# Homework 3

October 24, 2020

**Deadline**: November 15th 2020, the answer to the questions will be submitted via Canvas, and the code will be posted to github, which should include instructions to run the code. No late homework will be accepted.

**Rules**: You are strongly encouraged to discuss the homework with your peers, in particular, piazza is a very good environment for discussing the homework. However, you need to write your own homework and you need disclose you sources.

A In this exercise you will solve the one-dimensional advection equation using the method of characteristics and basic Fourier analysis

$$\begin{cases} \partial_t u(x,t) + a\partial_x u(x,t) &= 0, \text{ for } x \in \mathbb{R}, t \in \mathbb{R}^+ \\ u(x,0) &= u_0(x), \text{ for } x \in \mathbb{R} \end{cases} \tag{1}$$

(a) (10 points) A characteristic curve can be seen as a curve in time-space $(x(s), t(s))$, such that $u$, the solution to the PDE, is constant along the trajectory of that curve (or integrable, i.e., the solution of the PDE is reduced to the ODE along that curve). For simplicity we will asume that the curve is linear in time. Thus, the solution in that curve needs to satisfy

$$u(x(t), t) = c. \tag{2}$$

i. Using the fact that $u$ is constant in the characteristic curve, and assuming enough regularity of $u$ and $x(s)$, show that

$$\partial_t u(x(t), t) + \partial_x u(x(t), t)x'(t) = 0. \tag{3}$$

ii. Using the question above to show that

$$x'(t) = a. \tag{4}$$

iii. Using the initial condition $x(0) = x_0$, find an expression for $x(t)$ and show that

$$u(x(t), t) = u_0(x_0). \tag{5}$$

iv. Conclude that

$$u(x,t) = u_0(x - at). \tag{6}$$

(b) (10 points) Now we solve the same problem using Fourier Analysis.

i. What are the hypothesis on $u_0(x)$ so its Fourier transform defined as

$$\hat{u}_0(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u_0(x) dx, \tag{7}$$

exists? What is the regularity of $\hat{u}_0(\xi)$? What happens with $\hat{u}_0(\xi)$ as $|\xi| \to \infty$? (You should be able to find these answers in any introductory Fourier Analysis textbook)

ii. One can extend the Fourier transform to an operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, whose inverse is given by

$$u_0(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{u}_0(\xi) d\xi. \tag{8}$$

Assuming that a function $v \in L^2(\mathbb{R})$ is differentialy and $v' \in L^2(\mathbb{R})$ show that

$$v'(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} i\xi \hat{v}(\xi) d\xi. \tag{9}$$

**Hint:** you can use the Leibniz formula to differentiate inside the integral.

iii. Show that if $v \in L^2(\mathbb{R})$, and $h \in \mathbb{R}$ is fixed, then

$$\mathcal{F}[v(\cdot + h)](\xi) = e^{-i\xi h}\hat{v}(\xi), \tag{10}$$

i.e., translation in space by $h$, equals modulation in Fourier space by a factor $e^{-i\xi h}$.
**Hint:** Use the definition of Fourier transform coupled with a change of variable.

iv. Define the Fourier tranform in space

$$\hat{u}(\xi, t) = \int_{\mathbb{R}} e^{-i\xi x} u(x,t)dx, \tag{11}$$

and show that

$$\partial_t \hat{u}(\xi, t) = -ai\xi\hat{u}(\xi, t). \tag{12}$$

v. Using the parts above show that

$$\hat{u}(\xi, t) = e^{-ai\xi}\hat{u}_0(\xi). \tag{13}$$

vi. Using Plancherel's Theorem, show that if $u_0(\cdot) \in L^2(\mathbb{R})$ then for any fixed $t$, the function $u(\cdot, t) \in L^2(\mathbb{R})$, i.e.,

$$\int_{\mathbb{R}} |u(x,t)|^2 dx < \infty. \tag{14}$$

vii. Using the inversion formula shown above, show that

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x-at)}\hat{u}_0(\xi)d\xi. \tag{15}$$

viii. Using the properties of the Fourier transform shows that

$$u(x,t) = u_0(x - at). \tag{16}$$

B (20 pts) Consider the PDE

$$u_t = \pm u_{xxxx}$$

with initial condition $u(x, 0) = u_0(x)$. Disregard boundary conditions in what follows.

(a) What sign should you choose in $\pm$ to obtain stable solutions? Justify.

(b) Assume the proper sign is taken to obtain a stable evolution. What is the stability condition on $\Delta t$ for the explicit Euler method?

(c) Your answer to the question above should indicate that the explicit Euler method is highly inefficient for this problem. In one sentence, how would remedy this situation?

Biharmonic diffusion is sometimes used as a physical dissipation mechanism in fluid mechanics, to damp high frequencies in higher proportion to lower frequencies than regular diffusion does.

C (60 pts) Consider the same $2D$ wave equation as in homework 2 , with the same initial and boundary conditions. In this question we benchmark spectral vs. FD methods.

(a) (10 pts) Consider the PDE discretized in time only, using the second order in time method of homework 2. Write the corresponding modified equation. The leading-order correction to the PDE should be a term proportional to $(\Delta t)^2\Delta^2 u$, where $\Delta^2$ is the square of the Laplacian. Explain how this observation can be used to formulate a fourth order in time method.

**Hint:** this trick is not unlike the Lax-Wendroff trick. In more generality, it involves calculus of FD operators, which was covered at the begining of this semester, and is credited to Tal-Ezer (1986).

(b) (10 pts) Explain how you would initialize your scheme in (a) for $u(\Delta t)$, (i.e. the first time-step after the initialization) while maintaining fourth order accuracy.

(c) (20 pts) Combine your method in (a), (b) with a $N$-by-$N$ tensor Chebyshev grid for the spatial derivatives. Choose the right numbers of points such that the different Chebyshev grids are nested. Show a log-log plot of the error vs. $N$, and check from this plot that your method is (at least) fourth order accurate.

**Hint:** programs 19 and 20 in Trefethen's book solve the problem, but with the basic second order in time method. His programs can be downloaded.

(d) (10 pts) What is the CFL condition for your scheme? Justify by an analysis of the eigenvalues of the matrix that implements the spatial derivatives.

**Hint:** you have already found the region of stability for the 3-point time-stepping rule in homework 2. For the eigenvalues of a spectral differentiation matrix, see the discussion around p.108 in Trefethen's book.

(e) (10 pts) A better comparison between spectral and FD methods is as follows. Fix $N$, the number of grid points per dimension, to be a reasonable power of 2 like 64 or 128. Consider the initial condition

$$u_t(x, y, 0) = \sin(B\pi x)\sin(B\pi y), \quad u(x, y, 0) = 0$$

for various integer values of $B$, so that the explicit solution is a known standing wave. Produce a plot of the log of the error of the solution after some time (e.g. $T = 0.75$ ), as a function of $B$, both for the spectral method and a finite difference method. Draw a conclusion concerning the number of points per wavelength that each method requires in order to get a fixed error level like $10^{-3}$(3 digits )

D (Bonus, 10 pts). Show that if a function $f$ is integrable and of bounded variation (possibly discontinuous), then its Fourier transform obeys

$$|\hat{f}(\xi)| \leq \|f\|_{TV}|\xi|^{-1}$$