Mathematics: analysis and approaches
Higher level
Paper 1

Topic: Integral

Student name

75 minutes

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[54 marks]**.

Q:	1	2	3	4	5	6	7		Total
Marks:	/7	/5	/8	/7	/7	/10	/10		**/54**

10 pages

Please **do not** write on this page.

Answers written on this page
will not be marked.

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\displaystyle\int \frac{\sin x \cos x}{\cos^2 x + \sin x + 1}\, dx$. [7]

2. [Maximum mark: 5]

Find the real number a such that $\displaystyle\int_a^{2a} \frac{e^x}{x}\, dx = \int_a^{2a} \frac{e^x}{x^2}\, dx.$ [5]

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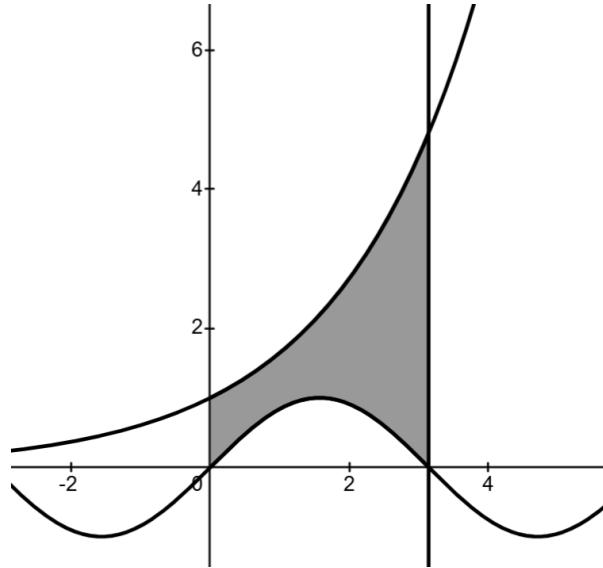$$\int_a^{2a} \frac{e^x}{x}\, dx = \int_a^{2a} \frac{e^x}{x^2}\, dx.$$

3. [Maximum mark: 8]

Using the substitution $x = a \sec \theta$, show that $\displaystyle\int_{a\sqrt{2}}^{2a} \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{1}{24a^3}(3\sqrt{3} + \pi - 6).$ [8]

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Turn over

4. [Maximum mark: 7]

Let $f(x) = e^{x/2}$ and $g(x) = \sin(x)$. Consider the region R in the first quadrant bounded by the graph of $f(x)$, the graph of $g(x)$, the y-axis, and the vertical line $x = \pi$. The region R is shaded in the graph below.

(a) Set up, but do not evaluate, a definite integral that represents the volume of the solid formed when the region R is revolved 360° about the x-axis. [1]

(b) Calculate the exact volume of the solid formed when the region R is revolved 360° about the line $y = -1$. [6]

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5. [Maximum mark: 7]

Consider the function $f(x) = \frac{x}{x^2+2}$ for $x \geq 0$.

(a) Find all axes intercepts of f. [1]

(b) Given that the region bounded by $f(x)$, the x-axis, and the line $x = c$ is $\ln 3$, find the value of c.

[6]

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6. [Maximum mark: 10]

(a) Find $\int (\ln x)^2$. [5]

(b) Hence, evaluate the improper integral $\int_0^1 (\ln x)^2$. [5]

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

7.   [Maximum mark: 10]

Consider the family of functions $f_n(x) = \cos^n x$, where $x \in \mathbb{R}, n \in \mathbb{N}$.

(a)   Show that:

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \text{ for } n > 1.$$

[4]

(b)   Hence, show that $\displaystyle\int f_n(x) \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int f_{n-2}(x) \, dx$ for $n > 1$.   [2]

(c)   Hence, find an expression for $\displaystyle\int \cos^4 x \, dx$, giving your answer in the form
$p \cos^3 x \sin x + q \cos x \sin x + rx + c$ where $p, q, r \in \mathbb{Q}^+$.   [4]

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