1 Dynamical systems

This course is about the **qualitative** approach to studying dynamical systems.

- 'Qualitative' means we are interested in finding geometrical descriptions of a system's behaviour, in describing its general features rather than solving it exactly.

- Very few systems can be solved exactly. Even computational solutions can be difficult to interpret, and sometimes misleading. A qualitative understanding of a system is vital before rushing into computer simulations. And it can tell you everything you need to understand without solving its equations at all.

A dynamical system is any system of equations that tells you how something evolves in time.

- An equation $\dot{x} = f(x)$ defines an *ordinary differential equation* (ODE) for the dependent variable x, in terms of the independent variable t, where $\dot{x}$ is shorthand for $\frac{dx}{dt}$.

- An equation $x_{n+1} = f(x_n)$ defines a *difference equation* or *map* in the discrete variable x_n.

- Both of these are types of dynamical system. Their solutions are functions $x(t)$ (for the ODE) or sequences $x_0, x_1, x_2, ...$ (for the map) that tell us how a system behaves over time, from an initial condition $x(0)$ or x_0.

- We can have sets of such equations defining an n-dimensional system, e.g. a set of ODEs in continuous variables $x, y, ...$

  $$(\dot{x}, \dot{y}, ...) = \{f(x, y, ...), g(x, y, ...), ...\}$$

  or a set of equations in discrete variables $x_n, y_n, ...$

  $$(x_{n+1}, y_{n+1}, ...) = \{f(x_n, y_n, ...), g(x_n, y_n, ...), ...\}$$

- There are other kinds of dynamical systems such as partial differential equations, cellular automata, delay differential equations, integral equations, renewal equations, stochastic differential equations, hybrid systems, piecewise smooth dynamical systems, . . . A lot of what we will study here provide ideas that can be extended to these.

2 An example of population growth

Take a population that has N individuals at time t, evolving as

$$\dot{N} \;=\; B \;-\; D \;+\; M \tag{2.1}$$

where B = births, D = deaths, M = migrations, per unit time.

A simple model is to say these changes are proportional to the number of individuals, so define a birth rate β such that $B = \beta N$, and death rate δ such that $D = \delta N$. These are defined per individual per unit time, with $\beta > 0$ and $\delta > 0$. For now say $M = 0$ (no migration, i.e. closed borders).

Then we have

$$\dot{N} \;=\; (\beta - \delta)N\ . \tag{2.2}$$

- This is easy to solve, e.g. by separation of variables

  $$\frac{dN}{N} = (\beta - \delta)dt \quad \Rightarrow \quad \ln \tfrac{N(t)}{N_0} = (\beta - \delta)t \quad \Rightarrow \quad N(t) = N_0 e^{(\beta-\delta)t} \tag{2.3}$$

  with initial condition $N(0) = N_0$.

- If $N_0 = 0$ then $N(t) = 0$ for all times t. We call this an *equilibrium*, i.e. a state where $\frac{dN}{dt} = 0$ so the system feels no impulse to change.

- If $\beta > \delta$ the population $N(t)$ grows exponentially away from $N = 0$, without bound. This makes sense as births outweigh deaths. We say the equilibrium $N = 0$ is *unstable* or *repelling*.

- If $\beta < \delta$ the population $N(t)$ shrinks *asymptotically* (in infinite time) towards $N = 0$. This makes sense as deaths overwhelm births. We say the equilibrium $N = 0$ is *stable* or *attracting*.

- We could re-define the crucial parameter as $\alpha = \beta - \delta$. Then there is only one parameter in the system, α = the difference between the birth and death rates. The actual values of β and δ don't matter, only their difference α (so $\beta = 2$, $\delta = 1$, behaves the same as $\beta = 10$, $\delta = 9$, as in both cases $\alpha = 1$).

A more realistic model is to say that the death rate increases if the population gets too large (typical in a contained environment), so it becomes $\delta = \gamma N$ for a constant γ, then

$$\dot{N} = (\beta - \gamma N)N \tag{2.4}$$

with $\beta, \gamma > 0$.

In this case to minimize the number of parameters we can re-scale (re-scaling is like non-dimensionalization):

- Let $N = x/\gamma$ for a scaled population x, then

  $$\dot{x} = (\beta - x)x \tag{2.5}$$

  which has only one parameter. Now the constant γ just acts like a scale for measuring the population, a system of units for N if you like.

- You can see immediately that this now has two *equilibria*, as there are two solutions to $\dot{x} = 0$. There is still an equilibrium at $x = 0$, and now a new one at $x = \beta$.

- We can still solve the nonlinear system (2.5) by separation of variables

  $$\begin{aligned} dt = \tfrac{dx}{(\beta - x)x} = \left(\tfrac{1}{x} + \tfrac{1}{\beta - x}\right)\tfrac{dx}{\beta} \quad &\Rightarrow \quad t = \tfrac{1}{\beta}\ln\tfrac{(\beta - x_0)x(t)}{(\beta - x(t))x_0} \\ &\Rightarrow \quad x(t) = \frac{\beta x_0 e^{\beta t}}{\beta - x_0 + x_0 e^{\beta t}} \end{aligned} \tag{2.6}$$

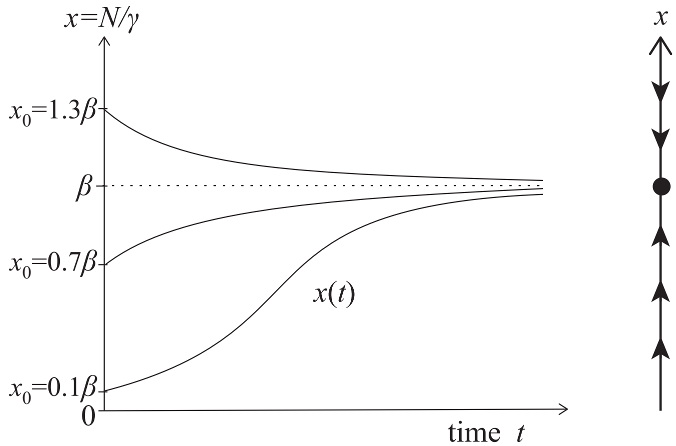
- The long time behaviour is now bounded as

  $$x(t) \to \frac{\beta x_0 e^{\beta t}}{x_0 e^{\beta t}} \to \beta \qquad \text{as} \quad t \to \infty\,. \tag{2.7}$$

- The nonlinear term stops the population exploding to infinity and instead cuts it off at $x = \beta$ (or $N = \beta/\gamma$). So
  - the equilibrium $x = 0$ is unstable,
  - the equilibrium $x = \beta$ is stable.

- Note this all assumed $\beta, \gamma > 0$. What would happen for $\beta < 0$ or $\gamma < 0$?

- Here's what these solutions look like, graphing $x(t)$ for different x_0 values...



  Note that changing the parameters β and γ would just change the scale on the vertical axis.

- On the right we've done away with the time axis, and just represented the flow of time by arrows on the x-axis. This is called the **phase portrait** of the system. It will be much more useful than the graph when we study systems with multiple variables $x, y, z, \ldots$.

These are about the last systems we'll be able to solve exactly . . .

- From hereon we'll need something smarter — more qualitative — to study how things behave.
- We'll keep using the population model to illustrate more general and powerful ways to find the behaviour of systems, especially when we cannot solve them like we did above.

A small but extremely important thing we did above was to reduce the number of constants, which works like *non-dimensionalization*.

[Side Notes:] **Rescaling / Non-dimensionalization**

Given a system $\dot{X} = F(X; a, b, ...)$ in terms of a variable x and parameters $a, b, ...$, try to define new scaled quantities to reduce the number of parameters in the equation.

- You can scale any of the variables and/or parameters, say $x = AX$ and let some $\alpha, \beta, ...$ be new combinations of the old parameters $a, b, ...$, to give some $\dot{x} = f(x; \alpha, \beta, ...)$
- The object is for the number of parameters $\alpha, \beta, ...$ to be less than the number of the original parameters $a, b, ...$
- This can be an incredibly powerful tool, and is *vitally important* to do in mathematical modeling. The behaviour in the population models above only really depended on one parameter. The other just behaved like a scaling or 'set of units' for the system.
- Sometimes we start off with a physical or biological problem with many rate constants, material coefficients, and so on, which can be reduced to just one or two parameters that define the system's behaviour.
- When you've studied the system in the scaled quantities, *after* you've understood its behaviour, then you just work out what that all looks like back in the original unscaled variables and parameters.

3 ODEs and their flows

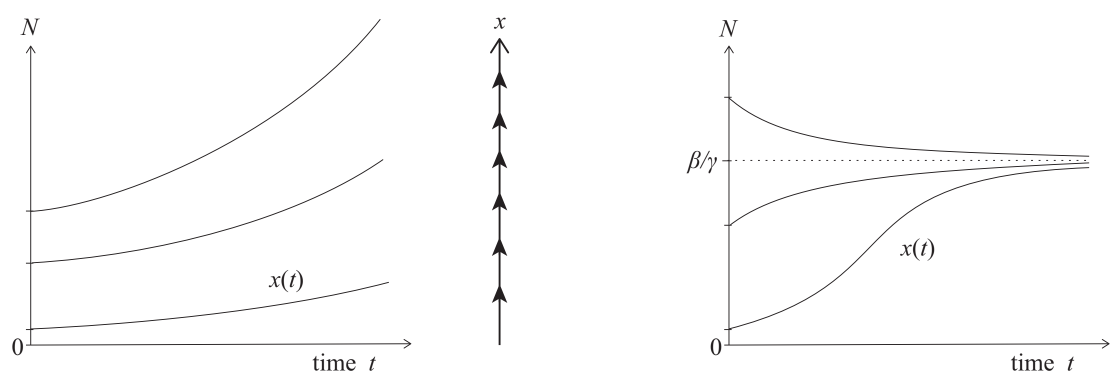
A lot of what we do will be **qualitative** dynamics, forming a conceptual sketch of a system.

That means not having an exact solution like we've plotted in the graphs above, but forming a picture of the geometry of solutions using things like equilibria, as in the phase portraits above.

This can actually be *more* powerful than having exact solutions.

Take the population models:

- Look at the arrows I've drawn on the graphs from the population models, showing the direction of travel according to the ODE, i.e. the vector $\dot{N}$ or $\dot{x}$. In higher dimensions we can't draw the graph (left of each picture below), but we can still understand things in terms of these vectors, drawn just in the space of x (right of each picture below), which are the **phase portraits** of the system.



- The phase portraits are often not only easier to sketch, and actually more useful, than the graphs of solutions. Still, as we get to more than two dimensions even these will be difficult to sketch, but fortunately we'll learn concepts to understand them that will work just as well in higher dimensions.

[Side Notes:] Flows, orbits and phase portraits

Consider an ODE

$$\dot{x} = f(x) \tag{3.1}$$

for $x \in D$ and $f \in R$, from the **domain** $D \subset \mathbb{R}^n$ to the **range** $R \subset \mathbb{R}^n$, that is $f : D \mapsto R$. If we are given an initial condition $x(0) = x_0$ we call this an **initial value problem**.

- The solution $x(t)$ to (3.1) traces out a **trajectory** through D as t changes.
- Often the solution we get depends crucially on the initial condition x_0, so it can help to write a solution of the initial value problem as

$$x(t) = \Phi_t(x_0) \quad s.t. \quad \frac{d}{dt}\Phi_t(x_0) = f\left(\Phi_t(x_0)\right) \quad \& \quad \Phi_0(x_0) = x_0 \; . \tag{3.2}$$

- The function $\Phi_t(x_0)$ is called the **flow operator** of the ODE.
- A complete trajectory $\{\Phi_t(x_0) \; : \; t \in [0, T]\}$ is called an **orbit** of the ODE through the point x_0.
- The collection of all orbits is called the **flow field** (or simply the **flow**).
- Its depiction in the **state space** or **phase space** of x is called the **phase portrait**.
- The system (3.1) is **autonomous** (time-independent).
  If instead time appears on the righthand side, say $\dot{x} = f(x, t)$, then the system is **non-autonomous**.
- An **equilibrium** is a point x_* where the system is stationary, i.e. where

$$f(x_*) = 0 \; . \tag{3.3}$$

All this is just the same if instead of $x \in \mathbb{R}$, we have a multivariable system with a vector $\mathbf{x} = (x, y, z, ...) \in \mathbb{R}^n$. Let's start with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ . . .

To use these methods we have to work with **first order differential equations**.

- Typically we can turn a one-dimensional n^{th} order ODE, into an n-dimensional first order ODE, just by associating each derivative with a spatial coordinate, so . . .
- the ODE $\dot{x} = f(x)$ is a first order ODE.
- the second order ODE $\ddot{x} + b(x)\dot{x} + a(x) = 0$ becomes a first order ODE by letting $y = \dot{x}$, giving

$$\dot{x} = y \ , \qquad \dot{y} = -b(x)y - a(x) \ .$$

- the third order ODE $\dddot{x} + c(x)\ddot{x} + b(x)\dot{x} + a(x) = 0$ becomes a first order ODE by letting $y = \dot{x}$ and $z = \ddot{x}$, giving

$$\dot{x} = y \ , \qquad \dot{y} = z \ , \qquad \dot{z} = -c(x)z - b(x)y - a(x) \ .$$

- and so on. These are quite easy to understand, as y is then the speed, z the acceleration, etc. and in a first order system we include these to form the system's **state space**.
- Particularly with high order (large dimensional) systems, we sometimes prefer indexed variables, so for the last example $\dddot{x} + c(x)\ddot{x} + b(x)\dot{x} + a(x) = 0$ we might instead let $x_1 = x$, $x_2 = \dot{x}$, and $x_3 = \ddot{x}$, giving

$$\dot{x}_1 = x_2 \ , \qquad \dot{x}_2 = x_3 \ , \qquad \dot{x}_3 = -c(x_1)x_3 - b(x_1)x_2 - a(x_1) \ .$$

  This **index form** (called by some the **state space** form) is particularly useful for computer simulations.
- The **state space** is the space occupied by the variables $(x, y, z, ...)$ or $(x_1, x_2, x_3, ...)$ of the n-dimensional first order ODE, typically $\mathbb{R}^n$ or some subset of it (e.g. the population model's state space is $\mathbb{R}_+$ (the positive part of $\mathbb{R}$), the predator-prey model's state space is $\mathbb{R}^2_+$).