PERFECTOID COVERS OF ABELIAN VARIETIES

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Abstract. For an abelian variety A over an algebraically closed non-archimedean field of residue characteristic p, we show that there exists a perfectoid space which is the tilde-limit of $\varprojlim_{[p]} A$.

1. Introduction

Let p be a prime and let K be an algebraically closed non-archimedean field of residue characteristic p. For an abelian variety A over K we consider the inverse system of A under the p-multiplication morphism:

$$\dots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A.$$

Via the adic analytification functor, we may see this as an inverse system of analytic adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. The primary goal of this article is to show that the "inverse limit" of this tower exists in some way and is a perfectoid space: Since inverse limits rarely exist in the category of adic spaces, in [17] the authors introduce the weaker notion of tilde-limits to remedy this problem. This is the notion of "limits" we are going to use. More precisely, we prove:

Theorem 1. *Let A be an abeloid variety over K, for instance an abelian variety seen as a rigid space. Then there is a unique perfectoid space A_∞ over K such that $A_\infty \sim \varprojlim_{[p]} A$ is a tilde-limit.*

The possibility of results in this direction is mentioned in §7 and §13 of [15], and in the case of abelian varieties A with good reduction, this theorem was proven already in [8] (Lemme A.16). In order to motivate our strategy for the general case, let us sketch the proof in the good reduction case (we follow Exercise 4 – 6 in [1], the proof is spelt out in detail in §2.3, Corollary 2.17 below):

Let A be an abelian variety of good reduction over K. Let $\mathcal{O}_K$ be the ring of integers of K and let $\pi \in \mathcal{O}_K$ be a pseudo-uniformiser such that $p \in \pi\mathcal{O}_K$. Then we can consider A as an abelian scheme over $\mathcal{O}_K$. Let $\mathcal{A}$ be its π-completion, a formal scheme over $\mathcal{O}_K$. Its adic generic fibre is the rigid analytification of A that we denote by the same letter. The mod π special fibre $\tilde{A} = A \times \mathcal{O}_K/\pi$ is a group scheme over $\mathcal{O}_K/\pi$, so the map $[p] : \tilde{A} \to \tilde{A}$ factors through the relative Frobenius map. The inverse limit $\varprojlim_{[p]} \tilde{A}$ in the category of schemes is thus relatively perfect over $\mathcal{O}_K/\pi$. We can similarly form the inverse limit $\mathcal{A}_\infty = \varprojlim_{[p]} \mathcal{A}$ in the category of formal schemes. Its adic generic fibre A_∞ is a tilde-limit of $\varprojlim_{[p]} A$, and it is perfectoid since $\varprojlim_{[p]} \tilde{A}$ is relatively perfect.

If A is an abelian variety with bad reduction, the assumption that K is algebraically closed assures that A is then semi-stable. In this case, the theory of Raynaud extensions provides us with a short exact sequence

$$0 \to T \to E \to B \to 0$$

of rigid groups, where $T = (\mathbb{G}_m^{\mathrm{an}})^d$ is a split rigid torus and B is the analytification of an abelian variety with good reduction, such that $A = E/M$ for a discrete lattice $M \subset E$. This short exact sequence is split locally on B, allowing us to locally write E as a product of T and an open subspace of B. Our strategy for the proof of Theorem 1, which more generally applies to any abeloid variety over K, is now similar to the good reduction case:

(1) Use formal models to construct a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$. This is easy.
(2) Use the formal models from (1) to construct a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.
(3) Study the quotient map $E \to A$ in the limit over $[p]$ to construct the desired space A_∞.

More precisely, this article is organised as follows: In §2 we develop a notion of a $[p]$-F-formal tower for a commutative rigid group G over K, which is roughly an inverse system of formal models for $[p] : G \to G$ that factor through the relative Frobenius mod π. This is an axiomatisation of the data that one uses to construct A_∞ in the case of good reduction. In particular, the same proof shows that if G admits $[p]$-F-formal tower, then there is a unique perfectoid tilde-limit $G_\infty \sim \varprojlim_{[p]} G$.

In chapter §3 we give a $[p]$-F-formal tower for a split rigid torus T in terms of a family of formal models $\mathfrak{T}_{q^{1/n}}$, thus showing that a perfectoid tilde-limit T_∞ of the inverse system of $[p]$ on T exists. This formal model receives a natural action by the formal torus $\overline{T}$.

We then in §4 use the language of fibre bundles to construct a $[p]$-F-formal tower for E: The Raynaud extension of A arises from a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \to B \to 0$$

by taking generic fibres and forming the pushout with respect to the open immersion $\overline{T}_\eta \to T$. Since the sequence is locally split, we can see $\overline{E} \to B$ as a principal $\overline{T}$-bundle and formation of E amounts to a change of fibre from $\overline{T}_\eta$ to T. We now obtain a $[p]$-F-formal tower for E from the formal models of §4 by the change of fibre from $\overline{T}$ to the formal models $\mathfrak{T}_{q^{1/n}}$. This gives the desired perfectoid tilde-limit E_∞.

In §5 we finish the proof of the main theorem by constructing A_∞ from E_∞ as follows: After choosing lattices $M \subseteq M_n \subseteq E$ that map isomorphically to M under $[p^n] : E \to E$, the $[p]$-multiplication tower of $A = E/M$ naturally factors into two separate towers: One is the tower of maps $E/M_{n+1} \to E/M_n$ induced from $[p]$-multiplication of E, the other is induced from the projection maps $v_n : E/M \to E/M_n$. Using local splittings, one can construct a perfectoid tilde-limit $A'_\infty \sim \varprojlim_n E/M_n$ of the first tower from E_∞. It fits into a short exact sequence

$$0 \to M \to E_\infty \to A'_\infty \to 0$$

The desired space $A_\infty \sim \varprojlim_{[p]} A$ can then be constructed from this using that the quotient maps $v_n E/M \to E/M_n$ are locally split in the analytic topology. This construction also gives the following analogue of Raynaud uniformisation for A_∞: Write D_n for the kernel of v_n, then there is a profinite perfectoid tilde-limit $D_\infty \sim \varprojlim_{[p]} D_n$. There is then a short exact sequence of perfectoid groups

$$0 \to M \to D_\infty \times E_\infty \to A_\infty \to 0$$

which we regard as an analogue of the sequence $0 \to M \to E \to A \to 0$.

At the end there is an appendix on fibre bundles and associated fibre bundle constructions in the context of formal, rigid and perfectoid spaces.

Acknowledgements

This work started as a group project at the 2017 Arizona Winter School. We would like to thank Bhargav Bhatt for proposing the project, for his guidance and for his constant encouragement, and we would like to thank Matthew Morrow for his help during the Arizona Winter School. In addition we would like to thank the organizers of the Arizona Winter School for setting up a great environment for us to participate in this project.

During this work Damián Gvirtz and Ben Heuer were supported by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. During the Arizona Winter School, Daria Shchedrina was supported by Peter Scholze and the DFG. Peter Wear was supported by NSF grant DMS-1502651 and UCSD and would like to thank Kiran Kedlaya for helpful discussions.

Notation

Let $\mathcal{O}_K$ be the ring of integers of K and a fixed pseudo-uniformiser $\pi \in \mathcal{O}_K$ such that $p \in \pi\mathcal{O}_K$.

By an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$, we mean an adic spaces in the sense of [17]. We use perfectoid spaces in the sense of §2 *ibid.* In their language, adic spaces in the sense of Huber are referred to as *honest* adic spaces.

Throughout the article, we make no distinction between rigid spaces and their corresponding honest adic spaces. In particular, by an open cover of a rigid space we shall always mean a cover of the associated adic space, and therefore the cover of the rigid space will automatically be an admissible cover in the sense of rigid analytic geometry. If a rigid space is obtained from a K-scheme X via rigid-analytification $X \mapsto X^{\mathrm{an}}$, we will often denote both by the same symbol X in order to simplify the notation.

For a π-adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$ with the π-adic topology, we denote by $\mathfrak{X}_\eta := \mathfrak{X} \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$ its adic generic fibre.

Finally, we recall some standard terminology: If X be a rigid space over K, a formal model of X is an admissible topologically finite type formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ together with an isomorphism of its generic fibre $\mathfrak{X}_\eta \xrightarrow{\sim} X$. If $\phi : X \to Y$ is a morphism of rigid spaces over K, with formal models $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, then a morphism of formal schemes $\Phi : \mathfrak{X} \to \mathfrak{Y}$ is a formal model of ϕ if it agrees with ϕ on the adic generic fiber via the identifications $\mathfrak{X}_\eta \xrightarrow{\sim} X$ and $\mathfrak{Y}_\eta \xrightarrow{\sim} Y$.

2. Tilde-limits of rigid groups

2.1. **Tilde-limits and formal models.** Inverse limits often do not exist in the category of adic spaces, and neither do they in rigid spaces. Instead we use the notion of tilde-limits from [17]:

Definition 2.1. Let $(X_i)_{i \in I}$ be a filtered inverse system of adic spaces with quasi-compact and quasi-separated transition maps, let X be an adic space with a compatible system of morphisms $f_i : X \to X_i$. We write $X \sim \varprojlim X_i$ and say that X is a **tilde-limit** of the inverse system $(X_i)_{i \in I}$ if the map of underlying topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and there exists an open cover of X by affinoids $\mathrm{Spa}(A, A^+) \subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(A_i, A_i^+) \subset X_i} A_i \to A$$

has dense image, where the direct limit runs over all $i \in I$ and all open affinoid subspaces $\mathrm{Spa}(A_i, A_i^+) \subset X_i$ through which the morphism $\mathrm{Spa}(A, A^+) \subseteq X \to X_i$ factors.

Remark 2.2. As pointed out after Proposition 2.4.4 of [17], tilde-limits (if they exist) are in general not unique. However, Corollary 2.6 below says that perfectoid tilde-limits are unique.

We recall a few results from [17], §2.4 on tilde-limits that we will use frequently throughout:

Proposition 2.3 ([17], Proposition 2.4.2). *Let (A_i, A_i^+) be a direct system of affinoids over $(\mathcal{O}_K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ carrying the π-adic topology. Let $(A, A^+) = (\varinjlim A_i, \varinjlim A_i^+)$ be the affinoid algebra equipped with the topology making $\varinjlim A_{i,0}$ with the π-adic topology a ring of definition. Then*

$$\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+).$$

Proposition 2.4 ([17], Proposition 2.4.3). *Let $X \sim \varprojlim_{i \in I} X_i$ be a tilde-limit and let $U_i \hookrightarrow X_i$ be an open immersion for some $i \in I$. Let $U_j := U_i \times_{X_i} X_j$ for $j \geq i$ and $U := U_i \times_{X_i} X$, then*

$$U \sim \varprojlim_{j \geq i} U_j.$$

Proposition 2.5 ([17], Proposition 2.4.5). *Let $(X_i)_{i \in I}$ be an inverse system of adic spaces over $(K, \mathcal{O}_K)$ and assume that there is a perfectoid space X such that $X \sim \varprojlim_{i \in I} X_i$. Then for any perfectoid $(K, \mathcal{O}_K)$-algebra (B, B^+),*

$$X(B, B^+) = \varprojlim_{i \in I} X_i(B, B^+)$$

Corollary 2.6. *Any two perfectoid tilde-limits $X, X' \sim \varprojlim X_i$ are canonically isomorphic.*

In the situation of the Corollary, we will also refer to X as *the* perfectoid tilde-limit of $\varprojlim X_i$. Of course perfectoid tilde-limits do not always exists. An example for a basic situation in which they do is the following:

Corollary 2.7. *Let $(S_i)_{i \in I}$ be an inverse system of finite sets. Let $S = \varprojlim_{i \in I} S_i$. Then the system of constant groups $\underline{S_i} = \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S_i, K), \mathrm{Map}_{\mathrm{cts}}(S_i, K^+))$ has a perfectoid tilde-limit*

$$\underline{S} := \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S, K), \mathrm{Map}_{\mathrm{cts}}(S, K^+)) \sim \varprojlim_{i \in I} \underline{S_i}.$$

Proof. Since S is compact, $\mathrm{Map}_{\mathrm{cts}}(S, K) = \mathrm{Map}_{\mathrm{cts}}(S, K^+)[\frac{1}{\pi}]$. Perfectoidness now follows from $\mathrm{Map}_{\mathrm{cts}}(S, K^+)/\pi = \mathrm{Map}_{\mathrm{lc}}(S, K^+/\pi)$. The tilde-limit property follows from 2.3. □

In the situation that the X_i are rigid space, one way to construct tilde-limits is by constructing well-behaved formal models. The reason for this is the combination of the following two Lemmas:

Lemma 2.8. *Let $(\mathfrak{X}_i, \phi_{ij})_{i \in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps $\phi_{ij} : \mathfrak{X}_j \to \mathfrak{X}_i$. Then the inverse limit $\mathfrak{X} = \varprojlim \mathfrak{X}_i$ exists in the category of formal schemes over $\mathcal{O}_K$. If all the $\mathfrak{X}_i$ are flat formal schemes, so is $\mathfrak{X}$.*

Proof. Since the transition maps are affine, we may reduce to the case that $X_i = \mathrm{Spf} A_i$ is affine. Let A be the π-adic completion of $\varinjlim A_i$, then $\mathrm{Spf}\, A$ is the inverse limit of the $\mathrm{Spf}\, A_i$. If the A_i are flat, then so is A because it is torsion-free over the valuation ring $\mathcal{O}_K$. □

The proof also shows that in the situation of the Lemma, $\mathfrak{X}$ is in particular a tilde-limit $\mathfrak{X} \sim \varprojlim \mathfrak{X}_i$ when considered as adic spaces. This remains true after passing to adic generic fibres:

Lemma 2.9. *Let $(\mathfrak{X}_i, \phi_{ij})_{i \in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps ϕ_{ij} and let $\mathfrak{X} = \varprojlim_{\phi_j} \mathfrak{X}_i$ be the limit. Let $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$ and $\mathcal{X} = (\mathfrak{X})_\eta$ be the adic generic fibres. Then $\mathcal{X} \sim \varprojlim \mathcal{X}_i$.*

Proof. This is a consequence of Proposition 2.3: The transition maps in the system are affine, hence quasi-compact quasi-separated. In order to prove the Lemma, we can restrict to an affine open subset $\mathrm{Spf}(A)$ of $\mathfrak{X}$ that arises as the inverse limit of affine open subsets $\mathrm{Spf}(A_i) \subseteq \mathfrak{X}_i$. Here all formal schemes are considered with the π-adic topology and A is the π-adic completion of $\varinjlim A_i$. On the generic fibre, A_i with the π-adic topology is an open subring of definition of $A_i[1/\pi]$. Therefore, Proposition 2.3 applies and shows that $\mathrm{Spf}(A)_\eta \sim \varprojlim \mathrm{Spf}(A_i)_\eta$ as desired. $\square$

Remark 2.10. This lemma essentially says that one can always construct a tilde-limit of an inverse system of rigid spaces $\mathcal{X}_i$ if it arises from an inverse system of formal schemes $\mathfrak{X}_i$ with affine transition maps. This is precisely what Scholze uses in [13] to construct the space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ (see Corollary III.2.19 in [13] and its proof).

We will need the following basic lemma later on.

Lemma 2.11. *Let (A_i, A_i^+) and (B_i, B_i^+) be direct systems of affinoids over $(K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ and $B_{i,0}$ carrying the π-adic topology. Assume that there are perfectoid tilde-limits $\mathrm{Spa}(A, A^+) \sim \varinjlim \mathrm{Spa}(A_i, A_i^+)$ and $\mathrm{Spa}(B, B^+) \sim \varinjlim \mathrm{Spa}(B_i, B_i^+)$. Then*

$$\mathrm{Spa}(A, A^+) \times \mathrm{Spa}(B, B^+) \sim \varprojlim(\mathrm{Spa}(A_i, A_i^+) \times \mathrm{Spa}(B_i, B_i^+))$$

is also a perfectoid tilde-limit.

Proof. The fibre product $\mathrm{Spa}(A, A^+) \times \mathrm{Spa}(B, B^+)$ exists and is perfectoid by [10, Prop 6.18].

We first check the condition on topological spaces: Since fibre products commute with limits in the category of sheaves, it follows from Proposition 2.5 that for any perfectoid field (C, C^+) over (K, K^+), we have $(\mathrm{Spa}(A, A^+) \times \mathrm{Spa}(B, B^+))(C, C^+) = \varprojlim(\mathrm{Spa}(A_i, A_i^+) \times \mathrm{Spa}(B_i, B_i^+))(C, C^+)$. Since the topological space can be reconstructed from this data, it follows that the underlying topological spaces of both sides coincide.

It remains to check that if $\varinjlim A_i \to A$ has dense image and $\varinjlim B_i \to B$ has dense image, then $\varinjlim(A_i \otimes B_i) \to A \otimes B$ has dense image. As direct limits commute with tensor products, we have $\varinjlim(A_i \otimes B_i) = (\varinjlim A_i) \otimes (\varinjlim B_i)$. Now density can be checked directly on elements. $\square$

2.2. **Tilde-limits for rigid groups.** One reason why perfectoid limits along group morphisms are particularly interesting is that the perfectoidness ensures that the limit has again a group structure:

Definition 2.12. A **perfectoid group** is a group object in the category of perfectoid spaces.

Note that the category of perfectoid spaces over K has finite products, so the notion of a group object makes sense.

Proposition 2.13. *Let $(G)_{i\in I}$ be a system of adic group with perfectoid tilde-limit $G_\infty \sim \varprojlim_{i\in I} G_i$.*

1. *There is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms*
2. *Given a morphism of inverse systems of adic groups $(G)_{i\in I} \to (H_j)_{j\in J}$ and a perfectoid tilde-limit $H_\infty \sim \varprojlim_{j\in J} H_j$, there is a unique morphism of perfectoid groups $H_\infty \to G_\infty$ commuting with all projection maps.*

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, Proposition 2.5, which shows that one can argue like in the case of categorical limits. $\square$

Let G be a rigid group over K, that is, a group object in the category of rigid spaces over K. Throughout we will always consider commutative rigid groups. The main topic of study of this work is the $[p]$-multiplication tower

$$\dots \xrightarrow{[p]} G \xrightarrow{[p]} G.$$

We will usually assume that G is p-divisible, i.e. that $[p] : G \to G$ is surjective.

Question 2. When is there a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$ is a tilde-limit?

We are primarily interested in the following examples:

(1) Analytifications of finite type group schemes over K. Examples include the analytification of an abelian variety A over K and of tori T over K.
(2) Generic fibres of topologically finite type formal group schemes over $\mathcal{O}_K$.
(3) Raynaud's covering space E of an abelian variety with semi-stable reduction.

Lemma 2.8 and 2.9 motivate the following definition.

Definition 2.14. Let G be a rigid group. A $[p]$**-formal tower** for G is a family of formal models $\{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ of G, together with affine transition maps $[\mathfrak{p}]_{n+1} : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$ which are formal models of $[p] : G \to G$. Sometimes we suppress notation and write $[\mathfrak{p}]$ for $[\mathfrak{p}]_n$ on $\mathfrak{G}_n$.

For example, for any formal group scheme $\mathfrak{G}$ for which $[p]$ is affine, the $[p]$-multiplication tower of $\mathfrak{G}$ gives rise to a $[p]$-formal tower for its generic fibre $\mathfrak{G}_\eta$.

2.3. **Perfectoidness.** We now introduce the formalism of $[p]$-F-formal towers to axiomatise the approach mentioned in the introduction which shows that A_∞ exists and is perfectoid in the case of good reduction.

Definition 2.15. A $[p]$**-F-formal tower** for a rigid analytic group G is a $[p]$-formal tower

$$(\{\mathfrak{G}_n\}_{n\in\mathbb{N}}, [\mathfrak{p}]_{n+1} : \mathfrak{G}_{n+1} \to \mathfrak{G}_n)$$

such that on the mod π special fibre $\tilde{G}_n$, each $[\mathfrak{p}]_{n+1}$ factors through the relative Frobenius map:

$$\begin{tikzcd}
& \tilde{G}^{(p)}_{n+1} \arrow[dr, dashed] & \\
\tilde{G}_{n+1} \arrow[ur, "F_{rel}"] \arrow[rr, "{[\mathfrak{p}]_{n+1}}"] & & \tilde{G}_n
\end{tikzcd}$$

Proposition 2.16. *Let G be a rigid group over a perfectoid field K. If G admits a $[p]$-F-formal tower, then there exists a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$.*

Proof. Let $(\{\mathfrak{G}_n\}_{n\in\mathbb{N}}, [\mathfrak{p}]_{n+1} : \mathfrak{G}_{n+1} \to \mathfrak{G}_n)$ be a $[p]$-F-formal tower for G. By Lemma 2.9 we have

$$G_\infty := (\varprojlim_{[\mathfrak{p}]} \mathfrak{G})_\eta \sim \varprojlim_{[p]} G.$$

To see that G_∞ is perfectoid, we proceed as the proof of [13], Corollary III.2.19. It suffices to prove that $\mathfrak{G}_\infty = \varprojlim_{[\mathfrak{p}]} \mathfrak{G}$ can be covered by formal schemes of the form $\mathrm{Spf}(S)$ where S is a flat $\mathcal{O}_K$-algebra such that the Frobenius map

$$S/\pi^{1/p} \to S/\pi$$

is an isomorphism. Lemma 5.6 of [10] then shows that $S[1/\pi]$ is perfectoid.

By assumption, on the mod π special fibre $\tilde{G}_n$, $[\mathfrak{p}]_{n+1}$ factors through the relative Frobenius. Now take any affine open subspace $\mathrm{Spf}(S_0) \subseteq \mathfrak{G}_0$. Let $[\mathfrak{p}^i] : \mathfrak{G}_i \to \mathfrak{G}_0$ be the composition $[\mathfrak{p}]_i \circ \cdots \circ [\mathfrak{p}]_1$, and let $\mathrm{Spf}\, S_i \subset \mathfrak{G}_i$ be the pullback of $\mathrm{Spf}\, S_0$ via $[p^i]$. Then we have a commutative diagram:

$$\begin{tikzcd}
& & \tilde{S}_i^{(p)} \arrow[dr, "F_{rel}"] & & \tilde{S}_{i+1}^{(p)} \arrow[dr, "F_{rel}"] & & \\
\dots \arrow[r] & \tilde{S}_{i-1} \arrow[ur, dashed] \arrow[rr] & & \tilde{S}_i \arrow[ur, dashed] \arrow[rr] & & \tilde{S}_{i+1} \arrow[r] & \dots
\end{tikzcd}$$

where the horizontal maps are induced from $[\mathfrak{p}] \mod \pi$.

From this we can check on elements that relative Frobenius is an isomorphism on $\tilde{S}_\infty := \varinjlim_i \tilde{S}_i$. Since K is perfectoid, we moreover have an isomorphism $\mathcal{O}_K/\pi^{1/p} \to \mathcal{O}_K/\pi$ from the absolute Frobenius on $\mathcal{O}_K/\pi$. Therefore absolute Frobenius on S_∞/π induces an isomorphism

$$S_\infty/\pi^{1/p} \xrightarrow{\sim} S_\infty/\pi.$$

Since each $\mathfrak{G}_i$ is flat, so are the S_i and thus so is S_∞. Thus $S_\infty[1/\pi]$ is a perfectoid K-algebra. Since G_∞ is covered by affinoids of the form $\mathrm{Spf}(S_\infty)_\eta$, this shows that G_∞ is perfectoid. □

2.4. **Examples.** Prop. 2.16 gives in more detail the proof of the result sketched in the introduction:

Corollary 2.17. *Let A be an abelian variety of good reduction over K. Then there is a perfectoid tilde-limit $A_\infty \sim \varprojlim A$.*

More generally, if $\mathfrak{G}$ is a flat commutative formal group scheme such that $[p]$-multiplication is affine, then the maps $[p] : \mathfrak{G} \to \mathfrak{G}$ define a $[p]$-F-formal tower for the rigid analytic group $G = \mathfrak{G}_\eta$, and $G_\infty := (\varprojlim_{[p]} \mathfrak{G})_\eta$ is a perfectoid tilde-limit of $\varprojlim_{[p]} G$.

Example. Let $\mathfrak{G}$ be the p-adic completion of the affine group scheme $\mathbb{G}_m$ over $\mathcal{O}_K$. The underlying formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, \mathcal{O}_K\langle X^{\pm 1}\rangle$. Multiplication by p on $\mathbb{G}_m$ gives a $[p]$-F-formal tower for G, so for the generic fibre $G = \mathfrak{G}_\eta$ we obtain the perfectoid tilde-limit $G_\infty := (\mathfrak{G}_\infty)_\eta \sim \varprojlim_{[p]} G$. More precisely, multiplication by p corresponds to the homomorphism

$$[p] : \mathcal{O}_K\langle X^{\pm 1}\rangle \to \mathcal{O}_K\langle X^{\pm 1}\rangle, \quad X \to X^p.$$

In the direct limit, we obtain $(\varinjlim_{[p]} \mathcal{O}_K\langle X^{\pm 1}\rangle)^\wedge = \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle$. Therefore, taking the generic fiber we get

$$G_\infty = \mathrm{Spa}(K\langle X^{\pm 1/p^\infty}\rangle, \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle)$$

and one can verify directly that we indeed have $G_\infty \sim \varprojlim_{[p]} G$.

Example. If G is not p-divisible, the tilde-limit of $\varprojlim_{[p]} G$ might exist for trivial reasons: For example, consider the p-adic completion $\mathfrak{G}_a$ of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Then one can check using formal models that the trivial group $\mathrm{Spa}(K, \mathcal{O}_K) \sim \varprojlim_{[p]} \mathfrak{G}_a$ is a perfectoid tilde-limit.

3. Formal models for tori

In this section we want to show that for a split rigid torus T over K, a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$ exists. This is easy to see directly, but in order to simplify the argument in the next chapter, we shall formulate the proof in the language of $[p]$-F-models.

3.1. **A family of explicit covers.** As a preparation, we briefly recall how $\mathbb{G}_m^{\mathrm{an}}$ is constructed: The following is taken from [2], §9.2 with slightly different notation. Throughout we use the following standard shorthand notation: For any $0 \neq a, b \in K$ with $|a| \leq |b|$, we write

$$K\langle a/X, X/b\rangle := K\langle Z_1, Z_2\rangle/(Z_1 Z_2 - \tfrac{a}{b})$$

and set $X := Z_2 b$. The associated affinoid space $\mathcal{B}(a, b) := \mathrm{Spa}(K\langle a/X, X/b\rangle)$ is the annulus of inner radius $|a|$ and outer radius $|b|$ inside $\mathbb{G}_m^{\mathrm{an}}$. We have natural open immersions $\mathcal{B}(a, a) \hookrightarrow B(a, b)$ and $\mathcal{B}(b, b) \hookrightarrow B(a, b)$, which we regard as the inner and outer "boundary", respectively. Explicitly,

$$\mathcal{B}(a, a) = \mathcal{B}(a, b)(|\tfrac{a}{X}| \geq 1), \quad \mathcal{B}(b, b) = \mathcal{B}(a, b)(|\tfrac{X}{b}| \geq 1).$$

Let us fix from now a choice of an element $q \in K^\times$ with $|q| < 1$ such that there is a compatible system of p^n-th roots $q^{1/p^n} \in K$. We can always find such a q since $K^\times$ is perfectoid.

We can then construct $\mathbb{G}_m^{\mathrm{an}}$ by glueing the annuli $\{\mathcal{B}(q^i, q^{i-1})\}_{i\in\mathbb{Z}}$ along the inner and outer boundary $\mathcal{B}(q^i, q^i)$ of $\mathcal{B}(q^{i+1}, q^i)$ and $\mathcal{B}(q^i, q^{i-1})$, respectively.

It is clear from this description that the $\mathcal{B}(a, b)$ as well as the glueing maps admit natural formal models, given by the topologically finite type $\mathcal{O}_K$-algebras

$$\mathfrak{B}(a, b) := \mathrm{Spf}(\mathcal{O}_K\langle \tfrac{a}{X}, \tfrac{X}{b}\rangle) := \mathrm{Spf}(\mathcal{O}_K\langle Z_1, Z_2\rangle/(Z_1Z_2 - \tfrac{a}{b})).$$

This is because the boundary $\mathfrak{B}(a, a) \subseteq \mathfrak{B}(a, b)$ is still the open subspace where $\frac{a}{X}$ becomes invertible, and similarly for $\mathfrak{B}(b, b) \subseteq \mathfrak{B}(a, b)$. We can use this to construct a formal model of $\mathbb{G}_m^{\mathrm{an}}$:

Lemma 3.1. *Let $q \in K^\times$ be as before.*

(1) The affine formal schemes $\{\mathfrak{B}(q^i, q^{i-1})\}_{i\in\mathbb{Z}}$ glue together to a formal model $\mathfrak{G}_q$ of $\mathbb{G}_m^{\mathrm{an}}$.
(2) There is a formal model $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ of $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$.
(3) The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ reduces mod π to the relative Frobenius map.

Proof. Part 1 follows from the above discussion. To construct $[\mathfrak{p}]$, consider the affinoid open subsets $\mathcal{B}(q^{i/p}, q^{(i-1/p)})$ of the source and $\mathcal{B}(q^i, q^{i-1})$ of the target. Then $[p]$ restricts to a map

$$\begin{gathered}\mathcal{B}(q^{i/p}, q^{(i-1)/p}) \xrightarrow{[p]} \mathcal{B}(q^i, q^{i-1}) \\ (q^{i/p}/X)^p, (X/q^{(i-1)/p})^p \leftarrow\!\shortmid\ q^i/X, X/q^{i-1}.\end{gathered} \tag{1}$$

It is clear from these formulas that this morphism has a natural formal model $[\mathfrak{p}] : \mathfrak{B}(q^i, q^{i-1}) \to \mathfrak{B}(q^{i/p}, q^{(i-1)/p})$. It is also evident that $[\mathfrak{p}]$ reduces to the relative Frobenius mod p. These formal models are compatible with the glueing maps because this is true generically. $\square$

Proposition 3.2. *The space $\mathbb{G}_m^{\mathrm{an}}$ has a $[p]$-F-formal tower. In particular, there exists a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ such that $(\mathbb{G}_m^{\mathrm{an}})_\infty \sim \varprojlim_{[p]} \mathbb{G}_m^{\mathrm{an}}$.*

Proof. This follows from Proposition 2.16 as by the Lemma, we have a $[p]$-F-formal tower

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_q. \qquad \square$$

3.2. **The action of $\overline{T}$.** While we do not have a formal model of the multiplication $\mathbb{G}_m^{\mathrm{an}} \times \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ in terms of $\mathfrak{G}_q$, we do have a formal model for the restriction of this multiplication to the action of the open annulus $(\hat{\mathbb{G}}_m)_\eta = \mathcal{B}(1, 1)$ on $\mathbb{G}_m^{\mathrm{an}}$.

Lemma 3.3. *The action of $(\hat{\mathbb{G}}_m)_\eta = \mathfrak{B}(1, 1)$ on $\mathbb{G}_m^{\mathrm{an}}$ has a formal model given by an action*

$$\mathfrak{m} : \hat{\mathbb{G}}_m \times \mathfrak{T}_q \to \mathfrak{T}_q.$$

The map $[\mathfrak{p}] : \mathfrak{T}_q^{1/p} \to \mathfrak{T}_q$ is semi-linear with respect to $[p] : \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m$ for these actions by $\hat{\mathbb{G}}_m$.

Proof. We can write $\mathfrak{B}(1, 1) = \mathrm{Spf}(\mathcal{O}_K\langle Z^{\pm 1}\rangle)$. The map $\mathfrak{m}$ can then be glued from the maps

$$\mathfrak{B}(1, 1) \times \mathfrak{B}(q^i, q^{i-1}) \to \mathfrak{B}(q^i, q^{i-1}), \quad Z \cdot \tfrac{a}{X}, Z \cdot \tfrac{X}{b} \leftarrow\!\shortmid \tfrac{a}{X}, \tfrac{X}{b}$$

The statement that $\mathfrak{m}$ is an action, as well as the semilinearity, can both be expressed in terms of diagrams, which commutes because they do generically. $\square$

3.3. **The case of general tori.** By taking products everywhere, all of the statements in this section immediately generalise to split tori over K:

Corollary 3.4. *Let $T \cong (\mathbb{G}_m^{\mathrm{an}})^d$ be a split torus over K.*

1. *The formal scheme $\mathfrak{T}_q := (\mathfrak{G}_q)^d$ is a formal model of T and the $[p]$-multiplication map has an affine formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ which is locally of the form $[\mathfrak{p}] : \mathfrak{B}(q^{i/p}, q^{(i-1)/p})^d \to \mathfrak{B}(q^i, q^{i-1})^d$. In particular, it reduces mod π to the relative Frobenius morphism.*
2. *The rigid group T has a $[p]$-F-formal tower. In particular, there exists a perfectoid space T_∞ such that $T_\infty \sim \varprojlim_{[p]} T$.*
3. *The action of the formal torus $\overline{T}$ on T has a formal model given by an action*
$$\mathfrak{m} : \overline{T} \times \mathfrak{T}_q \to \mathfrak{T}_q.$$
The map $[\mathfrak{p}] : \mathfrak{T}_q^{1/p} \to \mathfrak{T}_q$ is semi-linear with respect to $[p] : \overline{T} \to \overline{T}$ for the actions by $\overline{T}$.

4. A $[p]$-F-formal tower for Raynaud extensions

In this section we study the p-multiplication tower of the Raynaud extensions associated to abeloid varieties over an algebraically closed perfectoid field K. The main result of this section is Theorem 4.9, which says that the Raynaud extension E of an abeloid variety A over K admits a $[p]$-F-formal tower, and thus there exists a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

Remark 4.1. Everything in this section also works with minor modifications over a general perfectoid field. But we opt to work over an algebraically closed field to simplify the exposition.

4.1. **Raynaud extensions.** We briefly sketch the theory of Raynaud extensions here, and refer the readers to [7] for more details on the setup.

Let A be an abelian variety over K. There exists a unique connected open rigid analytic subgroup $\overline{A}$ of A which extends to a formal smooth $\mathcal{O}_K$-group scheme $\overline{E}$ with semi-abelian reduction. This $\overline{E}$ fits into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0 \tag{2}$$

where $\overline{B}$ is the completion of an abelian variety B over K of good reduction (as usual we also denote by B the rigid space associated to it), and $\overline{T}$ is the completion of a torus T of rank r over K. The rigid generic fibre $\overline{T}_\eta$ of the torus $\overline{T}$ canonically embeds into the rigid torus T^{an} which again we simply denote by T. This induces a pushout exact sequence in the category of rigid groups: More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout.

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{3}$$

The abelian variety A we started with can then be uniformized in terms of E as follows:

Definition 4.2. A subset M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}}^r$.

Theorem 4.3. *There exists a lattice $M \subseteq E$ of rank equal to the rank r of the torus for which the quotient E/M exists as a rigid space and has a group structure such that $E \to E/M$ is a rigid group homomorphism. Moreover, there is a natural isomorphism*

$$A = E/M.$$

The data of the extension (2) together with the lattice $M \subseteq E$ is what we refer to as a Raynaud uniformisation of A. This will in fact be the only input we need to construct the perfectoid tilde-limit A_∞. Consequently, our method generalises to the class of rigid groups which admit Raynaud uniformisation, namely to abeloid varieties:

Theorem 4.4 (Lütkebohmert, [7], Theorem 7.6.4)**.** *Let A be an abeloid variety, that is a connected smooth proper commutative rigid group over K. Then A admits a Raynaud uniformisation.*

In the situation of Raynaud uniformisation, since M is discrete, the local geometry of A is determined by the local geometry of E. We will therefore first study the $[p]$-multiplication tower of E and will then deduce properties of the $[p]$-multiplication tower of A.

Our strategy is to study the local geometry of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that one cannot direcly study short exact sequences locally on T, E or B. Instead, we have the following crucial Lemma, which says that one may regard Raynaud extensions as T-torsors of formal schemes.

Lemma 4.5. *The short exact sequence (2) admits local sections, that is there is a cover of $\overline{B}$ by formal open subschemes U_i such that there exist local sections $s : U_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times U_i \hookrightarrow E$.*

Proof. This is proved in Proposition A.2.5 in [7], where it is fomulated in terms of the group $\mathrm{Ext}(B, T)$. Also see [3], §1. □

Remark 4.6. In the following, we will freely work with fibre bundles of formal schemes and rigid and perfectoid spaces. For some background material on these we refer to the Appendix.

The sequence (2) from the last section gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}_\eta$ via push-out along $\overline{T}_\eta \to T$ can now conveniently be expressed in terms of the associated fibre bundle by saying that $E = T \times^{\overline{T}_\eta} \overline{E}_\eta$ in the sense of Definition A.7.

4.2. **A $[p]$-F-formal tower for E.** In this subsection we prove that E admits a $[p]$-F-formal tower. The first step is to construct a family of formal models for E. To do so, we fix now an element $q \in K^\times$ with $|q| < 1$ as well as a compatible system of p^n-th roots q^{1/p^n}.

Lemma 4.7. *Let $\mathfrak{T}_q$ be the formal model from Corollary 3.4. Then the formal scheme $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E}$ is a formal model of E. Furthermore, there exists a morphism*

$$\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$$

which is a fibre bundle and a formal model of $E \to B$.

Proof. Recall from Corollary 3.4 that $\mathfrak{T}_q$ has a $\overline{T}$-action that is a model of the $\overline{T}_\eta$-action on T. In particular, the associated fibre construction for the principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$ gives a fibre bundle $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$. Since $\mathfrak{T}_q$ is a formal model of T, one sees by Lemma A.12 that $\mathfrak{E}_q$ is a formal model of $T \times^{\overline{T}_\eta} \overline{E}_\eta$ which by definition is equal to E. Finally, since $\mathfrak{T}_q$ and $\overline{B}$ are flat, so is $\mathfrak{E}_q$. □

Lemma 4.8. *There is an affine morphism*

$$[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$$

which is a formal model of $[p] : E \to E$*.*

Proof. Recall that the multiplication map $[p] : T \to T$ has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ by Corollary 3.4. Functoriality of the associated fibre bundle construction, Proposition A.9, applied to the below diagram then gives a natural map $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}$ making the diagram commute:

$$\begin{tikzcd}
 & \mathfrak{T}_q \arrow[rr] & & \mathfrak{E}_q \\
\mathfrak{T}_{q^{1/p}} \arrow[ur, "{[\mathfrak{p}]}"] \arrow[rr] & & \mathfrak{T}_{q^{1/p}} \times^{\overline{T}} \overline{E} \arrow[ur, dotted, "\exists"] & \\
 & \overline{T} \arrow[uu] \arrow[rr] & & \overline{E} \arrow[uu] \\
\overline{T} \arrow[uu] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E} \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd} \tag{4}$$

The generic fibre of $[\mathfrak{p}]$ is $[p] : E \to E$. Indeed, Lemma A.10 and Remark A.11 say that $[p] \times^{[p]} [p]$ is the unique morphism of fibre bundles $E \to E$ making the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 1 \\
1 \arrow[r] & T \arrow[r] \arrow[u, "{[p]}"] & E \arrow[r] \arrow[u, dotted] & B \arrow[r] \arrow[u, "{[p]}"] & 1
\end{tikzcd} \tag{5}$$

But $[p] : E \to E$ is such a map, and thus coincides with the generic fibre of $[\mathfrak{p}]$.

It remains to see that $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ is affine. We first note that $[p] : \overline{B} \to \overline{B}$ is an affine morphism. The map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ is affine by construction, namely by Corollary 3.4 it is locally on $\mathfrak{T}_q$ of the form $[\mathfrak{p}] : \prod_{j=1}^{d} \mathfrak{B}(q^{i_j/p}, q^{(i_j-1)/p}) \to \prod_{j=1}^{d} \mathfrak{B}(q^{i_j}, q^{i_j-1})$. Note that both of these affine open subsets are fixed by the action of $\overline{T}$. We conclude from the construction in the proof of Proposition A.9 that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ locally on the target is of the form

$$[\mathfrak{p}] : \prod_{j=1}^{d} \mathfrak{B}(q^{i_j/p}, q^{(i_j-1)/p}) \times U' \to \prod_{j=1}^{d} \mathfrak{B}(q^{i_j}, q^{i_j-1}) \times U$$

for an affine open formal subscheme $U \subseteq \overline{B}$ with affine preimage U' under $[p] : \overline{B} \to \overline{B}$. $\square$

We are now ready to prove the main result of this section, namely that E_∞ is perfectoid:

Theorem 4.9. *The adic group E admits a $[p]$-F-formal tower of the form*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q.$$

In particular, there is a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$*.*

Proof. The described $[p]$-tower exists by iterating Lemma 4.8. We must show that the reduction mod π of the map $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through relative Frobenius.

Let us denote reduction mod π of a formal scheme by a $\sim$ over the respective symbol, for example the reductions of $\overline{T}$, $\overline{E}$ and $\mathfrak{T}$ are denoted by $\tilde{T}$, $\tilde{E}$ and $\tilde{\mathfrak{T}}$.

Recall that $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ was constructed using the cube in diagram (4) and functoriality of the associated bundle. Since formation of the associated bundle commutes with reduction mod π,

$$\tilde{\mathfrak{E}}_q = \tilde{\mathfrak{T}}_q \times^{\tilde{T}} \tilde{E}.$$

By Corollary 3.4, the model of the multiplication map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ reduces to relative Frobenius over p. In particular, we have a natural isomorphism $\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \cong \tilde{\mathfrak{T}}_q$ and we can identify $\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} = \tilde{\mathfrak{T}}_q$ in the following. The same is true for $\tilde{T}^{(p)} = \tilde{T}$. Proposition A.9 gives us a natural morphism

$$F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}} : \tilde{\mathfrak{T}}_{q^{1/p}} \times^{\tilde{T}} \tilde{E} \to \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \times^{\tilde{T}^{(p)}} \tilde{E}^{(p)}$$

and by functoriality of the relative Frobenius it is clear from uniqueness in the Proposition that is just the relative Frobenius of $\tilde{\mathfrak{E}}_{q^{1/p}}$.

Since $\tilde{E}$ and $\tilde{T}$ are group schemes, the reduction of $[p]$ on them factors through the relative Frobenius maps F_E and F_T respectively. Again by Proposition A.9, the reduction of the formal model of the p-multiplication cube in diagram (4) admits the following factorisation:

$$\begin{tikzcd}
 & & & \tilde{\mathfrak{T}}_q \arrow[rr] & & \tilde{\mathfrak{E}}_q \\
 & \tilde{\mathfrak{T}}^{(p)}_q \arrow[urr, equal] \arrow[rrr] & & & \tilde{\mathfrak{E}}^{(p)}_{q^{1/p}} \arrow[ur, dotted] & \\
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[ur, "F"'] \arrow[rr] & & \tilde{\mathfrak{E}}_{q^{1/p}} \arrow[urr, "F"'] & & & \\
 & & & \tilde{T} \arrow[uuu] \arrow[rr] & & \tilde{E} \arrow[uuu] \\
 & \tilde{T}^{(p)} \arrow[uuu] \arrow[urr, equal] \arrow[rrr] & & & \tilde{E}^{(p)} \arrow[uuu] \arrow[ur] & \\
\tilde{T} \arrow[uuu] \arrow[ur] \arrow[rr, "F"] & & \tilde{E} \arrow[uuu] \arrow[urr, "F"'] & & &
\end{tikzcd}$$

Since the outer map is the formal model $[\mathfrak{p}]$ by construction in diagram (4.8), this shows that the reduction of $[\mathfrak{p}]$ mod p factors through the relative Frobenius, as desired.

The conclusion that E_∞ is perfectoid then follows from Proposition 2.16. $\square$

Remark 4.10. With some work, the arguments in this section can be extended to any perfectoid base field. For instance, the Raynaud uniformisation of Theorem 4.3 might only be defined over a finite extension L of K. Our argument then gives a perfectoid space over the (necessarily perfectoid) field L. We can then use Galois descent to get a perfectoid space over our original field K. This uses that the quotient of a perfectoid space by a finite group often remains perfectoid: see Theorem 1.4 of [5] for details. Finally, one checks that this Galois descent commutes with tilde-limits.

5. The case of abeloid varieties

We now prove the main result, building on the preceding chapters. Let us recall some notation:

(1) g is the dimension of A.
(2) E is the Raynaud extension associated to A from Proposition 4.4, which is an extension of a split rigid torus T of rank r by an abelian variety B of good reduction.
(3) $M \subseteq E$ is a lattice of rank r such that $A = E/M$.

Recall from [] that the quotient map $\pi : E \to A$ is locally split in the analytic topology on A: As the action of M on E is totally discontinuous, for every point $x \in A$ there is an open neighbourhood U of E that maps U isomorphically onto x. The pullback of $\pi(U)$ along $[p] : A \to A$, however, will in general be bigger than the pullback of U along $[p] : E \to E$: The first is an étale $A[p]$-torsor, whereas the latter is an étale $E[p]$-torsor, and by the Snake Lemma we have a short exact sequence

$$1 \to E[p] \to A[p] \to M/M^p \to 1$$

In order to make it easier to relate these pullbacks, we therefore now subdivide the tower $\cdots A \xrightarrow{[p]} A$ into to partial towers. For this we make some auxiliary choices: Since K is algebraically closed, we can choose lattices $M_n \subseteq E$ such that $M_0 = M$ and $[p] : E \to E$ restricts to isomorphisms $M_{n+1} \to M_n$ for all n.

Remark 5.1. Such a choice is equivalent to the choice of subgroups $D_n \subseteq A[p^n]$ of rank p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$. Namely, given the lattices M_{n+1}, we obtain the desired torsion subgroups by setting $D_n := M_{n+1}/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/M^{p^n} \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ with properties as above, we recover M_n as the kernel of $E \to A \to A/D_n$.

One might call the choice of D_n for all n a partial anticanonical $\Gamma_0(p^\infty)$-structure, because if B admits a canonical subgroup (that is, if B has bad reduction or if it has good reduction and the reduction satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A always has a partial anticanonical subgroup even if B does not have a canonical subgroup.

Following the remark, denote by D_n the torsion subgroup $M_n/M \subseteq A$. The quotient $A_n := A/D_n = E/M_n$ is then another abeloid variety over K and the quotient map $v^n : A = E/M \to A_n = E/M_n$ is an isogeny of degree p^{rn} through which $[p^n] : A \to A$ factors: The $[p]$-multiplication tower now splits into two towers, one written vertically, the other horizontally:

(6)
$$\begin{tikzcd}
\ddots \arrow[dr] & & \vdots & \vdots \\
 & A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[r, "v"] \arrow[d, "{[p]}"] & A_2 \arrow[d, "{[p]}"] \\
 & & A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[d, "{[p]}"] \\
. & & & A
\end{tikzcd}$$

Since each $D_n = M_n/M$ is finite étale, all horizontal maps are finite étale. The vertical tower on the other hand fits into a commutative diagram which compares it to the $[p]$-tower of E:

(7)
$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & A_1 \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Definition 5.2. Let $M_\infty := \varprojlim M_n$. Since the transition maps are all isomorphisms, the projections $M_\infty \to M$ are isomorphisms as well. By the diagram, we have a natural map $M_\infty \to E_\infty$.

Proposition 5.3. *There is a perfectoid tilde-limit $A'_\infty \sim \varprojlim A_n$. It fits into a short exact sequence of perfectoid groups*
$$0 \to M_\infty \to E_\infty \to A'_\infty \to 0$$
that is locally split in the analytic topology on $A'_\infty \to A$.

Proof. Since the rows in (7) are exact, and the transition maps on the left are isomorphisms, it follows that for each $n \in \mathbb{N}$, the pullback $[p^n]^{-1}(U)$ maps isomorphically onto its image on A_n. Write $U_n := [p^n]^{-1}(U) \subseteq E$, and write $\pi_n : E \to A_n$ for the quotient map. Then (7) is locally of the form

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & M_1 \times U_1 \arrow[r] \arrow[d, "{[p]}"] & \pi_1(U_1) \arrow[r] \arrow[d, "{[p]}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \times U \arrow[r] & U \arrow[r] & 0 \end{tikzcd} \tag{8}$$

Let now U_∞ be the pullback of U along $E_\infty \to E$, then we have $U_\infty \sim \varprojlim U_n \cong \varprojlim \pi_n(U_n)$. In particular, the system $(\pi_n(U_n))_{n\in\mathbb{N}}$ has a perfectoid tilde-limit. This shows that $A'_\infty = \varprojlim A_n$ has a perfectoid tilde-limit. Finally, we obtain the desired short exact sequence in the limit over diagram (7) since the transition maps in (8) respect the splitting. ☐

In order to construct a tilde-limit for $\varprojlim A$, we can now use that the horizontal maps in diagram (6) are all finite étale. They are even finite covering maps, in the following sense:

Lemma 5.4. *For any $0 \leq n$, the preimage of U under the horizontal map $v^n : A \to A_n$ is isomorphic to p^{rn} disjoint copies of U_n. More canonically, it can be described as the isomorphic image of $D_n \times U_n$ under the multiplication map $A \times E \to A$, where $D_n = M_n/M$ (see Remark 5.1).*

Proof. For the first part, we observe that v^{n-m} fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r] & A_n \arrow[r] & 0 \end{tikzcd} \tag{9}$$

where the map on the left is the natural inclusion. Upon restriction to U_n, this diagram becomes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & M_n \times U_n \arrow[r] \arrow[d, equal] & (v^n)^{-1}(U_n) \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & M_n \times U_n \arrow[r] & U_n \arrow[r] & 0 \end{tikzcd} \tag{10}$$

and the claim follows from the fact that M is a discrete lattice of rank r. ☐

Definition 5.5. The $[p]$-multiplication on E maps M_{n+1} onto M_n and therefore the $[p]$-multiplication tower of A induces a tower
$$\dots \xrightarrow{[p]} D_{n+1} \xrightarrow{[p]} D_n \to \dots.$$
Since K is algebraically closed, the finite étale groups D_n are already constant. By Lemma 2.7, there is therefore a perfectoid group D_∞ such that
$$D_\infty \sim \varprojlim_n D_n.$$

Theorem 5.6. *(1) There is a perfectoid space A_∞ which is a tilde-limit of $\varprojlim_{[p]} A$.*

(2) *It is independent up to canonical isomorphism of the auxiliary choice of lattices M_n with $D_n = M_n/M$, but it remembers the choice as a pro-finite étale closed subgroup $D_\infty \subseteq A_\infty$.*
(3) *The preimage of $U \subseteq A$ under the projection $A_\infty \to A$ is isomorphic to $D_\infty \times U_\infty$.*
(4) *We have a commutative diagram of short exact sequences of perfectoid groups*

$$\begin{tikzcd} 0 \arrow[r] & M_\infty \arrow[r] \arrow[d, hook] & E_\infty \arrow[r] \arrow[d, hook] & A'_\infty \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D_\infty \arrow[r] & A_\infty \arrow[r] & A'_\infty \arrow[r] & 0 \end{tikzcd}$$

both of which are locally split in the analytic topology.
(5) *One can describe A_∞ as the associated fibre bundle*

$$A_\infty = D_\infty \times^{M_\infty} E_\infty.$$

In particular, we have an analytic-locally split short exact sequence of perfectoid groups

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the antidiagonal embedding of M_∞ into $D_\infty \times E_\infty$.

Remark 5.7. We think of part 5 as the analogue of the Raynaud uniformisation

$$0 \to M \to E \to A \to 0.$$

Here we note that while the map $E \to A$ is a quotient, in the limit over $[p]$ it becomes an immersion $E_\infty \hookrightarrow A_\infty$: The reason is that there projective system $(M, [p])$ has vanishing lim but non-vanishing Rlim^1, for instance when considered as abelian sheaves on perfectoid spaces for the pro-étale topology in the sense of [14]. A toy example of this phenomenon would be the inverse system over $[p]$ on the short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ whose limit yields an exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \varprojlim^1_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

We therefore think the quotient D_∞/M_∞ implicit in part 5 as being an incarnation of $\mathrm{Rlim}^1_{[p]} M_\infty$.

Proof of Theorem 5.6. Recall that by construction of A'_∞ we have a cover of A by open subsets U that pull back to perfectoid open subspaces U_∞ for which $U_\infty \sim \varprojlim U_n$. When we go along the diagonal tower, we obtain the inverse system

$$\cdots \to D_{n+1} \times U_{n+1} \to D_n \times U_n \to \dots.$$

By Lemma 2.11 this inverse system has perfectoid tilde-limit $D_\infty \times U_\infty$. These local tilde-limits glue together to give the desired tilde-limit A_∞. This proves 1, 2 and 3, and shows that the second row of 4 is locally split and in particular exact.

The first row in 4 is part of Proposition 5.3. Part of 5 then follows immediately from 4. □

Remark 5.8. When working over a general perfectoid base field, the lattices M_n may no longer be defined over K. Instead, one can show that the natural map $A[p^n] \times U_n \to V_n$ is an étale $E[p^n]$-torsor for the diagonal action where V_n is the pullback of U along $[p^n] : A \to A$. The point is that this torsor is split when C is algebraically closed.

6. Applications

In this section, we give three applications of our main result. For all of these, we assume that K is of characteristic 0, i.e. K is an algebraically closed non-archimedean field extension of $\mathbb{Q}_p$.

6.1. **Uniformisation.** As a first application of our main result, we note the following "p-adic uniformisation" of abelian varieties: Recall that any abelian variety A over $\mathbb{C}$ of dimension d has a uniformisation in terms of a complex torus $A \cong \mathbb{C}^g/\Lambda$ for some $2d$-dimensional lattice $\Lambda \subseteq \mathbb{C}^g$.

We have the following analogue of this over K: Let A be an abeloid variety over K of dimension d, considered as a rigid space. Then in the limit over n, the short exact sequences

$$0 \to A[p^n] \to A \to A \to 0$$

give rise to a short exact sequence of sheaves on perfectoid K-algebras with the pro-étale topology

$$0 \to T_pA \to A_\infty \to A \to 0.$$

Using the language of diamonds from [14], we then have:

Corollary 6.1. *The diamond $A^\diamond$ associated to A has a canonical presentation*

$$A^\diamond = A_\infty/T_pA$$

given by the perfectoid space A_∞ and the pro-étale subgroup T_pA.

Of course this p-adic uniformisation is very closely related to the uniformisation of the associated p-divisible groups described in [17] and [11, §4]: Indeed, in the language used there, we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_p(A[p^\infty]) \arrow[r] \arrow[d, equal] & \widetilde{A[p^\infty]} \arrow[r] \arrow[d, hook] & A[p^\infty] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_pA \arrow[r] & A_\infty \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

We note that for two abelian varieties A and B of dimension g, the universal covers A_∞ and B_∞ are different in general, so that this is only a "uniformisation" in a rather weak sense. However, they are canonically isomorphic if A and B are abelian varieties of good reduction with the same special fibre, or if A and B are p-power isogeneous, so that in these cases we can really think of T_pA as a a $2g$-dimension $\mathbb{Z}_p$-lattice determining A.

6.2. **Stein property.** As a second application, we can combine our main theorem with work of Reineke to deduce the following Artin vanishing result:

Corollary 6.2. *Let A be an abeloid variety over K of dimension d. Let L be a constructible sheaf of $\mathbb{F}_p$-modules on $A_{\text{ét}}$. Then for any $i > d$,*

$$\varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^*L) = 0.$$

Proof. Due to Theorem 1, we can apply [9, Theorem 3.3], to the inverse system $\cdots \to A \xrightarrow{[p]} A$. $\square$

The point of this result is that while for an affine algebraic variety X over K of dimension d, it is a Theorem of Artin and Grothendieck that we have $H^i_{\text{ét}}(X, L) = 0$ for any constructible $\mathbb{F}_p$-module L and any $i > d$, the rigid analogue of this statement is false in general. The basic idea is now that an analogue of this statement is true for the pullback of L to A_∞, in the following sense: Consider the morphism of sites $\nu : A_{\text{proét}} \to A_{\text{ét}}$. Then by regarding A_∞ as an object in $A_{\text{proét}}$ via the pro-étale morphism $A_\infty \to A$, one can show that

$$H^i_{\text{proét}}(A_\infty, \nu^*L) = \varinjlim H^i_{\text{ét}}(A, [p^n]^*L).$$

By results from [12], the space A_∞ has a "Stein space"-like property in the sense that $H^j(V, \nu^*L) = 0$ for any affinoid perfectoid $V \subseteq A_\infty$ and any $j > 0$. One can use this to reduce to a computation in Čech cohomology, which shows that the left hand side vanishes for $i > d$.

6.3. **covers of curves.** As a third application, we describe how one can obtain functorial perfectoid pro-étale covers of curves over K. This was observed by Hansen [6]:

Let C be a smooth projective curve of genus $g \geq 1$ over K, which we consider as an analytic adic space. We fix a geometric point $x \in C$ and consider the étale fundamental group $\pi_1(C, x)$. This is a profinite group, and for every open subgroup $H \subseteq \pi_1(C, x)$, there is a corresponding finite étale morphism $C_H \to C$. For any two open subgroups $H_1 \subseteq H_2 \subseteq \pi_1(C, x)$, there is a natural morphism $C_{H_1} \to C_{H_2}$. For varying H, one therefore has a filtered inverse system $(C_H)_{H \subseteq \pi_1(C,x)}$ which we may regard as an object in $C_{\text{proét}}$.

Corollary 6.3. *There is a perfectoid tilde-limit $C_\infty \sim \varprojlim_H C_H$ where H ranges over the open subgroups of $\pi_1(C, x)$.*

Appendix A. Fibre bundles of formal and rigid spaces

In this chapter we review the theory of fibre bundles in the setting of formal and rigid geometry.

Notation A.1. In the following, if $\pi : E \to B$ is a morphism of formal schemes, then for a formal open subscheme $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.2. Let T be a formal group scheme. Let F be a formal scheme with an action $m : T \times F \to F$. A morphism $\pi : E \to B$ of formal schemes is called a **fibre bundle with fibre F and structure group T** if there is a cover $\mathfrak{U}$ of B of open formal subschemes $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl, "p_2"] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i : F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that

$$\phi_{ij} = F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}$$

Definition A.3. When $F = T$ with the action on itself by left multiplication, then a fibre bundle $\pi : E \to B$ with fibre T and structure group T is called a T-torsor.

Example. The short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0$ from §4 yields a T-torsor $\overline{E} \xrightarrow{\pi} \overline{B}$ by Lemma 4.5. Moreover, for any formal open subscheme $U \subseteq \overline{B}$, the map $E|_U \to U$ is still a T-torsor.

The ϕ_{ij} from condition (b) are determined by the maps $\psi_{ij} : U_{ij} \to T$. By glueing, one sees:

Lemma A.4. *Suppose we are given formal schemes F and B and a formal group scheme T with an action on F. Then fibre bundles $\pi : E \to B$ with fibre F and structure group T are equivalent to the data (up to refinement) of a cover $\mathfrak{U}$ of B by formal open subschemes and morphisms $\psi_{ij} : U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk} \cdot \psi_{ij} = \psi_{ik}$.*

Lemma A.5. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. Then the natural T-action on $F \times U_i$ for each i via the first factor glue to a natural T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.6. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T. Then a **morphism of fibre bundles** $f : (E', B', \pi') \to (E, B, \pi)$ is a commutative diagram of formal schemes

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & B' \\ \downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f_B} \\ E & \xrightarrow{\pi} & B \end{array}$$

in which the morphism f_E is also T-linear. We often abbreviate this by writing $f : E' \to E$.

Definition A.7. Let $\pi : E \to B$ be a T-torsor. Let F be a formal scheme with an action by T. Since the data in Lemma A.4 are completely independent of the fibre, the morphisms $\psi_{ij} : U_{ij} \to T$ by Lemma A.4 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel-Weil construction.

A.1. **The semi-linear case.**

Definition A.8. Let $g : T' \to T$ be a homomorphism of formal group schemes. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T' . Then a g-linear morphism of torsors is a diagram

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & B' \\ \downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f_B} \\ E & \xrightarrow{\pi} & B \end{array}$$

such that f_E is g-linear.

For a fixed $f_B : B' \to B$, one can equivalently characterise a morphism of torsors $f_E : E' \to E$ over f_B by the data of maps $f_B^{-1}(U_i) \to T'$ on some cover of B. Using this description, one sees:

Proposition A.9. *Given any homomorphism of group schemes $g : T' \to T$ and a g-linear homomorphism $h : F' \to F$ of formal schemes with T' and T-actions respectively, and a homomorphism $f : E' \to E$ of principal T' and T-bundles over g, one obtains a morphism*

$$h \times^g f : F' \times^{T'} E' \to F \times^T E$$

of fibre bundles over g. This makes $- \times^- -$ into a fibered bifunctor from the category of pairs (F, T) of formal schemes F with an action by T, fibered over the category of formal group schemes T with the category of T-torsors E, to the category of formal fibre bundles.

The associated bundle construction has the following universal property:

Lemma A.10. *In the context of Proposition A.9, assume moreover that F', F are formal group schemes and that the respective actions come from group homomorphisms $T' \to F'$ and $T \to F$. Then $h \times^g f$ is the unique h-linear morphism of fibre bundles making the following diagram commute:*

$$\begin{tikzcd} F' \times^{T'} E' \arrow[r, "h\times^f g"] & F \times^T E \\ E' \arrow[u] \arrow[r, "f"] & E. \arrow[u] \end{tikzcd}$$

Proof. The vertical maps in the diagram exist by functoriality via $E = T \times^T E \to H \times^T E$. On any compatible trivialisation $T' \times U' \to T \times U$ of $f : E' \to E$ there is then clearly only one way to extend this to $F' \times U' \to F \times U$ in a h-linear way. □

Remark A.11. All that we have done in this chapter can be done in completely the same way with formal schemes replaced by rigid or adic spaces. The different categories of fibre bundles are well-behaved with respect to the usual functors between these categories: For instance, by functoriality of fibre products there are natural rigidification and reduction functors from formal principal T-bundles over $\mathcal{O}_K$ to rigid principal T_η-bundles over K on the generic fibre. Moreover, these generic fibre and reduction functors commute with the associated fibre construction:

Lemma A.12. *Let T be a formal group scheme and let $\pi : E \to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Then $(F \times^T E)_\eta = F_\eta \times^{T_\eta} E_\eta$.*

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