Linear Filters and Convolution

Ahmed Ashraf

Linear Time(Shift) Invariant (LTI) Systems

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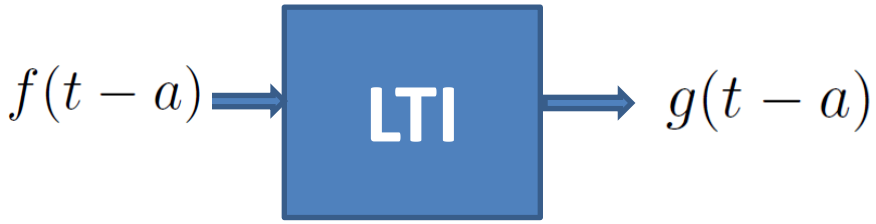

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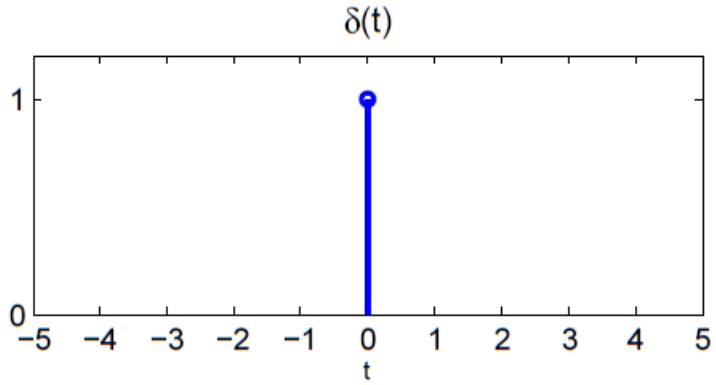
i) If $f_1(t)$ produces $g_1(t)$, *AND* $f_2(t)$ produces $g_2(t)$,
then for a linear system,
$f_1(t) + f_2(t)$ *produces* $g_1(t) + g_2(t)$.

(This property is called SUPERPOSITION)

Impulse Functions



Impulse Centered at the Origin

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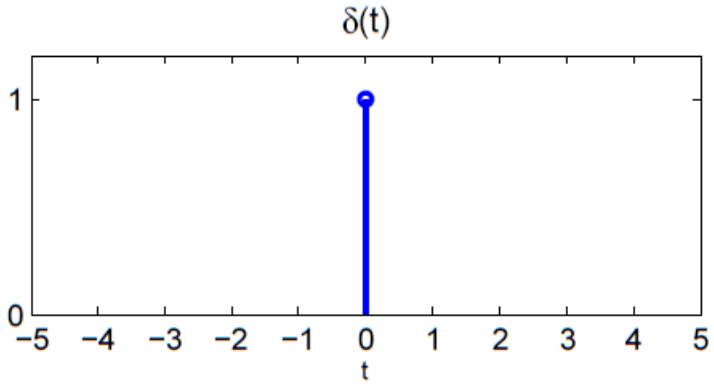

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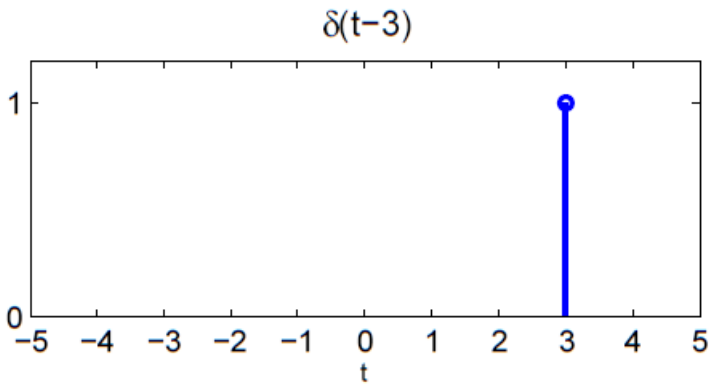

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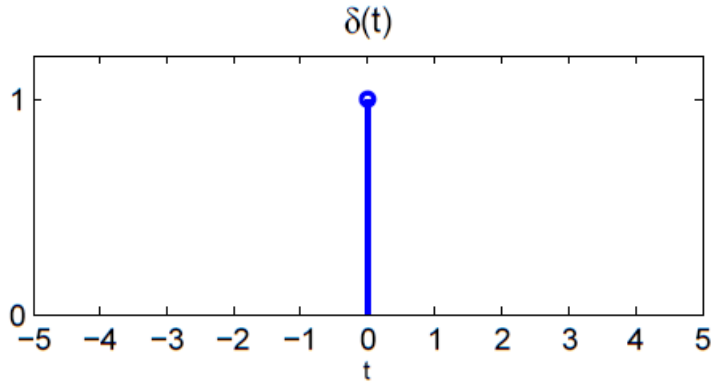

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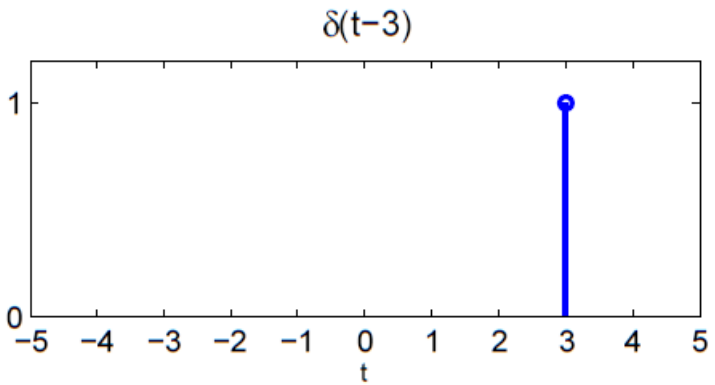

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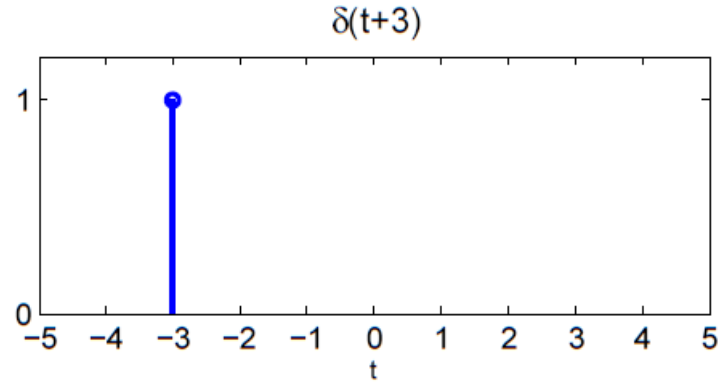

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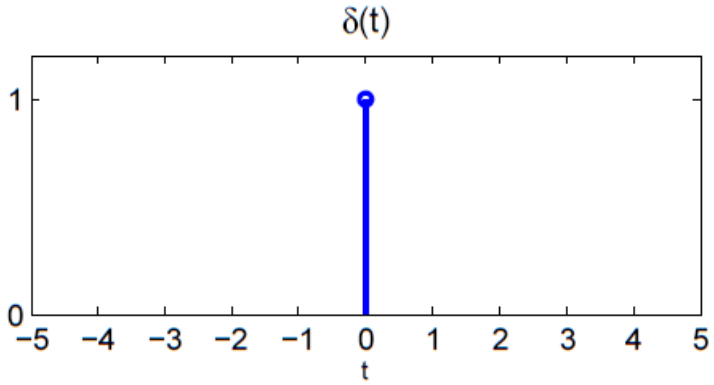

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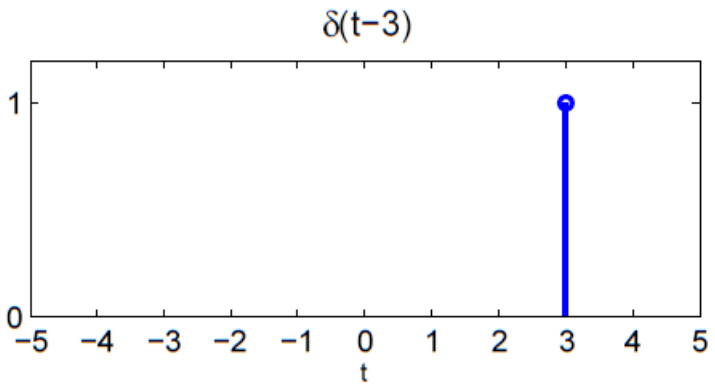

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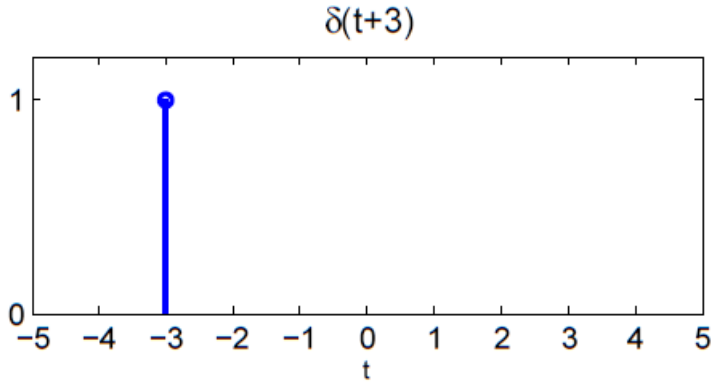

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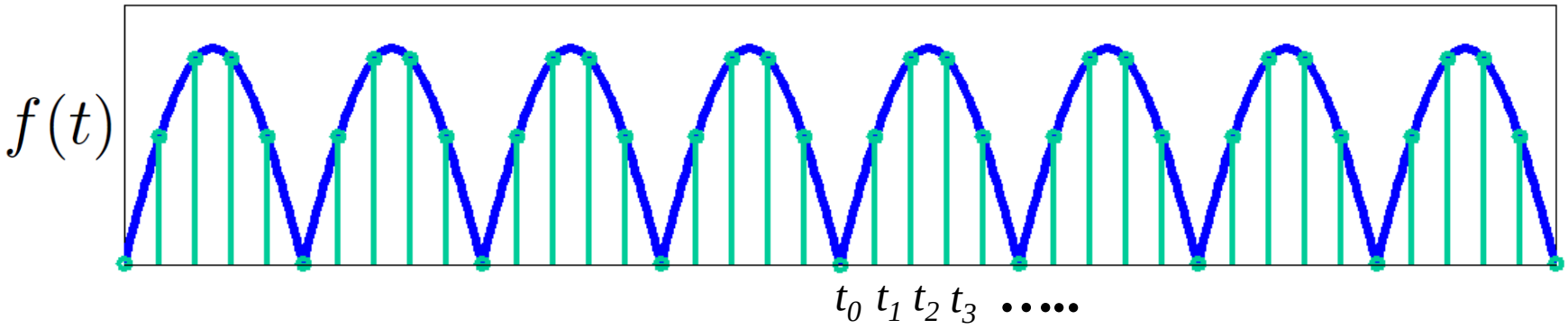
To figure out the direction of the shift, imagine this as a shift in the origin, and see where the argument of the impulse function is zero, i.e t-3 is zero for t=+3, and t+3 is zero for t=-3

Arbitrary Function as a weighted sum of shifted impulses



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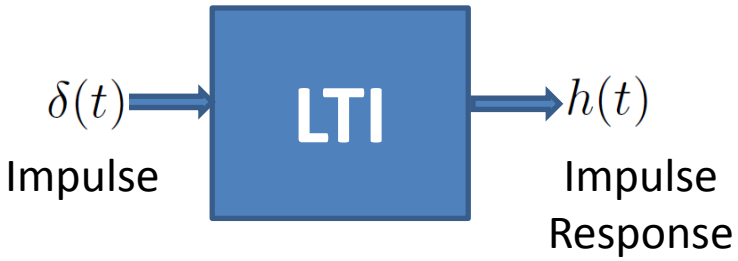

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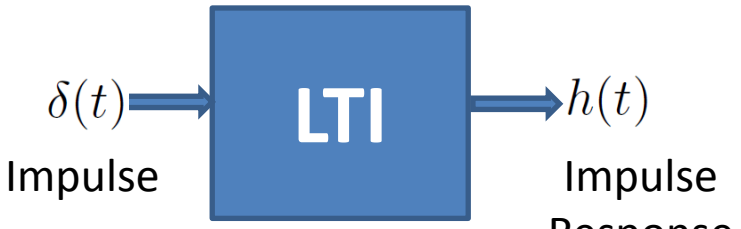

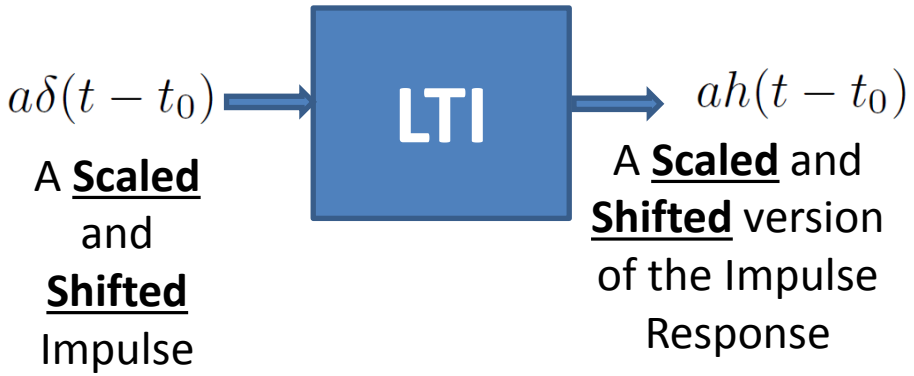

Given Impulse Response, Computing output to arbitrary function

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This sum is the convolution of f(t) and h(t), written as f(t)*h(t), and IS the output of the filter, i.e. the output is the correlation of the *flipped* h(t) with f(t)

Convolution between f(t) and h(t)

$$f(t) * h(t) = \sum_i f(t_i) h(t - t_i)$$

- *i.e., the concepts of convolution, flipping one signal, and then taking its correlation with the input to get the output are NOT handed to us by fiat. Rather they naturally emerge from the properties of Linearity and Time/Shift Invariance.*

- ***Homework**: Extend these concepts for 2D images and material covered in Lecture 3.*