LOCALLY CONSTANT VS. CONSTANT ON CONNECTED COMPONENT

ZIHENG HUANG

0. Introduction

This note serves to compare the notion of a locally constant function and a function that is constant on connected components. We derive a criterion for when the two notions coincide and give two applications. The applications are both classical results, but it is nice to discuss them using the language of locally constant functions.

1. When is a function that is constant on connected components locally constant?

Let X, Y be topological spaces and $f : X \to Y$ a function. We say that f is locally constant at $x \in X$ if there is some neighborhood of x on which f is constant. If f is locally constant at every $x \in X$, then f is said to be locally constant on X.

Recall that the connected component C of $x \in X$ is the largest connected subset of X containing x. Equivalently, whenever $S \subset X$ is a connected subset containing x, we have $S \subset C$.

A basic property of locally constant functions that gets used all the time is the following.

Proposition 1.1. *Let X be connected and $f : X \to Y$ be locally constant. Then f is constant.*

Proof. Let $y \in Y$ be in the image of f. We show that y must be the only point in the image of f, so f is constant.

The set $f^{-1}(\{y\})$ is open because at each $x \in f^{-1}(\{y\})$, there is a neighborhood U of x which f maps identically to the point y. Similarly, the set $f^{-1}(Y \setminus \{y\})) = \bigcup_{z \in Y \setminus \{y\}} f^{-1}(\{z\})$ is open.

Since $X = f^{-1}(\{y\}) \cup f^{-1}(Y \setminus \{y\})$ is connected, we must have $f^{-1}(Y \setminus \{y\}) = \phi$. □

Proposition 1.2. *Let $X = \bigcup C_i$ where each C_i is a connected component of X and $f : X \to Y$ be locally constant. Then f is constant on each C_i.*

It feels intuitive that the converse to proposition 1.2 should be true too. However, that is not the case as illustrated by the next example.

Example 1.1. Let $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ be given the subspace topology from $\mathbb{R}$. The connected components are simply all the singletons. The function $\mathrm{id}_X : X \to X$ is constant on each connected component. It is not locally constant at $0 \in X$ because every neighborhood of 0 contains a point $\frac{1}{n}$ for some $n \in \mathbb{N}$.

The example can be generalized. A space is called **totally disconnected** if its connect components are precisely the singletons. Every function on a totally disconnected space is constant on connected components. The converse to proposition 1.2 fails for all spaces that are totally disconnected but not discrete, such as X. We can see this by considering the identity function on this kind of spaces. Some other examples of spaces that are totally disconnected but not discrete include $\mathbb{Q}$ with its standard topology and the cantor set.

Note that the space X in example 1.1 is in fact a normal space and id_X is continuous, so imposing separation axioms on the domain and codomain or continuity condition on the function is unlikely to yield a converse to proposition 1.2.

There is, nevertheless, a converse of some kind. The statement is fairly natural and the proof is not hard, but the condition might be slightly tricky to come up with in the first place.

Proposition 1.3. *If $x \in X$ has a connected neighborhood and $f : X \to Y$ is constant on connected components, then f is locally constant at x.*

It is not at all difficult to apply proposition 1.3.

Proposition 1.4. *Let X be locally connected. Then $f : X \to Y$ is locally constant if and only if X is constant on connected components.*

Both directions of proposition 1.4 are useful in practice. We give an application for each.

2. Connected components of open sets in locally connected spaces

When one first learns the definition of a connected component, one might think that connected components are always clopen. It is indeed true that every connected component is closed. This is due to the fact that the closure of a connected subset is connected. But it is not true that connected components have to open. Take the space X from example 1.1. The point $0 \in X$ is not open.

We give an easy proof that connected components of an open subset of $\mathbb{R}^n$ is open using proposition 1.4. We need a standard property of locally constant functions.

Proposition 2.1. *Let $f : X \to Y$ be locally constant at $x \in X$. Then f is continuous at x.*

Proposition 2.2. *Let X be locally connected and $U = \cup_{i \in I} U_i$ be the connected components of the open set $U \subset X$. Then each U_i is open.*

Proof. Equip the set I with the discrete topology and consider the function $f : U \to I$ given by $f(x) = i$ if $x \in U_i$. Then f is constant on each connected component of U, so it is locally constant by proposition 1.4.

By proposition 2.1, we see that f is continuous. Therefore, $U_i = f^{-1}(\{i\})$ is open. □

Corollary. *Open subsets of $\mathbb{R}^n$ have open connected component. More generally, open subsets of manifolds have open connected component.*

One might think that functions that are constant on connected components are continuous, so we could have dispensed with proposition 2.1 in the proof of proposition 2.2. However, recall from example 1.1 that any function, continuous or not, on a totally disconnected spaces that is not discrete must be constant on connected components. We cannot take a shortcut.

3. The zeroth de Rham cohomology of a manifold

As an application of proposition 1.2, which is the "usual" direction of proposition 1.4, we compute the zeroth de Rham cohomology of a manifold X. Some notation first.

Take an n-manifold X and $p \in X$. The local coordinates of a chart (U, ϕ) are $x^1, \dots, x^n$, where each $x^i = r^i \circ \phi$ and r^i is the standard ith coordinate in $\mathbb{R}^n$.

The ith partial derivative of a smooth function f at $a \in U \subset \mathbb{R}^n$ is denoted $\frac{\partial f}{\partial r^i}(a)$. This is only a notation; we are not "differentiating with respect to a function".

Relative to a chart (U, ϕ) on a manifold, the ith partial derivative of a smooth function f is

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Let us recall the definition of the zeroth de Rham cohomology.

Definition 3.1. Let X be an n-manifold. The zeroth de Rham cohomology, denoted $H^0_{DR}(X)$, is the collection of C^∞ functions $f : X \to \mathbb{R}$

such that at each $p \in X$, we have a chart (U, ϕ) containing p with local coordinates $x^1, \dots, x^n$ and $\frac{\partial f}{\partial x^i}(p) = 0$ for each $i = 1, \dots, n$. Note that $H^0_{DR}(X)$ is an $\mathbb{R}$-vector space.

We first look at smooth functions on an open ball in $\mathbb{R}^n$. For each $a \in \mathbb{R}^n$, the open ball of radius $r > 0$ is denoted $B_r(a)$.

Proposition 3.1. *Let $a \in \mathbb{R}^n$ and $f : B_r(a) \to \mathbb{R}$ be smooth. Then $\frac{\partial f}{\partial r^i}(x) = 0$ for every $x \in B_r(a)$ if and only if f is constant on $B_r(a)$.*

Proof. Fix a point $b \in B_r(a)$. The line $l : [0, 1] \to B_r(a)$ joining a to b given by $l(t) = tb + (1 - t)a$ for $t \in [0, 1]$ is differentiable on $(0, 1)$ and continuous at the end points. Applying the mean value theorem and then the chain rule to $f \circ l$ shows that $f(b) = f(a)$. The other direction is obvious. $\square$

Proposition 3.1 can be extended in the obvious manner to arbitrary open subsets of $\mathbb{R}^n$.

Proposition 3.2. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be smooth. Then $\frac{\partial f}{\partial r^i}(x) = 0$ for every $x \in U$ if and only if f is locally constant.*

We can extend proposition 3.2 further to manifolds.

Proposition 3.3. *Suppose that f is any smooth function on a manifold X. Then $f \in H^0_{DR}(X)$ if and only if f is locally constant.*

The de Rham cohomology $H^0_{DR}(X)$ is now obvious.

Proposition 3.4. *Let $X = \bigcup_{i \in I} C_i$ be the connected components of a manifold X. Then we have an isomorphism of vector spaces*

$$H^0_{DR}(X) \cong \mathbb{R}^I$$

where $\mathbb{R}^I$ is the $\mathbb{R}$-vector space of functions $I \to \mathbb{R}$.

For another nice usage of locally constant functions, see the first chapter of [1] for a proof of the fundamental theorem of algebra using differential topology.

References

[1] John Willard Milnor and David W Weaver. **Topology from the differentiable viewpoint**, volume 21. Princeton university press, 1997.
[2] Loring W Tu. **An Introduction to Manifolds**. Springer, 2011.