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CS 714 HOMEWORK 3

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1 Problem A

1.1 (a)

1. Note that $u(x(t), t) = c$, we could take derivative over t on both side $c' = 0$ and get the desired equation:

$$\partial_t u(x(t), t) + \partial_x u(x(t), t) x'(t) = 0$$

2. Combine the equation we get in the previous question and the condition for the problem:

$$\partial_t u(x, t) + \partial_x u(x, t) a = 0$$

we can get $x'(t) = a$.

3. Since $x'(t) = a$ and the boundary condition $x(0) = x_0$, we can get $x(t) = at + x_0$. Furthermore, we can conclude $u(x(t), t) = u_0(x_0)$ by

$$c = u(x(t), t) = u(x(0), 0) = u(x_0, 0) = u_0(x_0)$$

4. Since we know that $u(x(t), t) = u_0(x_0)$ and also $x(t) = at + x_0$, we can get

$$u(at + x_0, t) = u_0(x_0)$$

the desired result $u(x, t) = u_0(x - at)$ yields immediately by setting $x = at + x_0$ in this equation.

1.2 (b)

1. For the existence of Fourier transform and the regularity of Fourier transform, we can refer to the Fourier inversion theorem. The Fourier inversion theorem holds for all continuous functions that are absolutely integrable (i.e.$L^1(\mathbb{R}^n)$) with absolutely integrable Fourier transform. Furthermore, $\hat{u}_0(\xi) \to 0$ when $|\xi| \to \infty$. The regulation condition also holds for L^2. In other words, if the function has finite 2-norm, then its Fourier transform is well-defined and also belongs to L^2. With a proper fraction $\frac{1}{\sqrt{2\pi}}$, the original function and the Fourier transformed function has the same 2-norm (See LeVeque's book Appendix E). More generally, it can be generalized to L^p where $1 \leq p \leq 2$.

2. By the Fourier tranform, we have:

$$v(x) = \frac{1}{2\pi} \int_R e^{i\xi x} \hat{v}(\xi) d\xi$$

[1] Codes can be found at https://github.com/ZihaoZheng-Stat/CS714

Since all regularity conditions are satisfied, we can apply Leibniz formula and take derivative with respect to x on both side, we have the desired result:

$$v'(x) = \frac{1}{2\pi}\int_R e^{i\xi x} i\xi \hat{v}(\xi) d\xi$$

3. By definition of Fourier tranform, we have:

$$\hat{v}(\xi) = \int_R e^{-i\xi x} v(x) dx$$

When translation in space by h, we apply the change of variable by $y = x + h$, and the desired result holds since the factor $e^{-i\xi h}$ comes from $e^{-i\xi(x+h)} = e^{-i\xi x} e^{-i\xi h}$.

4. By definition:

$$\hat{u}(\xi, t) = \int_R e^{-i\xi x} u(x, t) dx$$

Taking derivative on both side with respect to t (regulation conditions satisfied), we have:

$$\partial_t \hat{u}(\xi, t) = \int_R e^{-i\xi x} \partial_t u(x, t) dx = \int_R e^{-i\xi x}(-a\partial_x u(x, t)) dx = -ai\xi \hat{u}(\xi, t)$$

where the second last equality holds by definition of PDE of $u(x, t)$ and the last equality holds by previous observation:

$$v'(x) = \frac{1}{2\pi}\int_R e^{i\xi x} i\xi \hat{v}(\xi) d\xi$$

5. By the previous conclusion:

$$\partial_t \hat{u}(\xi, t) = -ai\xi \hat{u}(\xi, t)$$

solving this ODE will yield:

$$\hat{u}(\xi, t) = e^{-ai\xi t} c_0$$

where the constant c_0 can be calculated by the boundary condition $c_0 = \hat{u}(\xi, 0) = \hat{u}_0(\xi)$. Therefore, we have:

$$\hat{u}(\xi, t) = e^{-ai\xi t} \hat{u}_0(\xi)$$

6. The desired result is true by observing that:

$$\int_R |u(x, t)|^2 dx = \int_R |\hat{u}(\xi, t)|^2 d\xi = \int_R |e^{-ai\xi t} \hat{u}_0(\xi)|^2 d\xi < \infty$$

where the first equality holds by Plancherel's Theorem, the second equality holds by conclusion in previous part of $\hat{u}(\xi, t)$ and the last inequality holds by the definition of $\hat{u}_0(\xi)$ (from u_0 to $\hat{u}_0$) and the second-integrability of u_0.

7. The desired result holds by:

$$u(x, t) = \frac{1}{2\pi}\int_R e^{i\xi x} \hat{u}(\xi, t) d\xi = \frac{1}{2\pi}\int_R e^{i\xi x} e^{-ai\xi t} \hat{u}_0(\xi) d\xi = \frac{1}{2\pi}\int_R e^{i\xi(x-at)} \hat{u}_0(\xi) d\xi$$

where the first equality holds by Fourier inversion formula, the second equality holds by previous conclusion of $\hat{u}(\xi, t)$ and the last equality holds directly.

8. By previous conclusion:

$$u(x, t) = \frac{1}{2\pi}\int_R e^{i\xi(x-at)} \hat{u}_0(\xi) d\xi = \frac{1}{2\pi}\int_R e^{i\xi(x-at)} \int_R e^{-i\xi x} u_0(x) dx d\xi = \frac{1}{2\pi}\int_R \int_R e^{-i\xi at} u_0(x) dx d\xi$$

where the first equality holds by previous conclusion, the second equality holds by Fourier formula of $\hat{u}_0(\xi)$ and the last equality holds directly. Then the desired result $u_0(x - at)$ holds by applying change of variable where we used in (3).

2 Problem B

2.1 (a)

This is only true for negative sign. When using the negative sign, we have $|u(.,t)|$ exponentially decay, which yields stability. However, when using positive sign, $|u(.,t)|$ grows also exponentially, which is not stable.

I would use Fourier transform to illustrate this problem. The Fourier transform for u_t is $\hat{u}_t(\xi, t)$ and the Fourier tranform for u_{xxxx} is $(i\xi)^4\hat{u}(\xi, t) = \xi^4\hat{u}(\xi, t)$.

Therefore, for the negative sign $u_t = -u_{xxxx}$, we have $\hat{u}(\xi, t) = e^{-\xi^4 t}\hat{\eta}(\xi)$, which yields $|\hat{u}(\xi, t)| \leq |\hat{\eta}(\xi)||e^{-\xi^4 t}|$. $\hat{\eta}(\xi)$ could be solved by boundary condition. This suggests immediately the exponentially decay for $\hat{u}(\xi, t)$ and thus also for $u(x, t)$ by Fourier transform.

However, for the positive sign, we have $\hat{u}(\xi, t) = e^{\xi^4 t}\hat{\eta}(\xi)$, which grows exponentially fast.

2.2 (b)

Let's consider the stable version:

$$u_t + u_{xxxx} = 0$$

For the Euler explicit version for $y' = \lambda_y$, we have the following:

$$y^{n+1} = y^n + \Delta_t \lambda y^n$$

By defining ρ, we have the condition $|\rho| = |1 + \Delta_t\lambda| \leq 1$. Suppose the real part of $1 + \lambda\Delta_t$ is a and the imaginary part of $1 + \lambda\Delta_t$ is b, then the condition for $\lambda\Delta_t$ is $(a+1)^2 + b^2 \leq 1$, which is a circle centered at $(-1, 0)$ with radius 1 at the complex plane.

Now let's use the method of line method and get more information about λ, the eigenvalue of matrix for finite differencing approximation of u_{xxxx}:

$$U_i'(t) = -\frac{1}{\Delta_x^4}(U_{i-2}(t) - 4U_{i-1}(t) + 6U_i(t) - 4U_{i+1}(t) + U_{i+2}(t))$$

Regarding this as $y' = Ay$, then A is the matrix with the following structure:

- The diagonal element is $-6/\Delta_x^4$.
- The immediate off-diagonal element is $4/\Delta_x^4$.
- The further off-diagonal element is $-1/\Delta_x^4$.
- The rest are 0.

Numerically, I calculated the eigenvalues of matrix A (can also solve analytically). The eigenvalues are all real and the range between the absolute value of λ is from 0 to $16/\Delta_x^4$. Therefore we can get the CFL condition for Δ_t given Δ_x. The most important finding here is that Δ_t should be of $O(\Delta_x^4)$ (I think should be $\Delta_t \leq \frac{1}{8\Delta_x^4}$ by $\lambda\Delta_t \geq -2$).

2.3 (c)

The above indicates that Euler explicit is not good enough because it is so strict on the stability region of Δ_t (of order $O(\Delta_x^4)$).

However, we can remedy this issue by using Trapezoidal rule where the stability region of $\lambda\Delta_t$ is the full left half space. Then the order of λ (O(Δ_x^4)) does not involve.

3 Problem C

In the following problem, I will always denote k be the length of grid in time t and h be the length of grid in either x and y.

3.1 (a)

For the modified equation, consider the Taylor expansion in time $u(x,y,t+k)$ and $u(x,y,t-k)$:

$$u(x,y,t+k) = u(x,y,t) + ku_t(x,y,t) + \frac{k^2}{2}u_{tt}(x,y,t) + \frac{k^3}{6}u_{ttt}(x,y,t) + \frac{k^4}{24}u(x,y,t)$$

$$u(x,y,t-k) = u(x,y,t) - ku_t(x,y,t) + \frac{k^2}{2}u_{tt}(x,y,t) - \frac{k^3}{6}u_{ttt}(x,y,t) + \frac{k^4}{24}u(x,y,t)$$

Therefore for the second derivative approximation, we have the following modified equation:

$$\frac{u(x,y,t+k) - 2u(x,y,t) + u(x,y,t-k)}{k^2} = u_{tt}(x,y,t) + \frac{1}{12}k^2 u_{tttt}(x,y,t)$$

Since we know that $u_{tt}(x,y,t) = \Delta u(x,y,t)$ and $u_{tttt}(x,y,t) = \Delta^2 u(x,y,t) = u_{xxxx}(x,y,t) + 2u_{xxyy}(x,y,t) + u_{yyyy}(x,y,t)$, we have the leading term in the correction of modified equation by:

$$\frac{1}{12}k^2\Delta^2 u(x,y,t)$$

We can borrow the idea from here to get the fourth order approximation for the second derivative.

Suppose for the original second order approximation, we have the equation:

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{k^2} = AU^n$$

where A is the operator for Δu. It could be in finite difference scheme and can also be in Chebyshev spectral matrix scheme.

In order the get the fourth order, motivated by the Lax-Wendroff trick, we can consider:

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{k^2} = AU^n + \frac{1}{12}k^2 A^2 U^n$$

Another choice is to use the finite difference operator, the fourth-order accuracy approximation for the second order derivative is the following, using five points:

$$\frac{1}{12k^2}[-U^{n-2} + 16U^{n-1} - 30U^n + 16U^{n+1} - U^{n+2}]$$

3.2 (b)

The above update formula, obtaining the fourth order accurate approximation, is only true except for a few initial value. We also need a fourth order accuracy for the boundary derivative condition, in other words:

$$u_t(x,y,0) = f(x)f(y)$$

The basic idea here is to use the fourth order accuracy to approximate the first derivative u_t that helps us get information of u^1 (note that U^0 is already known). Then we could run the algorithm for the wave equation starting from U^2 (by U^0 and U^1).

Denote the "ghost" term U^{-1} that could help us to center the first order derivative at U^0. Then we could have the following two equations by Taylor expansion:

$$u(x,y,k) = u(x,y,0) + ku_t(x,y,0) + \frac{k^2}{2}u_{tt}(x,y,0) + \frac{k^3}{6}u_{ttt}(x,y,t)$$

$$u(x,y,-k) = u(x,y,0) - ku_t(x,y,0) + \frac{k^2}{2}u_{tt}(x,y,0) - \frac{k^3}{6}u_{ttt}(x,y,t)$$

Then we have the second order accurate approximation:

$$\frac{u(x,y,k) - u(x,y,-k)}{2k} = u_t(x,y,0)$$

In order to get the fourth order accurate approximation, we need additional correction:

$$\frac{u(x,y,k) - u(x,y,-k)}{2k} = u_t(x,y,0) + \frac{k^2}{6}u_{ttt}(x,y,0)$$

where we know actually $u_{ttt} = \partial_t \Delta u$ since $u_{tt} = \Delta_u$. Therefore we get a equation with unknown $U^{-1} = u(x,y,-k)$ and unknown $U^1 = u(x,y,k)$.

Remember that we can use the fact that the solution satisfies the wave equation at time 0. Therefore, we get another equation using the discretized version of $u_{tt} = \Delta u$ centered at 0 (this is also an equation with the same unknown U^1 and U^{-1}).

The above two equations can help us solve for U^1 and the ghost term U^{-1} and this finished the initialization (first time step) while maintaining the fourth order accuracy.

3.3 (c)

To implement the chebyshev spectral method, we specified the grid on x, y be h and the grid on t be k using the following, where x, y is on the chebyshev grid $\cos 2\pi jh$:

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{k^2} = L_N v + \frac{1}{12}k^2 L_N^2 v$$

where L_N is the chebyshev matrix as the following definition:

$$L_N = D_N^2 \otimes I_x + I_y \otimes D_N^2 + b.c.$$

where we have the closed-form formula for the operator D_N in the textbook. v is the vectorized of matrix sized $(N+1)(N+1)$ for the chebyshev grid point x, y. The boundary condition $b.c.$ is also inserted as the previous problem justified.

In order to evaluate the error, we use the nested grid and suppose the finest one as the "true answer". Figure 1 shows the result and it is clear that the slope is slightly larger than 4, which yileds the fourth order accuracy is guaranteed.

3.4 (d)

In order for the CFL condition, we also want to solve first the ODE:

$$y'' = \lambda y$$

We could write y'' as $y'' = \frac{y^{n+1} - 2y^n + y^{n-1}}{k^2}$. Following the same argument as in the previous homework, we get the condition for $|\rho| \leq 1$ where ρ is the solution for the following equation:

$$\frac{\rho - 2 + \frac{1}{\rho}}{k^2} = \lambda$$

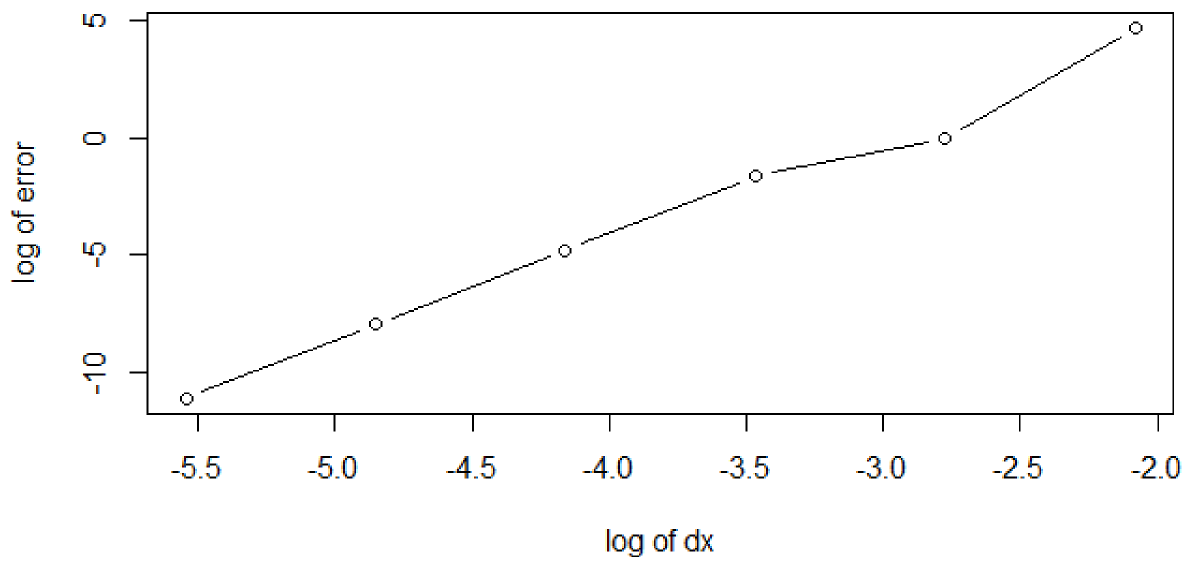

Figure 1: Figure of (c) in Problem C: log-log plot for error evaluation.

This yields the condition for $k^2\lambda$:

$$k^2\lambda \geq -4$$

For the second order accuracy (the original scheme) approach, we can replace λ the eigenvalues of matrix L_N, which is twice of the eigenvalues of matrix of D_N^2. Note that the eigenvalue of D_N^2 are all negative real and the largest is $-0.048N^4$. This yields the extreme eigenvalue of λ, which is the extreme eigenvalue of L_N, that we need to consider, is $-0.096N^4$. This together will yield the CFL condition (noted this is for the second order accuracy), which is:

$$k \leq 6.5N^{-2}$$

This is much strict than the finite differencing CFL condition since we need the higher order of N or equivalently h to guarantee k to be stable.

For the higher order accuracy (fourth order accuracy here), what we just need to modify is the fill-in of λ. Here λ should be filled with the extreme eigenvalue of $L_N + \frac{1}{12}k^2L_N^2$ where the eigenvalue of L_N^2 is just the squared of eigenvalues of L_N. Combining together we get the CFL condition for fourth order accuracy by:

$$k^2\lambda_{L_N+\frac{1}{12}k^2L_N^2} \geq -4$$

desired result follows immediately by replacing with $-0.096N^4$ and solving the equation.

3.5 (e)

Let's first derive the analytical solution.

We guess the analytical solution has the form:

$$u(x,y,t) = \alpha \sin B\pi x \sin B\pi y \sin \beta t$$

Then by the condition, we need the following:

$$\begin{aligned}
u_t(x,y,0) &= \alpha\beta \sin B\pi x \sin B\pi y \cos \beta 0 = \alpha\beta \sin B\pi x \sin B\pi y = \sin B\pi x \sin B\pi y \\
u(x,y,0) &= 0 \\
u_{tt} &= -\alpha\beta^2 \sin B\pi x \sin B\pi y \sin \beta t \\
\Delta u &= -2\alpha B^2\pi^2 \sin B\pi x \sin B\pi y \sin \beta t
\end{aligned}$$

Therefore we have $\alpha\beta = 1$ and $\beta^2 = 2B^2\pi^2$, which yields the analytical solution to be:

$$u(x,y,t) = \frac{1}{\sqrt{2}B\pi} \sin B\pi x \sin B\pi y \sin \sqrt{2}B\pi t$$

In order to evaluate the error and make the figure, we fixed $N = 64$ which indicates $\Delta_x = \frac{1}{64}$. For FD and chebyshev spectral method, we use Δ_t that satisfies the CFL condition.

Due to the time complexity, especially when using Chebyshev spectral method, We run the algorithm until $T = 0.25$ and make the figure of error (between numerical solution and analytical solution) using different B's. Also due to the time complexity, I can only provide a few points for chebyshev spectral methods but more for finite differencing. However, Figure 2 should provide enough details.

It is clear that at low level of B, chebyshev spectral method is much better than FD. When B increases, the smoothness of the function increases and the numerical approach provides slightly larger error. That's well interpreted the error of number of points per wavelength.

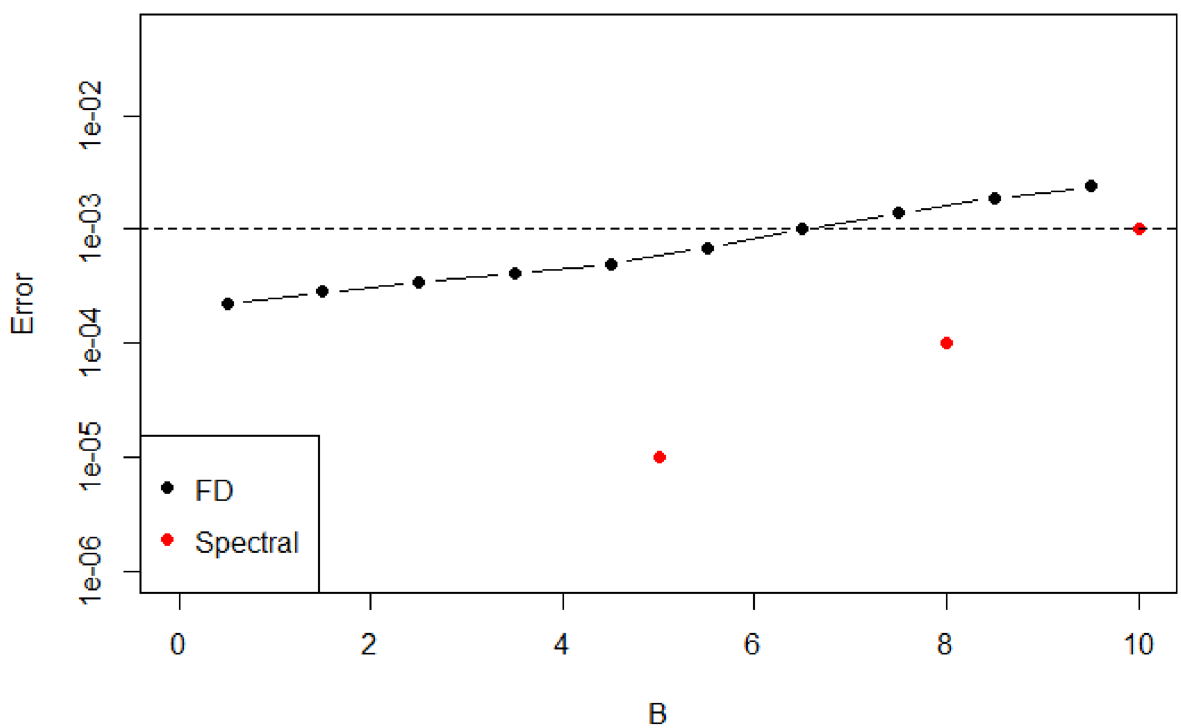

Figure 2: Figure of (e) in Problem C

4 Problem D

We would like to prove the following:

$$|\xi||\hat{f}(\xi)| \leq |f|_{TV} \tag{1}$$

Before going into the main proof, we clarify the following two statements:

1. We have different definitions of total variation for function f is continuous and not continuous. Therefore, we will discuss separately.
2. We consider $f \in \mathbb{R}$. For high dimensional function f, we just only need to replace $\frac{d}{dx}$ by the j-th directional derivative.

Proof of statement:

1. For function $f \in L_1$, the total variation has the following definition:

$$|f|_{TV} = \sup\{\int f(x)\frac{d}{dx}\phi(x)dx : \forall \phi s.t. |\phi|_\infty \leq 1\}$$

   Therefore we have

$$\begin{aligned} LHS = i\xi\hat{f}(x) &= \hat{\frac{d}{dx}}f(\xi) \\ &= \int_R \frac{d}{dx}f(x)e^{-2\pi i\xi x}dx \\ &\leq \sup|\int_R \frac{d}{dx}f(x)\phi(x)dx| \\ &= \sup|\int_R f(x)\frac{d}{dx}\phi(x)dx| := |f|_{TV} \end{aligned}$$

   where the inequality holds by observing that $||e^{-2\pi\xi x}||_\infty = 1$ and the definition of total variation (for any ϕ with norm less than or equal to 1). The second last equality holds by integration by part.

2. For a more general f, we copy another definition of Fourier transform, which might be more convenient to prove the statement:
   $\hat{f}$ is a Fourier transform for any $\phi \in S(\mathbb{R})$ (schwarz space), where:

$$(\hat{f}, \phi) = (f, \hat{\phi})$$

$$(\frac{d}{dx}f, \phi) = (f, \frac{d}{dx}\phi)$$

   Therefore, we have the following:

$$\begin{aligned} (i\xi\hat{f}, \phi) = i\xi(\hat{f}, \phi) &= i\xi(f, \hat{\phi}) \\ &= i\xi \int\int f(x)\phi(\xi)e^{-i\xi x}d\xi dx \\ &= \int\int f(x)\frac{d}{dx}(\phi(\xi)e^{-i\xi x})d\xi dx \\ &= (f, \frac{d}{dx}\hat{\phi}) \end{aligned}$$

   Therefore we have the statement to be true:

$$
\begin{aligned}
|\xi \hat{f}| &\leq \frac{1}{||\phi||} \int \int f(x) \frac{d}{dx} (\phi(\xi) e^{-i\xi x}) d\xi dx \\
&= \int \int f(x) \frac{d}{dx} (\frac{\phi(\xi)}{||\phi||} e^{-i\xi x}) d\xi dx \\
&\leq \sup | \int \int f(x) \frac{d}{dx} \phi(x) dx | := |f|_{TV}
\end{aligned}
$$

where the last inequality holds by observing that $||\frac{\phi(\xi)}{||\phi||} e^{-i\xi x}||_\infty = 1$