PERFECTOID COVERS OF ABELIAN VARIETIES

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Abstract. For an abelian variety A over an algebraically closed non-archimedean field of residue characteristic p, we show that there exists a perfectoid space which is the tilde-limit of $\varprojlim_{[p]} A$. Our proof also works for the larger class of abeloid varieties.

1. Introduction

Let p be a prime and let K be an algebraically closed non-archimedean field of residue characteristic p. For an abelian variety A over K we consider the inverse system of A under the p-multiplication morphism:

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A.$$

Via the adic analytification functor, we may see this as an inverse system of analytic adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. The primary goal of this article is to show that the "inverse limit" of this tower exists in some way and is a perfectoid space: Since inverse limits rarely exist in the category of adic spaces, in [SW13] the authors introduce the weaker notion of tilde-limits to remedy this problem. This is the notion of "limits" we are going to use. More precisely, we prove:

Theorem 1. *Let A be an abeloid variety over K, for instance an abelian variety seen as a rigid space. Then there is a unique perfectoid space A_∞ over K such that $A_\infty \sim \varprojlim_{[p]} A$ is a tilde-limit.*

The possibility of results in this direction is mentioned in §7 and §13 of [Sch14], and in the case of abelian varieties with good reduction, this theorem was proven already in [PS16] (Lemme A.16). In order to motivate our strategy for the general case, let us sketch the proof in the good reduction case (we follow Exercise 4 – 6 in [Bha19] and refer to Corollary 2.13 below for some more details):

Let A be an abelian variety of good reduction over K and let $A_{\mathcal{O}_K}$ denote its Néron model over $\mathcal{O}_K$. Let $\pi \in \mathcal{O}_K$ be a pseudo-uniformiser such that $p \in \pi\mathcal{O}_K$. Denote by $\mathcal{A}$ the π-adic completion of $A_{\mathcal{O}_K}$, a formal scheme over $\mathcal{O}_K$. Its adic generic fibre is the rigid analytification of A that we denote by the same letter. The mod π special fibre $\tilde{A} = A_{\mathcal{O}_K} \times \mathrm{Spec}(\mathcal{O}_K/\pi)$ is a group scheme over $\mathcal{O}_K/\pi$, so the map $[p]\colon \tilde{A} \to \tilde{A}$ factors through the relative Frobenius map. The inverse limit $\varprojlim_{[p]} \tilde{A}$ then exists in the category of schemes and it is relatively perfect over $\mathcal{O}_K/\pi$. We can similarly form the inverse limit $\mathcal{A}_\infty = \varprojlim_{[p]} \mathcal{A}$ in the category of formal schemes. Its adic generic fibre A_∞ is a tilde-limit of $\varprojlim_{[p]} A$, and it is perfectoid since $\varprojlim_{[p]} \tilde{A}$ is relatively perfect.

If A is an abelian variety with bad reduction, the assumption that K is algebraically closed assures that A has semi-stable reduction. In this case, the theory of Raynaud extensions provides us with a short exact sequence

$$0 \to T \to E \to B \to 0$$

of rigid groups, where $T = (\mathbb{G}_m^{\mathrm{an}})^d$ is a split rigid torus and B is the analytification of an abelian variety with good reduction, such that $A = E/M$ for a discrete lattice $M \subset E$. This short exact sequence is split locally on B, allowing us to locally write E as a product of T and an open subspace of B. Our strategy for the proof of Theorem 1, which more generally applies to any abeloid variety over K, is now similar to the good reduction case:

(1) Construct a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$. This is easy.
(2) Use T_∞ to construct a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.
(3) Study the quotient map $E \to A$ in the limit over $[p]$ to construct the desired space A_∞.

More precisely, this article is organised as follows: In §3 we use the language of fibre bundles to construct a perfectoid tilde-limit E_∞: The Raynaud extension of A arises from a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \to \overline{B} \to 0$$

by taking generic fibres and forming the pushout with respect to the open immersion $\overline{T}_\eta \to T$. Since the sequence is locally split, we can see $\overline{E} \to B$ as a principal $\overline{T}$-bundle and formation of E amounts to a change of fibre from $\overline{T}_\eta$ to T. We get the desired tilde-limit by tracing locally splitting through the tower of multiplication by $[p]$. This will also show that there is a short exact sequence of perfectoid groups

$$1 \to T_\infty \to E_\infty \to B_\infty \to 1.$$

In §4 we finish the proof of Theorem 1 by constructing A_∞ from E_∞ as follows: After choosing lattices $M \subset M_n \subset E$ that map isomorphically to M under $[p^n]\colon E \to E$, the $[p]$-multiplication tower of $A = E/M$ naturally factors into two separate towers: One is the tower of maps $E/M_{n+1} \to E/M_n$ induced from $[p]$-multiplication of E, and the other is induced from the projection maps $v^n\colon E/M \to E/M_n$. Using local splittings, one can construct a perfectoid tilde-limit $A'_\infty \sim \varprojlim_n E/M_n$ of the first tower from E_∞. It fits into a short exact sequence

$$0 \to M \to E_\infty \to A'_\infty \to 0.$$

The desired space $A_\infty \sim \varprojlim_{[p]} A$ can then be constructed from this using that the quotient maps $v^n\colon E/M \to E/M_n$ are locally split in the analytic topology. This construction also gives the following analogue of Raynaud uniformisation for A_∞: Write D_n for the kernel of v^n. Then there is a profinite perfectoid tilde-limit $D_\infty \sim \varprojlim_{[p]} D_n$ and a short exact sequence of perfectoid groups

$$0 \to M \to D_\infty \times E_\infty \to A_\infty \to 0,$$

which we regard as an analogue of the sequence $0 \to M \to E \to A \to 0$.

We give three applications of Theorem 1 in §6. The paper ends with an appendix on fibre bundles and associated fibre bundle constructions in the context of formal, adic and perfectoid spaces.

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Notation

Let K be an algebraically closed non-archimedean field, let $\mathcal{O}_K$ be the ring of integers of K and fix a pseudo-uniformiser $\pi \in \mathcal{O}_K$ such that $p \in \pi\mathcal{O}_K$.

We will use adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Huber, and perfectoid spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Scholze [Sch12]. We denote by $X \mapsto X^{\mathrm{an}}$ the analytification functor from schemes of finite type over X to analytic adic spaces over $(K, \mathcal{O}_K)$.

By a rigid spaces, we shall always mean an analytic adic space of topologically finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$. In particular, by an open cover of a rigid space we shall always mean a cover of the associated adic space, so that we do not need the notion of admissible covers.

For a π-adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$, we denote by $\mathfrak{X}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O}_K,\mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$ its adic generic fibre. Here X^{ad} is the adification in the sense of [SW13].

2. Tilde-limits of rigid groups

2.1. **Tilde-limits.** We begin with some lemmas on tilde-limits that we will need throughout.

Inverse limits often do not exist in the category of adic spaces, and neither do they in rigid spaces. Instead we use the notion of tilde-limits from [Hub96, Definition 2.4.2]:

Definition 2.1. Let $(X_i)_{i\in I}$ be a filtered inverse system of adic spaces with quasi-compact and quasi-separated transition maps, and let X be an adic space with a compatible system of morphisms $f_i \colon X \to X_i$. We write $X \sim \varprojlim X_i$ and say that X is a **tilde-limit** of the inverse system $(X_i)_{i\in I}$ if the map of underlying topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and there exists an open cover of X by affinoids $\mathrm{Spa}(A, A^+) \subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(A_i,A_i^+)\subset X_i} A_i \to A$$

has dense image, where the direct limit runs over all $i \in I$ and all open affinoid subspaces $\mathrm{Spa}(A_i, A_i^+) \subset X_i$ through which the morphism $\mathrm{Spa}(A, A^+) \subseteq X \to X_i$ factors.

Remark 2.2. As pointed out after Proposition 2.4.4 of [SW13], tilde-limits (if they exist) are in general not unique. However, Corollary 2.6 below says that perfectoid tilde-limits are unique.

We recall a few results from [SW13], §2.4 on tilde-limits that we will use frequently throughout:

Proposition 2.3 ([SW13], Proposition 2.4.2)**.** *Let (A_i, A_i^+) be a direct system of affinoids over $(K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ carrying the π-adic topology. Let $(A, A^+) = (\varinjlim A_i, \varinjlim A_i^+)$ be the affinoid algebra equipped with the topology making $\varinjlim A_{i,0}$ with the π-adic topology a ring of definition. Then*

$$\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+).$$

Proposition 2.4 ([SW13], Proposition 2.4.3)**.** *Let $X \sim \varprojlim_{i\in I} X_i$ be a tilde-limit and let $U_i \hookrightarrow X_i$ be an open immersion for some $i \in I$. Set $U_j := U_i \times_{X_i} X_j$ for $j \geq i$ and $U := U_i \times_{X_i} X$. Then*

$$U \sim \varprojlim_{j\geq i} U_j.$$

Proposition 2.5 ([SW13], Proposition 2.4.5)**.** *Let $(X_i)_{i\in I}$ be an inverse system of adic spaces over $(K, \mathcal{O}_K)$ and assume that there is a perfectoid space X such that $X \sim \varprojlim_{i\in I} X_i$. Then for any perfectoid K-algebra (B, B^+),*

$$X(B, B^+) = \varprojlim_{i\in I} X_i(B, B^+).$$

Corollary 2.6. *Any two perfectoid spaces that are tilde-limits of the same inverse system of adic spaces over $(K, \mathcal{O}_K)$ are canonically isomorphic.*

In the situation of the corollary, we will also refer to such a perfectoid space as *the* perfectoid tilde-limit of the inverse system. Of course perfectoid tilde-limits do not always exists. An example for a basic situation in which they do is the following:

Corollary 2.7. *Let $(S_i)_{i\in I}$ be an inverse system of finite sets. Let $S = \varprojlim_{i\in I} S_i$. Then the system of constant groups $\underline{S_i} = \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S_i, K), \mathrm{Map}_{\mathrm{cts}}(S_i, \mathcal{O}_K))$ has a perfectoid tilde-limit*

$$\underline{S} := \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S, K), \mathrm{Map}_{\mathrm{cts}}(S, \mathcal{O}_K)) \sim \varprojlim_{i\in I} \underline{S_i}.$$

Proof. Since S is compact, $\mathrm{Map}_{\mathrm{cts}}(S, K) = \mathrm{Map}_{\mathrm{cts}}(S, \mathcal{O}_K)[\frac{1}{\pi}]$. Perfectoidness now follows from $\mathrm{Map}_{\mathrm{cts}}(S, \mathcal{O}_K)/\pi = \mathrm{Map}_{\mathrm{lc}}(S, \mathcal{O}_K/\pi)$. The tilde-limit property follows from Proposition 2.3. □

We will need the following basic lemma later on.

Lemma 2.8. *Let (A_i, A_i^+) and (B_i, B_i^+) be direct systems of affinoids over $(K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ and $B_{i,0}$ carrying the π-adic topology. Assume that there are perfectoid tilde-limits $\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+)$ and $\mathrm{Spa}(B, B^+) \sim \varprojlim \mathrm{Spa}(B_i, B_i^+)$. Then*

$$\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B, B^+) \sim \varprojlim (\mathrm{Spa}(A_i, A_i^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B_i, B_i^+))$$

is also a perfectoid tilde-limit.

Proof. The fibre product $\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B, B^+)$ exists and is perfectoid by [Sch12, Prop 6.18]. In fact, it is represented by $\mathrm{Spa}(C, C^+)$, where $C = A\widehat{\otimes}_K B$ and C^+ is the π-adic completion of the integral closure of the image of $A^+ \otimes_{\mathcal{O}_K} B^+$.

We first check the condition on topological spaces: Since fibre products commute with limits in the category of sheaves, it follows from Proposition 2.5 that for any perfectoid field (D, D^+) over $(K, \mathcal{O}_K)$, we have

$$(\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B, B^+))(D, D^+) = \varprojlim (\mathrm{Spa}(A_i, A_i^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B_i, B_i^+))(D, D^+).$$

Since the topological space can be reconstructed from this data, it follows that the underlying topological spaces of both sides coincide.

It remains to check that if $\varinjlim A_i \to A$ has dense image and $\varinjlim B_i \to B$ has dense image, then $\varinjlim (A_i \otimes B_i) \to A \otimes B$ has dense image. As direct limits commute with tensor products, we have $\varinjlim (A_i \otimes B_i) = (\varinjlim A_i) \otimes (\varinjlim B_i)$. Now density can be checked directly on elements. □

2.2. **Perfectoid tilde-limits for rigid groups.** One reason why perfectoid tilde-limits along group homomorphisms are particularly interesting is that these again have a group structure:

Definition 2.9. A **perfectoid group** is a group object in the category of perfectoid spaces. Note that the category of perfectoid spaces over K has finite products, so this is a well-defined notion.

Proposition 2.10. *Let $(G_i)_{i\in I}$ be an inverse system of adic groups with perfectoid tilde-limit $G_\infty \sim \varprojlim_{i\in I} G_i$.*

1. *There is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms*
2. *Given a morphism of inverse systems of adic groups $(G_i)_{i\in I} \to (H_j)_{j\in J}$ and a perfectoid tilde-limit $H_\infty \sim \varprojlim_{j\in J} H_j$, there is a unique morphism of perfectoid groups $G_\infty \to H_\infty$ commuting with all projection maps.*

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, Proposition 2.5, which shows that one can argue like in the case of categorical limits. □

Let G be an adic group locally of finite type over $(K, \mathcal{O}_K)$, that is, a group object in the category of adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. Throughout we will always consider commutative groups. The main topic of study of this work is the $[p]$-multiplication tower

$$\cdots \xrightarrow{[p]} G \xrightarrow{[p]} G.$$

We will usually assume that G is p-divisible, i.e. that $[p]\colon G \to G$ is surjective.

Question 2.11. When is there a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$ is a tilde-limit?

We are primarily interested in the following examples:

1. Analytifications over $\mathrm{Spa}(K, \mathcal{O}_K)$ of finite type group schemes over K. Examples include analytifications of abelian varieties A over K and of tori T over K.
2. Generic fibres of locally topologically finite type formal group schemes over $\mathcal{O}_K$.
3. Raynaud's covering space E of an abelian variety with semi-stable reduction.

Remark 2.12. More generally, one could ask Question 2.11 for families of abelian varieties over $\mathrm{Spec}(R)$ where R is any perfectoid ring. Considering the fibers of such a family in any point of $\mathrm{Spa}(R, R^\circ)$ motivates to also study analytifications over $\mathrm{Spa}(K, K^+)$ where K^+ is any open bounded integrally closed subring of $\mathcal{O}_K$. However, one can reduce this case to the one of $K^+ = \mathcal{O}_K$.

Indeed, this follows from the following technical observation: Let $(X_i)_{i\in I}$ be an inverse system of adic spaces X_i of finite type over (K, K^+) with finite transition maps. Let $X_{i,\eta} := X_i \times_{\mathrm{Spa}(K,K^+)} \mathrm{Spa}(K, \mathcal{O}_K)$. Then the following are equivalent:

1. There is a perfectoid tilde-limit $X_\infty \sim \varprojlim_{i\in I} X_i$.
2. There is a perfectoid tilde-limit $X_{\infty,\eta} \sim \varprojlim_{i\in I} X_{i,\eta}$.

We will therefore restrict attention to the case of $K^+ = \mathcal{O}_K$ without loss of generality.

As we have already mentioned in the introduction, Question 2.11 has an affirmative answer in the case of abelian varieties of good reduction:

Proposition 2.13. *Let A be an abelian variety of good reduction over K. Then there is a perfectoid tilde-limit $A_\infty \sim \varprojlim A$.*

More generally, if $\mathfrak{G}$ is a flat commutative formal group scheme such that $[p]$-multiplication is affine, then the same proof shows that $G_\infty := (\varprojlim_{[p]} \mathfrak{G})_\eta$ is a perfectoid tilde-limit of $\varprojlim_{[p]} G$.

Lemma 2.14. *Let T be a torus over K. Then there is a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$.*

Proof. Since we assume K algebraically closed, we may choose a splitting $T \cong (\mathbb{G}_m^{\mathrm{an}})^d$ for some $d \in \mathbb{N}$. By Lemma 2.8, it suffices to consider the case of $d = 1$. For this, we may use the open embedding $\mathbb{G}_m^{\mathrm{an}} = \mathbb{P}^1 \setminus \{\infty\} \subseteq \mathbb{P}^1$. Sending $(x : y) \mapsto (x^p : y^p)$ defines a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$. The pullback of φ to $\mathbb{G}_m^{\mathrm{an}}$ is precisely $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$. $\square$

Example 2.15. If G is not p-divisible, the tilde-limit of $\varprojlim_{[p]} G$ might exist for trivial reasons: For example, consider the p-adic completion $\mathfrak{G}_a$ of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Then one can check using formal models that the trivial group $\mathrm{Spa}(K, \mathcal{O}_K) \sim \varprojlim_{[p]} \mathfrak{G}_a$ is a perfectoid tilde-limit.

3. Perfectoid tilde-limits of Raynaud extensions

In this section we study the p-multiplication tower of the Raynaud extensions associated to abeloid varieties over an algebraically closed perfectoid field K. The main result of this section is Theorem 3.7, which says that the Raynaud extension E of an abeloid variety A over K admits a $[p]$-F-formal tower, and thus there exists a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

Remark 3.1. Everything in this section also works with minor modifications over a general perfectoid field. But we opt to work over an algebraically closed field to simplify the exposition.

3.1. **Raynaud extensions.** We briefly sketch the theory of Raynaud extensions here, and refer the readers to [BL84, Lüt09, Lüt16] for more details on the setup.

Let A be an abelian variety over K. There exists a unique open rigid analytic subgroup $\overline{A}$ of A such that $\overline{A}$ admits a formal model $\overline{E}$ that is a connected smooth $\mathcal{O}_K$-group scheme fitting into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0, \tag{1}$$

where $\overline{B}$ is a formal abelian scheme over $\mathcal{O}_K$ with rigid generic fibre $B := \overline{B}_\eta$, and $\overline{T}$ is the completion of a torus $T_{\mathcal{O}_K}$ of rank r over $\mathcal{O}_K$. We set $T := T_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K$ and denote its analytification also by T. Then the rigid generic fibre $\overline{T}_\eta$ of the formal torus $\overline{T}$ canonically embeds into T. This induces a pushout exact sequence in the category of rigid groups: More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout:

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0. \end{tikzcd} \tag{2}$$

The abelian variety A can be uniformized in terms of E as follows:

Definition 3.2. A subset M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}}^r$.

Theorem 3.3. *There exists a lattice $M \subset E$ of rank equal to the rank r of the torus such that the quotient E/M exists as a rigid space and such that there is a natural isomorphism*

$$A = E/M$$

making $E \to E/M = A$ a rigid group homomorphism.

The data of the extension (1) together with the lattice $M \subset E$ is what we refer to as a Raynaud uniformisation of A. This will be the only input we need to construct the perfectoid tilde-limit A_∞. Consequently, our method generalises to the class of rigid groups which admit Raynaud uniformisation, namely to abeloid varieties:

Theorem 3.4 (Lütkebohmert, [Lüt16], Theorem 7.6.4)**.** *Let A be an abeloid variety, that is, a connected smooth proper commutative rigid group over K. Then A admits a Raynaud uniformisation.*

Note that if A is an abelian variety, then so is B ([Lüt16, Theorem 6.4.8]).

In the situation of Raynaud uniformisation, since M is discrete, the local geometry of A is determined by the local geometry of E. We will therefore first study the $[p]$-multiplication tower of E in the rest of this section and will then deduce properties of the $[p]$-multiplication tower of A in the next section.

Our strategy is to study the local geometry of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that one cannot direcly study short exact sequences locally on T, E or B. Instead, we have the following crucial lemma, which says that one may regard Raynaud extensions as T-torsors of formal schemes.

Lemma 3.5. *The short exact sequence (1) admits local sections, that is there is a cover of $\overline{B}$ by formal open subschemes U_i such that there exist local sections $s : U_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times U_i \hookrightarrow E$.*

Proof. This is proved in Proposition A.2.5 in [Lüt16], where it is fomulated in terms of the group $\mathrm{Ext}(B, T)$. Also see [BL91], §1. $\square$

Remark 3.6. In the following, we will freely work with fibre bundles of formal schemes and rigid and perfectoid spaces. For some background material on these we refer to the appendix.

The sequence (1) gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}_\eta$ via push-out along $\overline{T}_\eta \to T$ can be expressed in terms of the associated fibre bundle by saying that $E = T \times^{\overline{T}_\eta} \overline{E}_\eta$ in the sense of Definition A.8.

We can use this to prove the main result of this section, namely that E_∞ is perfectoid:

Theorem 3.7. *There is a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$. It fits into short exact sequence of perfectoid groups*

$$1 \to T_\infty \to E_\infty \to B_\infty \to 1$$

that is analytic-locally split.

Proof. We can split $[p^n] : E \to E$ into two morphisms as follows:

$$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] \arrow[d, "{[p^n]}"] & E \arrow[r, "\pi"] \arrow[d, "{[p^n]\times\pi}"] & B \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r] \arrow[d, equal] & E \times_{B,[p^n]} B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "{[p^n]}"] & 1 \\
1 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 1
\end{tikzcd}$$

Here the bottom morphism of short exact sequences is the base-change of the T-bundle $E \to B$ along $[p^n] : B \to B$, whereas the top morphism of short exact sequences is the change of fibre along $[p^n] : T \to T$. It is clear that the vertical maps in the middle compose to $[p^n] : E \to E$.

Let now $U \subseteq B$ be an open subspace on which $E \to B$ is split, then $E|_U = T \times U$. Let U_n be the pullback of U along $[p^n]$. Then it follows from the universal property of the fibre product that $E \times_{B,[p^n]} B|_{U_n} \to U_n$ is again split. Consequently, the pullback of the entire diagram to U becomes

$$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] \arrow[d, "{[p]}"] & T \times U_n \arrow[r, "\pi"] \arrow[d, "{[p]\times\mathrm{id}}"] & U_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r] \arrow[d, equal] & T \times U_n \arrow[r] \arrow[d, "{\mathrm{id}\times[p]}"] & U_n \arrow[r] \arrow[d, "{[p]}"] & 1 \\
1 \arrow[r] & T \arrow[r] & T \times U \arrow[r] & U \arrow[r] & 1.
\end{tikzcd}$$

It follows that the inverse system

$$\cdots \xrightarrow{[p]} E \xrightarrow{[p]} E$$

restricts over the open $E|_U \subseteq E$ on the base to the inverse system

$$\xrightarrow{[p]\times[p]} T \times U_n \xrightarrow{[p]\times[p]} T \times U_{n-1} \xrightarrow{[p]\times[p]} \cdots \xrightarrow{[p]\times[p]} E_{|U}.$$

To prove the theorem, it suffices to show that this system has a perfectoid tilde-limit. As there is a perfectoid tilde-limit $B_\infty \sim \varprojlim_{[p]} B$, by pullback to U there is also a perfectoid tilde-limit $U_\infty \sim \varprojlim U_n$. Since there is also a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$, Lemma 2.8 says that the above system has perfectoid tilde-limit $T_\infty \times U_\infty$, as desired.

By Lemma 2.5, the map $T \to E \to B$ in the limit over $[p]$ now produces maps $T_\infty \to E_\infty \to B_\infty$. By the above local description of E_∞, the latter morphism is locally split, which shows that this sequence is exact. □

Remark 3.8. With some work, the arguments in this section can be extended to any perfectoid base field. For instance, the Raynaud uniformisation of Theorem 3.3 might only be defined over a finite extension L of K. Our argument then gives a perfectoid space over the (necessarily perfectoid) field L. We can then use Galois descent to get a perfectoid space over our original field K. This uses that the quotient of a perfectoid space by a finite group often remains perfectoid: see Theorem 1.4 of [Han16] for details. Finally, one checks that this Galois descent commutes with tilde-limits.

4. The case of abeloid varieties

We now prove Theorem 1, building on the preceding sections. Recall our setup: Let A be an abeloid variety over K. Let E be the Raynaud extension associated to A from Proposition 3.4, which is an extension of an abeloid variety B of good reduction by a split rigid torus T of rank r, and $M \subset E$ is a lattice of rank r such that $A = E/M$.

By Proposition 3.4, the quotient map $\pi \colon E \to A$ is locally split in the analytic topology on A: As the action of M on E is totally discontinuous, for every point $x \in A$ there is an open neighbourhood U' of E such that π maps isomorphically onto an open $U := \pi(U')$ containing x. Here we are careful to distinguish $U' \subset E$ and $U \subset A$, even though the two are isomorphic via π.

We fix from now on a cover $\mathfrak{U}$ of A by opens U of this form.

The pullback of U' along $[p]\colon A \to A$ will in general be bigger than the pullback of U along $[p] : E \to E$: e.g. in characteristic 0, the first is an étale $A[p]$-torsor, whereas the latter is an étale $E[p]$-torsor, and by the Snake Lemma we have a short exact sequence

$$1 \to E[p] \to A[p] \to M/M^p \to 1$$

To relate the pullbacks, we subdivide the tower

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$$

into two partial towers. For this we make some auxiliary choices: Since K is algebraically closed, we can choose lattices $M_n \subseteq E$ such that $M_0 = M$ and $[p]\colon E \to E$ restricts to isomorphisms $M_{n+1} \to M_n$ for all n.

Remark 4.1. Such a choice is equivalent to the choice of subgroups $D_n \subseteq A[p^n]$ of order p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$. Namely, given the lattices M_n, we obtain the desired torsion subgroups by setting $D_n := M_n/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/p^nM \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ with properties as above, we recover M_n as the kernel of $E \to A \to A/D_n$.

One might call the choice of D_n for all n a partial anticanonical $\Gamma_0(p^\infty)$-structure, because if B admits a canonical subgroup (that is, if it satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A always has a partial anticanonical subgroup even if B does not have a canonical subgroup.

Following the remark, denote by D_n the torsion subgroup $M_n/M \subset A$. The quotient $A_n := A/D_n = E/M_n$ is then another abeloid variety over K and the quotient map $v^n\colon A = E/M \to A_n = E/M_n$ is an isogeny of degree p^{rn} through which $[p^n]\colon A \to A$ factors. The $[p]$-multiplication tower now splits into two towers, one written vertically, the other horizontally:

(3)
$$\begin{tikzcd}
\ddots \arrow[dr] & & \vdots & \vdots \\
& A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[r, "v"] \arrow[d, "{[p]_E}"] & A_2 \arrow[d, "{[p]_E}"] \\
& & A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[d, "{[p]_E}"] \\
& & & A.
\end{tikzcd}$$

Since each $D_n = M_n/M$ is finite étale, all horizontal maps are finite étale. The vertical tower on the other hand fits into a commutative diagram which compares it to the $[p]$-tower of E:

(4)
$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & A_1 \arrow[r] \arrow[d, "{[p]_E}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Definition 4.2. Let $M_\infty := \varprojlim M_n$. Since the transition maps are all isomorphisms, the projections $M_\infty \to M$ are isomorphisms as well. By Proposition 2.5, we get a natural map $M_\infty \to E_\infty$.

Proposition 4.3. *There is a perfectoid tilde-limit $A'_\infty \sim \varprojlim A_n$. It fits into a short exact sequence of perfectoid groups*

$$0 \to M_\infty \to E_\infty \to A'_\infty \to 0$$

that is locally split in the analytic topology on $A'_\infty \to A$.

Proof. We work locally on opens $U' \subset E$ mapping isomorphically to U in our cover $\mathfrak{U}$ of A. Write $\pi_n \colon E \to A_n$ for the quotient map. Since the rows in (4) are exact, and the transition maps on the left are isomorphisms, it follows that for each $n \in \mathbb{N}$, the quotient map π_n sends the pullback $U'_n := [p^n]^{-1}(U')$ isomorphically onto $U_n := \pi_n(U'_n) \subseteq A_n$. Thus (4) is locally of the form

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & M_1 \times U'_1 \arrow[r] \arrow[d, "{[p]}"] & U_1 \arrow[r] \arrow[d, "{[p]_E}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \times U' \arrow[r] & U \arrow[r] & 0. \end{tikzcd} \tag{5}$$

Let U_∞ be the pullback of U' along $E_\infty \to E$. We have $U_\infty \sim \varprojlim U'_n \cong \varprojlim U_n$. The system $(U_n)_{n\in\mathbb{N}}$ thus has a perfectoid tilde-limit. This shows that $\varprojlim A_n$ has a perfectoid tilde-limit. We can therefore apply Proposition 2.5 to get a morphism $E_\infty \to A'_\infty$, obtaining the desired short exact sequence in the limit over diagram (4) since the transition maps in (5) respect the splitting. $\square$

We will keep the notation introduced in the above proof: U' is an open of E mapping isomorphically to $U \subset A$. The open $U'_n := [p^n]^{-1}(U') \subset E$ maps isomorphically to its image $U_n \subset A_n$ and we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_n \arrow[r] \arrow[d, equal] & M_n \times U'_n \arrow[r] \arrow[d, hook] & U_n \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r, "\pi_n"] & A_n \arrow[r] & 0. \end{tikzcd}$$

To construct a tilde-limit for $\varprojlim A$, we use the fact that the horizontal maps in diagram (3) are all finite étale. They are even finite covering maps, in the following sense:

Lemma 4.4. *For any $n \geq 0$, the preimage of $U_n \subset A_n$ under the horizontal map $v^n \colon A \to A_n$ is isomorphic to p^{rn} disjoint copies of U_n. More canonically, it is isomorphic to $D_n \times U_n$, where $D_n = M_n/M$ (see Remark 4.1).*

Proof. For the first part, we observe that the map v^n fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r] & A_n \arrow[r] & 0 \end{tikzcd} \tag{6}$$

where the map on the left is the natural inclusion. Upon restriction to $U_n \subset A_n$, this becomes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & M_n \times U'_n \arrow[r] \arrow[d, equal] & (v^n)^{-1}(U_n) \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & M_n \times U'_n \arrow[r] & U_n \arrow[r] & 0 \end{tikzcd} \tag{7}$$

and the claim follows the fact that M is a discrete lattice of rank r, and from $U'_n \cong U_n$. $\square$

Definition 4.5. The $[p]$-multiplication on E maps M_{n+1} onto M_n and therefore the $[p]$-multiplication tower of A induces a tower

$$\cdots \xrightarrow{[p]} D_{n+1} \xrightarrow{[p]} D_n \to \cdots .$$

Since K is algebraically closed, the finite étale groups D_n are already constant. By Lemma 2.7, there is a profinite perfectoid group D_∞ such that

$$D_\infty \sim \varprojlim_n D_n.$$

Theorem 1 is part of the following theorem:

Theorem 4.6. *(1) There is a perfectoid space A_∞ which is a tilde-limit of $\varprojlim_{[p]} A$.*

(2) It is independent up to canonical isomorphism of the auxiliary choice of lattices M_n with $D_n = M_n/M$, but it remembers the choice as a pro-finite étale closed subgroup $D_\infty \subseteq A_\infty$.

(3) The preimage of any $U \in \mathfrak{U}$ under the projection $A_\infty \to A$ is isomorphic to $D_\infty \times U_\infty$.

(4) We have a commutative diagram of short exact sequences of perfectoid groups

$$\begin{tikzcd}
0 \arrow[r] & M_\infty \arrow[r] \arrow[d, hook] & E_\infty \arrow[r] \arrow[d, hook] & A'_\infty \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D_\infty \arrow[r] & A_\infty \arrow[r] & A'_\infty \arrow[r] & 0
\end{tikzcd}$$

both of which are locally split in the analytic topology.

(5) One can describe A_∞ as the associated fibre bundle

$$A_\infty = D_\infty \times^{M_\infty} E_\infty.$$

In particular, we have an analytic-locally split short exact sequence of perfectoid groups

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the antidiagonal embedding of M_∞ into $D_\infty \times E_\infty$.

Remark 4.7. We think of part (5) as the analogue of the Raynaud uniformisation

$$0 \to M \to E \to A \to 0.$$

Here we note that while the map $E \to A$ is a quotient, in the limit over $[p]$ it becomes an immersion $E_\infty \hookrightarrow A_\infty$: The reason is that the projective system $(M, [p])$ has vanishing lim but non-vanishing Rlim^1, for instance, when considered as abelian sheaves on perfectoid spaces for the pro-étale topology in the sense of [Sch17] (assuming that K is of characteristic 0). A toy example of this phenomenon would be the inverse system over $[p]$ on the short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ whose limit yields an exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \varprojlim^1_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

We therefore think the quotient D_∞/M_∞ implicit in part (5) as being an incarnation of $\mathrm{Rlim}^1_{[p]} M_\infty$.

Proof of Theorem 4.6. We keep the notation from the proof of Proposition 4.3: We have a cover of A_n by open subsets U_n and a perfectoid open subspace $U_\infty \subseteq E_\infty$ for which $U_\infty \sim \varprojlim U_n$.

By Lemma 4.4, the restriction of diagram (3) to the open U of the bottom A becomes

$$\begin{tikzcd}
\ddots & \vdots & \vdots \\
D_2 \times U_2 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & v^{-1}(U_2) \arrow[r, "v"] \arrow[d, "{[p]_E}"] & U_2 \arrow[d, "{[p]_E}"] \\
& D_1 \times U_1 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & U_1 \arrow[d, "{[p]_E}"] \\
& & U.
\end{tikzcd}$$

Hence the restriction of the tower $\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$ to U becomes the inverse system

$$\cdots \to D_{n+1} \times U_{n+1} \to D_n \times U_n \to \cdots .$$

By Lemma 2.8 this inverse system has perfectoid tilde-limit $D_\infty \times U_\infty$. These local tilde-limits glue together to give the desired tilde-limit A_∞. This proves parts (1), (2) and (3), and shows that the second row of part (4) is locally split and in particular exact.

The first row in part (4) is from Proposition 4.3. Part (5) follows immediately from part (4). □

Remark 4.8. When working over a general perfectoid base field, the lattices M_n may no longer be defined over K. Instead, one can show that the natural map $A[p^n] \times U_n \to V_n$ is an étale $E[p^n]$-torsor for the diagonal action where V_n is the pullback of U along $[p^n]\colon A \to A$. The point is that this torsor is split when K is algebraically closed.

5. Applications

In this section, we give three applications of our main result. For all of these, we assume that K is of characteristic 0, i.e. K is an algebraically closed non-archimedean field extension of $\mathbb{Q}_p$.

5.1. **Uniformisation.** Our first application is a "p-adic uniformisation" of abelian varieties. Recall that any abelian variety A over $\mathbb{C}$ of dimension d has a uniformisation in terms of a complex torus $A \cong \mathbb{C}^d/\Lambda$ for some $2d$-dimensional lattice $\Lambda \subseteq \mathbb{C}^d$. More canonically, it admits the presentation

$$A \cong \mathrm{Lie}A/H_1(A, \mathbb{Z}).$$

We have the following analogue of this over K: Let A be an abeloid variety over K of dimension d, considered as a rigid space. Then in the limit over n, the short exact sequences

$$0 \to A[p^n] \to A \to A \to 0$$

give rise to a short exact sequence of sheaves on perfectoid K-algebras with the pro-étale topology

$$0 \to T_pA \to A_\infty \to A \to 0.$$

Using the language of diamonds from [Sch17], we then have:

Corollary 5.1. *The diamond $A^\diamond$ associated to A has a canonical presentation*

$$A^\diamond = A_\infty/T_pA$$

given by the perfectoid space A_∞ and the pro-étale subgroup T_pA.

Of course this p-adic uniformisation of A is very closely related to the uniformisation of the associated p-divisible group $A[p^\infty]$ described in [SW13] and [Sch13b, §4]: Indeed, in the language used there, we have a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & T_p(A[p^\infty]) \arrow[r] \arrow[d, equal] & \widetilde{A[p^\infty]} \arrow[r] \arrow[d, hook] & A[p^\infty] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_pA \arrow[r] & A_\infty \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

We note that for two abelian varieties A and B of dimension d, the universal covers A_∞ and B_∞ are different in general, so that this is only a "uniformisation" in a rather weak sense. However, they are canonically isomorphic if A and B are abelian varieties of good reduction with the same special fibre, or if A and B are p-power isogeneous, so that in these cases we can really think of T_pA as a a $2d$-dimension $\mathbb{Z}_p$-lattice determining A.

5.2. **Stein property.** As a second application, we can combine our main theorem with work of Reineke to deduce the following Artin vanishing result:

Corollary 5.2. *Let A be an abeloid variety over K. Let L be a constructible sheaf of $\mathbb{F}_p$-modules on $A_{\text{ét}}$. Then for any $i > \dim A$,*

$$\varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^*L) = 0.$$

Proof. Due to Theorem 1, we can apply [Rei19, Theorem 3.3] to the system $\cdots \to A \xrightarrow{[p]} A$. □

A theorem of Artin and Grothendieck states if X is an affine algebraic variety over K, then $H^i_{\text{ét}}(X, L) = 0$ for any constructible $\mathbb{F}_p$-module L and any $i > \dim A$. However, the rigid analogue of this statement is false in general. The point of the Corollary is that an analogue of this vanishing statement is true for the pullback of L to A_∞ in the following sense: Consider the morphism of sites $\nu\colon A_{\text{proét}} \to A_{\text{ét}}$. Then by regarding A_∞ as an object in $A_{\text{proét}}$ via the pro-étale morphism $A_\infty \to A$, one can show

$$H^i_{\text{proét}}(A_\infty, \nu^*L) = \varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^*L).$$

By results from [Sch13a], the space A_∞ has a "Stein space"-like property in the sense that we have $H^j(V, \nu^*L) = 0$ for any affinoid perfectoid $V \subseteq A_\infty$ and any $j > 0$. One can use this to reduce to a computation in Čech cohomology, which shows that the left hand side vanishes for $i > \dim A$.

5.3. **Universal perfectoid covers of curves.** As a third application, we describe how one can obtain functorial perfectoid pro-étale covers of curves over K. This was observed by Hansen [Han15].

Let C be a smooth projective curve of genus $g \geq 1$ over K, which we consider as an analytic adic space. By [Lüt93, Theorem 3.1], GAGA induces an equivalence of categories between finite étale covers of the scheme C and finite étale covers of the adic space C. We can therefore fix a base point $x \in C$ and study the usual étale fundamental group $\pi_1(C, x)$ using the language of adic spaces. This is a profinite group, and for every open subgroup $G \subseteq \pi_1(C, x)$, there is a corresponding finite étale morphism $C_G \to C$. For any two open subgroups $G_1 \subseteq G_2 \subseteq \pi_1(C, x)$, there is a natural morphism $C_{G_1} \to C_{G_2}$. For varying G, one therefore has a filtered inverse system $(C_G)_{G\subseteq\pi_1(C,x)}$ which we may regard as an object in $C_{\text{proét}}$.

Corollary 5.3. *There is a perfectoid tilde-limit $C_\infty \sim \varprojlim_G C_G$ where G ranges over the open subgroups of $\pi_1(C, x)$.*

Sketch. We construct C_∞ in two steps. The choice of the base point x gives $\iota\colon C \to A$, an embedding of C into its Jacobian. Let C_n be the pullback of C along the map $[p^n]\colon A \to A$. Combining our main theorem with [Sch15, Lemma II.2.2], we can pull back perfectoid tilde-limits along closed immersions and hence get a perfectoid space $C'_\infty \sim \varprojlim C_n$ with a Zariski closed embedding $C'_\infty \to A_\infty$.

We can now use the fact that pro-étale covers of perfectoid spaces are again perfectoid to construct a perfectoid cover C_∞ of C'_∞ that packages up the entire étale fundamental group of C. As we are assuming that K has characteristic 0, the maps $[p^n]\colon A \to A$ are finite étale, so the induced covers $C_n \to C$ are finite étale. The inverse system

$$\cdots \to C_n \to \cdots \to C_1 \to C$$

therefore corresponds to a chain of subgroups

$$\cdots < H_n < \cdots < H_1 < \pi_1(C,x).$$

For any open subgroup G of $\pi_1(C,x)$ corresponding to the finite étale cover $C_G \to C$, we have a decreasing sequence of positive integers

$$\cdots \leq [H_n : H_n \cap G] \leq \cdots \leq [H_1 : H_1 \cap G] \leq [\pi_1(C,x) : \pi_1(C,x) \cap G].$$

So there is an integer d such that for n sufficiently large, we have $[H_n : H_n \cap G] = d$. Translating back to the language of finite étale covers, we see that for such n, the map

$$C_{H_{n+1}\cap G} \to C_{H_n \cap G} \times_{C_{H_n}} C_{H_{n+1}}$$

coming from the universal property of fibre product is an isomorphism: Both spaces are finite étale covers of $C_{H_{n+1}}$ of degree d, so the map is a finite étale cover of degree 1. This implies that the natural morphism $\varprojlim C_{H_n\cap G} \to \varprojlim C_{H_n}$ of objects of $C_{\text{proét}}$ is finite étale in the sense of [Sch13a, Definition 3.9]. To simplify notation, we write this morphism as $C_{G,\infty} \to C'_\infty$ (one can also think of this as the corresponding map of perfectoid spaces obtained by applying the functor from perfectoid objects in $C_{\text{proét}}$ to perfectoid spaces described just after [Sch13a, Lemma 4.5]).

We can now rewrite the pro-étale object $\varprojlim_{G\to 1} C_G$ as

$$\varprojlim_{G\to 1} \varprojlim_{n\to\infty} C_{H_n\cap G} = \varprojlim_{G\to 1} C_{G,\infty}.$$

As the $C_{G,\infty}$ have compatible finite étale maps to C'_∞, we obtain a pro-étale map (again in the sense of [Sch13a, Definition 3.9])

$$\varprojlim_{G\to 1} C_{G,\infty} \to C'_\infty.$$

By [Sch13a, Lemma 4.6], pro-étale covers of perfectoid objects are again perfectoid, giving us the desired perfectoid space

$$C_\infty \sim \varprojlim_{G\to 1} C_G. \qquad \square$$

Appendix A. Fibre bundles of formal and rigid spaces

In this appendix we review the theory of fibre bundles in the setting of formal and rigid geometry.

Notation A.1. In the following, if $\pi\colon E \to B$ is a morphism of formal schemes, then for a formal open subscheme $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.2. Let T be a formal group scheme. Let F be a formal scheme with an action $m\colon T \times F \to F$. A morphism $\pi\colon E \to B$ of formal schemes is called a **fibre bundle with fibre F and structure group T** if there is a cover $\mathfrak{U}$ of B of open formal subschemes $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl, "p_2"] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i \colon F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that ϕ_{ij} coincides with the composite

$$F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}.$$

Definition A.3. When $F = T$ with the action on itself by left multiplication, then a fibre bundle $\pi \colon E \to B$ with fibre T and structure group T is called a **T-torsor**.

Example A.4. The short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0$ from §3 yields a T-torsor $\overline{E} \xrightarrow{\pi} \overline{B}$ by Lemma 3.5. Moreover, for any formal open subscheme $U \subseteq \overline{B}$, the map $E|_U \to U$ is a T-torsor.

The ϕ_{ij} from condition (b) are determined by the maps $\psi_{ij} \colon U_{ij} \to T$. By glueing, one sees:

Lemma A.5. *Suppose we are given formal schemes F and B and a formal group scheme T with an action on F. Then fibre bundles $\pi \colon E \to B$ with fibre F and structure group T are equivalent to the data (up to refinement) of a cover $\mathfrak{U}$ of B by formal open subschemes and morphisms $\psi_{ij} \colon U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk}|_{U_{ijk}} \cdot \psi_{ij}|_{U_{ijk}} = \psi_{ik}|_{U_{ijk}}$ on the intersection $U_{ijk} := U_i \cap U_j \cap U_k$.*

Lemma A.6. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. Then the natural T-action on $F \times U_i$ for each i via the first factor glue to a natural T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.7. Let $\pi \colon E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' \colon E' \to B'$ be a fibre bundle with fibre F' and structure group T. Then a **morphism of fibre bundles** $f \colon (E', B', \pi') \to (E, B, \pi)$ is a commutative diagram of formal schemes

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

in which the morphism f_E is also T-linear. We often abbreviate this by writing $f \colon E' \to E$.

Definition A.8. Let $\pi \colon E \to B$ be a T-torsor. Let F be a formal scheme with an action by T. Since the data in Lemma A.5 are completely independent of the fibre, the morphisms $\psi_{ij} \colon U_{ij} \to T$ by Lemma A.5 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel-Weil construction.

A.1. **The semi-linear case.**

Definition A.9. Let $g\colon T' \to T$ be a homomorphism of formal group schemes. Let $\pi\colon E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi'\colon E' \to B'$ be a fibre bundle with fibre F' and structure group T' . Then a g-linear morphism of torsors is a diagram

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & B' \\ \downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f_B} \\ E & \xrightarrow{\pi} & B \end{array}$$

such that f_E is g-linear.

For a fixed $f_B\colon B' \to B$, one can equivalently characterise a morphism of torsors $f_E\colon E' \to E$ over f_B by the data of maps $f_B^{-1}(U_i) \to T'$ on some cover of B. Using this description, one sees:

Proposition A.10. *Given any homomorphism of group schemes $g\colon T' \to T$ and a g-linear homomorphism $h\colon F' \to F$ of formal schemes with T' and T-actions respectively, and a homomorphism $f\colon E' \to E$ of principal T' and T-bundles over g, one obtains a morphism*

$$h \times^g f\colon F' \times^{T'} E' \to F \times^T E$$

of fibre bundles over g. This makes $- \times^- -$ into a fibered bifunctor from the category of pairs (F, T) of formal schemes F with an action by T, fibered over the category of formal group schemes T with the category of T-torsors E, to the category of formal fibre bundles.

The associated bundle construction has the following universal property:

Lemma A.11. *In the context of Proposition A.10, assume moreover that F', F are formal group schemes and that the respective actions come from group homomorphisms $T' \to F'$ and $T \to F$. Then $h \times^g f$ is the unique h-linear morphism of fibre bundles making the following diagram commute:*

$$\begin{array}{ccc} F' \times^{T'} E' & \xrightarrow{h \times^f g} & F \times^T E \\ \uparrow & & \uparrow \\ E' & \xrightarrow{\quad f \quad} & E. \end{array}$$

Proof. The vertical maps in the diagram exist by functoriality via $E = T \times^T E \to F \times^T E$. On any compatible trivialisation $T' \times U' \to T \times U$ of $f : E' \to E$ there is then clearly only one way to extend this to $F' \times U' \to F \times U$ in a h-linear way. $\square$

Remark A.12. All that we have done in this section can be done in completely the same way with formal schemes replaced by rigid or adic spaces. The different categories of fibre bundles are well-behaved with respect to the usual functors between these categories: For instance, by functoriality of fibre products there are natural rigidification and reduction functors from formal principal T-bundles over $\mathcal{O}_K$ to rigid principal T_η-bundles over K on the generic fibre. Moreover, these generic fibre and reduction functors commute with the associated fibre construction:

Lemma A.13. *Let T be a formal group scheme and let $\pi\colon E \to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Then $(F \times^T E)_\eta = F_\eta \times^{T_\eta} E_\eta$.*

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