|^2 ds\right].\end{aligned} \tag{14}$$

Combining equations (8) to (10), we have

$$E[\sup_{0\leq s\leq t}|X_s^{n+1}-X_s^n|^2] \leq 3(K^2T+C_2||j||_{L^\infty}+T||g||_{L^1}^2)\int_0^t E[\sup_{0\leq u\leq s}|X_u^n-X_u^{n-1}|^2 ds]. \tag{15}$$

Thus, by induction, we can conclude that

$$E[\sup_{[0,T]}|X_s^{n+1}-X_s^n|^2] \leq \frac{C(TD_T)^n}{(n!)}.$$

Therefore, X^n indeed converges to some integrable adapted process X, which is the global solution we are looking for. □

Now we have proven the existence of global solutions when f is globally Lipschitz. By using the truncation method, we can modify f to be globally Lipschitz, and when $|x|\leq N$, $f^N = f$. Through similar discussions in class, we know that local solutions exist and are unique. Next, we prove the existence of global solutions.

For convenience, we introduce the following definition:

Definition 1.1. *We say that the equation* (1) *satisfies the monotonicity condition if there exists a constant C such that for all $x\in\mathbb{R}$,*

$$xf(x)+\frac{1}{2}\sigma^2(t,x)\leq C(1+|x|^2).$$

It is easy to see that because the polynomial $f(x) = -x^{2n-1}+a_1x^{2n-2}+\ldots+a_0$ is controlled by the highest degree term when x is sufficiently large, $xf(x)$ will be negative for sufficiently large x. When x is small, there clearly exists C in a bounded disk, and since σ satisfies linear growth, the equation indeed satisfies the monotonicity condition.

Theorem 2. *The equation* (1) *has a unique global solution when $g \in L^1([-1,0],\mathbb{R})$, $j(t)$ is bounded, $\sigma(t,x)$ has linear growth, and $k \in C([-1,0])$.*

Proof. Previously, we proved the existence of local solutions, and thus the existence of maximal local solutions. Let the maximal local solution be (X,τ), and the approximating sequence be (X,τ_n), where $\tau_n \nearrow \tau$. For convenience, let $X_t^n := X(t\wedge\tau_n)$.
By Itô's formula, we have

$$\begin{aligned}(X_t^n)^2 = k(0)^2 + 2\int_0^t X_s^n f(X_s^n) I_{[0,\tau_n]} ds + 2\int_0^{t\wedge\tau_n} X_s^n \sigma(s,B_S) dB_S \\ + 2\int_0^{t\wedge\tau_n} X_s^n \left(\int_{-1}^0 g(r) X_{r+s}^n dr\right) ds + \int_0^{t\wedge\tau_n} \sigma^2(s,X_s^n) ds.\end{aligned} \tag{16}$$

Taking the supremum on both sides, and using the monotonicity condition for the second and last terms, we have:

$$\begin{aligned}\sup_{0\le s\le t} |X_s^n|^2 \le k(0)^2 + 2C\int_0^t \left(1+\sup_{0\le l\le s}|X_u^n|^2\right) ds \\ + 2||g||_{L^1}\int_0^t \sup_{-1\le u\le s}|X_u^n|^2 ds + \sup_{0\le s\le t}\int_0^{s\wedge\tau_n} X_u^n\sigma(u,X_u^n) dB_u.\end{aligned} \tag{17}$$

For the second-to-last term, we estimate $\sup_{-1\le u\le s}|X_s^n|^2 \le \sup_{-1\le t\le 0}|k(t)|^2 + \sup_{0\le u\le s}|X_u^n|^2$. For the last term, when taking the expectation, we use the BDG inequality:

$$\begin{aligned}E\left[\sup_{0\le s\le t}\int_0^{s\wedge\tau_n} X_u^n\sigma(u,X_u^n)dB_u\right] &\le E\left[\left(\int_0^t |X_s^n|^2|\sigma(s,X_s^n)|^2|I_{[0,\tau_n]}|^2 ds\right)^{\frac{1}{2}}\right] \\ &\le E\left[\sup_{0\le s\le t}|X_s^n|\left(\int_0^t |\sigma(s,X_s)|^2 ds\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{2}E\left[\sup_{0\le s\le t}|X_s^n|^2\right] + C_T\int_0^t E\left[1+\sup_{0\le u\le s}|X_u^n|^2\right] ds.\end{aligned} \tag{18}$$

Combining the above estimates, we get:

$$\begin{aligned}\frac{1}{2}E\left[1+\sup_{0\le s\le t}|X_s^n|^2\right] \le \frac{1}{2} + k(0)^2 + 2||g||_{L^1}||\sup_{[-1,0]}|k||_{L^2} \\ + (2C + C_T + 2||g||_{L^1})\int_0^t E\left[1+\sup_{0\le u\le s}|X_u^n|^2\right] ds.\end{aligned} \tag{19}$$

Thus, by applying Gronwall's inequality, we have the estimate:

$$E\left[\sup_{0\le t\le T}|X_t^n|^2\right] \le E\left[1+\sup_{0\le t\le T}|X_t^n|^2\right] \le Ce^{C_T T}. \tag{20}$$

By Fatou's lemma, letting $n\to\infty$, we have

$$E[|X(t\wedge\tau)|^2] < \infty.$$

Thus, the global solution exists and is unique. □

At this point, we have completed the proof.