APPENDIX

This section includes the proofs of *Lemma 1* and *Proposition 1*.
The numbering cited in this section (e.g., (1)) corresponds to the numbered equations in the main text.

Proof of *Lemma 1*

Lemma 1:

Suppose μ is a positive regular Borel measure, and f, e are Simple Functions that $f(x) = \sum_{i=1}^{n} \alpha_i \chi_{I_i^f}(x)$ and $e(x) = \sum_{j=1}^{m} \beta_j \chi_{I_j^e}(x)$ then the distance $d(f, e)$ defined under (1) can be expressed as

$$d(f,e) = \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(\alpha_i, \beta_j) \cdot \mu(I_i^f \cap I_j^e).$$

Proof:

Follow (1), we have

$$d(f,e) = \int_{\mathbb{X}} \rho(\sum_{i=1}^{n} \alpha_i \cdot \chi_{I_i^f}(x), \sum_{j=1}^{m} \beta_j \cdot \chi_{I_j^e}(x))\, d\mu.$$

Under the definition of Simple Function, $\{I_i^f\}_{i=1}^{n}$ and $\{I_i^e\}_{i=1}^{m}$ are both measurable partitions of $\mathbb{X}$.
Therefor

$$\sum_{i=1}^{n} \chi_{I_i^f}(x) \equiv 1\,, \sum_{j=1}^{m} \chi_{I_j^e}(x) \equiv 1.$$

Consequently,

$$\sum_{i=1}^{n} \alpha_i \chi_{I_i^f}(x) = \sum_{i=1}^{n} \alpha_i \cdot (\chi_{I_i^f}(x) \cdot \sum_{j=1}^{m} \chi_{I_i^e}(x)) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \cdot \chi_{I_i^f \cap I_i^e}(x).$$

Similarly,

$$\sum_{j=1}^{m} \beta_j \cdot \chi_{I_j^e}(x) = \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \cdot \chi_{I_j^e \cap I_i^f}(x).$$

Hence, we obtain

$$d(f,e) = \int_{\mathbb{X}} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(\alpha_i, \beta_j) \cdot \chi_{I_i^f \cap I_j^e}(x)\, d\mu.$$

Notably, $\{I_i^f \cap I_j^e\}$ is also a measurable partition of $\mathbb{X}$, and μ is a positive regular Borel measure. Thus, we can further simplify the expression as

$$\begin{aligned} d(f,e) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{I_i^f\cap I_j^e}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho(\alpha_i,\beta_j)\cdot\chi_{I_i^f\cap I_j^e}(x)\,d\mu \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{I_i^f\cap I_j^e}\rho(\alpha_i,\beta_j)\cdot\chi_{I_i^f\cap I_j^e}(x)\,d\mu \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\rho(\alpha_i,\beta_j)\ \cdot\mu(I_i^f\cap I_j^e). \end{aligned}$$

□

Proof of *Proposition 1*

Proposition 1:

Under *Problem 2* and (4), (5) , let the function $\rho(p_1,p_2)$ is linear depend on p_1 for a fixed p_2, then the $d(\mathcal{F},\mathcal{E})$ in (8) is equivalent to

$$d(\mathcal{F},\mathcal{E}) = \sum_{k=1}^{K}\sum_{i=1}^{L^{(k)}}\sum_{j=1}^{m}\rho(p_i^{(k)},p_j^{\mathcal{E}})\cdot\mu(I_i^{(k)}\cap I_j^{\mathcal{E}}).$$

Proof:

If $\rho(p_1,p_2)$ is linear depend on p_1 for a fixed p_2, under (5) and (6), we can rewrite the ρ in (8) as:

$$\rho(p_i^{\mathcal{F}},p_j^{\mathcal{E}}) = \frac{1}{K}\sum_{k=1}^{K}\rho(p_{i^k}^{(k)},p_j^{\mathcal{E}}),$$

where $I_i^{\mathcal{F}} \subseteq I_{i^k}^{(k)}$ for every k.
Therefor,

$$\begin{aligned} d(\mathcal{F},\mathcal{E}) &= \sum_{i=1}^{L^{\mathcal{F}}}\sum_{j=1}^{m}\left(\frac{1}{K}\sum_{k=1}^{K}\rho(p_{i^k}^{(k)},p_j^{\mathcal{E}})\right)\cdot\mu(I_i^{\mathcal{F}}\cap I_j^{\mathcal{E}}) \\ &= \frac{1}{K}\sum_{i=1}^{L^{\mathcal{F}}}\sum_{j=1}^{m}\sum_{k=1}^{K}\rho(p_{i^k}^{(k)},p_j^{\mathcal{E}})\cdot\mu(I_i^{\mathcal{F}}\cap I_{i^k}^{(k)}\cap I_j^{\mathcal{E}}). \end{aligned}$$

Furthermore, for every $l \neq i^k$, we have $I_i^{\mathcal{F}} \cap I_l^{(k)} = \emptyset$, which implies:

$$\begin{aligned} d(\mathcal{F},\mathcal{E}) &= \frac{1}{K}\sum_{i=1}^{L^{\mathcal{F}}}\sum_{j=1}^{m}\sum_{k=1}^{K}\sum_{l=1}^{L^{(k)}}\rho(p_l^{(k)},p_j^{\mathcal{E}})\cdot\mu(I_i^{\mathcal{F}}\cap I_l^{(k)}\cap I_j^{\mathcal{E}}) \\ &= \frac{1}{K}\sum_{k=1}^{K}\sum_{l=1}^{L^{(k)}}\sum_{j=1}^{m}\rho(p_l^{(k)},p_j^{\mathcal{E}})\cdot\sum_{i=1}^{L^{\mathcal{F}}}\mu(I_i^{\mathcal{F}}\cap I_l^{(k)}\cap I_j^{\mathcal{E}}). \end{aligned}$$

As we defined in *Lemma 1*, μ is a positive regular Borel measure and $\{I_i^{\mathcal{F}}\}_{i=0}^{L^{\mathcal{F}}}$ is a measurable partition of $\mathbb{X}$. Hence, we obtain

$$\sum_{i=1}^{L^{\mathcal{F}}} \mu(I_i^{\mathcal{F}} \cap I_l^{(k)} \cap I_j^{\mathcal{E}}) = \mu((\bigcup_{i=1}^{L^{\mathcal{F}}} I_i^{\mathcal{F}}) \cap I_l^{(k)} \cap I_j^{\mathcal{E}}) = \mu(I_l^{(k)} \cap I_j^{\mathcal{E}}).$$

Using these results, we can simplify the expression for $d(\mathcal{F}, \mathcal{E})$ as follows:

$$\begin{aligned} d(\mathcal{F}, \mathcal{E}) &= \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{L^{(k)}} \sum_{j=1}^{m} \rho(p_l^{(k)}, p_j^{\mathcal{E}}) \cdot \mu(I_l^{(k)} \cap I_j^{\mathcal{E}}) \\ &= \frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{L^{(k)}} \sum_{j=1}^{m} \rho(p_i^{(k)}, p_j^{\mathcal{E}}) \cdot \mu(I_i^{(k)} \cap I_j^{\mathcal{E}}). \end{aligned}$$

□