PERFECTOID COVERS OF ABELIAN VARIETIES

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Abstract. For an abelian variety A over an algebraically closed non-archimedean field of residue characteristic p, we show that there exists a perfectoid space which is the tilde-limit of $\varprojlim_{[p]} A$. Our proof also works for the larger class of abeloid varieties.

Keywords: abelian varieties, abeloid varieties, universal covers, perfectoid spaces

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1. Introduction

Let p be a prime and let K be an algebraically closed non-archimedean field of residue characteristic p. For an abelian variety A over K we consider the inverse system of A under the p-multiplication morphism:

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A.$$

Via the adic analytification functor, we may see this as an inverse system of analytic adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. The primary goal of this article is to show that the "inverse limit" of this tower exists in some way and is a perfectoid space: Since inverse limits rarely exist in the category of adic spaces, in [9] Huber introduced the weaker notion of tilde-limits to remedy this problem. This is the notion of "limits" we are going to use. More precisely, we prove the following slightly more general result:

Theorem 1. *Let A be an abeloid variety over K, for instance an abelian variety seen as a rigid space. Then there is a unique perfectoid space A_∞ over K such that $A_\infty \sim \varprojlim_{[p]} A$ is a tilde-limit.*

The possibility of results in this direction is mentioned in §7 and §13 of [19], and in the case of abelian varieties with good reduction, this theorem was proven already in [14, Lemme A.16]. We recall the argument in Lemma 2.13 below.

In general, A has semi-stable reduction by the assumption that K is algebraically closed. Consequently, the theory of Raynaud extensions provides us with a short exact sequence

$$0 \to T \to E \to B \to 0$$

of rigid groups, where $T = (\mathbb{G}_m^{\mathrm{an}})^r$ is a split rigid torus and B is the analytification of an abelian variety with good reduction, such that $A = E/M$ for a discrete lattice $M \subset E$. This short exact sequence is split locally on B in the analytic topology, allowing us to locally write E as a product of T and an open subspace of B. Our strategy for the proof of Theorem 1, which more generally applies to any abeloid variety over K, is now similar to the good reduction case:

(1) Construct a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$. This is easy.
(2) Use T_∞ and B_∞ to construct a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

(3) Study the quotient map $E \to A$ in the limit over $[p]$ to construct the desired space A_∞.

More precisely, this article is organised as follows: In §2 we recall the definition of tilde-limits and collect some useful lemmas about tilde-limits and perfectoid spaces. In particular, we construct the perfectoid tilde-limit T_∞. In §3 we use local splittings to construct a perfectoid tilde-limit E_∞: The Raynaud extension of A mentioned earlier arises from a short exact sequence of formal group schemes over $\mathcal{O}_K$

$$0 \to \overline{T} \to \overline{E} \to \overline{B} \to 0$$

by taking generic fibres and forming the pushout with respect to the open immersion $\overline{T}_\eta \to T$. We get the desired tilde-limit by tracing the local splitting through the tower of multiplication by $[p]$. This will also show that there is a short exact sequence of perfectoid groups in the analytic topology

$$0 \to T_\infty \to E_\infty \to B_\infty \to 0.$$

In §4 we finish the proof of Theorem 1 by constructing A_∞ from E_∞ as follows: After choosing lattices $M \subset M_n \subset E$ that map isomorphically to M under $[p^n]\colon E \to E$, the $[p]$-multiplication tower of $A = E/M$ naturally factors into two separate towers: One is the tower of maps $E/M_{n+1} \to E/M_n$ induced from $[p]$-multiplication of E, and the other is induced from the projection maps $v^n\colon E/M \to E/M_n$. Using local splittings, one can construct a perfectoid tilde-limit $A'_\infty \sim \varprojlim_n E/M_n$ of the first tower from E_∞. It fits into a short exact sequence

$$0 \to M \to E_\infty \to A'_\infty \to 0.$$

The existence of $A_\infty \sim \varprojlim_{[p]} A$ then follows as the quotient maps $v^n\colon E/M \to E/M_n$ are étale. In fact, they are locally split in the analytic topology, from which one can deduce the following analogue of Raynaud uniformisation for A_∞: Write D_n for the kernel of v^n. Then there is a profinite perfectoid tilde-limit $D_\infty \sim \varprojlim_{[p]} D_n$ and a short exact sequence of perfectoid groups

$$0 \to M \to D_\infty \times E_\infty \to A_\infty \to 0,$$

which we regard as an analogue of the sequence $0 \to M \to E \to A \to 0$.

We give three applications of Theorem 1 in §5: As observed by Hansen, one can deduce from Theorem 1 the existence of certain universal covers of curves by embedding them into their Jacobian:

Corollary 1.1 (Hansen, [8])**.** *Let C be a connected smooth projective curve of genus $g \geq 1$ over K. Fix a geometric point $x\colon \mathrm{Spec}(K) \to C$ and for each open subgroup H of $\pi_1(C,x)$, let C_H denote the finite étale cover of C corresponding to H. We regard C and C_H as analytic adic spaces.*

(1) There is a perfectoid tilde-limit $\widetilde{C} \sim \varprojlim_H C_H$ where H ranges over the open subgroups of $\pi_1(C,x)$.

(2) The morphism $\widetilde{C} \to C$ is a pro-étale $\pi_1(C,x)$-torsor. It is universal with this property in the sense that it represents the fibre functor sending pro-finite-étale perfectoid covers $X \to C$ to the $\pi_1(C,x)$-module $F(X) = \mathrm{Hom}_C(x,X)$.

(3) For any pro-finite-étale morphism $X \to C$, there is a natural isomorphism

$$X = \underline{F(X)} \times^{\pi_1(C,x)} \widetilde{C} := (\underline{F(X)} \times \widetilde{C})/\pi_1(C,x).$$

Here the right hand side is the categorical quotient in adic spaces for the antidiagonal action.

Second, we note that the analogue of this corollary also works for C replaced by an abelian variety, in which case the pro-étale fundamental group is isomorphic to the adelic Tate module

$TA := \varprojlim_{N\in\mathbb{N}} A[N]$. In particular, one obtains from this two different natural ways to uniformise the diamond $A^\diamondsuit$ attached to A: On the one hand, as a consequence of Theorem 1, we can write

$$A^\diamondsuit = A_\infty / T_p A.$$

On the other hand, one can deduce from Theorem 1 that there is also a perfectoid tilde-limit $\widetilde{A} \sim \varprojlim_{[N]} A$ which gives rise to a natural isomorphism

$$A^\diamondsuit = \widetilde{A}/TA.$$

Here the second equation describes A in terms of the universal connected pro-finite-étale cover $\widetilde{A} \to A$, whereas the first uses the universal connected pro-finite-étale pro-p-cover. Either may be seen as a sort of analogue of Riemann uniformisation of abelian varieties over $\mathbb{C}$.

Our third application of Theorem 1 states that in line with this analogy to the complex case, the cohomology of constructible $\mathbb{F}_p$-sheaves on A_∞ behaves like that of a Stein space: This follows in combination with a result of Reinecke [15]:

Corollary 1.2 (Reinecke)**.** *Let L be a constructible sheaf of $\mathbb{F}_p$-modules on $A_{\text{ét}}$. Then for $i > \dim A$,*

$$\varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^* L) = 0.$$

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Notation

Let K be an algebraically closed non-archimedean field, let $\mathcal{O}_K$ be the ring of integers of K and fix a pseudo-uniformiser $\varpi \in \mathcal{O}_K$ such that $p \in \varpi\mathcal{O}_K$.

We will use adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Huber, and perfectoid spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Scholze [16]. We denote by $X \mapsto X^{\mathrm{an}}$ the analytification functor from schemes of finite type over X to analytic adic spaces over $(K, \mathcal{O}_K)$.

By a rigid space, we shall always mean an analytic adic space of topologically finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$. In particular, by an open cover of a rigid space we shall always mean a cover of the associated adic space, so that we do not need the notion of admissible covers.

For a ϖ-adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$, we denote by $\mathfrak{X}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O}_K,\mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$ its adic generic fibre. This is the adic space representing the functor which is denoted by $\mathfrak{X}^{\mathrm{ad}}_\eta$ in [22].

2. Tilde-limits of rigid groups

2.1. **Tilde-limits.** We begin with some lemmas on tilde-limits that we will need throughout.

Inverse limits often do not exist in the category of adic spaces, and neither do they in rigid spaces. Instead we use the notion of tilde-limits from [9, Definition 2.4.2]:

Definition 2.1. Let $(X_i)_{i\in I}$ be a filtered inverse system of adic spaces with quasi-compact and quasi-separated transition maps, and let X be an adic space with a compatible system of morphisms $f_i\colon X\to X_i$. We write $X\sim \varprojlim X_i$ and say that X is a **tilde-limit** of the inverse system $(X_i)_{i\in I}$ if

(1) the map of underlying topological spaces $|X|\to \varprojlim |X_i|$ is a homeomorphism, and
(2) there exists an open cover of X by affinoids $\mathrm{Spa}(A,A^+)\subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(A_i,A_i^+)\subset X_i} A_i\to A$$

has dense image, where the direct limit runs over all $i\in I$ and all open affinoid subspaces $\mathrm{Spa}(A_i,A_i^+)\subset X_i$ through which the morphism $\mathrm{Spa}(A,A^+)\subseteq X\to X_i$ factors.

Remark 2.2. As pointed out after Proposition 2.4.4 of [22], tilde-limits (if they exist) are in general not unique. However, Corollary 2.5 below says that perfectoid tilde-limits are unique.

We recall a few results from [22], §2.4 on tilde-limits that we will use frequently throughout:

Proposition 2.3 ([22, Proposition 2.4.3])**.** *Let $X\sim\varprojlim_{i\in I} X_i$ be a tilde-limit and let $U_i\hookrightarrow X_i$ be an open immersion for some $i\in I$. Set $U_j:=U_i\times_{X_i}X_j$ for $j\geq i$ and $U:=U_i\times_{X_i}X$. Then*

$$U\sim\varprojlim_{j\geq i} U_j.$$

Proposition 2.4 ([22, Proposition 2.4.5])**.** *Let $(X_i)_{i\in I}$ be an inverse system of adic spaces over $(K,\mathcal{O}_K)$ and assume that there is a perfectoid space X such that $X\sim\varprojlim_{i\in I}X_i$. Then for any perfectoid K-algebra (B,B^+),*

$$X(B,B^+)=\varprojlim_{i\in I}X_i(B,B^+).$$

Corollary 2.5. *Any two perfectoid spaces that are tilde-limits of the same inverse system of adic spaces over $(K,\mathcal{O}_K)$ are canonically isomorphic.*

In the situation of the corollary, we will also refer to such a perfectoid space as *the* perfectoid tilde-limit of the inverse system. Of course perfectoid tilde-limits do not always exist. An example of a basic situation in which they do is the following:

Lemma 2.6. *Let $(S_i)_{i\in I}$ be an inverse system of finite sets. Let $S=\varprojlim_{i\in I}S_i$. Then the system of adic spaces $\underline{S_i}:=\mathrm{Spa}(\mathrm{Map}(S_i,K),\mathrm{Map}(S_i,\mathcal{O}_K))$ has a perfectoid tilde-limit*

$$\underline{S}:=\mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S,K),\mathrm{Map}_{\mathrm{cts}}(S,\mathcal{O}_K))\sim\varprojlim_{i\in I}\underline{S_i}.$$

Proof. Since S is compact, $\mathrm{Map}_{\mathrm{cts}}(S,K)=\mathrm{Map}_{\mathrm{cts}}(S,\mathcal{O}_K)[\frac{1}{\varpi}]$. This is perfectoid since we have $\mathrm{Map}_{\mathrm{cts}}(S,\mathcal{O}_K)/\varpi=\varinjlim_{i\in I}\mathrm{Map}(S_i,\mathcal{O}_K/\varpi)$. The tilde-limit property follows from [22, Proposition 2.4.2]. □

We will need the following basic lemma later on.

Lemma 2.7. *Let $(A_i, A_i^+)_{i\in I}$ and $(B_i, B_i^+)_{i\in I}$ be direct systems of affinoids over $(K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ and $B_{i,0}$ carrying the ϖ-adic topology. Assume that there are perfectoid tilde-limits $\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+)$ and $\mathrm{Spa}(B, B^+) \sim \varprojlim \mathrm{Spa}(B_i, B_i^+)$. Then*

$$\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B, B^+) \sim \varprojlim_{i\in I} (\mathrm{Spa}(A_i, A_i^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B_i, B_i^+))$$

is also a perfectoid tilde-limit.

Proof. The fibre product $\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B, B^+)$ exists and is perfectoid by [16, Proposition 6.18]. In fact, it is represented by $\mathrm{Spa}(C, C^+)$, where $C = A\widehat{\otimes}_K B$ and C^+ is the ϖ-adic completion of the integral closure of the image of $A^+ \otimes_{\mathcal{O}_K} B^+$ in C.

We first check the condition on topological spaces: Since fibre products commute with limits in the category of sheaves, it follows from Proposition 2.4 that for any perfectoid field (D, D^+) over $(K, \mathcal{O}_K)$, we have

$$(\mathrm{Spa}(A, A^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B, B^+))(D, D^+) = \varprojlim (\mathrm{Spa}(A_i, A_i^+) \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(B_i, B_i^+))(D, D^+).$$

Since the topological space can be reconstructed from this data by [21, Proposition 12.7, Lemma 15.6], it follows that the underlying topological spaces of both sides coincide.

It remains to check that if $\varinjlim A_i \to A$ has dense image and $\varinjlim B_i \to B$ has dense image, then $\varinjlim (A_i \otimes B_i) \to A \otimes B$ has dense image. As direct limits commute with tensor products, we have $\varinjlim (A_i \otimes B_i) = (\varinjlim A_i) \otimes (\varinjlim B_i)$. Now density can be checked directly on elements. $\square$

2.2. **Perfectoid tilde-limits for rigid groups.** One reason why perfectoid tilde-limits along group homomorphisms are particularly interesting is that these again have a group structure:

Definition 2.8. A **perfectoid group** is a group object in the category of perfectoid spaces.

The category of perfectoid spaces over K has finite products, so this is a well-defined notion.

Lemma 2.9. *Let $(G_i)_{i\in I}$ be an inverse system of adic groups such that the transition maps are homomorphisms of adic groups. Assune that there is a perfectoid tilde-limit $G_\infty \sim \varprojlim_{i\in I} G_i$.*

(1) There is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms
(2) Given a homomorphism of inverse systems of adic groups $(G_i)_{i\in I} \to (H_j)_{j\in J}$ and a perfectoid tilde-limit $H_\infty \sim \varprojlim_{j\in J} H_j$, there is a unique homomorphism of perfectoid groups $G_\infty \to H_\infty$ commuting with all projection maps.

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, Proposition 2.4, which shows that one can argue like in the case of categorical limits. $\square$

Let G be an adic group locally of finite type over $(K, \mathcal{O}_K)$, that is, a group object in the category of rigid spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. Throughout we will always consider commutative groups. The main topic of study of this work is the $[p]$-multiplication tower

$$\cdots \xrightarrow{[p]} G \xrightarrow{[p]} G.$$

We will usually assume that $[p]\colon G \to G$ is surjective.

Question 2.10. When is there a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$ is a tilde-limit?

Example 2.11. If $[p] : G \to G$ is not surjective, $\varprojlim_{[p]} G$ might have a tilde-limit for trivial reasons: For example, let $\mathfrak{G}_a$ be the p-adic completion of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Then the trivial group $\mathrm{Spa}(K, \mathcal{O}_K) \sim \varprojlim_{[p]} (\mathfrak{G}_a)^{\mathrm{ad}}_\eta$ is a perfectoid tilde-limit.

Regarding Question 2.10, we are primarily interested in the following examples:

(1) Analytifications over $\mathrm{Spa}(K, \mathcal{O}_K)$ of finite type group schemes over K. Examples include analytifications of abelian varieties A over K and of tori T over K.
(2) Generic fibres of locally topologically finite type formal group schemes over $\mathcal{O}_K$.
(3) Raynaud's covering space E of an abelian variety with semi-stable reduction.

Remark 2.12. More generally, one could ask Question 2.10 for families of abelian varieties over $\mathrm{Spec}(R)$ where R is any perfectoid ring. Considering the fibers of such a family in any point of $\mathrm{Spa}(R, R^\circ)$ motivates to also study analytifications over $\mathrm{Spa}(K, K^+)$ where K^+ is any open bounded integrally closed subring of $\mathcal{O}_K$. However, one can reduce this case to the one of $K^+ = \mathcal{O}_K$.

Indeed, this follows from the following technical observation: Let $(X_i)_{i\in I}$ be an inverse system of reduced adic spaces X_i of finite type over (K, K^+) with finite transition maps. Let $X_{i,\eta} := X_i \times_{\mathrm{Spa}(K,K^+)} \mathrm{Spa}(K, \mathcal{O}_K)$. Then the following are equivalent:

(1) There is a perfectoid tilde-limit $X_\infty \sim \varprojlim_{i\in I} X_i$.
(2) There is a perfectoid tilde-limit $X_{\infty,\eta} \sim \varprojlim_{i\in I} X_{i,\eta}$.

We shall omit the proof of this equivalence, as this will not be relevant in the following. Instead, we simply take it as a motivation to restrict attention to the case of $K^+ = \mathcal{O}_K$.

As we have already mentioned in the introduction, Question 2.10 has an affirmative answer in the case of abelian varieties of good reduction by [14, Lemme A.16]. More generally:

Lemma 2.13. *Let $\mathfrak{G}$ be a commutative flat ϖ-adic formal group scheme over $\mathcal{O}_K$ such that the morphism $[p] : \mathfrak{G} \to \mathfrak{G}$ is affine. Let $G = \mathfrak{G}^{\mathrm{ad}}_\eta$ be the adic generic fibre. Then $G_\infty := (\varprojlim_{[p]} \mathfrak{G})^{\mathrm{ad}}_\eta$ is a perfectoid tilde-limit*

$$G_\infty \sim \varprojlim_{[p]} G.$$

In particular, if B is an abelian variety of good reduction over K, there is a perfectoid tilde-limit $B_\infty \sim \varprojlim B$.

Proof. This holds by the same proof as in [14, Lemme A.16], (see also Exercise $4-6$ in [1]): Let $\varpi \in \mathcal{O}_K$ be a pseudo-uniformiser such that $p \in \varpi\mathcal{O}_K$. The assumption that $[p] : \mathfrak{G} \to \mathfrak{G}$ is affine ensures that the limit $\mathfrak{G}_\infty := \varprojlim_{[p]} \mathfrak{G}$ exists.

The mod ϖ special fibre $\mathfrak{G}/\varpi := \mathfrak{G} \times \mathrm{Spec}(\mathcal{O}_K/\varpi)$ is a group scheme over $\mathcal{O}_K/\varpi$, so the map $[p]\colon \mathfrak{G}/\varpi \to \mathfrak{G}/\varpi$ factors through the relative Frobenius map [5, Exp. VII, 4.3]. Consequently, the fibre $\varprojlim_{[p]} \mathfrak{G}/\varpi$ of $\mathfrak{G}_\infty$ over $\mathcal{O}_K/\varpi$ is relatively perfect: Indeed, we have a commutative diagram

$$\begin{tikzcd}
 & \varprojlim_{[p]} \mathfrak{G}/\varpi \arrow[rr, "{[p]}", "\sim"'] \arrow[dr, "F"] & & \varprojlim_{[p]} \mathfrak{G}/\varpi \\
\varprojlim_{[p]} (\mathfrak{G}/\varpi)^{(p)} \arrow[ur] \arrow[rr, "{[p]}", "\sim"'] & & \varprojlim_{[p]} (\mathfrak{G}/\varpi)^{(p)} \arrow[ur] &
\end{tikzcd}$$

in which the horizontal maps are isomorphisms. Thus also F is an isomorphism. This implies that the adic generic fibre of $\mathfrak{G}_\infty$ is perfectoid by [16, Theorem 5.2]. $\square$

Lemma 2.14. *Let T be a torus over K. Then there is a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$.*

Proof. Since we assume K algebraically closed, we may choose a splitting $T \cong (\mathbb{G}_m^{\mathrm{an}})^r$ for some $r \in \mathbb{N}$. By Lemma 2.7, it suffices to consider the case of $r = 1$. For this, we may use the open embedding $\mathbb{G}_m^{\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}} \setminus \{0, \infty\} \subseteq \mathbb{P}^{1,\mathrm{an}}$. Sending $(x : y) \mapsto (x^p : y^p)$ defines a morphism $\varphi : \mathbb{P}^{1,\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$. The pullback of φ to $\mathbb{G}_m^{\mathrm{an}}$ is precisely $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$. We can therefore apply Proposition 2.3 to the perfectoid tilde-limit $\mathbb{P}_1^{\mathrm{perf}} \sim \varprojlim_\varphi \mathbb{P}^{1,\mathrm{an}}$ introduced in [16]. □

3. Perfectoid tilde-limits of Raynaud extensions

In this section we study the p-multiplication tower of the Raynaud extensions associated to abeloid varieties over an algebraically closed perfectoid field K. The main result of this section is Proposition 3.7, which shows that there exists a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$, which itself has an extension structure that one could call a "perfectoid Raynaud extension".

Remark 3.1. Everything in this section also works with minor modifications over a general perfectoid field. But we opt to work over an algebraically closed field to simplify the exposition.

3.1. **Raynaud extensions.** We briefly sketch the theory of Raynaud extensions here, and refer the readers to [2, 3, 12, 13] for more details on the setup.

Let A be an abelian variety over K. Then by [3, Theorem 1.1] there exists a unique open rigid analytic subgroup of A that admits a formal model $\overline{E}$ that is a connected smooth $\mathcal{O}_K$-group scheme fitting into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0, \tag{1}$$

where $\overline{B}$ is a formal abelian scheme over $\mathcal{O}_K$ with rigid generic fibre $B := \overline{B}_\eta$, and $\overline{T}$ is the completion of a torus $T_{\mathcal{O}_K}$ of rank r over $\mathcal{O}_K$. We set $T := T_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K$ and denote its analytification also by T. Then the rigid generic fibre $\overline{T}_\eta$ of the formal torus $\overline{T}$ canonically embeds into T. This induces a pushout exact sequence in the category of rigid groups: More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{T}_\eta & \longrightarrow & \overline{E}_\eta & \longrightarrow & \overline{B}_\eta & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0. \end{array} \tag{2}$$

The abelian variety A can be uniformized in terms of E as follows:

Definition 3.2. A subspace M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}^r}$.

Theorem 3.3 (Bosch–Lütkebohmert, [3, Theorem 1.2])**.** *With setup as in the previous paragraph, there exists a lattice $M \subset E$ of rank equal to the rank r of the torus T such that the quotient E/M exists as a rigid space and such that there is a natural isomorphism of rigid groups*

$$A = E/M.$$

The data of the extension (1) together with the lattice $M \subset E$ is what we refer to as a Raynaud uniformisation of A. This will be the only input we need to construct the perfectoid tilde-limit A_∞. Consequently, our method generalises to the class of rigid groups which admit Raynaud uniformisation, namely to abeloid varieties:

Theorem 3.4 (Lütkebohmert, [13, Theorem 7.6.4])**.** *Let A be an abeloid variety, that is, a connected smooth proper rigid group over K. Then A admits a Raynaud uniformisation.*

In the situation of Raynaud uniformisation, since M is discrete, the local geometry of A is determined by the local geometry of E. We will therefore first study the $[p]$-multiplication tower of E in the rest of this section and will then deduce properties of the $[p]$-multiplication tower of A in the next section.

Our strategy is to study the local geometry of E and $\overline{E}$ via T and B. In order to put ourselves in a situation where we can work in an abelian category, we shall do so in the category of abelian sheaves on the site of sheafy adic spaces with the analytic topology. The following crucial lemma says that Raynaud extensions are locally split in this topology:

Lemma 3.5. *The short exact sequence (1) admits local sections, that is there is a cover of $\overline{B}$ by formal open subschemes $\overline{U}_i$ such that there exist local sections $s : \overline{U}_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times \overline{U}_i \hookrightarrow \overline{E}$.*

Proof. This is proved in Proposition A.2.5 in [13], where it is fomulated in terms of the group $\mathrm{Ext}^1(\overline{B}, \overline{T})$. Also see [3], §1. □

As a consequence, in more topological terms, diagram (2) can be interpreted as follows: We may regard $\overline{E} \to \overline{B}$ as a principal $\overline{T}$-bundle, and the fact that E is obtained from $\overline{E}_\eta$ via push-out along $\overline{T}_\eta \to T$ can be expressed by saying that $E = T \times^{\overline{T}_\eta} \overline{E}_\eta$ is the associated fibre bundle obtained by change-of-fibre. But in the following, we choose to stick to sheaf-theoretic language:

Definition 3.6. We call a sequence of adic groups $0 \to T \to E \xrightarrow{\pi} B \to 0$ an **analytic-locally split** short exact sequence if it is a short exact sequence of abelian sheaves on the site of sheafy adic spaces with the analytic topology. Equivalently, this means that T is the kernel of π and $\pi : E \to B$ is a principal T-torsor in the analytic topology.

In particular, any Raynaud extension is an analytic-locally split short exact sequence. The main goal of this section is to use this to deduce the following from the existence of perfectoid tilde-limits $B_\infty \sim \varprojlim_{[p]} B$ and $T_\infty \sim \varprojlim_{[p]} T$:

Proposition 3.7. *Let $0 \to T \to E \to B \to 0$ be a rigid Raynaud extension. Then there is a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$. It fits into an analytic-locally split short exact sequence of perfectoid groups*

$$0 \to T_\infty \to E_\infty \to B_\infty \to 0.$$

Proof. The morphism $[p^n] : E \to E$ induces a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & E & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle [p^n]} & & \big\downarrow{\scriptstyle [p^n]} & & \big\downarrow{\scriptstyle [p^n]} & & \\ 0 & \longrightarrow & T & \longrightarrow & E & \xrightarrow{\pi} & B & \longrightarrow & 0. \end{array} \tag{3}$$

The basic idea is now to lift local splittings of π in this diagram:

Let $\mathfrak{U}$ be a cover of $\overline{B}$ by formal opens U over which $\overline{E} \to \overline{B}$ is split; this exists by Lemma 3.5. In particular, $E \to B$ is then also split over (the generic fibre of) U. Let U_n be the pullback of U along $[p^n]$. We now use the following Lemma, which is the main input in the proof of the Proposition.

Lemma 3.8. *For any $U \in \mathfrak{U}$, the morphism $\pi : \overline{E} \to \overline{B}$ is still split over $U_n = [p^n]^{-1}(U)$.*

Proof. We may reduce to the case of $n=1$. Choose $s \in \mathcal{O}_K$ such that $(s^p)=(p)$. In particular, if K has characteristic p, we have $s=0$.

We first claim that the reduction $\overline{E}/s \to \overline{B}/s$ over $\mathcal{O}_K/s$ admits a splitting over U_n/s. To see this, we first reduce mod p and use that in characteristic p, the morphism $[p]:\overline{B}/p \to \overline{B}/p$ decomposes into the Verschiebung $V:(\overline{B}/p)^{(p)} \to \overline{B}/p$ and the relative Frobenius $F:\overline{B}/p \to (\overline{B}/p)^{(p)}$.

Note that we have $(T/p)^{(p)} = T/p$, and Verschiebung is the identity on T/p. The multiplication by $[p]$ on $\overline{E}/p$ therefore gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{T}/p \arrow[r] \arrow[d, "F"] & \overline{E}/p \arrow[r] \arrow[d, "F"] & \overline{B}/p \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & (\overline{T}/p)^{(p)} \arrow[r] \arrow[d, equal, "V"] & (\overline{E}/p)^{(p)} \arrow[r] \arrow[d, "V"] & (\overline{B}/p)^{(p)} \arrow[r] \arrow[d, "V"] & 0 \\
0 \arrow[r] & \overline{T}/p \arrow[r] & \overline{E}/p \arrow[r] & \overline{B}/p \arrow[r] & 0.
\end{tikzcd}$$

Since the bottom left vertical morphism is an isomorphism, the middle row is obtained from the bottom row via pullback along $V:(\overline{B}/p)^{(p)} \to \overline{B}/p$. Consequently, the bottom right square is Cartesian, and the splitting over $U/p \subseteq \overline{B}/p$ in the bottom row therefore lifts to a splitting $V^{-1}(U/p) \to (\overline{E}/p)^{(p)}$ in the middle row by the universal property of the fibre product.

Recall now that the absolute Frobenius on $\mathcal{O}_K/p$ factors into the reduction $\mathcal{O}_K/p \to \mathcal{O}_K/s$ and

$$f:\mathcal{O}_K/s \xrightarrow{\sim} \mathcal{O}_K/p, \quad x \mapsto x^p$$

which is an isomorphism since K is perfectoid. Base change along this isomorphism induces an isomorphism $g:\overline{B}/s \to (\overline{B}/p)^{(p)}$, and similarly for $\overline{E}$ and $\overline{T}$. In particular, after base-change along the isomorphism f, the middle row of the diagram becomes

$$0 \to \overline{T}/s \to \overline{E}/s \to \overline{B}/s \to 0.$$

We claim that under this identification, the open subset $V^{-1}(U/p) \subseteq (\overline{B}/p)^{(p)}$ becomes U_n/s, showing that indeed $\overline{E}/s \to \overline{B}/s$ is split over U_n/s. To see this, consider the commutative diagram

$$\begin{tikzcd}
\overline{B}/p \arrow[r, "F"] & (\overline{B}/p)^{(p)} \arrow[r, "V"] & \overline{B}/p \\
\overline{B}/s \arrow[u, hook] \arrow[r, "F_{\mathrm{abs}}"] & \overline{B}/s. \arrow[u, "g"'] &
\end{tikzcd}$$

By definition, the pullback of U/p along first $V \circ F = [p]$ and then the reduction to $\overline{B}/s$ is U_n/s. Since the absolute Frobenius is the identity on the underlying topological spaces, this shows that the pullback of U/p along first V and then g is also equal to U_n, as we needed to see.

It remains to lift this splitting from the special fibre over $\mathcal{O}_K/s$ to $\mathcal{O}_K$. For this we use that $\pi:\overline{E} \to \overline{B}$ is formally smooth. The lifting diagram

$$\begin{tikzcd}
\overline{E} \arrow[d, "\pi"'] & \overline{E}_{|U_n}/s \arrow[l, hook'] \arrow[r] & U_n/s \arrow[d] \arrow[l, bend right] \\
\overline{B} & & U_n \arrow[ll, hook'] \arrow[ull, dotted]
\end{tikzcd}$$

therefore produces a splitting $U_n \to \overline{E}$. This shows that $\overline{E} \to \overline{B}$ is split over U_n. $\square$

Let us now also denote by U and U_n the respective generic fibres, then it follows from the Lemma that also $\pi : E \to B$ is split over U_n. Locally over U, the diagram (3) is therefore of the form

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "{[p^n]}"] & T \times U_n \arrow[r] \arrow[d, "{[p^n]\times[p^n]}"] & U_n \arrow[r] \arrow[d, "{[p^n]}"] & 0 \\ 0 \arrow[r] & T \arrow[r] & T \times U \arrow[r] & U \arrow[r] & 0. \end{tikzcd}$$

Let $U_\infty \sim \varprojlim U_n$ be the perfectoid tilde-limit that exists by Lemma 2.13 and Lemma 2.3. Then by Lemma 2.14 and Lemma 2.7, the inverse system in the middle has perfectoid tilde-limit $T_\infty \times U_\infty$. Glueing these for all $U \in \mathfrak{U}$ produces the desired space $E_\infty \sim \varprojlim_{[p]} E$ which by construction fits into the desired analytic-locally split short exact sequence

$$0 \to T_\infty \to E_\infty \to B_\infty \to 0. \qquad \square$$

Remark 3.9. There is an alternative proof of the tilde-limit property that also constructs a formal model $\mathfrak{E}_\infty$ of E_∞, like in Lemma 2.13. For this, one first takes a sequence of formal models

$$\cdots \to \mathfrak{T}_2 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{T}_1 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{T}_0$$

of $\cdots \xrightarrow{[p]} T \xrightarrow{[p]} T$. This can be done in such a way that each $[\mathfrak{p}]_i$ reduces to the relative Frobenius mod p. Then $\mathfrak{T}_\infty := \varprojlim_{[\mathfrak{p}]_i} \mathfrak{T}_i$ is a formal model of the perfectoid space T_∞ (giving an alternative proof that T_∞ is a perfectoid tilde-limit). When we set $\mathfrak{E}_i := \mathfrak{T}_i \times^{\overline{T}} \overline{E}$, we get an inverse system

$$\cdots \to \mathfrak{E}_2 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{E}_1 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{E}_0$$

with transition maps that factor through the relative Frobenius map mod p. Thus the generic fibre of $\mathfrak{E}_\infty := \varprojlim_{[\mathfrak{p}]_i} \mathfrak{E}_i$ is a perfectoid tilde-limit of $\cdots \xrightarrow{[p]} E \xrightarrow{[p]} E$.

However, this construction does not give the local splittings in Proposition 3.7.

Remark 3.10. With some work, the arguments in this section can be extended to any perfectoid base field. For instance, the Raynaud uniformisation of Theorem 3.3 might only be defined over a finite extension L of K. Our argument then gives a perfectoid space over the (necessarily perfectoid) field L. We can then use Galois descent to get a perfectoid space over our original field K. This uses that the quotient of a perfectoid space by a finite group often remains perfectoid: see Theorem 1.4 of [7] for details. Finally, one checks that this Galois descent commutes with tilde-limits.

4. The case of abeloid varieties

We now prove Theorem 1, building on the preceding sections. Recall our setup: Let A be an abeloid variety over K. Let E be the Raynaud extension associated to A from Proposition 3.4, which is an extension of an abeloid variety B of good reduction by a split rigid torus T of rank r, and $M \subset E$ is a lattice of rank r such that $A = E/M$.

By Proposition 3.4, the quotient map $\pi \colon E \to A$ is locally split in the analytic topology on A: As the action of M on E is totally discontinuous, for every point $x \in A$ there is an open neighbourhood U' of E such that π maps isomorphically onto an open $U := \pi(U')$ containing x. Here we are careful to distinguish $U' \subset E$ and $U \subset A$, even though the two are isomorphic via π.

We fix from now on a cover $\mathfrak{U}$ of A by opens U of this form.

The pullback of U' along $[p]\colon A \to A$ will in general be bigger than the pullback of U along $[p] : E \to E$: e.g. in characteristic 0, the first is an étale $A[p]$-torsor, whereas the latter is an étale $E[p]$-torsor, and by the Snake Lemma we have a short exact sequence

$$0 \to E[p] \to A[p] \to M/pM \to 0.$$

To relate the pullbacks, we subdivide the tower

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$$

into two partial towers. For this we make some auxiliary choices: Since K is algebraically closed, we can choose lattices $M_n \subseteq E$ such that $M_0 = M$ and $[p]\colon E \to E$ restricts to isomorphisms $M_{n+1} \to M_n$ for all n.

Remark 4.1. Such a choice is equivalent to the choice of subgroups $D_n \subseteq A[p^n]$ of order p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$. Namely, given the lattices M_n, we obtain the desired torsion subgroups by setting $D_n := M_n/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/p^nM \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ with properties as above, we recover M_n as the kernel of $E \to A \to A/D_n$.

One might call the choice of D_n for all n a partial anticanonical $\Gamma_0(p^\infty)$-structure, because if B admits a canonical subgroup (that is, if it satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A always has a partial anticanonical subgroup even if B does not have a canonical subgroup.

Following the remark, denote by D_n the torsion subgroup $M_n/M \subset A$. The quotient $A_n := A/D_n = E/M_n$ is then another abeloid variety over K and the quotient map

$$v^n\colon A = E/M \to A_n = E/M_n$$

is an isogeny of degree p^{rn} through which $[p^n]\colon A \to A$ factors. The $[p]$-multiplication tower now splits into two towers, one written vertically, the other horizontally:

(4)
$$\begin{tikzcd}
{} \arrow[dr, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] \\
& A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[r, "v"] \arrow[d, "{[p]_E}"] & A_2 \arrow[d, "{[p]_E}"] \\
& & A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[d, "{[p]_E}"] \\
& & & A.
\end{tikzcd}$$

Since each $D_n = M_n/M$ is finite étale, all horizontal maps are finite étale. The vertical tower on the other hand fits into a commutative diagram which compares it to the $[p]$-tower of E:

(5)
$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & A_1 \arrow[r] \arrow[d, "{[p]_E}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

By Proposition 3.7, the middle row has a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

Definition 4.2. Let $M_\infty := \varprojlim_{n\in\mathbb{N}} M_n$ be the limit of the left vertical tower.

We note that M_∞ is an actual limit, not just a tilde-limit, because the transition maps are isomorphisms. In particular, the projection $M_\infty \to M$ is an isomorphism as well. By Proposition 2.4, we get a natural map $M_\infty \to E_\infty$.

Proposition 4.3. *There is a perfectoid tilde-limit $A'_\infty \sim \varprojlim_{n\in\mathbb{N}} A_n$. It fits into an analytic-locally split short exact sequence of perfectoid groups*

$$0 \to M_\infty \to E_\infty \to A'_\infty \to 0.$$

Proof. We work locally on opens $U' \subset E$ mapping isomorphically to U in our cover $\mathfrak{U}$ of A. Write $\pi_n\colon E \to A_n$ for the quotient map. Since the rows in (5) are exact, and the transition maps on the left are isomorphisms, it follows that for each $n \in \mathbb{N}$, the quotient map π_n sends the pullback $U'_n := [p^n]^{-1}(U')$ isomorphically onto $U_n := \pi_n(U'_n) \subseteq A_n$. Thus (5) is locally of the form

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & M_1 \times U'_1 \arrow[r] \arrow[d, "{[p]}"] & U_1 \arrow[r] \arrow[d, "{[p]_E}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \times U' \arrow[r] & U \arrow[r] & 0. \end{tikzcd} \tag{6}$$

Let U_∞ be the pullback of U' along $E_\infty \to E$. We have $U_\infty \sim \varprojlim U'_n \cong \varprojlim U_n$. The system $(U_n)_{n\in\mathbb{N}}$ thus has a perfectoid tilde-limit. This shows that $\varprojlim A_n$ has a perfectoid tilde-limit. We can therefore apply Proposition 2.4 to get a morphism $E_\infty \to A'_\infty$, obtaining the desired short exact sequence in the limit over diagram (5) since the transition maps in (6) respect the splitting. □

We will keep the notation introduced above: U' is an open of E mapping isomorphically to $U \subset A$. The open $U'_n := [p^n]^{-1}(U') \subset E$ maps isomorphically to its image $U_n \subset A_n$ and we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_n \arrow[r] \arrow[d, equal] & M_n \times U'_n \arrow[r] \arrow[d, hook] & U_n \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r, "\pi_n"] & A_n \arrow[r] & 0. \end{tikzcd}$$

To construct a tilde-limit of $\varprojlim_{[p]} A$, we now use the fact that the horizontal maps in diagram (4) are all finite étale. They are even finite covering maps, in the following sense:

Lemma 4.4. *For any $n \geq 0$, the preimage of $U_n \subset A_n$ under the horizontal map $v^n\colon A \to A_n$ is isomorphic to p^{rn} disjoint copies of U_n. More canonically, it is isomorphic to $D_n \times U_n$, where $D_n = M_n/M$ (see Remark 4.1).*

Proof. For the first part, we observe that the map v^n fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r] & A_n \arrow[r] & 0 \end{tikzcd} \tag{7}$$

where the map on the left is the natural inclusion. Upon restriction to $U_n \subset A_n$, this becomes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & M_n \times U'_n \arrow[r] \arrow[d, equal] & (v^n)^{-1}(U_n) \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & M_n \times U'_n \arrow[r] & U_n \arrow[r] & 0 \end{tikzcd} \tag{8}$$

and the claim follows from the fact that M is a discrete lattice of rank r, and from $U'_n \cong U_n$. □

Definition 4.5. The $[p]$-multiplication on E maps M_{n+1} onto M_n and therefore the $[p]$-multiplication tower of A induces a tower

$$\cdots \xrightarrow{[p]} D_{n+1} = M_{n+1}/M \xrightarrow{[p]} D_n = M_n/M \to \cdots .$$

Since K is algebraically closed, the finite étale groups D_n are already constant. By Lemma 2.6, there is a profinite perfectoid group D_∞ such that

$$D_\infty \sim \varprojlim_n D_n.$$

The quotient maps $M_n \to D_n = M_n \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ in the limit give rise to a closed immersion of perfectoid groups $M_\infty \hookrightarrow D_\infty = M_\infty \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Theorem 1 is now part of the following theorem:

Theorem 4.6. *(1) There is a unique perfectoid space A_∞ which is a tilde-limit of $\varprojlim_{[p]} A$.*
(2) The auxiliary subgroups $D_n \subseteq A$ in the limit give rise to a pro-finite subgroup $D_\infty \subseteq A_\infty$.
(3) The preimage of any $U \in \mathfrak{U}$ under the projection $A_\infty \to A$ is isomorphic to $D_\infty \times U_\infty$.
(4) There is a natural map of analytic-locally split short exact sequences of perfectoid groups

$$\begin{tikzcd} 0 \arrow[r] & M_\infty \arrow[r] \arrow[d, hook] & E_\infty \arrow[r] \arrow[d, hook] & A'_\infty \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D_\infty \arrow[r] & A_\infty \arrow[r] & A'_\infty \arrow[r] & 0. \end{tikzcd}$$

(5) In particular, we have an analytic-locally split short exact sequence of perfectoid groups

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the antidiagonal embedding of M_∞ into $D_\infty \times E_\infty$.

We note that A_∞ is independent of the auxiliary choice of D_n up to isomorphism by Corollary 2.5, but the subgroup D_∞ and in particular the diagrams in (4) and (5) depend on this choice.

Remark 4.7. We think of part (5) as the analogue of the Raynaud uniformisation

$$0 \to M \to E \to A \to 0.$$

Here we note that while the map $E \to A$ is a quotient, in the limit over $[p]$ it becomes an injection $E_\infty \hookrightarrow A_\infty$: The reason is that the projective system $(M, [p])$ has vanishing $\varprojlim$ but non-vanishing $\mathrm{R}^1 \varprojlim$, for instance when considered as abelian sheaves on perfectoid spaces for the pro-étale topology in the sense of [21] (assuming that K is of characteristic 0). A toy example of this phenomenon would be the inverse system over $[p]$ on the short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ whose limit yields an exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{R}^1 \varprojlim_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

We therefore think of the quotient $D_\infty/M_\infty = M_\infty \otimes_{\mathbb{Z}} (\mathbb{Z}_p/\mathbb{Z})$ implicit in part (5) as being an incarnation of $\mathrm{R}^1 \varprojlim_{[p]} M_\infty$.

Proof of Theorem 4.6. We keep the notation from the proof of Proposition 4.3: We have a cover of A_n by open subsets U_n and a perfectoid open subspace $U_\infty \subseteq E_\infty$ for which $U_\infty \sim \varprojlim U_n$.

By Lemma 4.4, the restriction of diagram (4) to the open U of the bottom A becomes

$$\begin{tikzcd}
{} \arrow[dr, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] \\
& D_2 \times U_2 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & v^{-1}(U_2) \arrow[r, "v"] \arrow[d, "{[p]_E}"] & U_2 \arrow[d, "{[p]_E}"] \\
& & D_1 \times U_1 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & U_1 \arrow[d, "{[p]_E}"] \\
& & & U.
\end{tikzcd}$$

Hence the restriction of the tower $\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$ to U becomes the inverse system

$$\cdots \to D_{n+1} \times U_{n+1} \to D_n \times U_n \to \cdots .$$

By Lemma 2.7 this inverse system has perfectoid tilde-limit $D_\infty \times U_\infty$. These local tilde-limits glue together to give the desired tilde-limit A_∞. This proves parts (1), (2) and (3), and shows that the second row of part (4) is locally split and in particular exact.

The first row in part (4) is from Proposition 4.3. Part (5) follows immediately from part (4). $\square$

Remark 4.8. When working over a general perfectoid base field, the lattices M_n may no longer be defined over K. Instead, one can show that the natural map $A[p^n] \times U_n \to V_n$ is an étale $E[p^n]$-torsor for the diagonal action where V_n is the pullback of U along $[p^n]\colon A \to A$. The point is that this torsor is split when K is algebraically closed.

5. Applications

In this section, we give three applications of our main result. For all of these, we assume that K is of characteristic 0, i.e. K is an algebraically closed non-archimedean field extension of $\mathbb{Q}_p$.

5.1. **Uniformisation.** Our first application is a "p-adic uniformisation" of abelian varieties. Recall that any abelian variety A over $\mathbb{C}$ of dimension d has a uniformisation in terms of a complex torus $A \cong \mathbb{C}^d/\Lambda$ for some $2d$-dimensional lattice $\Lambda \subseteq \mathbb{C}^d$. More canonically, it admits the presentation

$$A \cong \operatorname{Lie} A/H_1(A, \mathbb{Z}).$$

We have the following analogue of this over K: Let A be an abeloid variety over K of dimension d, considered as a rigid space. Then in the limit over n, the short exact sequences

$$0 \to A[p^n] \to A \to A \to 0$$

give rise to a short exact sequence of sheaves on perfectoid K-algebras with the pro-étale topology

$$0 \to T_pA \to A_\infty \to A \to 0.$$

Using the language of diamonds from [21], we then have:

Corollary 5.1. *The diamond $A^\diamondsuit$ associated to A has the natural presentation*

$$A^\diamondsuit = A_\infty/T_pA$$

given by the perfectoid space A_∞ with its pro-étale subgroup T_pA.

Here we think of $T_pA = H_1^{\text{ét}}(A,\mathbb{Z}_p)$ as the analogue of $H_1(A,\mathbb{Z})$ in the complex setting.

Remark 5.2. This p-adic uniformisation of A is very closely related to the uniformisation of the associated rigid analytic p-divisible group $A[p^\infty]$ in the sense of Fargues [6], as described in [22] and [18, §4]. The precise connection is as follows: Using the Raynaud uniformisation, one can attach to A a unique p-divisible group over $\mathcal{O}_K$ with generic fibre $A[p^\infty]$ whose associated rigid analytic p-divisible group G in the sense of Fargues [6] is canonically an open subgroup $G \subseteq A$. The universal cover of G in the sense of [22, §3.1] fits into a pullback diagram of open adic subgroups

$$\begin{tikzcd}
0 \arrow[r] & T_pG \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T_pA \arrow[r] & A_\infty \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

In particular, we recover the statement that $\widetilde{G}$ is perfectoid.

However, we note that the translates of $\widetilde{G}$ do not cover all of A_∞, and therefore the fact that $\widetilde{G}$ is perfectoid does not show immediately that A_∞ is perfectoid. The issue is points of rank 2: Indeed, there are infinitely many disjoint translates of $\widetilde{G}$, but A_∞ is quasi-compact.

Remark 5.3. We note that for two abelian varieties A and B of dimension d, the universal covers A_∞ and B_∞ are different in general, so that this is only a "uniformisation" in a rather weak sense. However, in upcoming work of the third author, it is shown that A_∞ and B_∞ are isomorphic whenever A and B are "p-adically close" in some precise sense, so that at least locally in the moduli space of abelian varieties we can really think of T_pA as a $2d$-dimension $\mathbb{Z}_p$-lattice determining A.

There is a second closely related uniformisation which we discuss in §5.3 below.

5.2. **Stein property.** As a second application, we can combine our main theorem with work of Reinecke to deduce the following Artin vanishing result:

Corollary 5.4 (Reinecke, [15])**.** *Let A be an abeloid variety over K. Let L be a constructible sheaf of $\mathbb{F}_p$-modules on $A_{\text{ét}}$. Then for any $i > \dim A$,*

$$\varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A,[p^n]^*L) = 0.$$

Proof. Due to Theorem 1, we can apply [15, Theorem 3.3] to the system $\cdots \to A \xrightarrow{[p]} A$. ☐

A theorem of Artin and Grothendieck states if X is an affine algebraic variety over K, then $H^i_{\text{ét}}(X,L) = 0$ for any constructible $\mathbb{F}_p$-module L and any $i > \dim A$. However, the rigid analogue of this statement is false in general. The point of the Corollary is that an analogue of this vanishing statement is true for the pullback of L to A_∞ in the following sense: Consider the morphism of sites $\nu\colon A_{\text{proét}} \to A_{\text{ét}}$. Then by regarding A_∞ as an object in $A_{\text{proét}}$ via the pro-étale morphism $A_\infty \to A$, one can show

$$H^i_{\text{proét}}(A_\infty,\nu^*L) = \varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A,[p^n]^*L).$$

Let us sketch a proof that the left hand side vanishes for $i > \dim A$: One first reduces to the case that L is locally constant. Since A is proper and smooth, one can use the Primitive Comparison

Theorem [17, Theorem 5.1] [18, Theorem 3.13] to reduce to showing that $H^i_{\text{proét}}(A_\infty, \nu^* L \otimes_{\mathbb{F}_p} \mathcal{O}^+_{A_{\text{ét}}}/p)$ is almost zero for $i > d$. For this one can use that $H^j(V, \nu^* L \otimes_{\mathbb{F}_p} \mathcal{O}^+_{A_{\text{ét}}}/p)^a = 0$ for any small enough affinoid perfectoid open $V \subseteq A_\infty$ and any $j > 0$. This reduces the desired statement to a computation in Čech cohomology, which indeed vanishes in degree $> \dim A$.

5.3. **Universal perfectoid covers of curves.** As a third application, we describe how one can obtain universal perfectoid pro-étale covers of curves over K. This was first observed by Hansen [8], who on his blog sketched a strategy to prove this in the case that the Jacobian has good reduction. Due to our Theorem 1, this assumption can be removed, as we shall now demonstrate:

We start by recalling some background in a more general setting: Let C be a connected smooth proper rigid analytic curve over K. Any such curve arises as the analytification of some schematic smooth projective curve over K [13, Theorem 1.8.1], and by [11, Theorem 3.1], GAGA induces an equivalence of categories between finite étale covers of C considered as a scheme and as a rigid space, respectively. We can therefore fix a geometric base point $x : \mathrm{Spa}(K, \mathcal{O}_K) \to C$ and study the usual étale fundamental group $\pi_1(C, x)$ using the language of adic spaces.

To prepare our discussion, we recall from [17, §3] a few facts on the pro-finite-étale site of C: This is the category $C_{\text{profét}} = \text{pro}{-}C_{\text{fét}}$ of small cofiltered inverse systems $(X_i)_{i \in I}$ in the finite étale site $C_{\text{fét}}$. An object in $C_{\text{profét}}$ is called perfectoid if there is a perfectoid tilde-limit $X_\infty \sim \varprojlim X_i$. Let $C^{\text{perf}}_{\text{profét}}$ be the full subcategory of perfectoid objects, and let Perf_C be the category of perfectoid spaces over C. Then the argument in the proof of [23, Lemma 8.2.3] shows:

Lemma 5.5. *Sending perfectoid pro-étale objects to their tilde-limits defines a fully faithful functor*

$$C^{\text{perf}}_{\text{profét}} \to \mathrm{Perf}_C, \quad (X_i)_{i \in I} \mapsto X_\infty \sim \varprojlim_{i \in I} X_i.$$

We call the objects $X_\infty \to C$ in the essential image the pro-finite-étale perfectoid covers of C.

Proposition 5.6 ([17, Proposition 3.5])**.** *There is an equivalence of categories*

$$F : C_{\text{profét}} \to \pi_1(C, x){-}\text{pfSets}, \quad (Y_i)_{i \in I} \mapsto F(X) := \varprojlim_{i \in I} |Y_i \times_C x| = \varprojlim_{i \in I} \mathrm{Hom}_C(x, Y_i)$$

from the pro-finite-étale site of C to the category of profinite sets with continuous $\pi_1(C, x)$-action.

This restricts to the usual equivalence of finite étale covers to finite sets with continuous $\pi_1(C, x)$-action. In particular, for every open subgroup $H \subseteq \pi_1(C, x)$, there is a corresponding finite étale morphism $C_H \to C$ from a connected curve C_H, considered as an analytic adic space. For any two open subgroups $H_1 \subseteq H_2 \subseteq \pi_1(C, x)$, there is a natural map $C_{H_1} \to C_{H_2}$. For varying H, one therefore has a filtered inverse system $(C_H)_{H \subseteq \pi_1(C,x)}$ which we may regard as an object in $C_{\text{proét}}$.

Corollary 5.7 (Hansen, [8])**.** *Let C be a connected smooth projective curve of genus $g \geq 1$ over K, considered as an analytic adic space.*

1. *There is a perfectoid tilde-limit $\widetilde{C} \sim \varprojlim_H C_H$ where H ranges over the open subgroups of $\pi_1(C, x)$.*
2. *The morphism $\widetilde{C} \to C$ is a pro-étale $\pi_1(C, x)$-torsor. It is universal with this property in the sense that it represents the functor sending pro-finite-étale perfectoid covers $X \to C$ to the $\pi_1(C, x)$-module $F(X) = \mathrm{Hom}_C(x, X)$.*
3. *For any $X \in C_{\text{profét}}$, for example for any finite étale $X \to C$, there is a natural isomorphism*

$$X = \underline{F(X)} \times^{\pi_1(C,x)} \widetilde{C} := (\underline{F(X)} \times \widetilde{C})/\pi_1(C, x).$$

Here the right hand side is the categorical quotient in adic spaces for the antidiagonal action.

Remark 5.8. Parts (2) and (3) say that we may reasonably regard $\widetilde{C} \to C$ as the "universal cover" of C, in analogy with this notion in topology.

The proof also works in the case that C is an abelian variety. In this case, the étale fundamental group is simply the adelic Tate module $\pi_1(A,x) = TA := \varprojlim_{N\in\mathbb{N}} A[N](K)$. We then have:

Corollary 5.9. *Let A be an abelian variety over K, then there is a perfectoid tilde-limit*

$$\widetilde{A} \sim \varprojlim_{[N]} A$$

where N ranges over $N \in \mathbb{N}$ and the analogous statements of Corollary 5.7.(2) and (3) hold for the TA-torsor $\widetilde{A} \to A$. In particular, there is a natural isomorphism

$$A^{\diamondsuit} = \widetilde{A}/TA = \widetilde{A}/\pi_1(A,x).$$

Remark 5.10. There is a second, more topological universal property of A_∞ and $\widetilde{A}$: Namely, in upcoming work it will be shown that A_∞ is the universal cover of A with the property that $H^1_v(A_\infty, \mathbb{Z}_p) = 0$, whereas $\widetilde{A}$ is universal with the property that $H^1_v(\widetilde{A}, \widehat{\mathbb{Z}}) = 0$.

Proof of Corollary 5.7 and Corollary 5.9. To ease notation, let us abbreviate $G := \pi_1(C,x)$.

We construct $\widetilde{C}$ in two steps. The choice of the base point x gives an embedding $\iota\colon C \to A$ of C into its Jacobian. We can now argue like in [20, IV.1] to obtain a perfectoid pro-étale cover of C via pull-back: Let C_n be the pullback of C along the map $[p^n]\colon A \to A$. Combining our main theorem with [20, Lemma II.2.2], we obtain a strongly Zariski-closed subspace $C_\infty \to A_\infty$ that is the pullback of $C \to A$. It is then clear on affinoid subspaces that we have

$$C_\infty \sim \varprojlim C_n.$$

Indeed, the condition on topological spaces is immediate from [20, Lemma II.2.2]. The approximation condition follows since affinoid locally, $\mathcal{O}_{A_\infty} \to \mathcal{O}_{C_\infty}$ is surjective, hence any function $f \in \mathcal{O}_{C_\infty}$ can be lifted to $g \in \mathcal{O}_{A_\infty}$, and approximated by a convergent sequence of $g_n \in \varinjlim_{[p]} \mathcal{O}_A$. The images of the g_n in $\varinjlim_n \mathcal{O}_{C_n}$ then converge to f. This proves the displayed tilde-limit property.

We now use the fact that pro-étale covers of perfectoid spaces are again perfectoid to construct a perfectoid cover $\widetilde{C}$ of C_∞ that packages up the entire étale fundamental group of C. The exact same argument can be used to construct the tower $\widetilde{A} \to A_\infty$, proving Corollary 5.9.

As we are assuming that K has characteristic 0, the maps $[p^n]\colon A \to A$ are finite étale, so the induced covers $C_n \to C$ are finite étale. The inverse system

$$\cdots \to C_n \to \cdots \to C_1 \to C$$

therefore corresponds to a chain of subgroups

$$\cdots < G_n < \cdots < G_1 < G = \pi_1(C,x).$$

For any open subgroup H of G corresponding to the finite étale cover $C_H \to C$, we have a decreasing sequence of positive integers

$$\cdots \le [G_n : G_n \cap H] \le \cdots \le [G_1 : G_1 \cap H] \le [G : G \cap H]$$

which stabilises for $n \gg 0$. So there is an integer d such that for all $n \gg 0$, we have $[G_n : G_n \cap H] = d$. Translating back to the language of finite étale covers, we see that for such n, the map

$$C_{G_{n+1}\cap H} \to C_{G_n \cap H} \times_{C_{G_n}} C_{G_{n+1}}$$

coming from the universal property of the fibre product is an isomorphism: Both spaces are finite étale covers of $C_{G_{n+1}}$ of degree d, so the map is a finite étale cover of degree 1. This implies that the natural morphism $\varprojlim C_{G_n\cap H} \to \varprojlim C_{G_n}$ of objects of $C_{\text{profét}}$ is finite étale in the sense of [17, Definition 3.9]. To simplify notation, we write this morphism as $C_{H,\infty} \to C_\infty$ (via Lemma 5.5, one can also think of this as the corresponding map of perfectoid spaces).

We can now rewrite in $C_{\text{profét}}$:

$$\varprojlim_{H\to 1} C_H = \varprojlim_{H\to 1} \varprojlim_{n\to\infty} C_{G_n\cap H} = \varprojlim_{H\to 1} C_{H,\infty}.$$

As the $C_{H,\infty}$ have compatible finite étale maps to C_∞, we obtain a morphism in $C_{\text{profét}}$

$$\varprojlim_{H\to 1} C_{H,\infty} \to C_\infty.$$

By [17, Lemma 4.6], pro-finite-étale covers of perfectoid objects are again perfectoid, giving us the desired perfectoid space

$$\widetilde{C} \sim \varprojlim_{H\to 1} C_H.$$

This completes the construction of $\widetilde{C}$, and thus proves part (1).

To see part (2), we write $G = \varprojlim_N G/N$ where N ranges through the normal open subgroups. These are precisely the subgroups for which $C_N \to C$ is already a finite étale G/N-torsor. Concretely, this means that the following natural morphism is already an isomorphism:

$$G/N \times_C C_N \to C_N \times_C C_N.$$

We note that we also have $\widetilde{C} \sim \varprojlim_N C_N$, as normal open subgroups are cofinal in the inverse system of all open subgroups. In the limit, this shows that $\widetilde{C}$ is a pro-finite-étale G-torsor.

To see that $F(X) = \mathrm{Hom}_C(\widetilde{C}, X)$, we recall that for any Galois cover $C_N \to C$ with a finite Galois map $C_N \to X$, we have $F(X) = \mathrm{Hom}_C(C_N, X)$. It therefore suffices to see that

$$\mathrm{Hom}_C(\widetilde{C}, X) = \varinjlim_N \mathrm{Hom}_C(C_N, X).$$

But this follows from Lemma 5.5.

For (3), write $S = F(X)$, then it suffices to prove that the natural morphism

$$\rho : \underline{S} \times \widetilde{C} \to X$$

is a pro-finite-étale G-torsor for the antidiagonal action. Indeed, this implies that X is the categorical quotient by the action of G: This is because the torsor property implies $\mathcal{O}_X = (\rho_*\mathcal{O}_{\underline{S}\times\widetilde{C}})^G$ by combining [4, Lemma 2.24] and [10, Theorem 8.2.3].

Since connected components of X correspond to G-orbits of S, we may reduce to the case where X is connected and G acts transitively on $\underline{S}$. By writing X as a system of finite étale covers, we may further reduce to the case that S is finite. Fix $s \in S$ and let $H \subseteq G$ be the stabiliser of s, then $X = C_H$. It now suffices to show that for any normal open subgroup $N \subseteq G$ with $N \subseteq H$, the natural morphism

$$G/H \times C_N \to C_H$$

is a G/N-torsor, as the desired result will follow in the limit $N \to 1$. But this follows by Galois descent from the diagram

$$\begin{array}{ccc} G/H \times C_N & \longrightarrow & C_H \\ \uparrow & & \uparrow \\ G/N \times C_N & \longrightarrow & C_N \end{array}$$

which is Cartesian as $C_N \to C_H$ is a finite étale H/N-torsor. $\square$

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