# APPENDIX

This section includes the proofs of *Lemma 1* and *Proposition 1*.
The numbering cited in this section (e.g., (1)) corresponds to the numbered equations in the main text.

# Proof of *Lemma 1*

***Lemma 1:***

Suppose $\mu$ is a positive regular Borel measure, and $f, e$ are Simple Functions that $f(x) = \sum_{i=1}^{n} \alpha_i \chi_{I_i^f}(x)$ and $e(x) = \sum_{j=1}^{m} \beta_j \chi_{I_j^e}(x)$ then the distance $d(f, e)$ defined under (1) can be expressed as

$$d(f,e) = \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(\alpha_i, \beta_j) \cdot \mu(I_i^f \cap I_j^e).$$

***Proof:***

Follow (1), we have

$$d(f,e) = \int_{\mathbb{X}} \rho(\sum_{i=1}^{n} \alpha_i \cdot \chi_{I_i^f}(x), \sum_{j=1}^{m} \beta_j \cdot \chi_{I_j^e}(x)) \, d\mu.$$

Under the definition of Simple Function, $\{I_i^f\}_{i=1}^{n}$ and $\{I_i^e\}_{i=1}^{m}$ are both measurable partitions of $\mathbb{X}$.
Therefore,

$$\sum_{i=1}^{n} \chi_{I_i^f}(x) \equiv 1 \, , \sum_{j=1}^{m} \chi_{I_j^e}(x) \equiv 1.$$

Consequently,

$$\sum_{i=1}^{n} \alpha_i \chi_{I_i^f}(x) = \sum_{i=1}^{n} \alpha_i \cdot (\chi_{I_i^f}(x) \cdot \sum_{j=1}^{m} \chi_{I_i^e}(x)) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \cdot \chi_{I_i^f \cap I_i^e}(x).$$

Similarly,

$$\sum_{j=1}^{m} \beta_j \cdot \chi_{I_j^e}(x) = \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \cdot \chi_{I_j^e \cap I_i^f}(x).$$

Hence, we obtain

$$d(f,e) = \int_{\mathbb{X}} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(\alpha_i, \beta_j) \cdot \chi_{I_i^f \cap I_j^e}(x) \, d\mu.$$

Notably, $\{I_i^f \cap I_j^e\}$ is also a measurable partition of $\mathbb{X}$, and $\mu$ is a positive regular Borel measure. Thus, we can further simplify the expression as

$$
\begin{aligned}
d(f,e) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{I_i^f\cap I_j^e}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho(\alpha_i,\beta_j)\cdot\chi_{I_i^f\cap I_j^e}(x)\,d\mu \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{I_i^f\cap I_j^e}\rho(\alpha_i,\beta_j)\cdot\chi_{I_i^f\cap I_j^e}(x)\,d\mu \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\rho(\alpha_i,\beta_j)\ \cdot\mu(I_i^f\cap I_j^e).
\end{aligned}
$$

□

# Proof of *Proposition 1*

***Proposition 1:***

Under *Problem 2* and (4), (5) , let the function $\rho(p_1,p_2)$ is linear depend on $p_1$ for a fixed $p_2$, then the $d(\mathcal{F},\mathcal{E})$ in (8) is equivalent to

$$
d(\mathcal{F},\mathcal{E}) = \sum_{k=1}^{K}\sum_{i=1}^{L^{(k)}}\sum_{j=1}^{m}\rho(p_i^{(k)},p_j^{\mathcal{E}})\cdot\mu(I_i^{(k)}\cap I_j^{\mathcal{E}}).
$$

***Proof:***

If $\rho(p_1,p_2)$ is linear depend on $p_1$ for a fixed $p_2$, under (5) and (6), we can rewrite the $\rho$ in (8) as:

$$
\rho(p_i^{\mathcal{F}},p_j^{\mathcal{E}}) = \frac{1}{K}\sum_{k=1}^{K}\rho(p_{i^k}^{(k)},p_j^{\mathcal{E}}),
$$

where $I_i^{\mathcal{F}}\subseteq I_{i^k}^{(k)}$ for every $k$.
Therefore,

$$
\begin{aligned}
d(\mathcal{F},\mathcal{E}) &= \sum_{i=1}^{L^{\mathcal{F}}}\sum_{j=1}^{m}\Big(\frac{1}{K}\sum_{k=1}^{K}\rho(p_{i^k}^{(k)},p_j^{\mathcal{E}})\Big)\cdot\mu(I_i^{\mathcal{F}}\cap I_j^{\mathcal{E}}) \\
&= \frac{1}{K}\sum_{i=1}^{L^{\mathcal{F}}}\sum_{j=1}^{m}\sum_{k=1}^{K}\rho(p_{i^k}^{(k)},p_j^{\mathcal{E}})\cdot\mu(I_i^{\mathcal{F}}\cap I_{i^k}^{(k)}\cap I_j^{\mathcal{E}}).
\end{aligned}
$$

Furthermore, for every $l\neq i^k$, we have $I_i^{\mathcal{F}}\cap I_l^{(k)}=\emptyset$, which implies:

$$
\begin{aligned}
d(\mathcal{F},\mathcal{E}) &= \frac{1}{K}\sum_{i=1}^{L^{\mathcal{F}}}\sum_{j=1}^{m}\sum_{k=1}^{K}\sum_{l=1}^{L^{(k)}}\rho(p_l^{(k)},p_j^{\mathcal{E}})\cdot\mu(I_i^{\mathcal{F}}\cap I_l^{(k)}\cap I_j^{\mathcal{E}}) \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{l=1}^{L^{(k)}}\sum_{j=1}^{m}\rho(p_l^{(k)},p_j^{\mathcal{E}})\cdot\sum_{i=1}^{L^{\mathcal{F}}}\mu(I_i^{\mathcal{F}}\cap I_l^{(k)}\cap I_j^{\mathcal{E}}).
\end{aligned}
$$

As we defined in *Lemma 1*, $\mu$ is a positive regular Borel measure and $\{I_i^{\mathcal{F}}\}_{i=0}^{L^{\mathcal{F}}}$ is a measurable partition of $\mathbb{X}$. Hence, we obtain

$$\sum_{i=1}^{L^{\mathcal{F}}} \mu(I_i^{\mathcal{F}} \cap I_l^{(k)} \cap I_j^{\mathcal{E}}) = \mu((\bigcup_{i=1}^{L^{\mathcal{F}}} I_i^{\mathcal{F}}) \cap I_l^{(k)} \cap I_j^{\mathcal{E}}) = \mu(I_l^{(k)} \cap I_j^{\mathcal{E}}).$$

Using these results, we can simplify the expression for $d(\mathcal{F}, \mathcal{E})$ as follows:

$$\begin{aligned} d(\mathcal{F}, \mathcal{E}) &= \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{L^{(k)}} \sum_{j=1}^{m} \rho(p_l^{(k)}, p_j^{\mathcal{E}}) \cdot \mu(I_l^{(k)} \cap I_j^{\mathcal{E}}) \\ &= \frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{L^{(k)}} \sum_{j=1}^{m} \rho(p_i^{(k)}, p_j^{\mathcal{E}}) \cdot \mu(I_i^{(k)} \cap I_j^{\mathcal{E}}). \end{aligned}$$

□