Homework 1

September 9, 2020

Deadline: October 2nd 2019, the answer to the questions will be submitted via Canvas, and the code will be posted to github. No late homework will be accepted.

Rules: You are strongly encouraged to discuss the homework with your peers, in particular, piazza is a very good environment for discussing the homework. However, you need to write your own homework and you need disclose you sources.

A (20 points) In this question we generalize the centered difference scheme for the first derivative to higherorders. Assume that the samples of a function are given on a Cartesian grid with spacing h, and assume that the point x is on the grid. As in class, define the operator

$$\delta u(x) = u(x + h/2) - u(x - h/2)$$

Also define the averaging operator

$$\mu u(x) = \frac{u(x + h/2) + u(x - h/2)}{2}$$

Notice that μ is a handy way interpolate back to the grid, i.e., to fall back on the grid right after δ is taken. In fact, $\mu\delta u(x) = hD_c u(x)$, where D_c is the centered difference scheme

(a) Show that

$$\mu\left(1 + \frac{1}{4}\delta^2\right)^{-1/2} = 1$$

(b) Recall that we saw in class the hD and δ are related by

$$hD = \delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168} + \dots$$

Explain briefly, possible in only one sentence, why this equation does not produce proper FD schemes after truncation.

(c) Using the resutls above, derive a series for hD as a function of μ and δ, which gives the desired FD schemes upon truncation. You need to write at least the first two terms of the series. **Hint:** multiply the equations in the two parts above keeping the hD term constant. This will give you hD in terms of a function of μ and δ times a series in δ. Expand the function, and conclude by multiplying both series.

B (20 points) In this question we study the approximation properties of finite differences for functions of limited differentiability. Among those, piecewise polynomials are perhaps the most interesting for numerical computations. After subtracting the smooth component, any piecewise polynomial is, for some $n \geq 0$, locally of the form

$$u(x) = x_+^n = \begin{cases} x^n & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Consider the generic situation in which the stencil of D_c straddles the origin in an asymmetric manner,i.e., the two contentious differences are

$$D_c u(-\epsilon) = \frac{u(-\epsilon + h) - u(-\epsilon - h)}{2h}, \quad D_c u(-\epsilon + h) = \frac{u(-\epsilon + 2h) - u(-\epsilon)}{2h}$$

for some $0 < \epsilon < h$. Take $u(x)$ to be defined on an interval like $[-1, 1]$, thus we also consider $D_c(u - \epsilon + mh)$ for all relevant values of the integer m.

(a) For which $n \geq 0$, is D_c consistent for $u(x) = x_+^n$ in the maximum (i.e., uniform or ℓ_∞) norm? What is the order of accuracy of D_c as a function of $n \geq 0$.

(b) Repeat the same question as above, but with the ℓ_1 norm, which is defines as $\|E\|_1 = h \sum_j |E_j|$.

(c) For $n = 3$, we have the cubic splines, which are an importnat example. Is the Taylor expansion argument for the error of D_c in the maximum nomr too optimistic/ too pessimistic / just right?

C (50 points) In this question we solve the two-dimensional Poisson equation of electrostatics using finite differences. Consider $\Omega = [0,1]^2$, and suppose the Ω is made of a dialectric material with permitivity 1, which is subjected to *i)* a certain pattern of electric potential $f(y)$ on its left side, *ii)* it is grounded on its right side, and *iii)* is it insulated on its top and bottom sides.

The resulting equations that models the potential $u(x,y)$ inside Ω as a function of the excitation $f(y)$ is

$$\begin{aligned} -\Delta u(x,y) &= 0, \quad x \in \Omega \\ u(0,y) = f(y), &\quad u(1,y) = 0, \quad 0 \leq y \leq 1 \\ \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}&(x,1) = 0, \quad 0 \leq x \leq 1 \end{aligned}$$

As usual the electric current is ∇u, and $-\Delta$ is minus the two-dimensional Laplacian. In addition assume that

$$f(y) = \cos(2\pi y).$$

(a) Propose a second-order finite difference discretization for $-\Delta$, which takes into account the boundary conditions. Detail your choice for the number of grid points that you use in x vs. y, and whether your cells are square or rectangular. **Hint:** written as a block matrix, the result should be of the form

$$-\tilde{\Delta} = A \otimes I + I \otimes B$$

where $\tilde{\Delta}$ is your FD discretation, A is a matrix form of the discretization for the one-dimension Dirichlet Laplacian, B is a matrix form for the $1D$ Neumann Laplacian, and $\otimes$ is the Kronecker (or outer) product. Take special care identifiying the correct dimensions of A, B, and the identities.

(b) Using this FD scheme, implement a solver for $u(x,y)$. Solve the linear system by an iterative method of your choice. Do not explicitly form a block matrix for the Laplacian, and do not use a direct method such as Matlab's backslash or Python spsolve. Illustrate the convergence of your numerical schemein a log-log plot of the maximum norm of the error vs. h, the grid spacing. Check that the slope is approximately 2 in this graph. **Hint:** either find the exact solution as a basis for comparison, or use a numerical solution on a very fine grid for that purpose. You may find it useful to use nested grids for the different values of h, so that the points on a coarse grid are a subset of the point son a finer one.

(c) Argue (very briefly; in one sentence perhaps) the consistency of your scheme. It is fine to assume that the solution u is infinitely differentiable.

(d) What are the eigenvalues and eigenvectors of $A \otimes I + I \otimes B$ as a function of those of A and B?

(e) There is a good chance that if the top and bottom rows of your matrix B are scaled by an adequately small number (which does not change the original equation), then the eigenvalues of B are real and positive, except for a zero eigenvalue corresponding to the constant eigenvector. Feel free to check this numerically. If you take this fact for granted, and use the result in (d), to show that there is an eigenvalue gap:

$$\lambda_{\min}(-\tilde{\Delta}) \geq C > 0$$

with C independent of h.

(f) How many steps does the iterative method of your choice require for convergence to within agiven error tolerance? Does the theory match the numerics? **Hint:** see pages 74 and 75 of LeVeque 2007.

(g) Replace $f(y)$ given above by $\hat{f}(y) = sgn(cos(2\pi y))$, for the left boundary condition, where

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Run your code again. Empirically, what does the order of convergence become? How do you explain this behavior?

D (10 points) Implement a multigrid method of your choice to improve the convergence of the iterative method for question C. Illustrate the gain in convergence speed in the way you find most adequate.

E (Bonus, 20 points) Make your explanation in point (f) quantitative, i.e., repeat the error analysis with $\hat{f}(y)$ in place of $f(y)$.