PERFECTOID COVERS OF ABELIAN VARIETIES

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Abstract. For an abelian variety A over an algebraically closed non-archimedean field of residue characteristic p, we show that there exists a perfectoid space which is the tilde-limit of $\varprojlim_{[p]} A$.

1. Introduction

Let p be a prime and let K be a non-archimedean field which is either p-adic or of characteristic p. Let us moreover assume that K is algebraically closed. For an abelian variety A over K we consider the inverse system of A under the multiplication by p morphism:

$$\dots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$$

Via the rigid analytification functor, we may see this as an inverse system of analytic adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. The primary goal of this article is to show that the "inverse limit" of this tower exists in some way and is a perfectoid space: Since inverse limits rarely exist in the category of adic spaces (even for affinoid spaces), in [12] the authors introduce the weaker notion of tilde-limits to remedy this problem. This is the notion of "limits" we are going to use. More precisely, we prove:

Theorem 1. *Let A be an abelian variety over K, considered as a rigid space. Then there exists a unique perfectoid space A_∞ over K such that $A_\infty \sim \varprojlim_{[p]} A$ is a tilde-limit.*

The possibility of results in this direction is mentioned in §7 and §13 of [11], and in the case of abelian varieties A with good reduction, this theorem was proven already in [6] (Lemme A.16). In order to motivate our strategy for the general case, let us sketch the proof in the good reduction case (we follow Exercise 4 – 6 in [1], the proof is spelt out in detail in §2.3, Corollary 2.16 below):

Let A be an abelian variety of good reduction over K. Let $\mathcal{O}_K$ be the ring of integers of K and let $\pi \in \mathcal{O}_K$ be a pseudo-uniformiser such that $p \in \pi\mathcal{O}_K$. Then we can consider A as an abelian scheme over $\mathcal{O}_K$. Let $\mathcal{A}$ be its π-completion, a formal scheme over $\mathcal{O}_K$. Its adic generic fibre is the rigid analytification of A that we denote by the same letter. The mod π special fibre $\mathcal{A}_s = \mathcal{A} \times \mathcal{O}_K/\pi$ is a group scheme over $\mathcal{O}_K/\pi$, so the map $[p] : \mathcal{A}_s \to \mathcal{A}_s$ factors through the relative Frobenius map. The inverse limit $\varprojlim_{[p]} \mathcal{A}_s$ in the category of schemes is thus relatively perfect over $\mathcal{O}_K/\pi$. We can similarly form the inverse limit $\mathcal{A}_\infty = \varprojlim_{[p]} \mathcal{A}$ in the category of formal schemes. Its adic generic fibre A_∞ is a tilde-limit of $\varprojlim_{[p]} A$, and it is perfectoid since $\varprojlim_{[p]} \mathcal{A}_s$ is relatively perfect.

This gives the construction of A_∞ in the case of good reduction. The overall strategy of our proof of Theorem 1 is similar: We construct tilde-limits of various spaces via formal models and show that these tilde-limits are perfectoid via a Frobenius factorisation property on the special fibre.

As the case of good reduction is already settled, it remains to consider the case that A has bad reduction. The assumption that K is algebraically closed assures that A is then semi-stable. In this case, the theory of Raynaud extensions provides us with a short exact sequence

$$0 \to T \to E \to B \to 0$$

of rigid groups, where $T = (\mathbb{G}_m^{\mathrm{an}})^d$ is a split rigid torus and B is the analytification of an abelian variety with good reduction, such that $A = E/M$ for a discrete lattice $M \subset E$. This short exact sequence is split locally on B, allowing us to locally write E as a product of T and an open subspace of B. Our strategy is now the following:

(1) Use formal models to show that there exists a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$.
(2) Use the formal models from (1) to construct a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.
(3) Study the quotient map $E \to A$ in the limit over $[p]$ to construct the desired space A_∞.

More precisely, this article is organised as follows: In §2 we develop a notion of a $[p]$-F-formal tower for a rigid group G over K, which is roughly an inverse system of formal models for $[p] : G \to G$ that factor through the relative Frobenius mod π. This is an axiomatisation of the data that one uses in the construction of A_∞ in the case of good reduction. In particular, the same proof shows that if G admits $[p]$-F-formal tower, then there exists a unique perfectoid tilde-limit $G_\infty \sim \varprojlim_{[p]} G$.

In chapter §3 we give a $[p]$-F-formal tower for a split rigid torus T in terms of a family of formal models $\mathfrak{T}_{q^{1/n}}$, thus showing that a perfectoid tilde-limit T_∞ of the inverse system of $[p]$ on T exists. Crucially, this tower of formal models has the extra structure of an action by the formal torus $\overline{T}$.

We can therefore in §4 use the language of formal fibre bundles to construct a $[p]$-F-formal tower for E: The Raynaud extension of A arises from a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \to B \to 0$$

by taking generic fibres and forming the pushout with respect to the open immersion $\overline{T}_\eta \to T$. Equivalently, since the sequence is locally split, we can see $\overline{E} \to B$ as a principal $\overline{T}$-bundle and formation of E amounts to a change of fibre from $\overline{T}_\eta$ to T. Since the formal model tower of §3 receives a $\overline{T}$-action, we now obtain a $[p]$-F-formal tower for E from the formal models of §4 by the change of fibre from $\overline{T}$ to the formal models $\mathfrak{T}_{q^{1/n}}$. This gives the desired perfectoid tilde-limit E_∞.

In chapter §5 we finish the proof of the main theorem by constructing A_∞ from E_∞ as follows: After choosing lattices $M \subseteq M^{1/p^n} \subseteq E$ that are sent to M under $[p^n] : E \to E$, the $[p]$-multiplication tower of $A = E/M$ naturally factors into two separate towers: One is the tower of maps $E/M^{1/p^{n+1}} \to E/M^{1/p^n}$ induced from $[p]$-multiplication of E, the other is induced from the projection maps $E/M \to E/M^{1/p^n}$. By choosing certain charts of E that give local splittings of the projection $E \to E/M = A$, we show that the first of these towers has a perfectoid tilde-limit $E/M^{1/p^\infty}$ that is locally isomorphic to open subsets of E_∞. Since the second tower is easy to describe in terms of the chosen charts, it is then straightforward to find the desired space A_∞.

In chapter §6 we finally use our explicit construction to say something about the local geometry of A_∞. In particular, we describe the perfectoid group homomorphism $E_\infty \to A_\infty$ in terms of perfectoid principal bundles. The goal of this last section is to give an alternative description of A_∞ in terms of a quotient of perfectoid groups: There are a lattice $M_\infty \subseteq E_\infty$ and a pro-finite perfectoid group D_∞ over K for which there is a short exact sequence of perfectoid groups

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0.$$

This may be seen as an analogue in the limit of the sequence $0 \to M \to E \to A \to 0$.

At the end there is an appendix on fibre bundles and associated fibre bundle constructions in the context of formal, rigid and perfectoid spaces.

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Notation

We fix a prime p. Let K be a perfectoid field that is either p-adic or of characteristic p, with ring of integers $\mathcal{O}_K$ and a fixed pseudo-uniformiser $\pi \in \mathcal{O}_K$ such that $p \in \pi\mathcal{O}_K$. From §4 on we are going to assume that K is algebraically closed, but we don't need to make this assumption yet.

By an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$, we mean an adic spaces in the sense of [12], and we adopt the notion of perfectoid spaces defined in §2 *ibid.* In their language, adic spaces in the sense of Huber are referred to as *honest* adic spaces. Throughout the article, we make no distinction between rigid spaces and their corresponding honest adic spaces. In particular, by a cover of a rigid space we shall always mean a cover of the associated adic space, and therefore the cover of the rigid space will automatically be an admissible cover in the sense of rigid analytic geometry. If a rigid space is obtained from a K-scheme X via rigid-analytification $X \mapsto X^{\mathrm{an}}$, we will often denote both by the same symbol X in order to simplify the notation.

In a similar spirit, we are often going to see formal schemes as adic spaces via the fully faithful adification functor, and denote both by the same letter. For a formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$ with the π-adic topology, we denote by $\mathfrak{X}_\eta := \mathfrak{X} \times_{\mathrm{Spa}(\mathcal{O}_K,\mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$ its adic generic fibre. We denote by $\tilde{X} = \mathfrak{X} \times_{\mathrm{Spf}\,\mathcal{O}_K} \mathrm{Spf}\,\mathcal{O}_K/\pi$ its mod π special fibre, considered as a scheme over $\mathrm{Spec}\,\mathcal{O}_K/\pi$.

Finally, let us recall the following standard terminology:

(1) Let X be a rigid space over K. A **formal model** of X is an admissible topologically finite type formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ together with an isomorphism of its generic fibre $\mathfrak{X}_\eta \xrightarrow{\sim} X$ (which is often suppressed from notation).

(2) Let $\phi : X \to Y$ be a morphism of rigid spaces over K, with formal models $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. A morphism of formal schemes $\Phi : \mathfrak{X} \to \mathfrak{Y}$ is a **formal model** of ϕ if it agrees with ϕ on the adic generic fiber via the identifications $\mathfrak{X}_\eta \xrightarrow{\sim} X$ and $\mathfrak{Y}_\eta \xrightarrow{\sim} Y$.

2. Tilde-limits of rigid groups

2.1. **Tilde-limits and formal models.** Inverse limits often do not exist in the category of adic spaces, and neither do they in rigid spaces. Instead we use the notion of tilde-limits from [12]:

Definition 2.1. Let $(X_i)_{i\in I}$ be a filtered inverse system of adic spaces with quasi-compact and quasi-separated transition maps, let X be an adic space with a compatible system of morphisms $f_i : X \to X_i$. We write $X \sim \varprojlim X_i$ and say that X is a **tilde-limit** of the inverse system $(X_i)_{i\in I}$ if the map of underlying topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and there exists an open cover of X by affinoids $\mathrm{Spa}(A, A^+) \subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(A_i, A_i^+) \subset X_i} A_i \to A$$

has dense image, where the direct limit runs over all $i \in I$ and all open affinoid subspaces $\mathrm{Spa}(A_i, A_i^+) \subset X_i$ through which the morphism $\mathrm{Spa}(A, A^+) \subseteq X \to X_i$ factors.

Remark 2.2. As pointed out after Proposition 2.4.4 of [12], tilde-limits (if they exist) are in general not unique. For example, consider the inverse system consisting of a single affinoid pre-perfectoid space $X = \mathrm{Spa}(A, A^+)$ in the sense of Definition 2.3.4 in [12]. Then both the space X as well as its strong completion $\hat{X} = \mathrm{Spa}(\hat{A}, \hat{A}^+)$ are tilde-limits of X. However, we'll see in Remark **??** that the perfectoid tilde-limits we construct are unique.

Let us recall a few results from [12], §2.4 about the construction of tilde-limits that we will use frequently throughout:

Proposition 2.3 ([12], Proposition 2.4.2)**.** *Let (A_i, A_i^+) be a direct system of affinoids over $(\mathcal{O}_K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ carrying the π-adic topology. Let $(A, A^+) = (\varinjlim A_i, \varinjlim A_i^+)$ be the affinoid algebra equipped with the topology making $\varinjlim A_{i,0}$ with the π-adic topology a ring of definition. Then*

$$\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+).$$

Proposition 2.4 ([12], Proposition 2.4.3)**.** *Let $X \sim \varprojlim_{i \in I} X_i$ be a tilde-limit and let $U_i \hookrightarrow X_i$ be an open immersion for some $i \in I$. Let $U_j := U_i \times_{X_i} X_j$ for $j \geq i$ and $U := U_i \times_{X_i} X$, then*

$$U \sim \varprojlim_{j \geq i} U_j.$$

Proposition 2.5 ([12], Proposition 2.4.5)**.** *Let $(X_i)_{i \in I}$ be an inverse system of adic spaces over $(K, \mathcal{O}_K)$ and assume that there is a perfectoid space X such that $X \sim \varprojlim_{i \in I} X_i$. Then for any perfectoid $(K, \mathcal{O}_K)$-algebra (B, B^+),*

$$X(B, B^+) = \varprojlim_{i \in I} X_i(B, B^+)$$

Corollary 2.6. *If a perfectoid tilde-limit $X \sim \varprojlim X_i$ exists, it is unique up to unique isomorphism.*

In the situation of the Corollary, we will also refer to X as *the* perfectoid tilde-limit of $\varprojlim X_i$. Of course perfectoid tilde-limit don't always exists. One case in which this they exist is the following:

Proposition 2.7. *Let $(\mathrm{Spa}(A_i, A_i^+))_{i \in I}$ be an inverse system of affinoid perfectoid spaces over $(K, \mathcal{O}_K)$. Let $(A, A^+) = (\varinjlim A_i, \varinjlim A_i^+)$ be the affinoid algebra endowed with the topology making $A_0 = \varinjlim A_i^\circ$ with the π-adic topology an open subring of definition, and denote by $(\hat{A}, \hat{A}^+)$ the completion. Then $\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+)$ is a perfectoid tilde-limit.*

Proof. The space $\mathrm{Spa}(A, A^+)$ is a tilde-limit of (A, A^+) by Proposition 2.3. It is then clear from the definition that this remains true for $(\hat{A}, \hat{A}^+)$. It remains to see that this is a perfectoid $(K, \mathcal{O}_K)$-algebra: But $\hat{A}_0$ is a flat $\mathcal{O}_K$-algebra and in the direct limit the Frobenius isomorphisms $A_i^\circ/\pi^{1/p} \cong A_i^\circ/\pi$ induce an isomorphism

$$\hat{A}_0/\pi^{1/p} \cong \hat{A}_0/\pi.$$

Thus by Lemma 5.6 in [7], $(\hat{A}, \hat{A}^+)$ is a perfectoid $(K, \mathcal{O}_K)$-algebra. $\square$

In the situation that the X_i are rigid space, one way to construct tilde-limits is by constructing well-behaved formal models. The reason for this is the combination of the following two Lemmas:

Lemma 2.8. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps $\phi_{ij} : \mathfrak{X}_j \to \mathfrak{X}_i$. Then the inverse limit $\mathfrak{X} = \varprojlim \mathfrak{X}_i$ exists in the category of formal schemes over $\mathcal{O}_K$. If all the $\mathfrak{X}_i$ are flat formal schemes, so is $\mathfrak{X}$.*

Proof. In the affine case, if the inverse system is $\operatorname{Spf} A_i$, take A to be the π-adic completion of $\varinjlim A_i$, then $\operatorname{Spf} A$ is the inverse limit of the $\operatorname{Spf} A_i$. If the A_i are flat, then so is A because it is torsion-free over the valuation ring $\mathcal{O}_K$.

In general, ones uses that the transition maps are affine to reduce to the affine case. □

The proof also shows that in the situation of the Lemma, $\mathfrak{X}$ is in particular a tilde-limit $\mathfrak{X} \sim \varprojlim \mathfrak{X}_i$ when considered as adic spaces. This remains true after passing to adic generic fibres:

Lemma 2.9. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps ϕ_{ij} and let $\mathfrak{X} = \varprojlim_{\phi_j} \mathfrak{X}_i$ be the limit. Let $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$ and $\mathcal{X} = (\mathfrak{X})_\eta$ be the adic generic fibres. Then $\mathcal{X} \sim \varprojlim \mathcal{X}_i$.*

Proof. This is a consequence of Proposition 2.3: The transition maps in the system are affine, hence quasi-compact quasi-separated. In order to prove the Lemma, we can restrict to an affine open subset $\operatorname{Spf}(A)$ of $\mathfrak{X}$ that arises as the inverse limit of affine open subsets $\operatorname{Spf}(A_i) \subseteq \mathfrak{X}_i$. Here all formal schemes are considered with the π-adic topology and A is the π-adic completion of $\varinjlim A_i$. On the generic fibre, A_i with the π-adic topology is an open subring of definition of $A_i[1/\pi]$. Therefore, Proposition 2.3 applies and shows that $\operatorname{Spf}(A)_\eta \sim \varprojlim \operatorname{Spf}(A_i)_\eta$ as desired. □

Remark 2.10. This lemma essentially says that one can always construct a tilde-limit of an inverse system of rigid spaces $\mathcal{X}_i$ if it arises from an inverse system of formal schemes $\mathfrak{X}_i$ with affine transition maps. This is precisely what Scholze uses in [9] to construct the space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ (see Corollary III.2.19 in [9] and its proof).

Remark 2.11. Let us recall Raynaud's theory of formal models: Under mild assumptions, one can always find formal models of rigid spaces, and (possibly after admissible blow ups) of morphisms between them. More precisely, Raynaud's theorem [2, section 8.4] states that

(1) Let X be a quasi-separated quasi-paracompact rigid space over K. Then there exist an admissible quasi-paracompact formal model $\mathfrak{X}$ for X.
(2) If $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blow-up of admissible formal schemes, then it induces an isomorphism on the generic fibre $\mathfrak{X}'_\eta \xrightarrow{\sim} \mathfrak{X}_\eta$.
(3) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be admissible quasi-paracompact formal schemes over $\mathcal{O}_K$ and let $f : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ be a morphism of their associated rigid spaces. Then there exist an admissible blow-up $\pi : \mathfrak{X}' \to \mathfrak{X}$ and a map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{Y}$ such that $\mathfrak{f}_\eta = f \circ \pi_\eta$.

In particular, given an inverse system $(\mathcal{X}_i, \phi_{ij})$ of rigid spaces, one can always choose formal models $\mathfrak{X}_i$ for the $\mathcal{X}_i$, and by successive admissible blow-ups one can also find models for the transition maps ϕ_{ij}. This theorem doesn't immediately allow us to construct a tilde-limit using Lemma 2.9 as the morphisms might not be affine.

2.2. **Tilde-limits for rigid groups.** Let G be a rigid group over K, that is, a group object in the category of rigid spaces over K. We are primarily interested in the following examples:

(1) Analytifications of finite type group schemes over K. Examples include the analytification of an abelian variety A over K and of tori T over K.

(2) Generic fibres of topologically finite type formal group schemes over $\mathcal{O}_K$.
(3) Raynaud's covering space E of an abelian variety with semi-stable reduction.

Lemma 2.8 and 2.9 motivate the following definition.

Definition 2.12. Let G be a rigid group. A $[p]$**-formal tower** for G is a family of formal models $\{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ of G, together with affine transition maps $[\mathfrak{p}]_{n+1}:\mathfrak{G}_{n+1}\to\mathfrak{G}_n$ which are formal models of $[p]:G\to G$. Sometimes we suppress notation and write $[\mathfrak{p}]$ for $[\mathfrak{p}]_n$ on $\mathfrak{G}_n$.

The following Corollary is an immediate consequence of our discussion in the previous subsection:

Corollary 2.13. *Let G be a rigid group. If G admits a $[p]$-formal tower, then there exists an adic space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$.*

For example, for any formal group scheme $\mathfrak{G}$ for which $[p]$ is affine, the $[p]$-multiplication tower of $\mathfrak{G}$ gives rise to a $[p]$-formal tower for its generic fibre $\mathfrak{G}_\eta$.

2.3. **Perfectoidness.** We now introduce the formalism of $[p]$-F-formal towers to axiomatise the approach mentioned in the introduction which shows that A_∞ exists and is perfectoid in the case of good reduction.

Definition 2.14. Let G be a rigid analytic group, a $[p]$-F**-formal tower** for G is a $[p]$-formal tower

$$(\{\mathfrak{G}_n\}_{n\in\mathbb{N}},[\mathfrak{p}]_{n+1}:\mathfrak{G}_{n+1}\to\mathfrak{G}_n)$$

such that each $\mathfrak{G}_n$ is flat over $\mathcal{O}_K$, and on the mod π special fibre $\tilde{G}_n$, each $[\mathfrak{p}]_{n+1}$ factors through the relative Frobenius morphism:

$$\begin{tikzcd} & \tilde{G}_{n+1}^{(p)} \arrow[dr, dashed] & \\ \tilde{G}_{n+1} \arrow[ur, "F_{rel}"] \arrow[rr, "{[p]}"] & & \tilde{G}_n \end{tikzcd}$$

Proposition 2.15. *Let G be a rigid group over a perfectoid field K. If G admits a $[p]$-F-formal tower, then there exists a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$.*

Proof. Let $(\{\mathfrak{G}_n\}_{n\in\mathbb{N}},[\mathfrak{p}]_{n+1}:\mathfrak{G}_{n+1}\to\mathfrak{G}_n)$ be a $[p]$-F-formal tower for G. By Lemma 2.9 we have

$$G_\infty := (\varprojlim_{[\mathfrak{p}]}\mathfrak{G})_\eta \sim \varprojlim_{[p]} G.$$

To see that G_∞ is perfectoid, we proceed as the proof of [9], Corollary III.2.19. It suffices to prove that $\mathfrak{G}_\infty = \varprojlim_{[\mathfrak{p}]}\mathfrak{G}$ can be covered by formal schemes of the form $\mathrm{Spf}(S)$ where S is a flat $\mathcal{O}_K$-algebra such that the Frobenius map

$$S/\pi^{1/p}\to S/\pi$$

is an isomorphism. Lemma 5.6 of [7] then shows that $S[1/\pi]$ is perfectoid.

By assumption, on the mod π special fibre $\tilde{G}_n$, $[\mathfrak{p}]_{n+1}$ factors through the relative Frobenius. Now take any affine open subspace $\mathrm{Spf}(S_0)\subseteq\mathfrak{G}_0$. Let $[\mathfrak{p}^i]:\mathfrak{G}_i\to\mathfrak{G}_0$ be the composition $[\mathfrak{p}]_i\circ\cdots\circ[\mathfrak{p}]_1$, and let $\mathrm{Spf}\, S_i\subset\mathfrak{G}_i$ be the pullback of $\mathrm{Spf}\, S_0$ via $[p^i]$. Then we have a commutative diagram:

$$\begin{tikzcd}
 & & \tilde{S}_i^{(p)} \arrow[dr, "F_{rel}"] & & \tilde{S}_{i+1}^{(p)} \arrow[dr, "F_{rel}"] & & \\
\dots \arrow[r] & \tilde{S}_{i-1} \arrow[ur, dashed, "V"] \arrow[rr] & & \tilde{S}_i \arrow[ur, dashed, "V"] \arrow[rr] & & \tilde{S}_{i+1} \arrow[r] & \dots
\end{tikzcd}$$

where the horizontal maps are induced from $[\mathfrak{p}] \mod \pi$.

From this we can check on elements that relative Frobenius is an isomorphism on $\tilde{S}_\infty := \varinjlim_i \tilde{S}_i$. Since K is perfectoid, we moreover have an isomorphism $\mathcal{O}_K/\pi^{1/p} \to \mathcal{O}_K/\pi$ from the absolute Frobenius on $\mathcal{O}_K/\pi$. Therefore absolute Frobenius on S_∞/π induces an isomorphism

$$S_\infty/\pi^{1/p} \xrightarrow{\sim} S_\infty/\pi.$$

Since each $\mathfrak{G}_i$ is flat, so are the S_i and thus so is S_∞. Thus $S_\infty[1/\pi]$ is a perfectoid K-algebra. Since G_∞ is covered by affinoids of the form $\mathrm{Spf}(S_\infty)_\eta$, this shows that G_∞ is perfectoid. $\square$

In particular this gives in more detail the proof of the result that we sketched in the introduction:

Corollary 2.16. *Let A be an abelian variety of good reduction over a perfectoid field K. Then A_∞ exists and is perfectoid.*

More generally, if $\mathfrak{G}$ is a flat commutative formal group scheme such that $[p]$-multiplication is affine, then the maps $[p] : \mathfrak{G} \to \mathfrak{G}$ define a $[p]$-F-formal tower for the rigid analytic group $G = \mathfrak{G}_\eta$, and $G_\infty := (\varprojlim_{[p]} \mathfrak{G})_\eta$ is the perfectoid tilde-limit of $\varprojlim_{[p]} G$.

2.4. **Examples.**

Example. Let $\mathfrak{G}$ be the p-adic completion of the affine group scheme $\mathbb{G}_m$ over $\mathcal{O}_K$. The underlying formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, \mathcal{O}_K\langle X^{\pm 1}\rangle$. Multiplication by p on $\mathbb{G}_m$ gives a $[p]$-F-formal tower for G, so for the generic fibre $G = \mathfrak{G}_\eta$ we obtain the perfectoid tilde-limit $G_\infty := (\mathfrak{G}_\infty)_\eta \sim \varprojlim_{[p]} G$. More precisely, multiplication by p corresponds to the homomorphism

$$[p] : \mathcal{O}_K\langle X^{\pm 1}\rangle \to \mathcal{O}_K\langle X^{\pm 1}\rangle, \quad X \to X^p.$$

In the direct limit, we obtain $(\varinjlim_{[p]} \mathcal{O}_K\langle X^{\pm 1}\rangle)^\wedge = \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle$. Therefore, taking the generic fiber we get

$$G_\infty = \mathrm{Spa}(K\langle X^{\pm 1/p^\infty}\rangle, \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle)$$

and one can verify directly that we indeed have $G_\infty \sim \varprojlim_{[p]} G$.

Example. An example of a very different flavour is the p-adic completion $\mathfrak{G}$ of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Note that $G = \mathfrak{G}_\eta$ is not equal to $\mathbb{G}_a^{an}$, but is the closed unit disc in the latter.

The underlying formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, S$ with $S = \mathcal{O}_K\langle X\rangle$, and the $[p]$-multiplication is now given by

$$[p] : \mathcal{O}_K\langle X\rangle \to \mathcal{O}_K\langle X\rangle, \quad X \to pX.$$

In the direct limit, we first obtain the algebra $S'_\infty = \mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle$, consisting of power series $f = \sum_{n=0}^\infty a_n X^n \in \mathcal{O}_K[[X]]$ for which there exists an $m \in \mathbb{Z}_{\geq 0}$ so that $|p^{nm} a_n| \to 0$. Next we need to take the p-adic completion to form S_∞. But we have

$$p^n \mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle = p^n \mathcal{O}_K + X\mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle$$

and therefore $S'_\infty/\pi^n = \mathcal{O}_K/\pi^n\mathcal{O}_K$. Consequently, the completion is just $S_\infty = \mathcal{O}_K$ and thus its adic generic fiber is $G_\infty = \mathrm{Spa}(K, \mathcal{O}_K)$. This is still perfectoid but is just one point!

One can explain this observation geometrically: On the level of K-points, the formal scheme G is the closed unit disc and $[p]$ is scaling by p. A K-point in $\varprojlim_{[p]} G(K)$ therefore corresponds to a sequence of K-points of the closed unit disc of the form

$$(x, \frac{1}{p}x, \frac{1}{p^2}x, \dots).$$

But for this to be contained in the closed unit disc, we must have $x = 0$.

2.5. **group structure of the limit.** One reason why perfectoid limits along group morphisms are particularly interesting is that the perfectoidness ensures that the limit has again a group structure:

Definition 2.17. A **perfectoid group** is a group object in the category of perfectoid spaces.

Note that the category of perfectoid spaces over K has finite products, so the notion of a group object makes sense.

Proposition 2.18. *Let G be a rigid group with a perfectoid tilde-limit G_∞. Then*

1. *there is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms*
2. *given another rigid group H with perfectoid tilde-limit $H_\infty \sim \varprojlim_{[p]} H$ and a group homomorphism $H \to G$, there is a unique group homomorphism $H_\infty \to G_\infty$ commuting with all projection maps. In particular, formation of the perfectoid $\varprojlim_{[p]}$ is functorial.*

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, cf Proposition 2.4.5 of [12], which shows that one can argue like in the case of usual limits. □

3. Formal models for tori

In this section we want to show that for a split rigid torus T over K, a tilde-limit T_∞ exists and is perfectoid. We do this by exhibiting a $[p]$-F-model of T.

As a preparation, we consider the torus $\mathbb{G}_m^{\mathrm{an}}$ over K. Recall that it arises from rigid analytification of the affine torus $\mathbb{G}_m$ over K. Note however that $\mathbb{G}_m^{\mathrm{an}}$ is not affinoid (and not even quasi-compact). It contains the generic fibre of the p-adic completion of $\mathbb{G}_m$ as an open subspace. If we see $\mathbb{G}_m^{\mathrm{an}}$ as the rigid affine line with the origin removed, this subspace $\widehat{\mathbb{G}}_m$ can be identified with the open annulus of radius 1. In other words, on the level of points it corresponds to $\mathcal{O}_K^\times \subseteq K^\times$.

Finally, recall that for every $x \in K^\times$ we have a translation map

$$\mathbb{G}_m^{\mathrm{an}} \xrightarrow{x\cdot} \mathbb{G}_m^{\mathrm{an}}$$

that is an isomorphism of rigid spaces sending the point 1 to x.

3.1. **A family of explicit covers.** We briefly recall how $\mathbb{G}_m^{\mathrm{an}}$ is constructed: The following is taken from [2], §9.2 with slightly different notation. Throughout we use the following standard shorthand notation: For any $0 \neq a, b \in K$ with $|a| \leq |b|$ consider the topologically finite type K-algebra

$$K\langle X/b\rangle := \left\{ \sum_{n=0}^{\infty} c_n (X/b)^n \in K[[X]] \text{ where } c_n \in K \text{ such that } |c_n| \to 0 \right\}$$

which is isomorphic to $K\langle X\rangle$ via $X \mapsto X/b$, as well as

$$K\langle a/X, X/b\rangle := K\langle X/b\rangle\langle Z\rangle/(ZX - a).$$

The associated affinoid space $\mathcal{B}(a,b)$ is the annulus of radii $|a|$ and $|b|$ inside $\mathbb{A}^{\mathrm{an}}_K$:

$$\mathcal{B}(a,b) = \mathrm{Sp}(L_{a,b}), \qquad \text{where } L_{a,b} := K\langle a/X, X/b\rangle.$$

When $|a| < |b|$, then by the boundary of $\mathcal{B}(a,b)$ we mean the disjoint union of the affinoid subdomains $\mathcal{B}(a,a)$ and $\mathcal{B}(b,b)$, which we refer to as the inner and outer boundary, respectively. Explicitly,

$$\begin{aligned} \mathcal{B}(a,b) &\hookleftarrow \mathcal{B}(a,a) \\ L_{a,b} = K\langle a/X, X/b\rangle &\to K\langle a/X, X/a\rangle = L_{a,a} \\ a/X, X/b &\mapsto a/X, \frac{a}{b}(X/a) \end{aligned} \tag{1}$$

and similarly

$$\begin{aligned} \mathcal{B}(a,b) &\hookleftarrow \mathcal{B}(b,b) \\ L_{a,b} = K\langle a/X, X/b\rangle &\to K\langle b/X, X/b\rangle = L_{b,b} \\ a/X, X/b &\mapsto \frac{a}{b}(b/X), X/b. \end{aligned} \tag{2}$$

Since we are mainly interested in the p-multiplication map, we will construct a family of covers of $\mathbb{G}_m^{\mathrm{an}}$ on which $[p]$ can be seen directly. Fix $q \in K^\times$ with $|q| < 1$. Then the annuli $\{\mathcal{B}(q^i, q^{i-1})\}_{i\in\mathbb{Z}}$ form an admissible affinoid covering of $\mathbb{G}_m^{\mathrm{an}}$ and are glued together by identifying the inner boundary of $\mathcal{B}(q^i, q^{i-1})$ with the outer boundary of $\mathcal{B}(q^{i+1}, q^i)$. We denote this cover by $\mathfrak{U}_q$.

Assume now that q has a p-th root $q^{1/p}$ in K. The above then gives a finer cover $\mathfrak{U}_{q^{1/p}}$ of $\mathbb{G}_m^{\mathrm{an}}$. Using both covers $\mathfrak{U}_q$ and $\mathfrak{U}_{q^{1/p}}$, we can easily see the $[p]$-multiplication $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ as follows: Consider the affinoid open subsets $\mathcal{B}(q^{i/p}, q^{(i-1/p)})$ of the source and $\mathcal{B}(q^i, q^{i-1})$ of the target. Then $[p]$ restricts to

$$\begin{aligned} \mathcal{B}(q^i, q^{i-1}) &\xleftarrow{[p]} \mathcal{B}(q^{i/p}, q^{(i-1)/p}) \\ K\langle q^i/X, X/q^{i-1}\rangle &\to\ K\langle q^{i/p}/X, X/q^{(i-1)/p}\rangle \\ q^i/X, X/q^{i-1} &\mapsto\ (q^{i/p}/X)^p, (X/q^{(i-1)/p})^p. \end{aligned} \tag{3}$$

These further restrict to morphisms $B(q^{i/p}, q^{(i-1)/p}) \to B(q^i, q^i)$ on the inner boundary, and similarly for the inner boundary. These maps are by construction compatible with the glue maps (1) and (2). For the map (2), this is reflected by the fact that the following diagram commutes:

$$\begin{tikzcd} \mathcal{B}(q^i, q^{i-1}) & \mathcal{B}(q^i, q^i) \arrow[l, hook'] & X/q^{i-1} \arrow[r, mapsto] \arrow[d, mapsto] & q\cdot(X/q^i) \arrow[d, mapsto] \\ \mathcal{B}(q^{i/p}, q^{(i-1)/p}) \arrow[u, "{[p]}"] & \mathcal{B}(q^{i/p}, q^{i/p}) \arrow[l, hook'] \arrow[u, "{[p]}"'] & (X/q^{i-1})^p \arrow[r, mapsto] & q\cdot(X/q^i)^p, \end{tikzcd} \tag{4}$$

and similarly with $q^i/X \mapsto q^i/X$. The case of the inner boundary (1) is essentially identical.

3.2. **A family of formal models.** At this point we have constructed a cover $\mathfrak{U}_q$ of $\mathbb{G}_m^{\mathrm{an}}$ depending on a choice of $q \in K^\times$ with $|q| < 1$. The affinoid subspaces $\mathcal{B}(q^i, q^{i-1})$ that we have used for this admit natural formal models: Namely, consider the topologically finite type $\mathcal{O}_K$-algebras

$$\mathcal{O}_K\langle X/a\rangle = \left\{ \sum_{n=0}^{\infty} c_n (X/b)^n \in K[[X]] \text{ where } c_n \in \mathcal{O}_K \text{ such that } |c_n| \to 0 \right\},$$

$$L^\circ_{q^i,q^{i-1}} := \mathcal{O}_K\langle q^i/X, X/q^{i-1}\rangle = \mathcal{O}_K\langle X/q^{i-1}\rangle\langle Z\rangle/(YZ-q^i).$$

It is shown in [2] §9.2 that $L^\circ_{q^i,q^{i-1}}$ is flat over $\mathcal{O}_K$ and thus gives rise to a flat formal $\mathcal{O}_K$-scheme $\mathfrak{B}(q^i,q^{i-1}) = \mathrm{Spf}(L^\circ_{q^i,q^{i-1}})$ which is a formal $\mathcal{O}_K$-model of $\mathcal{B}(q^i,q^{i-1})$. The open formal subschemes

$$\mathfrak{B}(q^{i-1},q^{i-1}) = \mathfrak{B}(q^i,q^{i-1})((X/q^{i-1})^{-1}),$$
$$\mathfrak{B}(q^i,q^i) = \mathfrak{B}(q^i,q^{i-1})((q^i/X)^{-1})$$

are formal models for $\mathcal{B}(q^{i-1},q^{i-1}),\mathcal{B}(q^i,q^i)$ respectively. We conclude:

Lemma 3.1. *The affine formal schemes $\{\mathfrak{B}(q^i,q^{i-1})\}_{i\in\mathbb{Z}}$ glue together to a flat formal scheme $\mathfrak{G}_q$ such that $(\mathfrak{G}_q)_\eta = \mathbb{G}_m^{\mathrm{an}}$. In other words, $\mathfrak{G}_q$ is a flat formal model for $\mathbb{G}_m^{\mathrm{an}}$.*

3.3. **A family of formal models for p-multiplication.** As before choose $q \in K^\times$ such that $|q| < 1$ and such that there exists a p-th root $q^{1/p} \in K$. A closer look at the map (3) shows that the $[p]$-multiplication extends to a morphism of formal schemes

$$[\mathfrak{p}] : \mathfrak{B}(q^{i/p}, q^{(i-1)/p}) \xrightarrow{[p]} \mathfrak{B}(q^i,q^{i-1}).$$

The diagram (4) shows that these maps glue to a morphism

$$[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q.$$

By construction, after tensoring $-\otimes_{\mathcal{O}_K} K$ all morphisms on algebras coincide with those defined in (1), (2), (3), respectively. We conclude:

Proposition 3.2. *The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ is an affine formal model of $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$.*

We moreover see directly from the construction:

Proposition 3.3. *The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ reduces mod π to the relative Frobenius map.*

We now have everything together to finish our proof that $(\mathbb{G}_m^{\mathrm{an}})_\infty$ is perfectoid:

Proposition 3.4. *The space $\mathbb{G}_m^{\mathrm{an}}$ has a $[p]$-F-formal tower. In particular, there exists a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ such that $(\mathbb{G}_m^{\mathrm{an}})_\infty \sim \varprojlim_{[p]} \mathbb{G}_m^{\mathrm{an}}$.*

Proof. Since K is perfectoid, we can find $q \in K^\times$ such that $|q| < 1$ for which there exist arbitrary p^n-th roots. We choose such a q and roots q^{1/p^n} for all n. Then the two Propositions above combine to show that

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_q$$

is a $[p]$-F-formal tower. By Proposition 2.15 we obtain a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ as desired. ☐

3.4. **The action of $\overline{T}$.** The multiplication $\mathbb{G}_m^{\mathrm{an}} \times \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ restricts to give an action of the annulus $\mathcal{B}(1,1)$ on each of the affinoids in our rigid analytic cover as follows:

$$\begin{aligned} \mathcal{B}(q^i,q^{i-1}) &\xleftarrow{m} \mathcal{B}(1,1)\times\mathcal{B}(q^i,q^{i-1}) \\ K\langle q^i/X, X/q^{i-1}\rangle &\to K\langle Z,1/Z\rangle\widehat{\otimes}K\langle q^i/X, X/q^{i-1}\rangle \\ q^i/X &\mapsto 1/Z\otimes q^i/X \\ X/q^{i-1} &\mapsto Z\otimes X/q^{i-1} \end{aligned} \tag{5}$$

The same arguments as in the last section show that the map described in (5) has a flat formal model

$$\mathfrak{B}(q^i,q^{i-1}) \xleftarrow{m} \mathfrak{B}(1,1)\times\mathfrak{B}(q^i,q^{i-1}).$$

We therefore conclude:

Proposition 3.5. *For any $q \in K^\times$ with $|q| < 1$, the formal torus $\overline{T} := \mathfrak{B}(1,1)$ has a natural action on $\mathfrak{G}_q$ via a map*

$$\mathfrak{m} : \overline{T} \times \mathfrak{G}_q \to \mathfrak{G}_q.$$

This map is a formal model of the action of the annulus $\mathcal{B}(1,1)$ on $\mathbb{G}_m^{\mathrm{an}}$. Furthermore, this action is compatible with the models for $[p]$ in the sense that if there is a p-th root $q^{1/p} \in K$, then the following diagram commutes.

$$\begin{tikzcd} \overline{T} \times \mathfrak{G}_{q^{1/p}} \arrow[r, "\mathfrak{m}"] \arrow[d, "{[p]\times[\mathfrak{p}]}"'] & \mathfrak{G}_{q^{1/p}} \arrow[d, "{[\mathfrak{p}]}"] \\ \overline{T} \times \mathfrak{G}_q \arrow[r, "\mathfrak{m}"] & \mathfrak{G}_q. \end{tikzcd}$$

Proof. The existence of $\mathfrak{m}$ follows from the above consideration concerning the map (5). The rest is clear from the construction: All adic rings we have used in the construction are $\mathcal{O}_K$-subalgebras of the affinoid K-algebras used to define $\mathbb{G}_m^{\mathrm{an}}$, so the equalities hold because they hold for $\mathbb{G}_m^{\mathrm{an}}$. □

3.5. **The case of general tori.** By taking products everywhere, all of the statements in this section immediately generalise to split tori:

Corollary 3.6. *Let T be a split torus over K of the form $T = (\mathbb{G}_m^{\mathrm{an}})^d$. Then for any $q \in K^\times$ with $|q| < 1$ the formal scheme $\mathfrak{T}_q := (\mathfrak{G}_q)^d$ is a formal model of T. Assume that there is a p-th root $q^{1/p} \in K$, then the $[p]$-multiplication map has an affine formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ that locally on polyannuli is of the form $[\mathfrak{p}] : \prod_{j=1}^d \mathfrak{B}(q^{i_j/p}, q^{(i_j-1)/p}) \to \prod_{j=1}^d \mathfrak{B}(q^{i_j}, q^{i_j-1})$. Moreover this map reduces mod π to the relative Frobenius morphism.*

Corollary 3.7. *Let T be a split torus over K, considered as a rigid space. Then T has a $[p]$-F-formal tower. In particular, there exists a perfectoid space T_∞ such that $T_\infty \sim \varprojlim_{[p]} T$.*

Corollary 3.8. *Let T be any split torus over K. For any $q \in K^\times$ with $|q| < 1$, the formal completion $\overline{T}$ has a natural action on $\mathfrak{T}_q$ via a map*

$$\mathfrak{m} : \overline{T} \times \mathfrak{T}_q \to \mathfrak{T}_q.$$

This map is a formal model of the action of the annulus $\overline{T}$ on T. Furthermore, this action is compatible with the models for $[p]$ in the sense that if there is a p-th root $q^{1/p} \in K$, then the map $[\mathfrak{p}] : \mathfrak{T}_q^{1/p} \to \mathfrak{T}_q$ is semi-linear with respect to $[p] : \overline{T} \to \overline{T}$.

4. A $[p]$-F-formal tower for Raynaud extensions

In this section we study the p-multiplication tower of the Raynaud extensions associated to abelian varieties over an algebraically closed perfectoid field K. The main result of this section is Theorem 4.11, namely the Raynaud extension E of an abelian variety A over K admits a $[p]$-F-formal tower, showing that there exists a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

Remark 4.1. While we expect that our main theorem still holds over any perfectoid field, it is easier to work in the algebraically closed case since this simplifies the theory of Raynaud uniformisation, in particular Lemma 4.5 below.

4.1. **Raynaud extensions.** We briefly sketch the theory of Raynaud extensions here, and refer the readers to [5] for more details on the setup.

Let A be an abelian variety over K. There exists a unique connected open rigid analytic subgroup $\overline{A}$ of A which extends to a formal smooth $\mathcal{O}_K$-group scheme $\overline{E}$ with semi-abelian reduction. This $\overline{E}$ fits into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0 \tag{6}$$

where $\overline{B}$ is the completion of an abelian variety B over K of good reduction (as usual we also denote by B the rigid space associated to it), and $\overline{T}$ is the completion of a torus T of rank r over K. The rigid generic fibre $\overline{T}_\eta$ of the torus $\overline{T}$ canonically embeds into the rigid torus T^{an} which again we simply denote by T. One can show that this induces a pushout exact sequence in the category of rigid groups. More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout.

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{7}$$

The abelian variety A we started with can then be uniformized in terms of E as follows:

Definition 4.2. A subset M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}^r}$.

Proposition 4.3. *There exists a lattice $M \subseteq E$ of rank equal to the rank r of the torus for which the quotient E/M exists as a rigid space and has a group structure such that $E \to E/M$ is a rigid group homomorphism. Moreover, there is a natural isomorphism*

$$A = E/M.$$

Since M is discrete, the local geometry of A is thus determined by the local geometry of E. We will therefore first study the $[p]$-multiplication tower of E and will then deduce properties of the $[p]$-multiplication tower of A.

Remark 4.4. Our strategy is to study the local geometry of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that one cannot direcly study short exact sequences locally on T, E or B.

An important tool is therefore the following Lemma:

Lemma 4.5. *The short exact sequence (6) admits local sections, that is there is a cover of B by formal open subschemes U_i such that there exist local sections $s : U_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times U_i \hookrightarrow E$.*

Proof. This is proved in Proposition A.2.5 in [5], where it is fomulated in terms of the group $\mathrm{Ext}(B, T)$. Also see [3], §1. ☐

This Lemma suggests that instead of considering Raynaud extensions from the abelian category viewpoint, one should consider them as principal T-bundles of formal schemes.

Remark 4.6. This is the language we use in the rest of the paper: We will work with fibre bundles of formal schemes, rigid spaces and schemes. The main technical tool we need is the associated fibre construction in these settings. For a rigorous treatment of these we refer to the Appendix.

First of all, we note that the sequence (6) from the last section gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}_\eta$ via push-out along $\overline{T}_\eta \to T$ can now conveniently be expressed in terms of the associated fibre bundle by saying that $E = T \times^{\overline{T}_\eta} \overline{E}_\eta$ in the sense of Definition A.9. We then have the following description of $[p]$:

Lemma 4.7. *The map $[p] : E \to E$ coincides with the morphism*

$$[p] \times^{[p]} [p] : T \times^{\overline{T}_\eta} \overline{E}_\eta \to T \times^{\overline{T}_\eta} \overline{E}_\eta$$

induced by the different $[p]$-multiplication maps by Proposition A.18.

Proof. Lemma A.19 in light of Remark A.20 applied to the maps $g = [p] : \overline{T}_\eta \to \overline{T}_\eta$, $h = [p] : T \to T$ and $f = [p] : \overline{E}_\eta \to \overline{E}_\eta$ says that $[p] \times^{[p]} [p]$ is the unique morphism of fibre bundles $E \to E$ making the following diagram commute:

$$\begin{tikzcd}
 & T \arrow[rr] & & E \\
T \arrow[ur, "{[p]}"] \arrow[rr] & & E \arrow[ur, dotted, "\exists!"] & \\
 & \overline{T}_\eta \arrow[uu, hook] \arrow[rr] & & \overline{E}_\eta \arrow[uu] \\
\overline{T}_\eta \arrow[uu, hook] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E}_\eta \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd} \tag{8}$$

Since $[p] : E \to E$ is such a map, the Lemma follows. $\square$

4.2. A $[p]$-F-formal tower for E.

In this subsection we prove that E admits a $[p]$-F-formal tower. The first step is to construct a family of formal models for E.

Lemma 4.8. *Let $q \in K^\times$ with $|q| < 1$. Let $\mathfrak{T}_q$ be the formal model from Corollary 3.6. Then there is a formal scheme $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E}$ that is a flat formal model of the rigid space E. Furthermore, there exists a morphism*

$$\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$$

which is a fibre bundle and a formal model of $E \to B$.

Proof. Recall from Proposition 3.8 that $\mathfrak{T}_q$ has a $\overline{T}$-action that is a model of the $\overline{T}_\eta$-action on T. In particular, the associated fibre construction for the principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$ gives a fibre bundle $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$. Since $\mathfrak{T}_q$ is a formal model of T, one sees by Lemma A.21 that $\mathfrak{E}_q$ is a formal model of $T \times^{\overline{T}_\eta} \overline{E}_\eta$ which by definition is equal to E. Finally, since $\mathfrak{T}_q$ and $\overline{B}$ are flat, so is $\mathfrak{E}_q$. $\square$

Next we construct a model for the $[p]$-multiplication map. Here we can use again that $[p]$ exists on $\overline{E}$ and on $\mathfrak{T}_{q^{1/p}}$.

Lemma 4.9. *Let $q \in K^\times$ be such that $|q| < 1$ and let $q^{1/p} \in K$ be a p-th root of q. Then there is an affine morphism*

$$[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$$

which is a formal model of $[p] : E \to E$.

Proof. Recall that the multiplication map $[p] : T \to T$ has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ by Corollary 3.6. This fits into a commutative diagram

$$\begin{tikzcd}
\mathfrak{T}_{q^{1/p}} \arrow[r, "{[\mathfrak{p}]}"] & \mathfrak{T}_q \\
\overline{T} \arrow[u, hook] \arrow[r, "{[p]}"] & \overline{T}. \arrow[u, hook]
\end{tikzcd}$$

Functoriality of the associated fibre construction in the general case, Proposition A.18, applied to the diagram below then gives a natural map $\mathfrak{E}_{q^{1/p}} \to \mathfrak{E}$ making the diagram commute:

$$\begin{tikzcd}
 & \mathfrak{T}_q \arrow[rr] & & \mathfrak{E}_q \\
\mathfrak{T}_{q^{1/p}} \arrow[ur, "{[\mathfrak{p}]}"] \arrow[rr] & & \mathfrak{T}_{q^{1/p}} \times^{\overline{T}} \overline{E} \arrow[ur, dotted, "\exists"] & \\
 & \overline{T} \arrow[uu] \arrow[rr] & & \overline{E} \arrow[uu] \\
\overline{T} \arrow[uu] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E} \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd} \tag{9}$$

By Lemma 4.7 this diagram equals diagram (8) on the generic fibre.

To see that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ is affine, first note that $[p] : \overline{B} \to \overline{B}$ is an affine morphism. The map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ is affine by construction, namely by Corollary 3.6 it is locally on $\mathfrak{T}_q$ of the form $[\mathfrak{p}] : \prod_{j=1}^d \mathfrak{B}(q^{i_j/p}, q^{(i_j-1)/p}) \to \prod_{j=1}^d \mathfrak{B}(q^{i_j}, q^{i_j-1})$. Note that both of these affine open subsets are fixed by the action of $\overline{T}$. We conclude from the construction in the proof of Proposition A.18 that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ locally on the target is of the form

$$[\mathfrak{p}] : \prod_{j=1}^d \mathfrak{B}(q^{i_j/p}, q^{(i_j-1)/p}) \times U' \to \prod_{j=1}^d \mathfrak{B}(q^{i_j}, q^{i_j-1}) \times U$$

for an affine open formal subscheme $U \subseteq \overline{B}$ with affine preimage U' under $[p] : \overline{B} \to \overline{B}$. This shows that the morphism is affine locally on the target, and therefore is affine. □

We have thus proved the first part of what we want to show about tilde-limits of E:

Lemma 4.10. *With notation as above, the rigid group E admits a $[p]$-formal tower of the form*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

for some $q \in K^\times$. In particular, there exists an adic space E_∞ such that $E_\infty \sim \varprojlim_{[p]} E$.

Proof. By Lemma 4.9, any choice of $q \in K^\times$ with $|q| < 1$ and a compatible system of p^n-th roots $q^{1/p^n} \in K^\times$ gives a tower

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

that on the generic fibre equals $\dots \xrightarrow{[p]} E \xrightarrow{[p]} E$. This is the desired $[p]$-formal tower. □

We are now ready to prove the main result of this section, namely that E_∞ is perfectoid.

Theorem 4.11. *Let K be perfectoid. Then the $[p]$-formal tower from Lemma 4.10*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

is already a $[p]$-F-formal tower. In particular, the space E_∞ is perfectoid.

Proof. It suffices to prove that the reduction mod π of the map $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through relative Frobenius.

In the following we denote reduction of a formal scheme by a $\sim$ over the respective symbol, for example the reductions of $\overline{T}$, $\overline{E}$ and $\mathfrak{T}$ are denoted by $\tilde{T}$, $\tilde{E}$ and $\tilde{\mathfrak{T}}$.

Recall that $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ was constructed using the $[p]$-multiplication cube in diagram (9) and functoriality of the associated bundle. Also recall that all statements we have used about fibre bundles also hold when we replace formal schemes over $\mathcal{O}_K$ by schemes over $\mathcal{O}_K/\pi$, and formation of the associated bundle commutes with this reduction. In particular,

$$\tilde{\mathfrak{E}}_q = \tilde{\mathfrak{T}}_q \times^{\tilde{T}} \tilde{E}.$$

By Corollary 3.6, the model of the multiplication map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ reduces to relative Frobenius over p. In particular, we have a natural isomorphism

$$\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \cong \tilde{\mathfrak{T}}_q$$

and we can identify $\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} = \tilde{\mathfrak{T}}_q$ in the following. The same is true for $\tilde{T}^{(p)} = \tilde{T}$.

Since $\tilde{E}$ and $\tilde{T}$ are group schemes, the reduction of $[p]$ on them factors through the relative Frobenius maps F_E and F_T respectively. By functoriality of relative Frobenius ("Frobenius commutes with any map") we have a commutative diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "F_{\tilde{E}}"] & \tilde{E}^{(p)} \\ \tilde{T} \arrow[u, hook] \arrow[r, "F_{\tilde{T}}"] & \tilde{T}^{(p)}. \arrow[u, hook] \end{tikzcd}$$

In other words, $F_{\tilde{E}}$ is an $F_{\tilde{T}}$-linear morphism of fibre bundles. Again by functoriality of Frobenius we also have a commutative diagram

$$\begin{tikzcd} \tilde{\mathfrak{T}}_{q^{1/p}} \arrow[r, "F_{\tilde{\mathfrak{T}}}"] & \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \\ \tilde{T} \arrow[u, hook] \arrow[r, "F_{\tilde{T}}"] & \tilde{T}^{(p)}. \arrow[u, hook] \end{tikzcd}$$

By Proposition A.18, we thus obtain a natural morphism

$$F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}} : \tilde{\mathfrak{T}}_{q^{1/p}} \times^{\tilde{T}} \tilde{E} \to \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \times^{\tilde{T}^{(p)}} \tilde{E}^{(p)}.$$

Using the explicit description of $F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}}$ in the proof of Proposition A.18, we easily check that this morphism is just the relative Frobenius of $\tilde{\mathfrak{E}}_{q^{1/p}}$: This is a consequence of the fact that relative Frobenius on the fibre product $\tilde{T} \times \tilde{U}$ for any $\tilde{U} \subseteq \tilde{B}$ is just the product of the relative Frobenius morphisms of $\tilde{T}$ and $\tilde{U}$, and thus the morphisms θ_i from Lemma A.8 are all trivial.

But this means that again by Proposition A.18, the reduction of the formal model of the p-multiplication cube in diagram 9 admits the following factorisation:

$$\begin{tikzcd}
 & & & \tilde{\mathfrak{T}}_q \arrow[rr] & & \tilde{\mathfrak{E}}_q \\
 & \tilde{\mathfrak{T}}_q^{(p)} \arrow[urr, equal] \arrow[rrr] & & & \tilde{\mathfrak{E}}^{(p)}_{q^{1/p}} \arrow[ur, dotted] & \\
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[ur, "F"'] \arrow[rr] & & \tilde{\mathfrak{E}}_{q^{1/p}} \arrow[urr, "F"'] & & & \\
 & & & \tilde{T} \arrow[uuu] \arrow[rr] & & \tilde{E} \arrow[uuu] \\
 & \tilde{T}^{(p)} \arrow[uuu] \arrow[urr, equal] \arrow[rrr] & & & \tilde{E}^{(p)} \arrow[uuu] \arrow[ur] & \\
\tilde{T} \arrow[uuu] \arrow[ur, "F"'] \arrow[rr] & & \tilde{E} \arrow[uuu] \arrow[urr, "F"'] & & &
\end{tikzcd}$$

Since the composed maps $\tilde{E} \to \tilde{E}$ on the bottom right, $\tilde{T} \to \tilde{T}$ on the bottom left and $\tilde{\mathfrak{T}}_{q^{1/p}} \to \tilde{\mathfrak{T}}_q$ on the upper left by construction are the reductions of the respective p-multiplication maps $[p]$, the functoriality of the associated bundle construction in Proposition A.18 implies that the two maps on the upper right compose to the reduction of $[\mathfrak{p}] \times^{[p]} [p]$. But $[\mathfrak{p}] \times^{[p]} [p]$ is equal to $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ by definition of the latter. This completes the proof that the reduction of $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through the relative Frobenius on $\tilde{\mathfrak{E}}_{q^{1/p}}$.

The conclusion that E_∞ is perfectoid then follows from Proposition 2.15. $\square$

5. The case of abelian varieties with semi-stable reduction

We now want to prove the main result of this work, building on the preceding chapters:

Theorem 5.1. *Let A be an abelian variety over an algebraically closed perfectoid field K. Then there is a perfectoid space A_∞ such that*

$$A_\infty \sim \varprojlim_{[p]} A.$$

We have seen this already for abelian varieties with good reduction in Corollary 2.16. It thus suffices to deal with the bad reduction case. Since K is algebraically closed, this means that A has semi-stable reduction. Before we proceed with the proof of the main theorem which occupies the rest of the section, let us recall some notation:

(1) g is the dimension of A.
(2) E is the Raynaud extension associated to A from Proposition 4.3, which is an extension of a split rigid torus T of rank r by an abelian variety B of good reduction.
(3) $M \subseteq E$ is a lattice of rank r such that $A = E/M$.

5.1. Covering A by subspaces of E.

The first step towards towards the proof is to construct a cover of $A = E/M$ by subspaces of E that behaves well under $[p]$-multiplication.

As a first step we recall how to relate the lattice M to a Euclidean lattice in $\mathbb{R}^r$, cf §2.7 and §6.2 in [5]. On the level of points, $\mathbb{G}_m^{\mathrm{an}}$ has an absolute value map

$$|-| : \mathbb{G}_m^{\mathrm{an}}(K) = K^\times \to \mathbb{R}^\times, \quad x \mapsto |x|$$

which induces the following group homomorphism from the torus T:

$$|-| : T(K) = (K^\times)^r \to (\mathbb{R}^\times)^r, \quad (x_1, \dots, x_n) \mapsto (|x_1|, \dots, |x_n|)$$

Since when working with lattices we prefer additive notation, we also consider the map

$$\ell : T(K) = (K^{\times})^r \to \mathbb{R}^r, \quad x_1, \dots, x_n \mapsto (-\log|x_1|, \dots, -\log|x_n|).$$

Note that this map has dense image due to our assumptions on K.

The formal torus $\overline{T}$ corresponds on K-points to $\overline{T}_\eta(K) = (\mathcal{O}_K^{\times})^r$ and is thus in the kernel of ℓ. We can therefore extend ℓ to $E(K)$ as follows: Locally over an open subspace $U \subseteq B$ we have $E|_U = T \times^{\overline{T}_\eta} \overline{E}_\eta|_U$ and we define ℓ by projection from the first factor. The different $E|_U$ are then glued on intersections using the $\overline{T}_\eta$-action on T. But since ℓ on T is invariant under the $\overline{T}_\eta$-action, the maps glue together to a group homomorphism

$$\ell : E(K) \to \mathbb{R}^r.$$

Since $A = E/M$ is proper, the lattice M is sent by ℓ to a Euclidean lattice $\Lambda \subset \mathbb{R}^r$ of full rank r (see Proposition 6.1.4 in [5]). In particular, the map l restricts to an isomorphism of discrete torsion-free groups

$$\ell : M \xrightarrow{\sim} \Lambda \subseteq \mathbb{R}^r.$$

The idea is now that one can understand the quotient E/M in terms of the quotient $\mathbb{R}^r/\Lambda$. We are going to make this precise in the following:

For any $d \in \mathbb{R}^r_{>0}$, consider the cuboid with length $2d$ centered at the origin:

$$S(d) = \{(s_1, \dots, s_r) \in \mathbb{R}^r | |s_i| \leq d_i\}$$

Choose d small enough such that $S(d)$ intersects Λ only in $0 \in \Lambda$ and such that the translates of $S(d)$ by Λ are all disjoint. We may moreover assume that $\exp(-d_i) \in |K^{\times}|$ for all i. Choose $q_1, \dots, q_r \in K$ such that $|q_i| = \exp(-d_i)$. We denote by $\mathcal{B}(q, q^{-1})$ the affinoid open rigid multi-annulus in T centered at 1 of radii $|q_i| < 1 < |q_i|^{-1}$ in every direction.

Lemma 5.2. *The inverse image $\ell^{-1}(S(d)) \subseteq E(K)$ is the underlying set of the admissible open subspace $E(q) := \mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ of E considered as a rigid space.*

Proof. One shows this first for the map $T \to \mathbb{R}^r$, where it is clear that the preimage is $\mathcal{B}(q, q^{-1})$. This is also described in §6.4 of [4]. The statement for $\mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ follows by direct inspection on local trivialisations $\mathcal{B}(q, q^{-1}) \times U$ for $U \subseteq B$. $\square$

Note that by our choice of d, the map $\mathbb{R}^r \to \mathbb{R}^r/\Lambda$ maps $S(d)$ bijectively onto its image. Lemma 5.2 then says that we can use $\mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ as a chart for E/M around the origin.

In order to obtain charts around other points of E/M, we simply need to consider translations: Recall that for any $c \in T(K)$, the translation $c\cdot : T \to T$ is an isomorphism of rigid spaces that sends 1 to c. We denote the image of any admissible open subspace U under translation by $c \cdot U$.

For the next Lemma, recall that K is algebraically closed and hence we can identify the underlying set of E considered as a rigid space with the set of K-points $E(K)$.

Lemma 5.3. *With notation as before, let $c \in T(K)$ be any point and let $s = l(c)$. Then the inverse image $\ell^{-1}(s + S(d)) \subseteq E(K)$ of the translation of $S(d)$ by s is the underlying set of the admissible open subset $E(c, q) := (c \cdot \mathcal{B}(q, q^{-1})) \times^{\overline{T}_\eta} \overline{E}_\eta \subseteq E$.*

Proof. Since ℓ commutes with the translations

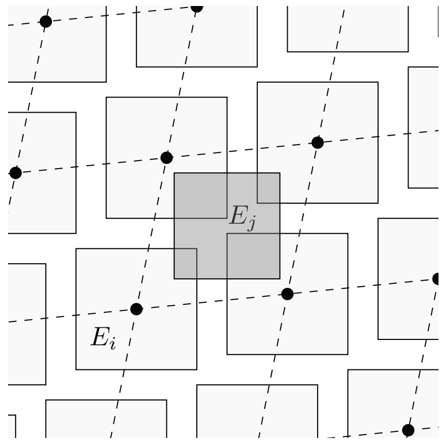

FIGURE 1. Given two charts E_i and E_j, the chart E_j is glued to E_i along intersections with all translates of E_i by $q \in M$.

$$\begin{array}{ccc} T(K) & \xrightarrow{\ell} & \mathbb{R}^r \\ {\scriptstyle \cdot c}\downarrow & & \downarrow{\scriptstyle +s} \\ T(K) & \xrightarrow{\ell} & \mathbb{R}^r, \end{array}$$

this is an immediate consequence of Lemma 5.2. □

Definition 5.4. We call the spaces $E(c,q) \subseteq E$ **cuboids** centered at c. More precisely, they are locally a cuboid $c \cdot \mathcal{B}(q,q^{-1}) \subseteq T$ times an admissible open subspace of the abelian variety B.

Lemma 5.5. *There exist finitely many admissible open cuboids $E_1,\dots,E_k \subseteq E$ which map isomorphically onto their images in $A = E/M$ and which cover A admissibly.*

One can reconstruct A from any such cover by glueing $E_1,\dots,E_k$ as follows: By construction, for any E_i the translates $m \cdot E_i$ by $m \in M$ are pairwise disjoint and we thus have a canonical projection π from the union $\cup_{m\in M}(m \cdot E_i) \subseteq E$ to E_i. Let $E_{ij} := (\bigcup_{m\in M} m \cdot E_i) \cap E_j \subseteq E$. Then we glue E_j to E_i via the map

$$E_{ij} \to \bigcup_{m\in M} m \cdot E_i \xrightarrow{\pi} E_i.$$

Proof. Since $\mathbb{R}^r/\Lambda$ is compact, we can find finitely many $s_1,\dots,s_k$ and $d_1,\dots,d_k \in \mathbb{R}^r_{>0}$ such that $\mathbb{R}^r/\Lambda$ is covered by the $s_i + S(d_i)$. When we choose corresponding $c_1,\dots,c_k \in T(K)$ and $q_1,\dots,q_k$ as in Lemma 5.3, then the corresponding $E_i := E(c_i,q_i)$ give an admissible cover of $A = E/M$ by admissible open subsets of E.

In order to reconstruct A, note that $\cup_{m\in M} m \cdot E_i$ is precisely the preimage of E_i under the projection $E \to E/M$. In particular, the subspace E_{ij} is precisely the preimage of E_i under the composition $E_j \hookrightarrow E \to E/M$. In other words, the subspace $E_{ij} \subseteq E_j$ is the intersection of E_i and E_j when considered as subspaces of A. This shows that as charts of A, the spaces E_i and E_j are glued via E_{ij} as described. □

Finally, we need some control about what happens to the cubes under $[p]$-multiplication. Recall from Lemma 5.3 that we can always assume that c admits p-th roots.

Lemma 5.6. *Let $c^{1/p}$ be a p-th root of c and let $q^{1/p}$ be a p-th root of q in $(K^\times)^r$. Then under $[p] : E \to E$, the admissible open $E(c_i, q)$ pulls back to the admissible open $E(c_i^{1/p}, q^{1/p})$.*

Proof. It is clear that under $[p] : T \to T$, the admissible open cuboid $c \cdot \mathcal{B}(q, q^{-1})$ centered at c pulls back to $c^{1/p} \cdot \mathcal{B}(q^{1/p}, q^{-1/p})$. Note that this is independent of the choices of $c^{1/p}$ and $q^{1/p}$. Now recall that in terms of fibre bundles, multiplication $[p] : E \to E$ is

$$[p] \times^{[p]} [p] : T \times^{\overline{T}_\eta} \overline{E}_\eta \to T \times^{\overline{T}_\eta} \overline{E}_\eta$$

by Lemma 4.7. Thus $(c \cdot \mathcal{B}(q, q^{-1})) \times^{\overline{T}_\eta} \overline{E}_\eta$ pulls back to $(c^{1/p} \cdot \mathcal{B}(q^{1/p}, q^{-1/p})) \times^{\overline{T}_\eta} \overline{E_\eta}$. □

5.2. **The two towers.** In this section we want to separate the $[p]$-multiplication of A into two different towers, which we think of as being a "ramified" tower and an "étale" tower. Of course in characteristic 0 both towers will actually be étale, and these words are only meant to describe the behaviour of the maps relative to the lattice M.

For the ramified tower, we first make an auxiliary choice of certain torsion subgroups of A: Since K is algebraically closed, we can choose lattices $M^{1/p^n} \subseteq E$ such that $[p] : E \to E$ restricts to isomorphisms $M^{1/p^{n+1}} \to M^{1/p^n}$ for all n.

Remark 5.7. Such a choice is equivalent to the choice of subgroups $D_n \subseteq A[p^n]$ of rank p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$. Namely, given the lattices $M^{1/p^{n+1}}$, we obtain the desired torsion subgroups by setting $D_n := M^{1/p^{n+1}}/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/M^{p^n} \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ with properties as above, we recover M^{1/p^n} as the kernel of $E \to A \to A/D_n$.

One might call the choice of D_n for all n a partial anticanonical $\Gamma_0(p^\infty)$-structure, because if B admits a canonical subgroup (that is, if B has bad reduction or if it has good reduction and the reduction satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A always has a partial anticanonical subgroup even if B does not have a canonical subgroup.

Note that in the case of K perfectoid but not necessarily algebraically closed, one can still carry out the constructions in the following using partial anticanonical $\Gamma_0(p^\infty)$-structures, whereas the lattices M^{1/p^n} might not be defined over K.

Following the remark, denote by D_n the torsion subgroup $M^{1/p^n}/M \subseteq A$. The quotient $A/D_n = E/M^{1/p^n}$ is then another abelian variety over K and the quotient map $v^n : E/M \to E/M^{1/p^n}$ is an isogeny of degree p^{2gn} through which $[p^n] : A \to A$ factors:

(10)
$$\begin{tikzcd} & E/M^{1/p} \arrow[dr, "{[p^n]}"] & \\ E/M \arrow[ur, "v^n"] \arrow[rr, "{[p^n]}"] & & E/M. \end{tikzcd}$$

We think of these maps as being an analogue of Frobenius and Verschiebung, which is why we denote the left map by v. Putting everything together, the $[p]$-multiplication tower splits into two towers

$$
\begin{tikzcd}
\ddots \arrow[dr, dashed] & & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] \\
& E/M \arrow[r, "v"] \arrow[dr, "{[p]}"'] & E/M^{1/p} \arrow[r, "v"] \arrow[d, "{[p]}"] & E/M^{1/p^2} \arrow[d, "{[p]}"] \\
& & E/M \arrow[r, "v"] \arrow[dr, "{[p]}"'] & E/M^{1/p} \arrow[d, "{[p]}"] \\
& & & E/M
\end{tikzcd} \tag{11}
$$
.

Since each quotient $M^{1/p^n}/M$ is a finite étale group scheme, all horizontal maps are finite étale. The vertical tower on the other hand fits into a second commutative diagram of rigid groups which compares it to the $[p]$-tower of E:

$$
\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M^{1/p^2} \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p^2} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M^{1/p} \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & E/M \arrow[r] & 0
\end{tikzcd} \tag{12}
$$

5.3. **Constructing a limit of the vertical tower.** Our first step is to show that the tower on the right has a perfectoid tilde-limit. Recall from Lemma 5.5 that E/M can be covered by admissible open subspaces $E_1, \dots, E_k \subseteq E$ which map isomorphically onto an admissible open via $E \to E/M$. Denote by $E_i^{1/p^n} \subseteq E$ the pullback along $[p^n] : E \to E$. Also denote by $E_{ij}^{1/p^n} \subseteq E$ the pullback of the intersection E_{ij}. We can then reconstruct the space $E/M^{1/p^n}$ from the E_i^{1/p^n} as follows:

Lemma 5.8.

(1) *The restriction to $E_i^{1/p^n} \subseteq E$ of $E \to E/M^{1/p^n}$ is an isomorphism onto its image. In particular, we can view E_i^{1/p^n} as a chart of $E/M^{1/p^n}$, and this is the preimage of E_i under $E/M^{1/p^n} \to E/M$.*

(2) *The collection of E_i^{1/p^n} is an atlas for $E/M^{1/p^n}$.*

(3) *We can reconstruct $E/M^{1/p^n}$ from glueing the E_i^{1/p^n} along the E_{ij}^{1/p^n}.*

(4) *The map $[p^n] : E/M^{1/p^n} \to E/M$ can be glued from the restrictions of $[p^n] : E \to E$ to $E_i^{1/p^n} \to E_i$, that is these maps commute with the glueing maps on E_{ij}^{1/p^n}.*

The situation is thus like in Figure 2.

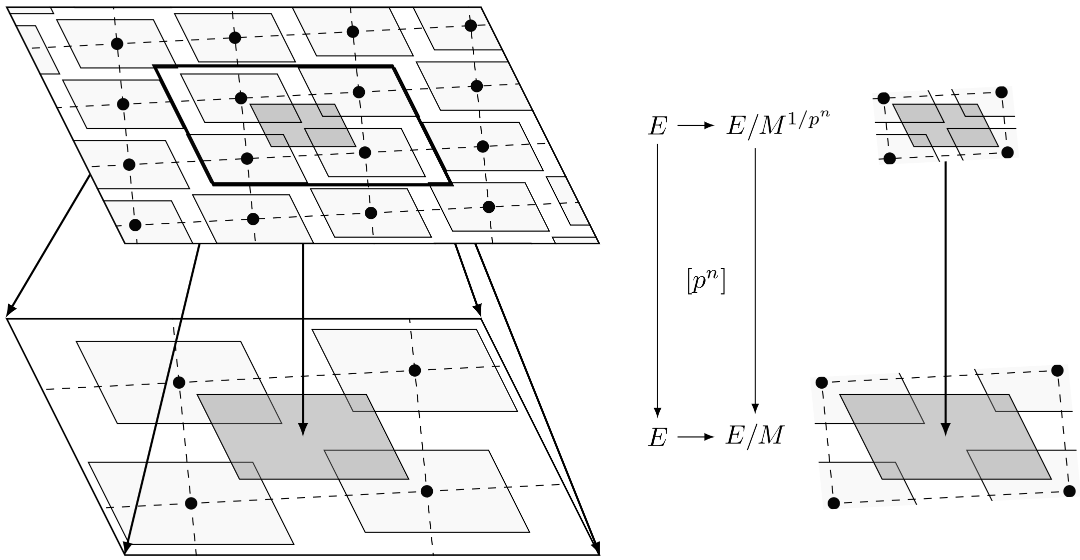

FIGURE 2. Illustration of how $[p^n] : E/M^{1/p^n} \to E/M$ can be glued from the maps $E_j^{1/p^n} \to E_j$. Here E_j on bottom and E_j^{1/p^n} on top are represented by the grey cuboids in the middle. On the left they are embedded into E whereas on the right they are considered as charts for E/M and $E/M^{1/p}$.

Proof. The first part follows because the map on the left of diagram 12 is an isomorphism. The second follows from the pullback of the E_i along $E/M^{1/p^n} \to E/M$, using that the diagram commutes. We thus obtain an admissible cover by cuboids $E_1^{1/p^n}, \dots, E_k^{1/p^n}$ of $E/M^{1/p^n}$. The second part of Lemma 5.5 applied to $E/M^{1/p^n}$ then shows that $E/M^{1/p^n}$ can be reconstructed by glueing along subspaces E_{ij}^{1/p^n}.

Finally, in order to see that one can glue together the map $[p^n] : E/M^{1/p^n} \to E/M$ from the E_i^{1/p^n}, use that intersection of cuboids are again cuboids, and so E_{ij}^{1/p^n} is a disjoint union of cuboids. It then follows from Lemma 5.6 that E_{ij} pulls back under $[p^n]$ to the intersection $E_{ij}^{1/p^n} \subseteq E/M^{1/p^n}$. That $[p^n]$ commutes with the glueing maps is clear because we know from diagram (10) that $[p^n] : E \to E$ induces a morphism $[p^n] : E/M^{1/p^n} \to E/M$. $\square$

We are now ready to prove:

Proposition 5.9. *There is a perfectoid space $E/M^{1/p^\infty}$ such that*

$$E/M^{1/p^\infty} \sim \varprojlim_n E/M^{1/p^n}.$$

Proof. Denote by E_i^{1/p^∞} the pullback of $E_i \subseteq E$ to E_∞. This is an open subspace of a perfectoid space and hence perfectoid. Moreover, by Proposition 2.4.3 in [12] we have

$$E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}.$$

Given two different E_i, E_j, we know by Lemma 5.8 that at every step in the tower, the pullbacks E_i^{1/p^n} and E_j^{1/p^n} to $E/M^{1/p^n}$ intersect in E_{ij}^{1/p^n}. We can thus glue the E_i^{1/p^∞} along pullbacks

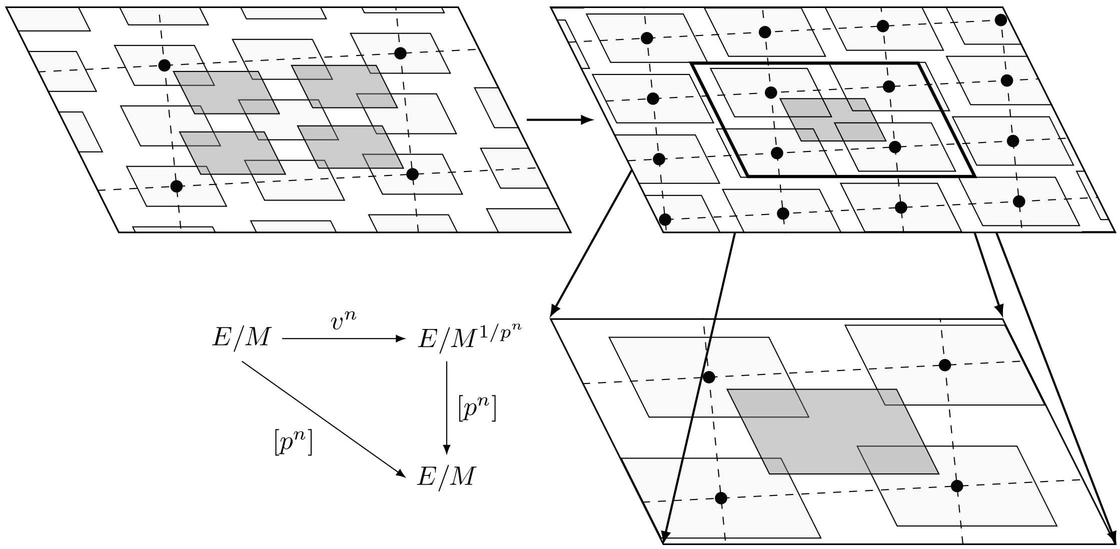

FIGURE 3. Illustration of how $[p] : E/M \to E/M$ factors in a part that is “ramified” (the vertical tower) and a part that is “étale” (the horizontal tower) with respect to our cover.

E_{ij}^{1/p^∞} of the intersections $E_{ij} = E_i \cap E_j$ to E_∞ and thus obtain a perfectoid space $E/M^{1/p^\infty}$. This is a tilde-limit for $\varprojlim_{[p]} E/M^{1/p^n}$ because by construction it is so locally, and the definition of tilde-limits in Definition 2.4.1 of [12] is local on the source. $\square$

5.4. **Constructing a limit of the horizontal tower.** In order to construct a tilde-limit for $\varprojlim A$, we can now use that the horizontal maps in diagram (11) are all finite étale. They are even finite covering maps, in the following sense:

Lemma 5.10. *For any $0 \le m \le n$, the preimage of E_i^{1/p^n} from Lemma 5.8 under the horizontal map $v^{n-m} : E/M^{1/p^m} \to E/M^{1/p^n}$ is isomorphic to $p^{r(n-m)}$ disjoint copies of E_i^{1/p^n}. More canonically, it can be described as the isomorphic image of $M^{1/p^n}/M^{1/p^m} \times E_i^{1/p^m}$ under the multiplication map $E/M^{1/p^m} \times E/M^{1/p^m} \to E/M^{1/p^m}$.*

Proof. By the first part of Lemma 5.8, we know that the preimage of E_i^{1/p^n} under the projection $E \to E/M^{1/p^n}$ is a disjoint union of translates of E_i^{1/p^n} by M^{1/p^n}. The result then follows because $M^{1/p^n}/M^{1/p^m} = \underline{(\mathbb{Z}/p^{n-m}\mathbb{Z})^r}$. $\square$

We also record the following immediate consequence:

Lemma 5.11. *The preimage of E_i under $[p^n] : A \to A$ is isomorphic to p^{rn} disjoint copies of E_i^{1/p^n}. More canonically, we can describe the preimage as the isomorphic image of $D_n \times E_i^{1/p^n}$ under the multiplication $A \times A \to A$. The situation is thus as in figure 3.*

Proof. This follows from the first part of Lemma 5.8 combined with Lemma 5.10 in the case of $m = 0$. $\square$

Lemma 5.12. *The squares in diagram (11) are all pullback diagrams.*

$$\begin{tikzcd} E/M^{1/p^n} \arrow[r, "v"] \arrow[d, "{[p]}"] & E/M^{1/p^{n+1}} \arrow[d, "{[p]}"] \\ E/M^{1/p^{n-1}} \arrow[r, "v"] & E/M^{1/p^n} \end{tikzcd}$$

Proof. This can for instance be checked locally: The admissible open subset $E_i^{1/p^n} \subseteq E/M^{1/p^n}$ from Lemma 5.8 is pulled back to $E_i^{1/p^{n+1}}$ under the vertical map $[p] : E/M^{1/p^{n+1}} \to E/M^{1/p^n}$. The preimage of E_i^{1/p^n} under the horizontal map $E/M^{1/p^{n-1}} \to E/M^{1/p^n}$ is p^r disjoint copies of E_i^{1/p^n} by Lemma 5.10. The pullback of E_i^{1/p^n} to the upper right is thus p^r disjoint copies of $E_i^{1/p^{n+1}}$, which is clearly the fibre product. □

Lemma 5.13. *The horizontal maps in diagram (11) induce natural finite étale morphisms $v : E/M^{1/p^\infty} \to E/M^{1/p^\infty}$ that fit into Cartesian diagrams*

$$\begin{tikzcd} E/M^{1/p^\infty} \arrow[r, "v^m"] \arrow[d] & E/M^{1/p^\infty} \arrow[d] \\ E/M^{1/p^{n-m}} \arrow[r, "v^m"] & E/M^{1/p^n} \end{tikzcd}$$

In particular, the preimage of E_i^{1/p^∞} under v^m is isomorphic to p^{rm} copies of E_i^{1/p^∞}.

Proof. Since $E/M \to E/M^{1/p}$ is finite étale, the fibre product with $E/M^{1/p^\infty} \to E/M^{1/p}$ exists and is perfectoid by Proposition 7.10 of [7].

The universal property of the fibre product then gives a unique map

$$E/M^{1/p^\infty} \to E/M \times_{E/M^{1/p}} E/M^{1/p^\infty}$$

making the natural diagrams commute. On the other hand, using Lemma 5.12 we see that the fibre product has compatible maps into the vertical inverse system over E/M. Since by Proposition 2.4.5 of [12] the perfectoid tilde-limit $E/M^{1/p^\infty}$ is universal for maps from perfectoid spaces to the inverse system, we obtain a unique map into the other direction. □

We thus obtain a pro-étale tower

$$\dots \xrightarrow{v} E/M^{1/p^\infty} \xrightarrow{v} E/M^{1/p^\infty} \xrightarrow{v} E/M^{1/p^\infty} \tag{13}$$

which we think of as being a kind of vertical "limit" of diagram 11.

We now want to explicitly construct a tilde-limit of this tower. Recall from Remark 5.7 the finite étale subgroups $D_n = M^{1/p^n}/M \subseteq E/M = A$ of rank p^{rn}. Since K is algebraically closed, they are constant and uncanonically isomorphism to $\underline{(\mathbb{Z}/p^n\mathbb{Z})}^r = \mathrm{Spa}(\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, K), \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, \mathcal{O}_K))$. The $[p]$-multiplication on E maps $M^{1/p^{n+1}}$ onto M^{1/p^n} and therefore the $[p]$-multiplication tower of E/M induces a tower

$$\dots \xrightarrow{[p]} D_{n+1} \xrightarrow{[p]} D_n \to \dots .$$

Lemma 5.14. *There exists a perfectoid space D_∞ such that $D_\infty \sim \varprojlim_{[p]} D_n$. Any choice of compatible isomorphisms $D_n \cong \underline{\mathbb{Z}/p^n\mathbb{Z}}^r$ induces an isomorphism*

$$D_\infty = \mathrm{Spa}(\mathrm{Map}_{cts}(\mathbb{Z}_p^r, K), \mathrm{Map}_{cts}(\mathbb{Z}_p^r, \mathcal{O}_K)).$$

Proof. The D_n are étale over $(K, \mathcal{O}_K)$ and thus affinoid perfectoid. Proposition 2.7 then shows that D_∞ exists. Given isomorphisms $D_n = \underline{(\mathbb{Z}/p^n\mathbb{Z})^r} = \mathrm{Spa}(\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, K), \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, \mathcal{O}_K))$, Proposition 2.7 moreover says that the tilde-limit is given by the π-adic completion of

$$\big(\varinjlim_n \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, K), \varinjlim_n \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, \mathcal{O}_K)\big) = (\mathrm{Map}_{lc}(\mathbb{Z}_p^r, K), \mathrm{Map}_{lc}(\mathbb{Z}_p^r, \mathcal{O}_K))$$

where Map_{lc} denotes the locally constant morphisms, for the p-adic topology on $\mathbb{Z}_p$. Since the π-adic completion of these are the continuous morphisms, this proves the Lemma. $\square$

Lemma 5.15. *Let $(X_n)_{n\in\mathbb{N}}$ be an inverse system of adic spaces with quasicompact and quasiseparated transition maps over $(K, \mathcal{O}_K)$ and let X be a perfectoid space such that $X \sim \varprojlim_n X_n$. Then $D_\infty \times X$ is a perfectoid tilde-limit*

$$D_\infty \times X \sim \varprojlim_{n\in\mathbb{N}} (D_n \times X_n).$$

Proof. Since D_∞ is perfectoid, the fibre product $D_\infty \times X$ exists and is perfectoid by [7], Proposition 6.18. Since the D_n are just a disjoint union of copies of $\mathrm{Spa}(K, \mathcal{O}_K)$, the fibre product $D_n \times X_n$ exists as an adic space. Let us choose compatible isomorphisms $D_n \cong \underline{(\mathbb{Z}/p^n\mathbb{Z})^r}$, then locally on an affinoid open $\mathrm{Spa}(A_n, A_n^+) \subseteq X_n$, the fibre product is of the form

$$D_n \times \mathrm{Spa}(A_n, A_n^+) = \mathrm{Spa}(\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_n), \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_n^+)).$$

Since $X \sim \varprojlim_n X_n$, we may cover X by open affinoids $U = \mathrm{Spa}(A, A^+)$ such that

$$\varinjlim_{\mathrm{Spa}(A_j, A_j^+)\subseteq X_n} A_j \to A$$

has dense image. Here the direct limit runs through all affinoid open subspaces $\mathrm{Spa}(A_j, A_j^+) \subseteq X_n$ for all n through which $U \subseteq X \to X_n$ factors, as in the definition of tilde-limits. Then by Lemma 5.14 and the construction of products of affinoid perfectoid spaces, we have

$$D_\infty \times U \cong \mathrm{Spa}(\mathrm{Map}_{cts}(\mathbb{Z}_p^r, A), \mathrm{Map}_{cts}(\mathbb{Z}_p^r, A^+)).$$

It thus suffices to show that $\varinjlim \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j)$ is dense in $\mathrm{Map}_{cts}(\mathbb{Z}_p^r, A)$. As a first step, even though in the above limit n depends on j, we may write this as two separate limits,

$$\varinjlim \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j) = \varinjlim_j \varinjlim_n \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j).$$

For fixed j, we then see that

$$\varinjlim_n \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j) = \mathrm{Map}_{lc}(\mathbb{Z}_p^r, A_j),$$

where the right hand side denotes locally constant morphisms. But it then follows from a pointwise approximation argument that $\varinjlim_j \mathrm{Map}_{lc}(\mathbb{Z}_p^r, A_j)$ has dense image in $\mathrm{Map}_{lc}(\mathbb{Z}_p^r, A)$. It is then clear that the latter has dense image in $\mathrm{Map}_{cts}(\mathbb{Z}_p^r, A)$. We conclude that $D_\infty \times X \sim \varprojlim D_n \times X_n$. $\square$

Proposition 5.16. *There exists a perfectoid space $(E/M^{1/p^\infty})_\infty$ such that*

$$(E/M^{1/p^\infty})_\infty \sim \varprojlim_v (E/M^{1/p^\infty}).$$

Moreover, the projection map $\pi : (E/M^{1/p^\infty})_\infty \to E/M^{1/p^\infty}$ is a profinite covering map, namely every point of $E/M^{1/p^\infty}$ has an open neighbourhood U such that $\pi^{-1}(U)$ is isomorphic to $D_\infty \times U$.

Proof. By Lemma 5.5, the preimage of E_i^{1/p^∞} under v^m of $E/M^{1/p^\infty}$ is isomorphic to $D_m \times E_i^{1/p^\infty}$. By Lemma 5.15 we then have

$$D_\infty \times E_i^{1/p^\infty} \sim \varprojlim_m D_m \times E_i^{1/p^\infty}.$$

Alternatively, we could without loss of generality assume that E_i^{1/p^∞} is affinoid perfectoid, and use Proposition 2.7 to see that a perfectoid tilde-limit exists. Using Proposition 2.5 and the universal property of the product, on then obtains an isomorphism of this tilde-limit to $D_\infty \times E_i^{1/p^\infty}$.

As before, these local tilde-limits glue using Propositions 2.4 and 2.5. □

5.5. **The diagonal tower: proof of the main theorem.** We now want to show that $(E/M^{1/p^\infty})_\infty$ is in fact a tilde-limit of the $[p]$-multiplication tower. In other words, this says that the horizontal limit of the vertical tilde-limits in diagram 11 is also a diagonal tilde-limit. This isn't just a formal consequence since tilde-limits aren't limits. But using the local geometry of the maps in the tower in terms of cuboids, it is still easy to see:

Proposition 5.17. *The perfectoid space $A_\infty := (E/M^{1/p^\infty})_\infty$ is a tilde-limit of $\varprojlim_{[p]} A$. It is independent up to unique isomorphism of the choice of partial anticanonical $\Gamma_0(p^\infty)$-structure, but it remembers the choice as a pro-finite étale closed subgroup $D_\infty \subseteq A_\infty$.*

The preimage of $E_i \subseteq A$ under the projection $A_\infty \to A$ is isomorphic to $D_\infty \times E_i^{1/p^\infty}$.

Proof. It is clear from $(E/M^{1/p^\infty})_\infty \sim \varprojlim_v E/M^{1/p^\infty}$ and $E/M^{1/p^\infty} \sim \varprojlim_{[p]} E/M^{1/p^n}$ that the underlying topological space of $(E/M^{1/p^\infty})_\infty$ is indeed isomorphic to $\varprojlim_{[p]} |E/M|$.

In order to show that it is a tilde-limit of $\varprojlim_{[p]} E/M$, it thus suffices to give an explicit cover of $(E/M^{1/p^\infty})_\infty$ by open affinoids satisfying the tilde-limit property.

Recall that by construction of $(E/M^{1/p^\infty})$ we have a cover of E/M by open subsets E_i that pull back to perfectoid open subspaces E_i^{1/p^∞} for which $E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}$. Moreover, by the second part of Proposition we know that the pullback of E_i^{1/p^∞} to $(E/M^{1/p^\infty})_\infty$ is $D_\infty \times E_i^{1/p^\infty}$.

On the other hand, when we go along the diagonal tower, we obtain the inverse system

$$\dots \to D_{n+1} \times E_i^{1/p^{n+1}} \to D_n \times E_i^{1/p^n} \to \dots.$$

By Lemma 5.15 this inverse system has perfectoid tilde-limit $D_\infty \times E_i^{1/p^\infty}$. These local tilde-limits glue together to give the desired tilde-limit A_∞.

That A_∞ is independent of the $\Gamma_0(p^\infty)$-structure up to unique isomorphism is a consequence of the universal property of the perfectoid tilde-limit. To see that D_∞ is a closed subgroup of A_∞, choose i such that the unit section of E/M lies in E_i. Then the unit section $\mathrm{Spa}(K, \mathcal{O}_K) \to E_i^{1/p^\infty}$ induces a closed immersion $D_\infty \hookrightarrow D_\infty \times E_i^{1/p^\infty} \hookrightarrow A_\infty$. □

This finishes the proof of Theorem 5.1.

Note that while the approach via cuboids E_i may look a bit technical on first glance, it has the advantage of giving an explicit description of A_∞ as being glued from pieces of $D_\infty \times E_\infty$ by glueing data that is controlled by the lattices M^{1/p^n}. This might be interesting for applications.

6. Limits of the covering maps

In this section we use the explicit constructions of the space A_∞ to study its geometry more closely. We retain notation and assumptions from the last chapter.

Over the course of the proof of Theorem 5.1, we have used three different towers: The tower $\cdots \to E \xrightarrow{[p]} E$, the tower $\cdots \to E/M \xrightarrow{[p]} E/M$ and the tower $\cdots \to E/M^{1/p} \xrightarrow{[p]} E/M$. The three are related by covering maps which fit into a commutative diagram of towers

$$\begin{tikzcd} E \arrow[r] \arrow[d, "{[p]}"] & E/M \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p^{n+1}} \arrow[d, "{[p]}"] \\ E \arrow[r] & E/M \arrow[r] & E/M^{1/p^n} \end{tikzcd}$$

As we have seen in the last sections, all three towers have perfectoid tilde-limits, that we have denoted by E_∞, A_∞ and $E/M^{1/p^\infty}$.

By Proposition 2.18 the map $\pi : E \to A = E/M$ in the limit induces a natural group homomorphism $\iota : E_\infty \to A_\infty$. A similar universal property argument shows that we obtain a natural group homomorphism $A_\infty \to E/M^{1/p^\infty}$. The composition of these two maps is the morphism $E_\infty \to E/M^{1/p^\infty}$, which is the limit of the maps $E \to E/M^{1/p^n}$ in the above diagram. We now want to look at these morphisms more closely one after the other.

We start with the morphism $E_\infty \to E/M^{1/p^\infty}$:

Proposition 6.1. *Denote by $M_\infty \cong M$ the perfectoid tilde-limit of the tower*

$$\dots \xrightarrow[{[p]}]{\sim} M^{1/p^2} \xrightarrow[{[p]}]{\sim} M^{1/p} \xrightarrow[{[p]}]{\sim} M.$$

There is a natural map $M_\infty \to E_\infty$ with respect to which we can interpret M_∞ as a lattice of rank r in E_∞. The map fits into a short exact sequence of perfectoid groups

$$0 \to M_\infty \to E_\infty \to E/M^{1/p^\infty} \to 0$$

that is locally split. In particular, we can view E_∞ as an M_∞-torsor over $E/M^{1/p^\infty}$.

Proof. The map $M_\infty \to E_\infty$ is induced by the universal property of the perfectoid tilde-limit as usual. In order to see that the sequence is exact, we need to see that the first map is a kernel of the second, and the second map is a categorical quotient of the first. To this end, we first analyse the morphism locally: The projections to the inverse system fit into a commutative diagram

$$\begin{tikzcd} M_\infty \arrow[r] \arrow[d, dashed] & E_\infty \arrow[r] \arrow[d, dashed] & E/M^{1/p^\infty} \arrow[d, dashed] \\ M^{1/p^n} \arrow[r] \arrow[d, "{[p^n]}"] & E \arrow[r] \arrow[d, "{[p^n]}"] & E/M^{1/p^n} \arrow[d, "{[p^n]}"] \\ M \arrow[r] & E \arrow[r] & E/M \end{tikzcd}$$

Let us consider the preimages of $E_i \subseteq E/M$ under these morphisms: By Lemma 5.5 we see that the pullback to E is $\bigsqcup_{q\in M} qE_i$. We can also see this as the isomorphic image of $M \times E_i$ under the multiplication map $E \times E \to E$. The pullback of E_i along $[p^n] : E/M^{1/p^n} \to E/M$ is E_i^{1/p^n} as we have seen in Lemma 5.8. The same Lemma shows that the pullback of this along $E \to E/M^{1/p^n}$ is $\bigsqcup_{q\in M^{1/p^n}} qE_i^{1/p^n} = M^{1/p^n} \times E_i^{1/p^n}$. We conclude that the pullback to E_∞ is $M_\infty \times E_i^{1/p^\infty}$. By construction of $E/M^{1/p^\infty}$ in the proof of Proposition 5.9, the pullback of E_i to $E/M^{1/p^\infty}$ is E_i^{1/p^∞}. All in all, we obtain a pullback diagram

$$\begin{tikzcd}
& E_\infty \arrow[rr] \arrow[dd] & & E/M^{1/p^\infty} \arrow[dd] \\
M_\infty \times E_i^{1/p^\infty} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & E_i^{1/p^\infty} \arrow[ur, hook] \arrow[dd] & \\
& E \arrow[rr] & & E/M \\
M \times E_i \arrow[ur, hook] \arrow[rr] & & E_i \arrow[ur, hook] &
\end{tikzcd}$$

We conclude that $E_\infty \to E/M^{1/p^\infty}$ is a principal M_∞-torsor of perfectoid groups. It is then clear that M_∞ is the preimage of $0 \in E/M^{1/p^\infty}$, from which one easily verifies that $M_\infty \hookrightarrow E_\infty$ has the universal property of the kernel. It remains to see that $E_\infty \to E/M^{1/p^\infty}$ has the universal property of the cokernel: Given any perfectoid group H and a group homomorphism $E_\infty \to H$ that is trivial on M_∞, the restriction $M_\infty \times E_i^{1/p^\infty} \to H$ gives a natural map $E_i^{1/p^\infty} \to H$. Since by construction of $E/M^{1/p^\infty}$ the spaces E_i^{1/p^∞} and E_j^{1/p^∞} are glued on E_{ij}^{1/p^∞} using translation by M_∞, these glue together to the desired morphism of $E/M^{1/p^\infty}$. □

The case of $\iota : A_\infty \to E/M^{1/p^\infty}$ is similar:

Proposition 6.2. *The subgroup $D_\infty \subseteq A_\infty$ gives rise to a short exact sequence of perfectoid groups*

$$0 \to D_\infty \to A_\infty \to E/M^{1/p^\infty} \to 0$$

that is locally split. In particular, we can view A_∞ as a D_∞-torsor over $E/M^{1/p^\infty}$.

Proof. By Proposition 5.16 the pullback of E_i^{1/p^∞} under $A_\infty \to E/M^{1/p^\infty}$ is

$$D_\infty \times E_i^{1/p^\infty} \to E_i^{1/p^\infty}$$

which shows that $A_\infty \to E/M^{1/p^\infty}$ is a D_∞-torsor. As in the last proof, this implies that the sequence in the Proposition is a short exact sequence. □

Finally, we consider the case of $\iota : E \to A = E/M$. While the limits of the last two towers were fibre bundles again, the map ι shows quite a different behaviour and on the opposite is an injective group homomorphism. This may seem strange at first, but it is actually what one might expect following the intuition of the following example:

Remark 6.3. Consider the following inverse system of abstract groups:

$$\begin{tikzcd}
& \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "{[p]}"] & \mathbb{R} \arrow[r] \arrow[d, "{[p]}"] & \mathbb{R}/\mathbb{Z} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

While at finite level the maps on the right are all covering maps, in the inverse limit the homological algebra of $\varprojlim$ produces a long exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \varprojlim^1_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

So in the limit the covering map becomes the kernel of a map to $\mathbb{Z}_p/\mathbb{Z}$.

For perfectoid groups the homological algebra argument of course doesn't apply. Nevertheless, we can again use the explicit covers of the last section to show that the situation is very similar as in the remark. In the following, we use the notion of injective morphism from [10], Definition 5.1.

Theorem 6.4. *The map $\iota : E_\infty \to A_\infty$ is an injective group homomorphism. It fits into the following commutative diagram of locally split short exact sequences of perfectoid groups:*

$$\begin{tikzcd} 0 \arrow[r] & M_\infty \arrow[r] \arrow[d, hook] & E_\infty \arrow[r] \arrow[d, hook] & E/M^{1/p^\infty} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D_\infty \arrow[r] & A_\infty \arrow[r] & E/M^{1/p^\infty} \arrow[r] & 0 \end{tikzcd}$$

The morphism ι is compatible with the splittings: Locally on open subspaces $U \subseteq E/M^{1/p^\infty}$ the morphism $E_\infty \hookrightarrow A_\infty$ is of the form $M_\infty \times U \to D_\infty \times U$. In particular, one can describe A_∞ as the associated fibre bundle

$$A_\infty = D_\infty \times^{M_\infty} E_\infty.$$

Proof. Recall that the map $\iota : E_\infty \to A_\infty$ arises by a universal property from the inverse system

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ E \arrow[r] \arrow[d, "{[p]}"'] & A \arrow[r, "v"] \arrow[d, "{[p]}"'] & E/M^{1/p} \arrow[d, "f"'] \\ E \arrow[r] & A \arrow[r, equal] & E/M. \end{tikzcd}$$

Using Lemmas 5.8 and 5.10 we see that the pullback of this diagram to $E_i \subseteq E/M$ is

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \bigsqcup_{q \in M^{1/p}} q \cdot E_i^{1/p^n} \arrow[r] \arrow[d] & \bigsqcup_{q \in D_n} E_i^{1/p^n} \arrow[r] \arrow[d] & E_i^{1/p^n} \arrow[d] \\ \bigsqcup_{q \in M} q \cdot E_i \arrow[r] & E_i \arrow[r, equal] & E_i \end{tikzcd}$$

We see from this description and from the last part of Proposition 5.17 that the pullback to infinite level is the sequence

$$M_\infty \times E_i^{1/p^\infty} \to D_\infty \times E_i^{1/p^\infty} \to E_i^{1/p^\infty}. \tag{14}$$

This shows that the diagram of short exact sequences commutes. Since the E_i^{1/p^∞} cover $E/M^{1/p^\infty}$ by construction, it also shows the description of A_∞ in terms of the associated fibre bundle.

It remains to prove that ι is injective: Choose a basis of M, and thus a trivialisation of all D_n. We then see that the map $M_\infty \to D_\infty$ on the level of the underlying topological spaces can be described as the inclusion $\mathbb{Z}^r \hookrightarrow \mathbb{Z}_p^r$. This shows that $M_\infty \hookrightarrow D_\infty$ is an injective group homomorphism of perfectoid spaces. Thus by equation (14), the morphism $E_\infty \to A_\infty$ is injective as well. $\square$

Corollary 6.5. *The injection $E_\infty \to A_\infty$ induces a short exact sequence of perfectoid groups*

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the diagonal embedding of M_∞ into $D_\infty \times E_\infty$. In particular, we can describe A_∞ as the quotient $(D_\infty \times E_\infty)/M_\infty$ in the category of perfectoid groups.

Proof. The map $D_\infty \times E_\infty \to A_\infty$ is just the composition of $D_\infty \times E_\infty \hookrightarrow A_\infty \times A_\infty$ with the multiplication of A_∞. It is then clear from the short exact sequences of Theorem 6.4 that M_∞ is the kernel of this map. That A_∞ has the universal property of the cokernel is a consequence of the universal property of the associated fibre bundle construction: Explicitly, this follows from the fact that locally over $U \subseteq E/M^{1/p^\infty}$, the map on the right is the projection

$$D_\infty \times M_\infty \times U \to D_\infty \times U.$$

This gives a local splitting, and thus the necessary map in the universal property of the cokernel. □

We can see the short exact sequence of 6.5 as the analogue at infinity of the short exact sequence of rigid groups

$$0 \to M \to E \to A \to 0.$$

In particular, while as a rigid analytic space A is locally isomorphic to open subspaces of E, the perfectoid space A_∞ is locally isomorphic to open subspaces of $D_\infty \times E_\infty$.

Appendix A. Fibre bundles of formal and rigid spaces

In this chapter we review the theory of fibre bundles with structure group T and in particular of principal T-bundles in the setting of formal and rigid geometry.

In this chapter we denote by T a commutative formal group scheme over $\mathcal{O}_K$. We denote the multiplication map by $m : T \times T \to T$. By a T-action on a formal scheme X we mean a morphism $m_X : T \times X \to X$ such that the usual associativity diagram commutes.

Definition A.1. By a **T-linear map** of schemes X and Y with T-actions we mean a morphism $\phi : X \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} T \times X & \xrightarrow{\mathrm{id}_T \times \phi} & T \times Y \\ \downarrow{\scriptstyle m_X} & & \downarrow{\scriptstyle m_Y} \\ X & \xrightarrow{\phi} & Y \end{array}$$

We denote by $\mathbf{FormAct}_T$ the category of formal schemes with action by T.

The definition of a principal T-bundle is just what we get when we take the definition of a principal G-bundle and replace the category of topological spaces by the category of formal schemes.

Notation A.2. In the following, if $\pi : E \to B$ is a morphism of formal schemes, then for a formal open subscheme $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.3. Let T be a formal group scheme. Let F be a formal scheme with an action $m : T \times F \to F$. A morphism $\pi : E \to B$ of formal schemes is called a **fibre bundle with fibre F and structure group T** if there is a cover $\mathfrak{U}$ of B of open formal subschemes $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl, "p_2"] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i : F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that

$$\phi_{ij} = F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}$$

Definition A.4. When we take F equal to the formal scheme T with the action on itself by left multiplication, then a fibre bundle $\pi : E \to B$ with fibre T and structure group T is called a **principal T-bundle**. This is also called a T-torsor.

Example. For the short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0$ from Section 4, $\overline{E} \xrightarrow{\pi} \overline{B}$ defines a principal $\overline{T}$-bundle by Lemma 4.5. Moreover, for any formal open subscheme $U \subseteq \overline{B}$, the map $E|_U \to U$ is still a principal $\overline{T}$-bundle. This is what we mean when we say that the notion of principal $\overline{T}$-bundles is better suited for studying $\overline{E}$ locally on $\overline{B}$ than the notion of short exact sequences.

The morphism ϕ_{ij} from condition (b) is fully determined by the morphism $\psi_{ij} : U_{ij} \to T$. By a glueing argument, one shows:

Lemma A.5. *Suppose we are given formal schemes F and B and a formal group scheme T with an action on F. Then fibre bundles $\pi : E \to B$ with fibre F and structure group T are equivalent to the data (up to refinement) of a cover $\mathfrak{U}$ of B by formal open subschemes and morphisms $\psi_{ij} : U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk} \cdot \psi_{ij} = \psi_{ik}$, by which we mean that the following diagram commutes:*

$$\begin{tikzcd} U_{ijk} \arrow[r, "\psi_{ij} \times \psi_{jk}"] \arrow[d, equal] & T \times T \arrow[d, "m"] \\ U_{ijk} \arrow[r, "\psi_{ik}"] & T. \end{tikzcd} \tag{15}$$

In order to define the category of fibre bundles, we also need the following:

Lemma A.6. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. With notations like in Definition A.3 we have a natural T-action on $F \times U_i$ for each i when we let T act trivially on U_i. These actions glue together to a T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.7. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T. Then a **morphism of fibre bundles** $f : (E', B', \pi') \to (E, B, \pi)$ is a commutative diagram of formal schemes

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

in which the morphism f_E is also T-linear (we often abbreviate this by writing $f : E' \to E$). We thus obtain the category of fibre bundles over T that we denote by $\mathbf{FormFibBun}_T$ and the full subcategory of principal T-bundles, that we denote by $\mathbf{FormPrinBun}_T$.

In the case of principal T-bundles, this data can be given as follows: Let $\mathfrak{U}$ be of B a cover over which E is trivialised. Then we can always refine U in such a way that for all $U \in \mathfrak{U}$ the fibre bundle E' is trivial over $U' := f_B^{-1}(U)$. The induced map $f_E : T \times U' \to T \times U$ is then T-linear and thus can be reconstructed from the induced map

$$\theta : U' = 1 \times U' \hookrightarrow T \times U' \xrightarrow{f_E} T \times U \xrightarrow{p_1} T.$$

Lemma A.8. *Fix a morphism $f_B : B' \to B$ and fix notation as above. Then the data of a morphism $f = (f_E, f_B)$ of principal T-bundles is equivalent to the data of morphisms $\theta_i : U'_i = f_B^{-1}(U_i) \to T$ for some cover $\mathfrak{U}$ of B by formal open subschemes U_i such that each θ_i induces a map $f_{E,i}$ such that for all i, j the following diagram commutes:*

$$\begin{tikzcd} T \times U'_{ij} \arrow[r, "\phi'_{ij}"] \arrow[d, "f_{E,i}"] & T \times U'_{ij} \arrow[d, "f_{E,j}"] \\ T \times U_{ij} \arrow[r, "\phi_{ij}"] & T \times U_{ij}. \end{tikzcd}$$

Moreover, commutativity of the above diagram is equivalent to commutativity of

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij} \times \theta_j"] \arrow[d, "(\psi_{ij} \circ f) \times \theta_i"'] & T \times T \arrow[d, "m"] \\ T \times T \arrow[r, "m"] & T. \end{tikzcd}$$

Or in short hand notation,

$$\psi'_{ij}(u)\theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u)$$

Proof. One direction is clear. For the other, the first part follows from glueing. The second part is a consequence of all maps in the first diagram being T-linear. $\square$

Definition A.9. Let $\pi : E \to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Since the data in the equivalent characterisation of Lemma A.5 is completely independent of the fibre, the morphisms $\psi_{ij} : U_{ij} \to T$ by Lemma A.5 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel-Weil construction.

Note that in many authors in differential geometry and topology denote the associated bundle by "$F \times_T E$" instead of $F \times^T E$. In our setting, however, this is slightly confusing since we often have natural maps from T to F and E, but $F \times^T E$ is usually *not* their fibre product. In fact it behaves more like a pushout, for instance in the case that E comes from a short exact sequence.

Proposition A.10. *The associated bundle construction is a bifunctor*

$$- \times^T - : \mathbf{FormAct}_T \times \mathbf{FormPrinBun}_T \to \mathbf{FormFibBun}_T$$

from the categories of formal schemes with T-action $\times$ the category of principal T-bundles to the category of fibre bundles with structure group T.

Proof. Let E and E' be principal T-bundles and let $f : E' \to E$ be a morphism of T-bundles. Let F and F' be formal schemes with T-action and let $h : F' \to F$ be a T-equivariant morphism. Then we can find compatible covers $\mathfrak{U}'$ of B' and $\mathfrak{U}$ of B such that locally we have diagrams like in Lemma A.8. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U_i'$ such that we obtain a natural map

$$F' \times U_i' \xrightarrow{h \times^T f} F \times U_i, \quad (x, u) \mapsto (h(x)\theta_i(u), f_B(u))$$

(of course this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersections Lemma A.8 implies that we have a commutative diagram

$$\begin{array}{ccc} F' \times U'_{ij} & \xrightarrow{h \times^T f} & F \times U_{ij} \\ {\scriptstyle \psi'_{ij} \times \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \psi_{ij} \times \mathrm{id}} \\ F' \times U'_{ij} & \xrightarrow{h \times^T f} & F \times U_{ij}. \end{array}$$

One easily checks that this is functorial in both components. □

Lemma A.11. *Let S be another formal group scheme that receives an action of T from a group homomorphism $g : T \to S$. Then for any principal T-bundle E, the Borel construction $S \times^T E$ is a principal S-bundle.*

Proof. This follows from Lemma A.5. The only thing we need to check is that the cocycle condition from diagram (15) also holds with respect to S. But g is a homomorphism and therefore the following diagram commutes:

$$\begin{array}{ccc} T \times T & \xrightarrow{g \times g} & S \times S \\ \downarrow {\scriptstyle m} & & \downarrow {\scriptstyle m} \\ T & \xrightarrow{g} & S. \end{array}$$

□

Lemma A.12. *The Borel construction is a functor $S \times^T -$ from principal T-bundles to principal S-bundles.*

Proof. This is a consequence of Lemma A.8. One obtains the necessary data by composing the morphisms $\theta' : U_i' \to T$ with the morphism $T \to S$. These morphisms glue together because the second diagram of Lemma A.8 commutes, as one easily sees from the fact that $T \to S$ is a morphism of formal groups. □

The Borel construction satisfies the following universal property:

Lemma A.13. *Let $g : T \to S$ be a homomorphism of formal group schemes and let $E \to B$ be a principal T-bundle. Let X be any principal S-bundle. Note that X receives a T-action from g. Then there is a functorial one-to-one correspondence between T-linear morphisms $E \to X$ and morphisms of principal S-bundles $S \times^T E \to X$.*

A.1. **The semi-linear case.** We later want to consider morphisms of fibre bundles that are induced from morphisms of short exact sequences. In this situation, in order to describe the morphism of the kernels, we need to incorporate morphisms of the structure group into the notion of morphisms of fibre bundles. For this we need semi-linear group actions.

Definition A.14. Let T and S be formal group schemes and let $g : T \to S$ be a homomorphism. Let X and Y be formal schemes with actions $m : T \times X \to X$ and $m : S \times Y \to Y$ respectively. Then by a g-linear morphism $f : X \to Y$ we mean a morphism of formal schemes such that the following diagram commutes

$$\begin{tikzcd} T \times X \arrow[r, "g\times f"] \arrow[d, "m"'] & S \times Y \arrow[d, "m"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Definition A.15. We denote by **FormAct** the category of pairs (T, X) where T is a formal group scheme and X is a formal scheme with T-action, and morphisms are pairs of (g, f) where g is a group homomorphism and f is a g-linear morphism. It has a natural forgetful functor to **FormGrp**, the category of formal group schemes.

Definition A.16. Let $g : T' \to T$ be a homomorphism of formal group schemes. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T' . Then a g-linear morphism of principal bundles is a diagram

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

such that f_E is g-linear. We denote by **FormFibBun** the category of fibre bundles (E, B, π, T, F) with arrows being the morphisms of principal bundles. It has a natural forgetful functor

$$(E, B, \pi, T, F) \mapsto T$$

to the category **FormGrp**. We denote by **FormPrinBun** the full subcategory of principal bundles.

We get the natural analogue of Lemma A.8:

Lemma A.17. *With the notations from Lemma A.8, a g-linear morphism of a principal T'-bundle to a principal T-bundle is equivalent to the data of morphisms $\theta : U'_i \to T$ such that the following diagram commutes on intersections:*

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij}\times\theta_j"] \arrow[d, "(\psi_{ij}\circ f)\times\theta_i"'] & T' \times T \arrow[r, "g\times \mathrm{id}"] & T \times T \arrow[d, "m"] \\ T \times T \arrow[rr, "m"] & & T \end{tikzcd}$$

Or in short hand notation,

$$g(\psi'_{ij}(u)) \cdot \theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u). \tag{16}$$

Similarly as in Proposition A.10 one can conclude from this that change of fibre is functorial in the following sense:

Proposition A.18. *Given any homomorphism of group schemes $g : T' \to T$ and a g-linear homomorphism $h : F' \to F$ of formal schemes with T' and T-actions respectively, and a homomorphism $f : E' \to E$ of principal T' and T-bundles over g, one obtains a morphism*

$$h \times^g f : F' \times^{T'} E' \to F \times^T E$$

of fibre bundles over g, in a way that is functorial in h, g, f. More precisely, the associated bundle construction is a fibered bifunctor

$$- \times^- - : \mathbf{FormAct} \times_{\mathbf{FormGrp}} \mathbf{FormPrinBun} \to \mathbf{FormFibBun}.$$

Proof. Let (E,B,π,T) and (E',B',π',T') be principal bundles. Let F and F' be formal schemes with T-action and T'-action respectively. Let $g : T \to T'$ be a group homomorphism and let $h : F' \to F$ be a g-equivariant morphism. Let $f : E' \to E$ be a morphism of principal fibre bundles over g. Then we can find compatible covers $\mathfrak{U}'$ of B' and $\mathfrak{U}$ of B such that locally we have diagrams like in Lemma A.8. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U_i'$ such that we obtain a natural map

$$F' \times U_i' \xrightarrow{(h \times^g f)} F \times U_i, \quad (x, u) \mapsto (h(x)\theta_i(u), f_B(u))$$

(as before this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersection Lemma A.8 implies that we have a commutative diagram

$$\begin{array}{ccc} F' \times U_{ij}' & \xrightarrow{h \times^g f} & F \times U_{ij} \\ {\scriptstyle \psi_{ij}' \times \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \psi_{ij} \times \mathrm{id}} \\ F' \times U_{ij}' & \xrightarrow{h \times^g f} & F \times U_{ij}. \end{array}$$

More precisely, by g-linearity of h one has

$$h(x \cdot \psi_{ij}'(u)) \cdot \theta_j(u) = h(x) \cdot g(\psi_{ij}'(u)) \cdot \theta_j(u) \overset{(16)}{=} h(x) \cdot \psi_{ij}(f(u)) \cdot \theta_i(u).$$

This shows that the maps glue to a morphism $h \times^g f$ as desired.

By refining covers, one easily checks that this is functorial in both components. □

We obtain a variant of Lemma A.13 in the semilinear case:

Lemma A.19. *Let E' be a principal T' bundle and let E be a principal T-bundle. Let H' and H be formal group schemes and assume there is a commutative diagram of group homomorphisms*

$$\begin{array}{ccc} H' & \xrightarrow{h} & H \\ \uparrow & & \uparrow \\ T' & \xrightarrow{g} & T. \end{array}$$

Let moreover $f : E' \to E$ be a g-linear morphism of fibre bundles. Then the map $h \times^g f$ from Proposition A.18 is the unique h-linear morphism of fibre bundles making the following diagram commute:

$$\begin{array}{ccc} H' \times^{T'} E' & \xrightarrow{h \times^f g} & H \times^T E \\ \uparrow & & \uparrow \\ E' & \xrightarrow{f} & E. \end{array}$$

Proof. The morphism exists by Proposition A.18. The vertical maps in the commutative diagram exist by functoriality via $E = T \times^T E \to H \times^T E$.

On the other hand, on any compatible trivialisation $T' \times U' \to T \times U$ of $f : E' \to E$ there is clearly only one way to extend this to $H' \times U' \to H \times U$ in a h-linear way. □

Remark A.20. All that we have done in this chapter can be done in completely the same way with formal schemes replaced by rigid spaces (covers being replaced by admissible covers) and also for schemes, or in fact for any site. We have preferred to use formal schemes to make things more explicit. The different categories of fibre bundles are well-behaved with respect to the usual functors between the different categories: For instance, by functoriality of fibre products there are natural rigidification and reduction functors from formal principal T-bundles over $\mathcal{O}_K$ to rigid principal T_η-bundles over K on the generic fibre, and to principal $\overline{T}$-bundles on the reduction $\mathcal{O}_K/\pi$. Moreover, these generic fibre and reduction functors commute with the associated fibre construction:

Lemma A.21. *Let T be a formal group scheme and let $\pi : E \to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Then*

$$(F \times^T E)_\eta = F_\eta \times^{T_\eta} E_\eta$$

Proof. This can be checked locally on any trivialising cover, where it is clear. □

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