MAT1856/APM466 Assignment 1

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Fundamental Questions - 25 points

1.

    (a) Governments issue bonds (rather than directly printing money) because increasing the monetary base outright can cause inflation and erode currency credibility, while bond issuance provides a controlled borrowing mechanism that aligns with monetary policy objectives.

    (b) For example, if investors foresee weaker growth and lower future short-term rates, they may buy longer-dated bonds (driving their yields down) even as shorter-term yields remain relatively higher, thereby flattening the yield curve from the long end.

    (c) Quantitative easing (QE) is an unconventional monetary policy in which a central bank buys large amounts of long-term securities (e.g., government bonds, mortgage-backed securities) to inject liquidity and lower long-term interest rates, and since COVID-19 began, the Federal Reserve has conducted extensive QE by purchasing such assets to stabilize financial markets and support economic activity.

2. The 10 bonds that I chose are listed below:

    - CAN 1.25 Mar 25, CAN 0.50 Sep 25, CAN 0.25 Mar 26, CAN 1.00 Sep 26, CAN 1.25 Mar 27, CAN 2.75 Sep 27, CAN 3.50 Mar 28, CAN 3.25 Sep 28, CAN 4.00 Mar 29, CAN 3.50 Sep 29

    Reasons for Selection:

    - **Even Maturity Distribution:** These bonds mature at regular intervals from March 2025 to September 2029, ensuring a smooth coverage of the 0–5 year range.
    - **Semi-Annual Coupon Structure:** All bonds use a standard March/September coupon schedule (constant spacing), making yield calculations and spot-rate bootstrapping straightforward.
    - **Representative Coupon Rates:** The range of coupons (0.25% to 4.00%) captures typical market conditions, providing a balanced set of yield points.
    - **Liquidity and Market Transparency:** Most of these issues are benchmarks or near-benchmark, improving reliability of observed market yields.
    - **Ease of Bootstrapping:** Avoiding zero-coupon or highly unusual coupon structures helps keep discount factor calculations more consistent.

3. The covariance matrix of a set of stochastic processes tells us how their variances (and co-variances) are related. The eigenvectors of this matrix identify the principal directions (or main patterns) in which the processes vary—often interpreted as a level shift, tilt, or curvature in a yield-curve context—and each eigenvalue measures how much of the total variance is captured by its associated eigenvector. Consequently, the largest eigenvalue corresponds to the most dominant source of variance (e.g., a parallel shift), the second largest indicates the next significant variation (e.g., a tilt), and so forth. In Principal Component Analysis (PCA), we use these eigenvectors to reduce dimensionality by focusing on only a few principal components that explain most of the variation, thereby simplifying how we model and interpret the underlying stochastic processes.

Empirical Questions - 75 points

4.

    (a) We adopt a continuously compounded model for the bond's yield. The generic pricing formula for a coupon bond is:

    $$P = \sum_{i=1}^{n} p_i \, e^{-r(t_i)\, t_i},$$

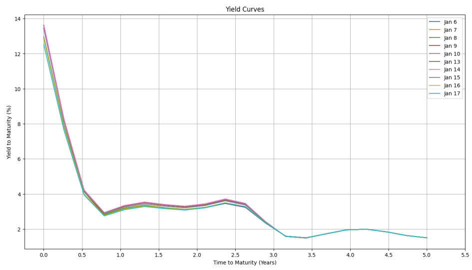
    where

    - P is the dirty price of the bond,

- n is the number of remaining coupon payments,
- p_i is the cash flow at the i-th coupon date (including face value in the last payment),
- t_i is the time (in years) until the i-th payment,
- $r(t_i)$ is the yield at t_i (modeled here as a constant r for the YTM),
- *face value* is the principal (e.g., \$100).

To produce a smooth yield curve, we apply an interpolation technique (e.g., a cubic spline) to the points in the range from 0 to 5 years (or longer, depending on the bonds' maturities).

By repeating this process for each trading day under consideration (e.g., Jan 6, Jan 7, etc.), we generate multiple sets of yield curves. Finally, we superimpose these curves in one plot to compare the day-to-day changes in yields across maturities. See the plot below.



(b)
- We start from the shortest-maturity bond, solve for its spot rate directly.
- Then, move to the next bond: discount early coupons with previously found spot rates, and solve for the new spot rate.
- Continue until we cover maturities up to 5 years.

Pseudocode

i. **Sort bonds** by ascending maturity (1,2,3,4,5 years).

ii. **Initialize** an empty list of spot rates, spotList.

iii. **For the first bond** (shortest maturity):

$$P = \frac{C+F}{(1+r_1)}, \quad \text{solve for } r_1 \text{ and append to spotList.}$$

iv. **For each subsequent bond** i:

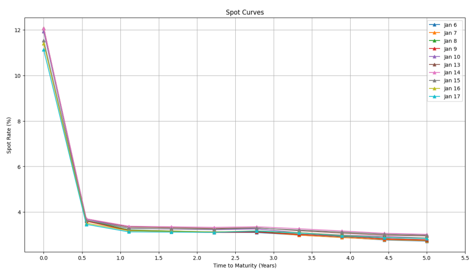
$$\text{discount earlier coupons using } r_1, \ldots, r_{i-1},$$

then solve

$$P - \sum_{\text{earlier}} \frac{C}{(1+r_k)^k} = \frac{C+F}{(1+r_i)^i},$$

for the unknown spot rate r_i. Append r_i to spotList.

v. **Repeat** for all bonds.



(c) We want the forward rates $f(1,2), f(1,3), f(1,4), f(1,5)$, where each $f(1,T)$ is the implied annual rate from year 1 to year T. Assuming we already have spot rates $r_1, r_2, \ldots, r_5$ (in years), we use:

$$e^{r_T \cdot T} = e^{r_1 \cdot 1} \times e^{f(1,T) \times (T-1)},$$

Pseudocode

i. **Obtain spot rates** r_1, r_2, r_3, r_4, r_5 for the day (from part 2).

ii. **Initialize an empty list** `forwardList`.

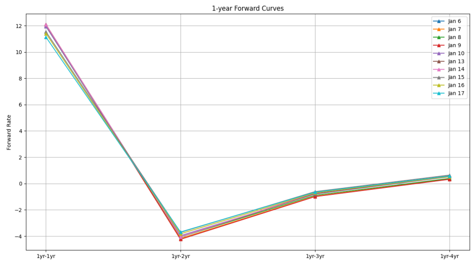
iii. **For each** T in $\{2, 3, 4, 5\}$:

A. Compute

$$f(1,T) \;=\; \frac{r_T \cdot T \;-\; r_1 \cdot 1}{T-1}.$$

B. Store $f(1,T)$ in `forwardList`.

iv. **Repeat for all days** to get forward rates for each date.



5. Results

Covariance Matrix of Yield Log-Returns

$$\begin{bmatrix} 9.21544382\times10^{-9} & -4.05011540\times10^{-8} & -8.39582626\times10^{-8} & -5.70967604\times10^{-8} & -8.71081299\times10^{-8} \\ -4.05011540\times10^{-8} & 3.21797617\times10^{-7} & 7.38225982\times10^{-7} & 2.60084512\times10^{-7} & 5.09208869\times10^{-7} \\ -8.39582626\times10^{-8} & 7.38225982\times10^{-7} & 1.71304531\times10^{-6} & 5.47648126\times10^{-7} & 1.12001623\times10^{-6} \\ -5.70967604\times10^{-8} & 2.60084512\times10^{-7} & 5.47648126\times10^{-7} & 8.19464330\times10^{-7} & 7.71434257\times10^{-7} \\ -8.71081299\times10^{-8} & 5.09208869\times10^{-7} & 1.12001623\times10^{-6} & 7.71434257\times10^{-7} & 1.04202552\times10^{-6} \end{bmatrix}$$

Covariance Matrix of Forward Rates Log-Returns

$$\begin{bmatrix} 7.19481051\times10^{-6} & 6.04018753\times10^{-6} & 5.14301127\times10^{-6} & 4.43200327\times10^{-6} \\ 6.04018753\times10^{-6} & 5.07091711\times10^{-6} & 4.31775321\times10^{-6} & 3.72086620\times10^{-6} \\ 5.14301127\times10^{-6} & 4.31775321\times10^{-6} & 3.67648428\times10^{-6} & 3.16826888\times10^{-6} \\ 4.43200327\times10^{-6} & 3.72086620\times10^{-6} & 3.16826888\times10^{-6} & 2.73032275\times10^{-6} \end{bmatrix}$$

Interpretation: Each covariance matrix entry $\mathrm{Cov}(X_i, X_j)$ (for yields) or $\mathrm{Cov}(F_k, F_\ell)$ (for forward rates) indicates how the respective daily log-returns move together over the 9 trading days observed. Larger positive values imply stronger positive co-movement, while negative values suggest inverse behavior.

6. Yield Log-Returns(Eigenvalues and eigenvectors):

$$\left[3.23537554\times10^{-6},\ 6.39320962\times10^{-7},\ 3.08517217\times10^{-8},\ -7.72671699\times10^{-23},\ 7.46436087\times10^{-24}\right]$$

$$\begin{bmatrix} -0.04296971 & -0.02237061 & -0.30773663 & 0.52843075 & -0.81297945 \\ 0.30310312 & -0.19421036 & 0.12011537 & -0.74119141 & -0.51943491 \\ 0.68563568 & -0.54175535 & -0.38052530 & 0.17754853 & 0.23678402 \\ 0.36485559 & 0.77228085 & -0.49206598 & -0.16332420 & 0.04811926 \\ 0.55051450 & 0.26807709 & 0.70988874 & 0.33644906 & -0.10425757 \end{bmatrix}$$

Forward Rates Log-Returns (Eigenvalues and eigenvectors):

$$\left[1.86723431\times10^{-5},\ 1.91581914\times10^{-10},\ -2.34016936\times10^{-22},\ 6.86089428\times10^{-16}\right]$$

$$\begin{bmatrix} -0.62073710 & 0.66639288 & -0.12865211 & 0.39249795 \\ -0.52112733 & 0.00595118 & 0.54728590 & -0.65488104 \\ -0.44372633 & -0.39346549 & -0.75452658 & -0.28103633 \\ -0.38238555 & -0.63330090 & 0.33855047 & 0.58145917 \end{bmatrix}$$

The largest eigenvalue and its corresponding eigenvector represent the principal component of variation in each covariance matrix, indicating the single direction along which the log-returns exhibit the greatest variance and hence capture most of the data's variability.

References and GitHub Link to Code

1 References

- Business Insider Bond Data Sources:
  - Short-term Bonds: https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=shortterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=&currency=184&rating=&country=19
  - Mid-term Bonds: https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=midterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=&currency=184&rating=&country=19
- Financial Definitions and Concepts:
  - Yield Curve: https://www.investopedia.com/terms/y/yieldcurve.asp
  - Spot Rate and Yield Curve: https://www.investopedia.com/terms/s/spot_rate_yield_curve.asp
  - Forward Rate: https://www.investopedia.com/terms/f/forwardrate.asp

2 GitHub Link

Ziheng Lin's GitHub: https://github.com/Ziheng-lin1030/APM466-Assignment-1