Assignment 6

EMATM0061: Statistical Computing and Empirical Methods, TB1, 2024

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Introduction

This is the sixth assignment for Statistical Computing and Empirical Methods. This assignment is mainly based on Lectures 15, 16, and 17 (see the Blackboard).

You can *optionally* submit this assignment by 13:00 Monday 4th November. This will help us understand your work but will *not* count towards your final grade. The submission point can be found under the assignment tab at Blackboards (click the title "Assignment 06" and upload a pdf file).

Load packages

Some of the questions in this assignment require the tidyverse package. If it hasn't been installed on your computer, please use "install.packages()" to install them first.

To load the tidyverse package:

```
library(tidyverse)
```

1. The Gaussian distribution

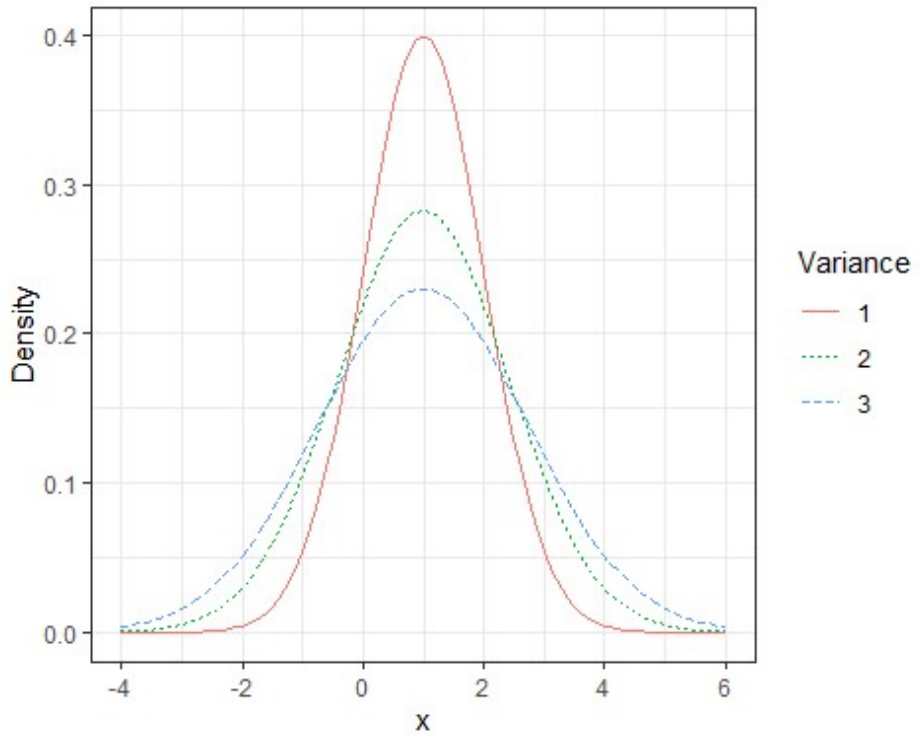
Gaussian random variables are an important family of continuous random variables. In this assignment, we will first explore several properties of the Gaussian distribution.

Use the help function to look up the following four functions: "dnorm()", "pnorm()", "qnorm()" and "rnorm()".

Also, the probability density function of a Gaussian random variable was introduced in Lecture 14.

(Q1) Generate a plot which displays the **probability density function** for three Gaussian random variables $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and $X_3 \sim \mathcal{N}(\mu_3, \sigma_3^2)$ with $\mu_1 = \mu_2 = \mu_3 = 1$ and $\sigma_1^2 = 1, \sigma_2^2 = 2, \sigma_3^2 = 3$.

Your plot should look like this:



(Q2) Generate a plot which displays the **cumulative distribution function** for three Gaussian random variables $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and $X_3 \sim \mathcal{N}(\mu_3, \sigma_3^2)$ with $\mu_1 = \mu_2 = \mu_3 = 1$ and $\sigma_1^2 = 1, \sigma_2^2 = 2, \sigma_3^2 = 3$.

(Q3) Generate a plot for the **quantile function** for the same three Gaussian distributions as above. Describe the relationship between the quantile function and the cumulative distribution function.

(Q4)

Now use "rnorm()" to generate a random independent and identically distributed sequence $Z_1, \cdots, Z_n \sim \mathcal{N}(0,1)$ so that each $Z_i \sim \mathcal{N}(0,1)$ has standard Gaussian distribution. Set $n = 100$. Make sure your code is reproducible by using the "set.seed()" function. Store your random sample in a vector called ""standardGaussianSample"".

(Q5)

Suppose $Z \sim \mathcal{N}(0,1)$ is a Gaussian random variable. Take $\alpha, \beta \in \mathbb{R}$ and let $W: \Omega \to \mathbb{R}$ be the random variable given by $W = \alpha Z + \beta$. Then W is also a Gaussian random variable. We will use this fact to create samples of Gaussian random variables from samples of standard Gaussian random variables.

为了满足 $Y \sim \mathcal{N}(1,3)$，我们可以使用以下公式：

- 目标分布的均值为 1，因此 $\beta = 1$。
- 目标分布的方差为 3，因此 $\alpha = \sqrt{3}$。

1. **均值计算**：
   - Y 的期望（均值）为：$E(Y) = E(\alpha Z + \beta) = \alpha \cdot E(Z) + \beta$。
   - 因为 $E(Z) = 0$，所以 $E(Y) = \alpha \cdot 0 + \beta = \beta$。
2. **目标均值为 1**：
   - 题目要求 Y 的均值为 1，因此我们需要让 $\beta = 1$ 来满足 $E(Y) = 1$。
1. **方差计算**：
   - Y 的方差 $\mathrm{Var}(Y)$ 可以表示为：
     $$\mathrm{Var}(Y) = \mathrm{Var}(\alpha Z + \beta)$$
   - 根据方差的性质，常数项 β 不影响方差，所以：
     $$\mathrm{Var}(Y) = \mathrm{Var}(\alpha Z) = \alpha^2 \mathrm{Var}(Z)$$
2. **已知 Z 的方差**：
   - 假设 $Z \sim \mathcal{N}(0,1)$，所以 $\mathrm{Var}(Z) = 1$。
3. **确定 α 的值**：
   - 题目要求 Y 的方差为 3，因此我们需要满足：
     $$\alpha^2 \cdot 1 = 3$$
   - 解得 $\alpha = \sqrt{3}$。

Use your existing sample stored in “standardGaussianSample” to generate a new sample of the form $Y_1, \cdots, Y_n \sim \mathcal{N}(1,3)$ with expectation $\mu = 1$ and population variance $\sigma^2 = 3$. The i-th observation in this sample should be of the form $Y_i = \alpha \cdot Z_i + \beta$, for appropriately chosen $\alpha, \beta \in \mathbb{R}$, where Z_i is the i-th observation in the sample “standardGaussianSample”. Store the generated sample of $Y_1, \cdots, Y_n$ in a vector called “mean1Var3GaussianSampleA”. So to answer this question you need to decide the value of α and β, such that Y_i has the required expectation and variance.

(Q6)

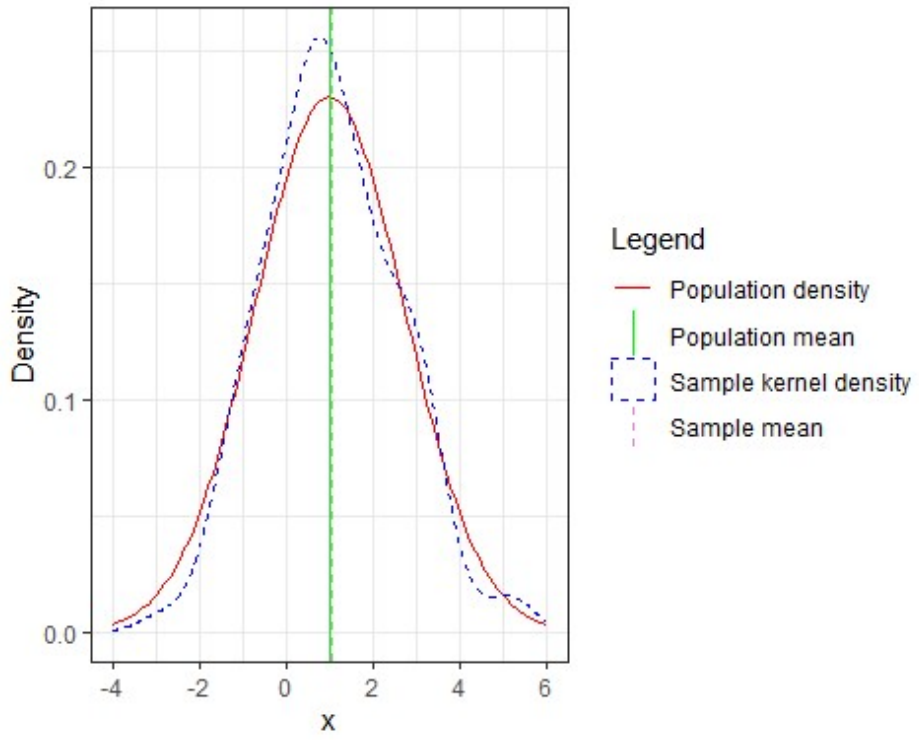
Reset the random seed to the same value as the one you used in (Q4) using the “set.seed()” function and generate an i.i.d. sample of the form $Y_1, \cdots, Y_n \sim \mathcal{N}(1,3)$ using the “rnorm()” function (instead of using “standardGaussianSample”). Store this sample in a vector called “mean1Var3GaussianSampleB”. Are the entries of the vectors “mean1Var3GaussianSampleA” and “mean1Var3GaussianSampleB” the same?

(Q7)

Now generate a graph which includes both a for your sample “mean1Var3GaussianSampleA” and a plot of the population density (the probability density function) generated using “dnorm()”. You can also include two vertical lines which display respectively the population mean and the sample mean.

Some guidance for creating the plot: It would be helpful to look at the example provided in Section 3.3(Q4) of Assignment 5. You may want to use the “geom_density()” and “geom_vline()” functions. In particular, both “geom_density()” and “geom_line()” have an argument called “data” that you may want to explore. Also, you can specify your own color by using the “scale_color_manual” and your own line type by “scale_linetype_manual”.

Your plot should look similar to the following:



(Q8) (*)

~~This is an optional question (*). If you are short on time you can work on the other questions first.~~

Recall that for a random variable $X: \Omega \to \mathbb{R}$ is said to be Gaussian with expectation μ and variance σ^2 (i.e., $X \sim \mathcal{N}(\mu, \sigma^2)$) if for any $a, b \in \mathbb{R}$, we have

$$\mathbb{P}(a \leq X \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2} dz.$$

Suppose $Z \sim \mathcal{N}(0,1)$ is a Gaussian random variable. Take $\alpha, \beta \in \mathbb{R}$ and let $W: \Omega \to \mathbb{R}$ be the random variable given by $W = \alpha Z + \beta$. In (Q5) we have assumed that W constructed in this way is a Gaussian random variable. Now, apply a change of variables to show that W is a Gaussian random variable with expectation β and variance α^2.

Hint: can you derive the expression of $\mathbb{P}(c \leq W \leq d)$?

2. Location estimators with Gaussian data

In this question we compare two estimators for the population mean μ_0 in a Gaussian setting in which we have independent and identically distributed data $X_1, \cdots, X_n \sim \mathcal{N}(\mu_0, \sigma_0^2)$.

The following code generates a data frame consisting of the mean squared error of the sample median as an estimator of μ_0.

```
set.seed(0)
num_trials_per_sample_size <- 1000
min_sample_size <- 30
max_sample_size <- 500
sample_size_inc <- 5
mu_0 <- 1
sigma_0 <- 3

# create data frame of all pairs of sample_size and trial
simulation_df<-crossing(trial=seq(num_trials_per_sample_size),
                        sample_size=seq(min_sample_size,

max_sample_size,sample_size_inc)) %>%
  # simulate sequences of Gaussian random variables
  mutate(simulation=pmap(.l=list(trial,sample_size),
                         .f=~rnorm(.y,mean=mu_0,sd=sigma_0))) %>%
  # compute the sample medians
  mutate(sample_md=map_dbl(.x=simulation,.f=median)) %>%
  group_by(sample_size) %>%
  summarise(msq_error_md=mean((sample_md-mu_0)^2))
```

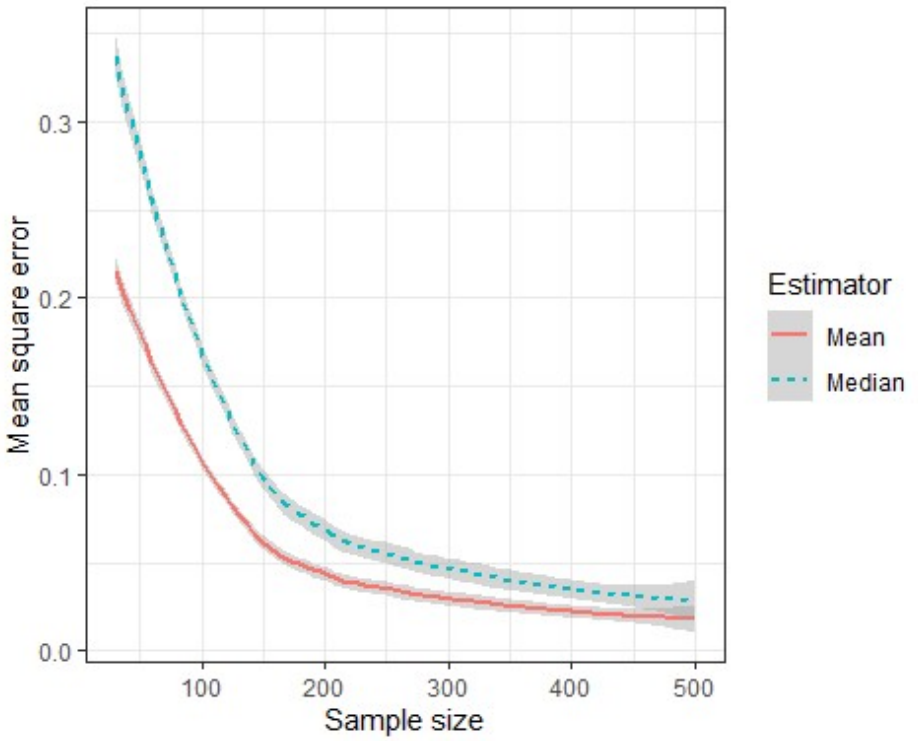
(Q1) Derive the mathematical expression for the population median of a Gaussian random variable $X_i \sim \mathcal{N}(\mu_0, \sigma_0^2)$.

(Q2)

Modify the above code to include estimates of the mean square error of the sample mean. Your data frame "simulation_df" should have a new column called "msq_error_mn" which estimates the mean squared error of the sample mean as an estimator of μ_0.

Then generate a plot which includes both the mean square error of the sample mean and the sample median as a function of the sample size.

Your plot might look like the following:



3. (**) The law of large numbers and Hoeffding's inequality

This is an optional question. . You can answer the other questions first before working on this one.

(Q1) Prove the following version of the weak law of large numbers.

Theorem (A law of large numbers). Let $X: \Omega \to \mathbb{R}$ be a random variable with a well-behaved expectation $\mu := \mathbb{E}(X)$ and variance $\sigma^2 := \mathrm{Var}(X)$. Let $X_1, \cdots, X_n: \Omega \to \mathbb{R}$ be a sequence of independent copies of X. Then for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) = 0.$$

You may want to begin by looking up the Chebyshev's inequality: For any random variable Z with finite expectation $\mathbb{E}(Z)$ and variance $\mathrm{Var}(Z)$, we have $\mathbb{P}(|Z - \mathbb{E}(Z)| \geq t) \leq \mathrm{Var}(Z)/t^2$ for any given number $t > 0$.

(Comparing the weak law of large numbers with Hoeffding's inequality): Below is some further information about Hoeffding's inequality. Hoeffding's inequality is the following important result:

Theorem (Hoeffding). Let $X: \Omega \to [0,1]$ be a **bounded** random variable with a well-behaved expectation $\mu := \mathbb{E}(X)$. Let $X_1, \cdots, X_n: \Omega \to \mathbb{R}$ be a sequence of independent copies of X. Then for all $\epsilon > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \leq e^{-2n\epsilon^2}.$$

Please note that in the Hoeffding Theorem we additionally assume X to be bounded.

We can view Hoeffding's inequality as a variant of the law of large numbers. However, Hoeffding's inequality gives us information about the rate of convergence. In particular, the sample average for bounded random variables converges **exponentially** fast to its expectation.

Hoeffding's inequality is a precursor to Vapnik-Chervonekis theory which serves as a foundation for the theory of Statistical Machine Learning.

4. Maximum likelihood estimates

In this section, we will explore maximum likelihood estimates that were introduced in Lecture 16.

4.1 Maximum likelihood estimates for Red tailed hawks

In this question we will fit a Gaussian model to a Red-Tailed hawk data set. First load the Hawks data set as follows:

```
library(Stat2Data)
data("Hawks")
```

(Q1) Now use your data wrangling skills to extract a subset of the Hawks data set so that every Hawk in the subset belongs to the ""Red-Tailed"" species, and extract the "Weight", "Tail" and "Wing" columns. The returned output should be a data frame called "RedTailedDf" with three numerical columns and 577 examples.

Display the first five rows of the "RedTailedDf". The resulting subset of the data frame should look as follows:

```
##   Weight Tail Wing
## 1    920  219  385
## 2    930  221  376
## 3    990  235  381
## 4   1090  230  412
## 5    960  212  370
```

(Q2)

We now model the vector of tail lengths from "RedTailedDf" as a sequence $X_1, \cdots, X_n \sim \mathcal{N}(\mu_0, \sigma_0^2)$ consisting of independent and identically distributed with unknown population mean μ_0 and population variance σ_0^2.

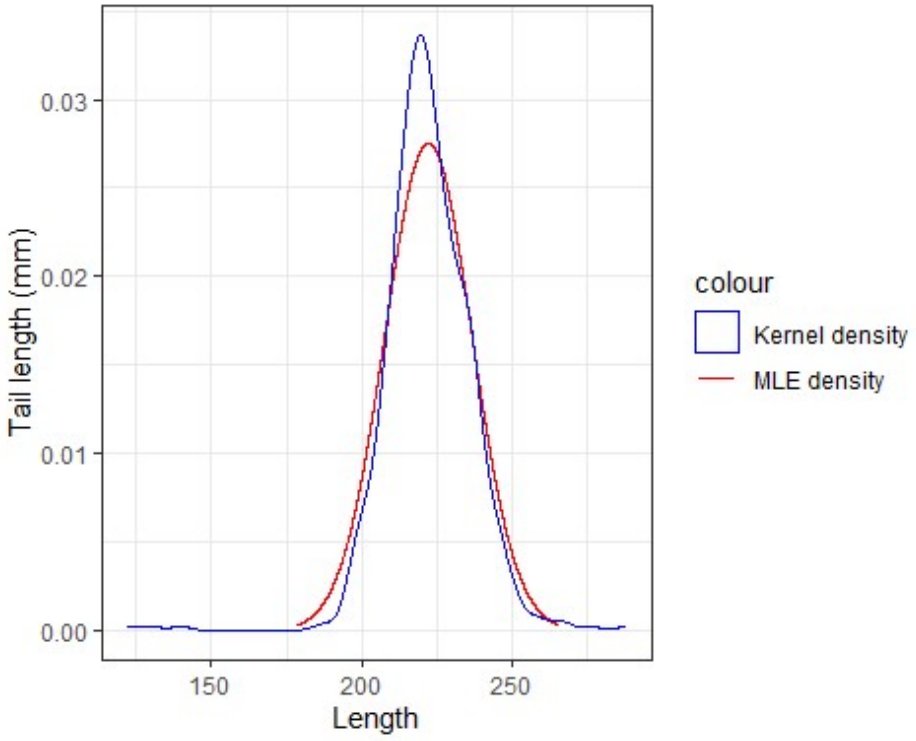
The maximum likelihood estimates for μ_0 is given by $\hat{\mu}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and the maximum likelihood estimate for σ_0^2 is given by $\hat{\sigma}^2_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu}_{\text{MLE}})^2$.

Apply the maximum likelihood method to compute the estimates $\hat{\mu}_{\text{MLE}}$ and $\hat{\sigma}^2_{\text{MLE}}$ for tail lengths using the sample in "RedTailedDf"

(Q3)

Next generate a plot which compares the probability density function for your fitted Gaussian model for the tail length of the Red-Tailed hawks with a kernel density plot.

Your plot should look as follows:



4.2 Unbiased estimation of the population variance

In this question we consider i.i.d. samples $X_1, \cdots, X_n \sim \mathcal{N}(\mu_0, \sigma_0^2)$ with unknown population mean μ_0 and unknown population variance σ_0^2.

Let $\overline{X}$ be the sample mean, Let $\hat{V}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ and let $\hat{V}_U := \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ (recall that this is an unbiased estimator for variance (see Lecture 15), which is different from the MLE $\hat{\sigma}^2_{\text{MLE}}$ discussed in the last question).

(Q1)

Conduct a simulation study which compares the bias of $\hat{V}_{\text{MLE}}$ as an estimate to the population variance σ_0^2 with the bias of $\hat{V}_U$ as an estimator for the population variance σ_0^2.

In your simulation study, you can consider different sample sizes ranging from 5 to 100 in increment of 5. For each sample size, conduct 1000 trials. For each trial, generate a samples $X_1, \cdots, X_n \sim \mathcal{N}(\mu_0, \sigma_0^2)$ with fixed parameters $\mu_0 = 1$ and $\sigma_0 = 3$, and then compute $\hat{V}_{\text{MLE}}$ and $\hat{V}_U$. Then create a plot which displays the computed bias of $\hat{V}_{\text{MLE}}$ and the bias of $\hat{V}_U$ as functions of the sample sizes.

(Q2)

Is $\sqrt{\hat{V}_U} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$ unbiased estimator for σ_0? You can conduct a simulation study (similar to in the last question) to answer this question.

(Q3) ~~(*optional) As an optional extra, give an analytic formula for the bias of $\hat{V}_{\text{MLE}}$ and the bias of $\hat{V}_U$ (not using R programming.)~~

4.3 Maximum likelihood estimation with the Poisson distribution

In this question we shall consider the topic of maximum likelihood estimation for an independent and identically distributed sample from a Poisson random variable. Recall that Poisson random variables are a family of discrete random variables with distributions supported on $\mathbb{N}_0 := \{0,1,2,\cdots,\}$

Poisson random variables are frequently used to model the number of events which occur at a constant rate in situations where the occurrences of individual events are independent. For example, we might use the Poisson distribution to model the number of mutations of a given strand of DNA per time unit, or the number of customers who arrive at the store over the course of a day. A classic example of statistical modelling based on a Poisson distribution is due to the statistician Ladislaus Josephovich Bortkiewicz. Bortkiewicz used the Poisson distribution to model the number of fatalities due to horse-kick per year for each group of cavalry. We shall apply maximum likelihood estimation to Bortkiewicz's data. First, let's explore maximum likelihood estimation for Poisson random variables.

A Poisson random variable has a probability mass function $p_\lambda: \mathbb{R} \to (0, \infty)$ with a single parameter $\lambda > 0$. The probability mass function $p_\lambda: \mathbb{R} \to (0, \infty)$ is defined for $x \in \mathbb{R}$ by

$$p_\lambda = \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!} & \text{for} \quad x \in \mathbb{N}_0, \\ 0 & \text{for} \quad x \neq \mathbb{N}_0. \end{cases}$$

Suppose that you have a sample of independent and identically distributed random variables $X_1, \cdots, X_n \sim p_{\lambda_0}$, i.e., $X_1, \cdots, X_n$ are independent and each has probability mass function p_{λ_0}. Complete the following tasks ((Q1) and (Q2) do not require R programming).

(Q1)

Show that for a sample $X_1, \cdots, X_n$, the likelihood function $l: (0, \infty) \to (0, \infty)$ is given by

$$l(\lambda) = e^{-n\lambda} \cdot \lambda^{n \cdot \overline{X}} \cdot \left(\prod_{i=1}^{n} \frac{1}{X_i!} \right),$$

where $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is the sample mean.

Then derive a formula for the derivative of the log-likelihood $\frac{\partial}{\partial \lambda} \log l(\lambda)$.

(Q2)

Show that $\lambda \to \log l(\lambda)$ reaches its maximum at the single point at which $\lambda = \overline{X}$. Hence, the maximum likelihood estimate for the true parameter λ_0 is $\hat{\lambda}_{\text{MLE}} = \overline{X}$.

(Q3)

Now conduct a simulation experiment which explores the behaviour of $\hat{\lambda}_{\text{MLE}}$ on simulated data. You may wish to consider a setting in which $\lambda_0 = 0.5$ and generate a plot of the mean squared error as a function of the sample size. To generate samples from Poisson distribution, you can consider the "rpois()" function. There is no specific requirement on the plot or the range of sample sizes etc, but you should make sure your results clearly display your conclusions.

(Q4)

Now that we have explored maximum likelihood estimation with a Poisson distribution for simulated data we shall return to Poisson modelling with real data. Let's take a look at the famous horse-kick fatality data set explored by Ladislaus

Josephovich Bortkiewicz. A csv file containing this data is available within Blackboard. The file name is "VonBortkiewicz.csv".

Download the csv file and load the file into an R data frame. You may wish to use the read.csv() function.

The count data for horse fatalities per year, per cavalry corps, are given in the "fatalities" column. Model the values in this column as independent random variables $X_1, \cdots, X_n$ from a Poisson distribution with parameter λ_0 and compute the maximum likelihood estimate $\hat{\lambda}_{\text{MLE}}$ for λ_0.

Use your fitted Poisson model to give an estimate for the probability that a single cavalry corps has no fatalities due to horse kicks in a single year. You may want to use the "dpois" function.

~~**(Q5)** (*optional)~~

~~As an optional extra give a formula for $\mathcal{I}(\lambda) := \mathbb{E}\left(\frac{\partial^2}{\partial\lambda^2}\log p_\lambda(X)\right)$ where $X \sim p_\lambda$ is a Poisson random variable with rate λ. Next generate a simulation involving random samples of size 1000 from a Poisson random variable with parameter $\lambda = 0.5$. Give a kernel density plot of $\sqrt{n\mathcal{I}(\lambda_0)}(\hat{\lambda}_{\text{MLE}} - \lambda_0)$.~~

4.4 Maximum likelihood estimation for the exponential distribution

Recall from our last assignment that given a positive real number $\lambda > 0$, an exponential random variable X with parameter λ is a continuous random variable with density $p_\lambda : \mathbb{R} \to (0, \infty)$ define by

$$p_\lambda(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \lambda e^{-\lambda x} & \text{if} \quad x \geq 0. \end{cases}$$

(Q1)

Suppose that $X_1, \cdots, X_n$ is an i.i.d sample from the exponential distribution with an unknown parameter $\lambda_0 > 0$. What is the maximum likelihood estimate for λ_0?

(Q2)

We shall now use the exponential distribution to model the differences in purchase times between customers at a large supermarket. In Blackboard you will find the "CustomerPurchase" csv file. Download the "CustomerPurchase" csv file and load the file into an R data frame. You may wish to use the read.csv() function. The first column is the purchase time given in seconds since the store opens.

Add a new column in your data frame called ""time_diffs"" which gives the time in seconds until the next customer's purchase. That is, letting $Y_1, Y_2, \cdots, Y_{n+1}$ denote the

sequence of arrival times in seconds, the "time_diffs" column contains $X_1, \cdots, X_n$ where $X_i = Y_{i+1} - Y_i$ for each $i = 1, \cdots, n$. You can let last row of ""time_diffs"" be NA (missing value). You may want to use the "lead()" function.

(Q3)

Model the sequence of differences in purchase times $X_1, \cdots, X_n$ as independent and identically distributed exponential random variables. Compute the maximum likelihood estimate of the rate parameter $\hat{\lambda}_{\text{MLE}}$ from your data.

(Q4)

Use your fitted exponential model to give an estimate of the probability of an arrival time in excess of one minute. You may wish to make use of the "pexp()" function.

5. Confidence intervals

5.1 Student's t-confidence intervals

In this problem we will discuss a parametric approach to obtaining confidence intervals based upon Student's t-distribution. In the code below ""adelie_flippers"" is a vector containing the flipper lengths of a sample of Adelie penguins. The following code computes confidence intervals based on ""adelie_flippers"" for the population mean of the flipper lengths for Adelie penguins using the Student's t-distribution method.

```
alpha <- 0.05
sample_size <- length(adelie_flippers) # adelie_flippers is a given
vector
sample_mean <- mean(adelie_flippers)
sample_sd <- sd(adelie_flippers)
t <- qt(1-alpha/2,df=sample_size-1)
# confidence interval
confidence_interval_l <- sample_mean-t*sample_sd/sqrt(sample_size)
confidence_interval_u <- sample_mean+t*sample_sd/sqrt(sample_size)
confidence_interval <- c(confidence_interval_l,confidence_interval_u)
confidence_interval
```

(Q1)

What would happen to the width of my confidence interval if the sample mean were higher? What would happen to the width of my confidence interval if the sample standard deviation were higher? What would happen to the width of my confidence interval if the sample size were larger (keeping the sample standard deviation the same)?

(Q2)

Use your data wrangling skills to extract a vector consisting of the weights of all the Red-Tailed hawks from the "Hawks" data set, with any missing values removed.

Now use the Student's t method to compute 99%-level confidence intervals for the population mean of the weights for the red-tailed hawks. Note that opting for confidence intervals with a confidence level of 99%, rather than a confidence level of 95%, requires a modified value of α.

(Q3)

What assumptions are made to derive confidence intervals based on Student's t-distribution? Check if these assumptions are justified using a kernel density plot with the "geom_density()" function and using a "QQ"-plot with the "stat_qq"() function.

5.2 Investigating coverage for Student's t intervals

In this question we shall assume that we have access to a sample $X_1, \cdots, X_n \sim \mathcal{N}(\mu_0, \sigma_0^2)$ consisting of i.i.d. Gaussian data. We are interested in determining the value of the unknown population mean μ_0 based on the sample $X_1, \cdots, X_n$.

Suppose we wish to compute confidence intervals for μ_0 with confidence level $(1-\alpha) \times 100\%$ for some $\alpha \in (0,1)$. For example, we could have $\alpha = 0.05$, in which cases we wish to compute confidence intervals with confidence level 95%.

Let $\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean and $S := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$ be the sample standard deviation. In addition, let $t_{\alpha/2,n-1}$ be the $\left(1-\frac{\alpha}{2}\right)$-quantile of the Student's t-distribution with $n-1$ degrees of freedom.

The Student's t confidence interval for μ_0 is given by $\left(L_\alpha(X_1, \cdots, X_n), R_\alpha(X_1, \cdots, X_n)\right)$ defined by

$$L_\alpha(X_1, \cdots, X_n) := \overline{X} - \frac{t_{\alpha/2,n-1}}{\sqrt{n}} \cdot S$$
$$R_\alpha(X_1, \cdots, X_n) := \overline{X} + \frac{t_{\alpha/2,n-1}}{\sqrt{n}} \cdot S$$

The following code generates a function "student_t_confidence_interval", which takes as input a sample $X_1, \cdots, X_n$ given as a vector along with a confidence level $\gamma = 1-\alpha$ and outputs a tuple containing $\left(L_\alpha(X_1, \cdots, X_n), R_\alpha(X_1, \cdots, X_n)\right)$

```
student_t_confidence_interval<-function(sample,confidence_level){
  sample<-sample[!is.na(sample)] # remove any missing values
```

```
  n<-length(sample) # compute sample size
  mu_est<-mean(sample) # compute sample mean
  sig_est<-sd(sample) # compute sample sd
  alpha = 1-confidence_level # alpha from gamma

  t<-qt(1-alpha/2,df=n-1) # get student t quantile
  l=mu_est-(t/sqrt(n))*sig_est # lower
  u=mu_est+(t/sqrt(n))*sig_est # upper
return(c(l,u))
}
```

Check that you understand this function and implement it for yourself.

The key property of a confidence interval for μ_0 at the confidence level of $(1-\alpha)\times 100\%$ is the following coverage property:

$$\mathbb{P}\{(L_\alpha(X_1,\cdots,X_n) \leq \mu_0 \leq R_\alpha(X_1,\cdots,X_n))\} \geq 1-\alpha.$$

This is known as a coverage property since it tells us that the confidence interval covers μ_0 with probability $1-\alpha$. The following simulation checks this property with $\mu_0 = 1, \sigma_0 = 3$ and a confidence level of 95% i.e. $\gamma = 0.95$.

```
num_trials <- 100000
sample_size <- 30
mu_0 <- 1
sigma_0 <- 3
alpha <- 0.05
set.seed(0) # set random seed for reproducibility

single_alpha_coverage_simulation_df <-
data.frame(trial=seq(num_trials)) %>%
  # generate random Gaussian samples:

mutate(sample=map(.x=trial,.f=~rnorm(n=sample_size,mean=mu_0,sd=sigma_0
))) %>%
  # generate confidence intervals:
  mutate(ci_interval=map(.x=sample,
.f=~student_t_confidence_interval(.x,1-alpha)))%>%
  # check if interval covers mu_0:
  mutate(cover=map_lgl(.x=ci_interval,
.f=~((min(.x)<=mu_0)&(max(.x)>=mu_0))))%>%
  # compute interval length:
  mutate(ci_length=map_dbl(.x=ci_interval, .f=~(max(.x)-min(.x))))

# estimate of coverage probability:
single_alpha_coverage_simulation_df %>%
  pull(cover) %>%
  mean()

## [1] 0.95003
```

(Q1)

Check that you understand the above code. Now modify the above code to conduct a simulation experiment to investigate how $\mathbb{P}\{(L_\alpha(X_1,\cdots,X_n) \leq \mu_0 \leq R_\alpha(X_1,\cdots,X_n))\}$ varies as a function of the confidence level $\gamma = 1-\alpha$.

(Q2)

How does the average length $\mathbb{E}(R_\alpha(X_1,\cdots,X_n) - L_\alpha(X_1,\cdots,X_n))$ vary as a function of the confidence level $\gamma = 1-\alpha$? You may want to conduct a simulation study to answer this question, similar to the one in the last question.