PERFECTOID LIMITS OF RIGID GROUPS VIA FORMAL MODELS

CLIFFORD BLAKESTAD, BEN HEUER, DAMIAN GVIRTZ, KOJI SHIMIZU, PETER WEAR, DARIA SCHEDRINA, AND ZIJIAN YAO

Abstract. For the Raynaud extension E of a semistable abelian variety A over a perfectoid field K, we show that there is a perfectoid space E_∞ such that $E \sim \varprojlim_{[p]} E$.

We first more generally consider a rigid group G over a non-archimedean field K. While limits don't exist in the rigid analytic category, limits are much better behaved in formal schemes over the ring of integers R of K. One can therefore give a simple criterion in terms of formal models that guarantees that a tilde-limit $G_\infty \sim \varprojlim_{[p]} G$ exists, namely that there is a well-behaved formal model of the $[p]$-multiplication tower. If K is perfectoid, we give a stronger criterion involving a Frobenius factorisation condition, which implies that G_∞ is perfectoid.

In the case of a rigid analytic split torus T, one can use a family of explicit covers by affinoids to construct formal models for which both of these conditions are satisfied.

For a Raynaud extension E one can use this as follows: One can construct E by extending the rigid fibre of a formal group scheme $\overline{E}$ by a rigid torus T in a certain way. In order to construct a formal model of E one just needs to extend $\overline{E}$ by a formal model of T. While this can be done explicitly using affinoid covers, the language of formal and rigid fibre bundles allows for a more conceptual treatment. Using the associated fibre construction we then show that there is a formal model of the $[p]$-multiplication tower of E which satisfies all the necessary criteria to show that E_∞ exists and is perfectoid.

Contents

1. Tilde limits of rigid groups

Let p be a prime. Let K be a complete non-archimedean field that is either an extension of $\mathbb{Q}_p$ or of characteristic p. Denote by R the ring of integers and let π denote a pseudo-uniformiser. Throughout we will study rigid analytic spaces over K. If such a space is obtained from a K-scheme X via rigid-analytification $X \mapsto X^{\mathrm{an}}$, we will often denote both by the same symbol X. Also, we will make no distinction between rigid analytic spaces and their corresponding adic spaces.

Often we start with a topologically finite type formal scheme $\mathfrak{X}$ over $\operatorname{Spf} R$ where $\mathfrak{X}$ has a π-adic topology. We then obtain a rigid space on the rigid generic fibre that we denote by $\mathfrak{X}_\eta \to \operatorname{Sp} K$. If we work with adic spaces, this is actually the fibre product of $\mathfrak{X}$ with the morphism $\operatorname{Spa}(K, R) \to \operatorname{Spa}(R, R)$. We also consider the reduction of $\mathfrak{X}$ mod π that we denote by $\tilde{X} = \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spf} R/\pi$ that we usually consider as a scheme over $\operatorname{Spec} R/\pi$.

Let G be a rigid group, that is a group object in the category of rigid space. For instance, we obtain rigid groups from analytification of finite type group schemes over K: We will be most interested in analytifications of an abelian variety A, but other important cases are the analytifications $\mathbb{G}_a^{\mathrm{an}}$ of $\mathbb{G}_a$ and $\mathbb{G}_m^{\mathrm{an}}$ of $\mathbb{G}_m$, or more generally of tori T over K. A second source of rigid groups are topologically finite type formal group schemes via the rigid generic fibre. A third important case is that of E the covering space in the sense of Raynaud of an abelian variety over K with semi-stable reduction.

This note is concerned with the following question:

Question. Given a rigid group G, when is there an adic space G_∞ such that

$$G_\infty \sim \varprojlim_{[p]} G$$

in the sense of [8]? If it exists, and K is perfectoid, when is G_∞ perfectoid?

The following example shows that the second question certainly doesn't have an affirmative answer for all rigid group varieties:

Example. For the additive group $\mathbb{G}_a^{\mathrm{an}}$, we know that $[p]$ is an isomorphism (on the level of schemes, hence also after rigid analytification) and therefore $\varprojlim_{[p]} \mathbb{G}_a = \mathbb{G}_a$ exists (even as an actual limit in the category of adic spaces!) but is certainly not perfectoid.

1.1. **A condition ensuring that the tilde-limit exists.**

But what about the first question? Should one expect a limit G_∞ to always exist? There is a chance that this is actually true, for the following reason: $\sim$-limits can be constructed from formal schemes, and rigid spaces can be studied via formal models via the strategy of "Raynaud's viewpoint".

Let us be more precise: The point of the $\sim$-limit is that inverse limits often don't exist in adic spaces, and neither do they in rigid spaces. They do however often exist in the category of formal schemes:

Lemma 1.1. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over R with affine transition maps $\phi_{ij} : \mathfrak{X}_j \to \mathfrak{X}_i$. Then the inverse limit $\mathfrak{X} = \varprojlim \mathfrak{X}_i$ exists in the category of formal schemes over R. If all the $\mathfrak{X}_i$ are flat formal schemes, so is $\mathfrak{X}$.*

Proof. I can't find a reference for this at the moment, but one should be able to do this like in the scheme case: In the affine case, if the inverse system is Spf A_i, take A to be the p-adic completion of $\varinjlim A_i$, then one shows that Spf A is the inverse limit of the Spf A_i. In the general case, use that the transition maps are affine to reduce to the affine case. □

In the situation of the lemma, since the transition maps are affine and hence quasi-compact and quasi-separated, it is clear from the construction that after passing to adic spaces, $\mathfrak{X}$ is also the tilde-limit $\mathfrak{X} \sim \varprojlim \mathfrak{X}_i$ in the sense of [8]. However, the situation is even better because this remains true after passing to the generic fibre $\mathrm{Spa}(K, R) \to \mathrm{Spa}(R, R)$.

Lemma 1.2. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over R with affine transition maps ϕ_{ij} and let $\mathfrak{X} = \varprojlim_{\phi_j} \mathfrak{X}_i$ be the limit. Let $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$ and $\mathcal{X} = (\mathfrak{X})_\eta$ be the adic generic fibres. Then*

$$\mathcal{X} \sim \varprojlim \mathcal{X}_i.$$

Proof. This is a consequence of [8], Proposition 2.4.2. □

What this means is that one can always construct the limit of an inverse system of rigid spaces $\mathcal{X}_i$ if it arises from an inverse system $\mathfrak{X}_i$ with affine transition maps. This is precisely what Scholze uses in [7] in order to construct the space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ (see the proof of Corollary III.2.19 in [7]).

If one starts with an inverse system of rigid spaces $\mathcal{X}_i$, a straightforward strategy to construct "the" tilde limit $\varprojlim \mathcal{X}_i$ is thus to look for formal models $\mathfrak{X}_i$, that is formal schemes over Spf R such that $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$, as well as affine formal models $\phi_{ji} : \mathfrak{X}_j \to X_i$ of the transition maps. If such data exists, Lemma 1.2 produces a tilde-limit $\mathcal{X} \sim \varprojlim \mathcal{X}_i$. Here we follow the following standard terminology:

Definition 1.3.
(1) Let $\mathcal{X}$ be a rigid space over K. Then a formal model of $\mathcal{X}$ is an admissible topologically finite type formal scheme $\mathfrak{X}$ over R together with an isomorphism of its generic fibre $\mathfrak{X}_\eta \xrightarrow{\sim} \mathcal{X}$ (which is often suppressed from notation).
(2) Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a morphism of rigid spaces over K. Let $\mathfrak{X}, \mathfrak{Y}$ be formal models of $\mathcal{X}, \mathcal{Y}$ respectively. Then a morphism of formal schemes $\Phi : \mathfrak{X} \to \mathfrak{Y}$ is a formal model of ϕ if the following diagram commutes:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\phi"] & \mathcal{Y} \\ \mathfrak{X}_\eta \arrow[u, "\cong"] \arrow[r, "\Phi"] & \mathfrak{Y}_\eta \arrow[u, "\cong"] \end{tikzcd}$$

Given a rigid space, how does one find formal models? In our applications we will always deal with explicit formal models, but there is a general theory of how this can be done. The following is merely there to further motivate the use of formal models, we won't use it in the rest of this write-up:

The theory of Raynaud's formal models explains under which circumstances formal models of rigid spaces and their maps exist. We need the following definition:

Definition 1.4 ([1], Def 8.2.12)**.** A topological (resp G-topological) space X is called **quasi-paracompact** if there exists an open (resp admissible open) cover $\mathfrak{U}$ of X such that

- each $U \in \mathfrak{U}$ is quasi-compact and
- the cover $\mathfrak{U}$ is of finite type, that is for each $U_i \in \mathfrak{U}$ there are only finitely many $U_j \in \mathfrak{U}$ such that $U_i \cap U_j \neq \emptyset$.

For instance, the spaces $\mathbb{G}_a^{\mathrm{an}}$ and $\mathbb{G}_m^{\mathrm{an}}$ are not quasi-compact, but they are quasi-paracompact since they can be covered using families of annuli that are admissible covers of finite type. Similarly, one should be able to show (replacing annuli by intersections of Laurent domains and Weierstrass domains) that if X is any quasi-compact space and $S \subseteq X$ is a Zariski-closed subset, then $X \setminus S$ is quasi-paracompact.

The main result of Raynaud's theory of formal models is then:

Theorem 1.5 ([1], section 8.4)**.**

(1) Let X be a quasi-separated quasi-paracompact rigid space over K. Then there exist an admissible quasi-paracompact formal scheme $\mathfrak{X}$ over R such that $X = \mathfrak{X}_\eta$.

(2) If $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blow-up of admissible formal schemes, then its generic fibre is an isomorphism $\mathfrak{X}'_\eta \xrightarrow{\sim} \mathfrak{X}_\eta$.

(3) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be admissible quasi-paracompact formal schemes over R and let $f : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ be a morphism of their associated rigid spaces. Then there exist an admissible blow-up $\pi : \mathfrak{X}' \to \mathfrak{X}$ and a map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{Y}$ such that $\mathfrak{f}_\eta = f \circ \pi_\eta$.

$$\begin{tikzcd} \mathfrak{X}' \arrow[d, "\pi"] \arrow[drr, "\mathfrak{f}"] & & \\ \mathfrak{X} & & \mathfrak{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{X}'_\eta \arrow[d, "\pi_\eta"', "\cong"] \arrow[dr, "\mathfrak{f}_\eta"] & \\ \mathfrak{X}_\eta \arrow[r, "f"] & \mathfrak{Y}_\eta \end{tikzcd}$$

The theorem implies that given an inverse system $(\mathcal{X}_i, \phi_{ij})$ of rigid spaces, one can always choose formal models $\mathfrak{X}_i$ and by successive admissible blow-ups while going along the inverse system one can also find models for the ϕ_{ij}. If it is possible to do this in such a way that transition maps are affine, this way one always obtains a construction of G_∞. More precisely, we can formalise this as follows:

Definition 1.6. For a rigid analytic group G, we call $[p]$-*model tower* the data of:

(1) a family of formal models $\mathfrak{G}_n$ of G for $n \in \mathbb{N}$,
(2) morphisms of formal schemes $[p] : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$ satisfying the following conditions:
   (a) $[p] : \mathfrak{G}_n \to \mathfrak{G}_{n+1}$ is a formal model of $[p] : G \to G$.
   (b) $[p]$ is an affine morphism

We can summarise our discussion in this chapter by the following Proposition:

Proposition 1.7. *Let G be a rigid analytic group. Then if G has a $[p]$-model tower, there exists a space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$.*

1.2. **A condition ensuring that the tilde-limit is perfectoid.** As a motivation for this question, and for what follows, let us revisit the case of $G = A$ an abelian variety with good reduction. The proof generalises to the following setting:

Proposition 1.8. *Assume that K is perfectoid. Let $\mathfrak{G}$ be a flat commutative formal group scheme of topologically finite type over R for which the p-multiplication map $[p] : \mathfrak{G} \to \mathfrak{G}$ is an affine morphism. Let $G = \mathfrak{G}_\eta$ be the rigid group obtained on the generic fibre. Then G_∞ exists and is perfectoid.*

Proof. We use the approach of the preceding section and work with the flat formal model $\mathfrak{G}$ of G. We are then in the nice situation that the map $[p] : \mathfrak{G} \to \mathfrak{G}$ is a formal model of the map $[p] : G \to G$. By Lemma 1.2 we therefore have

$$G_\infty = (\varprojlim_{[p]} \mathfrak{G})_\eta \sim \varprojlim_{[p]} G.$$

To see that G_∞ is perfectoid, we proceed exactly like in the proof of [7], Corollary III.2.19. It suffices to prove that $\mathfrak{G}_\infty = \varprojlim_{[p]} \mathfrak{G}$ can be covered by formal schemes of the form $\mathrm{Spf}(S)$ where S is a flat R-algebra such that the Frobenius map

$$S/p^{1/p} \to S/p$$

is an isomorphism. Lemma 5.6 of [6] then shows that $S[1/p]$ is perfectoid.

The key observation here is that upon reduction mod p, the p-multiplication factors through relative Frobenius. More precisely, denote by $\tilde{G}$ the reduction of $\mathfrak{G}$ mod p. Then $[p] : \tilde{G} \to \tilde{G}$ factors as

$$\begin{tikzcd} & \tilde{G} \arrow[dr, dashed] & \\ \tilde{G} \arrow[ur, "F_{rel}"] \arrow[rr, "{[p]}"] & & \tilde{G} \end{tikzcd}$$

This has the following consequence: Let $\mathrm{Spf}(S_1)$ be any affine open subspace of $\mathfrak{G}$ and let $\mathrm{Spf}\, S_n$ be the pullback via $[p^n] : \mathfrak{G} \to \mathfrak{G}$. Then we have a commutative diagram:

$$\begin{tikzcd} & & \tilde{S}_n^{(p)} \arrow[dr, "F_{rel}"] & & \tilde{S}_{n+1}^{(p)} \arrow[dr, "F_{rel}"] & & \\ \dots \arrow[r] & \tilde{S}_{n-1} \arrow[ur, dashed, "V"] \arrow[rr] & & \tilde{S}_n \arrow[ur, dashed, "V"] \arrow[rr, "{[p]}"] & & \tilde{S}_{n+1} \arrow[r] & \dots \end{tikzcd}$$

From this we can check on elements that relative Frobenius is an isomorphism on $\tilde{S}_\infty := \varinjlim_n \tilde{S}_n$. Since K is perfectoid, we moreover have an isomorphism $R/p^{1/p} \to R$ from the absolute Frobenius on R/p. Therefore absolute Frobenius on S_∞/p induces an isomorphism

$$S_\infty/p^{1/p} \xrightarrow{\sim} S_\infty/p.$$

Since $\mathfrak{G}$ is flat, so are the S_n and thus so is S_∞. Thus $S_\infty[1/p]$ is a perfectoid K-algebra. Since G_∞ is covered by affinoids of the form $\mathrm{Spf}(S_\infty)_\eta$, this shows that G_∞ is perfectoid. $\square$

Since the conditions are fulfilled for completions of abelian varieties over R, we conclude:

Corollary 1.9. *Let A be an abelian variety of good reduction over a perfectoid field K. Then A_∞ exists and is perfectoid.*

Before we proceed, we would like to mention two illustrative examples:

Example. Let $\mathfrak{G}$ be the p-adic completion of the affine group scheme $\mathbb{G}_m$ over R, that is the formal scheme of $\mathfrak{G}$ is $\operatorname{Spf} S$ where $S = R\langle X\rangle$. It is clear that $\mathbb{G}$ satisfies the conditions of Proposition 1.8, so for the generic fibre $G = \mathfrak{G}_\eta$ we obtain a perfectoid tilde-limit $G_\infty = \varprojlim_{[p]} G$. More precisely, the $[p]$-multiplication map corresponds to the homomorphism

$$[p] : R\langle X\rangle \to R\langle X\rangle, \quad X \to X^p.$$

In the direct limit, we obtain the algebra $S_\infty = (\varinjlim_{[p]} S)^\wedge = R\langle X^{1/p^\infty}\rangle$. On the generic fibre we thus obtain

$$G_\infty = \operatorname{Spa}(K\langle X^{1/p^\infty}\rangle, R\langle X^{1/p^\infty}\rangle)$$

and one can verify by hand that we indeed have $G_\infty = \varprojlim_{[p]} G$.

Example. An example of a very different flavour is $\mathfrak{G}$ the p-adic completion of the affine group scheme $\mathbb{G}_a$ over R. Note that $G = \mathfrak{G}_\eta$ is not equal to $\mathbb{G}_a^{an}$, but is the closed unit disc in the latter.

While the underlying formal scheme of $\mathfrak{G}$ is $\operatorname{Spf} S$ where $S = R\langle X\rangle$ as before, the $[p]$-multiplication is now given by

$$[p] : R\langle X\rangle \to R\langle X\rangle, \quad X \to pX.$$

In the direct limit, we first obtain the algebra $S'_\infty = \varinjlim_{[p]} S = R\langle \frac{1}{p^\infty} X\rangle$ of power series $f = \sum_{n=0}^\infty a_n X^n \in R[[X]]$ for which there is $m \in \mathbb{Z}_{\geq 0}$ such that $|p^{nm} a_n| \to 0$. In order to form S_∞, we need to p-adically complete S'_∞. But we have

$$p^n R\langle \frac{1}{p^\infty} X\rangle = p^n R + X R\langle \frac{1}{p^\infty} X\rangle$$

and therefore $S'_\infty/p^n = R/p^n R$. Consequently, the completion is $S_\infty = R$ and thus $G_\infty = \operatorname{Spa}(K, R)$ is perfectoid, but just one point!

Geometrically, this makes sense: On the level of K-points, the formal scheme $\mathbb{G}_a$ is the closed unit disc and $[p]$ is scaling points by p. A K-point of $\varprojlim_{[p]} \mathbb{G}_a(K)$ therefore corresponds to a sequence in the closed unit disc of the form $x, \frac{1}{p}x, \frac{1}{p^2}x, \dots$. But for this to be contained in the closed unit disc, we must have $x = 0$. Thus $\varprojlim_{[p]} \mathbb{G}_a(K) = 0$.

We now would like to see if the same strategy of proof also applies to other situations. For instance, if A is an abelian variety with bad reduction, there is no formal group scheme $\mathfrak{G}$ giving rise to A on the generic fibre (this follows from the theory the Néron model). But can we still employ a similar strategy? When we look at the proof more closely, we see that we didn't actually use the fact that G has a formal model that is also a formal group scheme. In fact, the only thing we needed was a model for the p-multiplication morphism

$[p] : G \to G$. Second, we never used that all the $\mathfrak{G}$ in the tower are copies of the same formal scheme. Weakening these two conditions, we arrive at the following definition:

Definition 1.10. For a rigid analytic group G, we call $[p]$-F-*model tower* the data of:

(1) a family of flat formal models $\mathfrak{G}_n$ of G for $n \in \mathbb{N}$,
(2) morphisms of formal schemes $[\mathfrak{p}] : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$ satisfying the following conditions:
   (a) the generic fibre of $[\mathfrak{p}] : \mathfrak{G}_n \to \mathfrak{G}_{n+1}$ coincides with $[p] : G \to G$.
   (b) $[\mathfrak{p}]$ is an affine morphism
   (c) Denote by $\tilde{G}_n$ the reduction of $\mathfrak{G}_n$ mod p. Then $[\mathfrak{p}]$ factors through the relative Frobenius morphism:

$$\begin{tikzcd}
 & \tilde{G}_n^{(p)} \arrow[dr, dashed] & \\
\tilde{G}_n \arrow[ur, "F_{rel}"] \arrow[rr, "{[p]}"] & & \tilde{G}_{n-1}
\end{tikzcd}$$

Example. If $\mathfrak{G}$ is a flat commutative formal group scheme such that p-multiplication is affine, then setting $\mathfrak{G}_n = \mathfrak{G}$ and taking for $[\mathfrak{p}]$ the actual p-multiplication maps $[p]$ defines a $[p]$-F-model tower for the rigid analytic group $G = \mathfrak{G}_\eta$.

The point of this definition is to extract from Proposition 1.8 precisely those properties that are needed to complete the proof. We therefore conclude

Proposition 1.11. *Let G be a rigid analytic group over a perfectoid field K. If G admits a $[p]$-F-model, then G_∞ exists and is perfectoid.*

What we aim to prove in the rest of this write-up is that for a Raynaud extension $0 \to T \to E \to B \to 0$, there is a $[p]$-F-model for T which induces a $[p]$-F-model for E. This will prove that tilde-limits T_∞ and E_∞ exist and are perfectoid if K is perfectoid.

2. Formal models for tori

Let K be perfectoid. In this section we want to show that for a split rigid torus T over K, a tilde-limit T_∞ exists and is perfectoid. We do this by exhibiting a $[p]$-F-model of T.

As a preparation, we consider the torus $\mathbb{G}_m^{\mathrm{an}}$ over K. Recall that it arises from rigid analytification of the affine torus $\mathbb{G}_m$ over K. Note however that $\mathbb{G}_m^{\mathrm{an}}$ is not affinoid (and not even quasi-compact). It contains the generic fibre of the p-adic completion of $\mathbb{G}_m$ as an open subspace. If we see $\mathbb{G}_m^{\mathrm{an}}$ as the rigid affine line with origin removed, this subspace $\widehat{\mathbb{G}}_m$ can be identified with the open annulus of radius 1. In other words, on the level of points it corresponds to $\mathcal{O}_K^\times \subseteq K^\times$[1].

[1] I think the theory of the Néron model tells us that this is the smallest open subgroup that admits a formal group scheme as a formal model. But I am not sure because the notion of Néron model for rigid spaces is a bit weird... Bosch and his various collaborators have papers on this.

2.1. **A family of explicit covers.** We briefly recall how $\mathbb{G}_m^{\mathrm{an}}$ is constructed: The following is inspired by [1], §9.2, although we choose slightly different constructions. Let $q \in K^\times$ with $|q| \leq 1$. Consider the annulus $B(q,1)$ of radii $|q|$ and 1 inside $\mathbb{A}_K^{\mathrm{an}}$:

$$B(q,1) = \mathrm{Sp}(L_q), \quad \text{where } L_q = K\langle X, Z\rangle/(XZ - q).$$

Similarly, for $q \in K^\times$ with $|q| \geq 1$ one constructs the annulus $B(1,q)$ by

$$B(1,q) = \mathrm{Sp}(L_q), \quad \text{where } L_q = K\langle q^{-1}X, Z\rangle/(XZ - 1)$$

where $K\langle q^{-1}X\rangle$ denotes the ring of those power series $f = \sum c_n X^m \in K[[X]]$ for which $|c_n/q| \to 0$ for $n \to \infty$. In particular, we have isomorphisms

$$K\langle X', Z\rangle/(X'Z - q^{-1}) \cong K\langle q^{-1}X, Z\rangle/(XZ - 1), \quad X' \mapsto q^{-1}X.$$

One can now construct $\mathbb{G}_m$ as follows: Choose sequences $a_n, b_n \in K^\times$ with $a_0 = 1 = b_0$ such that $|a_n| < |a_{n-1}| < ... < 1$ and $|a_n| \to 0$ and similarly $|b_n| > |b_{n-1}| > ... > 1$ and $|b_n| \to \infty$. Then one can glue the annuli $B(a_n, 1)$ and $B(1, b_n)$ using the following maps:

$$\begin{gathered} B(a_n,1) \hookleftarrow B(a_{n-1},1) \\ L_{a_n} = K\langle X, Z\rangle/(XZ - a_n) \to K\langle X, Z\rangle/(XZ - a_{n-1}) = L_{a_{n-1}} \\ X, Z \mapsto X, \frac{a_n}{a_{n-1}} Z \end{gathered} \tag{1}$$

and similarly

$$\begin{gathered} B(1,b_n) \hookleftarrow B(1,b_{n-1}) \\ L_{b_n} = K\langle X', Z\rangle/(X'Z - b_n^{-1}) \to K\langle X', Z\rangle/(X'Z - b_{n-1}^{-1}) = L_{b_{n-1}} \\ X', Z \mapsto \frac{b_{n-1}}{b_n} X', Z. \end{gathered} \tag{2}$$

Also, via the above maps, the annuli $B(a_n, 1)$ and $B(1, b_m)$ are glued along $B(a_0, 1) = B(1, 1) = B(1, b_0)$. This gives the desired space $\mathbb{G}_m^{\mathrm{an}}$.

Since we are mainly interested in the p-multiplication map, we will more precisely use the following cover on which $[p]$ can be seen directly: Choose $q \in K^\times$ with $|q| < 1$. Then for the sequences a_n and b_n from above we take $a_n = q^n$, $b_n = q^{-n}$. We call this cover $\mathfrak{U}_q$.

Assume now that q has a p-th root $q^{1/p}$ in K. The above then gives a finer cover $\mathfrak{U}_{q^{1/p}}$ of $\mathbb{G}_m^{\mathrm{an}}$. Using both covers $\mathfrak{U}_q$ and $\mathfrak{U}_{q^{1/p}}$, we can easily see the $[p]$-multiplication $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ as follows: Consider the affinoid open subsets $B(q^{1/p}, 1)$ of the source and $B(q^{1/p}, 1)$ of the target. Then $[p]$ restricts to

$$\begin{gathered} B(q,1) \xleftarrow{[p]} B(q^{1/p},1) \\ K\langle X, Z\rangle/(XZ - q) \to K\langle X, Z\rangle/(XZ - q^{1/p}) \\ X, Z \mapsto X^p, Z^p \end{gathered} \tag{3}$$

and similarly, on $B(1,q^{-1/p})$ and $B(1,q^{-1})$ the map is

$$\begin{aligned} B(1,q^{-1}) &\xleftarrow{[p]} B(1,q^{-1/p}) \\ K\langle X',Z\rangle/(X'Z-q^{-1}) &\to K\langle X',Z\rangle/(X'Z-q^{-1/p}) \\ X',Z &\mapsto X'^p, Z^p. \end{aligned} \tag{4}$$

The same works for the other affinoid open subspaces $B(q^n,1) \xleftarrow{[p]} B(q^{n/p},1)$ and for $B(1,q^{-n}) \xleftarrow{[p]} B(1,q^{-n/p})$. One can then show that the maps (3) and (4) are compatible with the glue maps (1) and (2). In the case of (3) this is basically because $a_n/a_{n-1}=q$ or $a_n/a_{n-1}=q^{1/p}$ depending on whether we work with $\mathfrak{U}_q$ or $\mathfrak{U}_{q^{1/p}}$ respectively, and the only thing to check is that

$$\begin{tikzcd} B(q^n,1) & B(q^{n-1},1) \arrow[l, hook'] & Z \arrow[r, mapsto] \arrow[d, mapsto] & qZ \arrow[d, mapsto] \\ B(q^{n/p},1) \arrow[u, "{[p]}"] & B(q^{(n-1)/p},1) \arrow[l, hook'] \arrow[u, "{[p]}"'] & Z^p \arrow[r, mapsto] & (q^{1/p}Z)^p = qZ^p. \end{tikzcd} \tag{5}$$

The case of (4) is very similar.

2.2. **A family of formal models.** Recall that we have constructed a cover $\mathfrak{U}_q$ of $\mathbb{G}_m^{\mathrm{an}}$ depending on a choice of $q \in K^\times$ with $|q|<1$. The affinoid subspaces $B(q^n,1)$ that we have used for this admit natural formal models: Namely, consider the R-algebra

$$L_q^\circ := R\langle X,Z\rangle/(XZ-q).$$

This is clearly of topologically finite type over R. It is moreover flat as an R-algebra (this should follow from Lemma 8.2.1 in [1]). For the same reason (or by $L_{q^{-1}} \cong L_q$) we see that

$$L_{q^{-1}}^\circ := R\langle X',Z\rangle/(X'Z-q)$$

is a flat topologically finite type R-algebra. Consequently, we have flat formal models

$$\begin{aligned} \mathfrak{B}(q,1) &:= \mathrm{Spf}(L_q^\circ), & \mathfrak{B}(q,1)_\eta &= B(q,1) \\ \mathfrak{B}(1,q) &:= \mathrm{Spf}(L_{q^{-1}}^\circ), & \mathfrak{B}(1,q)_\eta &= B(1,q) \end{aligned}$$

Looking at the glueing maps (1) and (2) it is clear from $a_n/a_{n-1} = b_{n-1}/b_n = q$ that these extend to glueing maps $\mathfrak{B}(q^n,1) \hookleftarrow B(q^{n-1},1)$ and $\mathfrak{B}(1,q^{-n}) \hookleftarrow B(1,q^{-(n-1)})$. We conclude:

Lemma 2.1. *The affine formal schemes $\mathfrak{B}(q^n,1)$ and $\mathfrak{B}(1,q^n)$ glue together to a flat formal scheme $\mathfrak{G}_q$ such that $(\mathfrak{G}_q)_\eta = \mathbb{G}_m^{\mathrm{an}}$. In other words, $\mathfrak{G}_q$ is a formal model for $\mathbb{G}_m^{\mathrm{an}}$.*

2.3. **A family of formal models for p-multiplication.** As before choose $q \in K^\times$ such that $|q| < 1$ and such that there exists a p-th root $q^{1/p} \in K$. A closer look at the maps (3) and (4) shows that the $[p]$-multiplication extends to a morphism of formal schemes

$$\mathfrak{B}(q,1) \xleftarrow{[p]} \mathfrak{B}(q^{1/p},1)$$

and similarly for $\mathfrak{B}(1,q^{-1})$. The diagram (5) shows that these maps glue to a morphism

$$[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q.$$

By construction, after tensoring $- \otimes_R K$ all morphisms on algebras coincide with those defined in (1), (2), (3), (4) respectively. We conclude:

Proposition 2.2. *The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ is a formal model of $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$.*

We moreover see directly from the construction:

Proposition 2.3. *The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ reduces mod p to the relative Frobenius map.*

We now have everything together to finish our proof that $(\mathbb{G}_m^{\mathrm{an}})_\infty$ is perfectoid:

Proposition 2.4. *The space $\mathbb{G}_m^{\mathrm{an}}$ has a $[p]$-F-model tower. In particular, there exists a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ such that $(\mathbb{G}_m^{\mathrm{an}})_\infty \sim \varprojlim_{[p]} \mathbb{G}_m^{\mathrm{an}}$.*

Proof. Since K is perfectoid, we can find $q \in K^\times$ such that $|q| < 1$ for which there exist arbitrary p^n-th roots. We choose such a q and roots q^{1/p^n} for all n. Then the two Propositions above combine to show that

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_q$$

is a $[p]$-F-model tower. Proposition 1.11 then gives the desired space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ and shows that it is perfectoid. $\square$

2.4. **The action of $\overline{T}$.** The multiplication $\mathbb{G}_m^{\mathrm{an}} \times \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ can locally be described in terms of the rigid analytic cover that we have defined above as follows : Let $a, b \in K^\times$ such that $|a|, |b| \leq 1$, then the multiplication map restricts to

$$\begin{aligned} B(a,1) \times B(b,1) &\xrightarrow{m} B(ab,1) \\ K\langle X,Z\rangle/(XZ-ab) &\to K\langle X,Z\rangle/(XZ-a)\widehat{\otimes} K\langle X,Z\rangle/(XZ-b) \\ X &\mapsto X \otimes X \\ Z &\mapsto bZ \otimes aZ \end{aligned} \tag{6}$$

and similarly on the $B(1,a) \times B(1,b)$ for $|a|, |b| \geq 1$. Multiplication on the $B(a,1) \times B(1,b)$ for $|a| < 1 < |b|$ is more difficult to see on the cover that we have chosen.

The same arguments as in the last section show that the map described in (6) has a flat formal model

$$\mathfrak{B}(a,1) \times \mathfrak{B}(b,1) \to \mathfrak{B}(ab,1).$$

This does *not* mean that multiplication has a formal model $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$. Indeed, the chosen description has different covers on source and target which in the formal case give rise to different formal schemes (the inversion map $i : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ on the other hand does have a formal model). Nevertheless, if we take $a = 1$ in the above , we see that we do have an action of the torus $\overline{T} := \mathfrak{B}(1,1)$ on each of $\mathfrak{B}(b,1)$ and $\mathfrak{B}(1,b)$. Using the formal models from the last section, we conclude:

Proposition 2.5. *For any $q \in K^\times$ with $|q| < 1$, the formal torus $\overline{T} := \mathfrak{B}(1,1)$ has a natural action on $\mathfrak{G}_q$ via a map*

$$\mathfrak{m} : \overline{T} \times \mathfrak{G}_q \to \mathfrak{G}_q.$$

This map is a formal model of the action of the annulus $\overline{T} = B(1,1)$ on $\mathbb{G}_m^{\mathrm{an}}$. Furthermore, this action is compatible with the models for $[p]$ in the sense that if there is a p-th root $q^{1/p} \in K$, then the following diagram commutes.

$$\begin{tikzcd}
\overline{T} \times \mathfrak{G}_{q^{1/p}} \arrow[r, "\mathfrak{m}"] \arrow[d, "{[p] \times [\mathfrak{p}]}"'] & \mathfrak{G}_{q^{1/p}} \arrow[d, "{[\mathfrak{p}]}"] \\
\overline{T} \times \mathfrak{G}_q \arrow[r, "\mathfrak{m}"] & \mathfrak{G}_q.
\end{tikzcd}$$

Proof. The existence of $\mathfrak{m}$ follows from the above consideration concerning the map (6). The rest is clear from the construction: All adic rings we have used in the construction where R-subalgebras of the affinoid K-algebras used to define $\mathbb{G}_m^{\mathrm{an}}$, so the equalities hold because they hold for $\mathbb{G}_m^{\mathrm{an}}$. $\square$

2.5. **The case of general tori.** By taking products everywhere, all of the statements in this section immediately generalises to split tori:

Corollary 2.6. *Let T be a split torus over K of the form $T = (\mathbb{G}_m^{\mathrm{an}})^d$. Then for any $q \in K^\times$ with $|q| < 1$ the formal scheme $\mathfrak{T}_q := (\mathfrak{G}_q)^d$ is a formal model of T. If there is a p-th root $q^{1/p} \in K$, the p-multiplication map has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ that reduces mod p to the relative Frobenius morphism.*

Corollary 2.7. *Let T be a split torus over K, considered as a rigid space. Then T has a $[p]$-F-model tower. In particular, there exists a perfectoid space T_∞ such that $T_\infty \sim \varprojlim_{[p]} T$.*

Corollary 2.8. *Let T be any split torus over K. For any $q \in K^\times$ with $|q| < 1$, the formal completion $\overline{T}$ has a natural action on $\mathfrak{T}_q$ via a map*

$$\mathfrak{m} : \overline{T} \times \mathfrak{T}_q \to \mathfrak{T}_q.$$

This map is a formal model of the action of the annulus $\overline{T}$ on T. Furthermore, this action is compatible with the models for $[p]$ in the sense that if there is a p-th root $q^{1/p} \in K$, then the map $\mathfrak{m}$ is semi-linear with respect to $[p] : \overline{T} \to T$.

3. Raynaud extensions as principal bundles of formal and rigid spaces

In the following we want consider the case of rigid groups arising from the Raynaud extensions associated to semi-stable abelian varieties over a non-archimedean complete field K. More precisely, let A be an abelian variety over K of semi-stable reduction. We denote by N the identity component of the Néron-model and by $\overline{E}$ its completion along the special fibre. Then by the theory of Raynaud, $\overline{E}$ is a formal group that fits into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} B \to 0 \tag{7}$$

where B is an abelian variety of good reduction, and $\overline{T}$ is the completion of a torus over K. After possibly passing to a finite extension of K, we can always assume that the torus is split. The rigid generic fibre $\overline{T}_\eta$ of the torus $\overline{T}$ canonically extends to the torus T^{an} which again we simply denote by T. One can show that this induces a pushout exact sequence in the category of rigid groups, see §1 of [3]. More precisely, there exists a rigid group variety E such that the following diagram commutes and the square on the left is a pushout.

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{8}$$

We would like to study properties of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that one cannot direcly study short exact sequences locally on T, E or B. An important tool is therefore the following Lemma:

Lemma 3.1 ([3], §1)**.** *The short exact sequence (7) admits local sections, that is there is a cover of B by formal open subschemes U_i such that there exist sections $s : U_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times U_i \hookrightarrow E$.*

Proof. This is proved in Proposition A.2.5 in [5], in terms of the group $\mathrm{Ext}(B, T)$. ☐

The last Lemma suggest that instead of considering Raynaud extensions from the abelian category viewpoint, one should consider them as fibre bundles of formal schemes with structure group T, or more precisely as principal T-bundles of formal schemes, which are also called torsors. This is the language we want to use in the following: We will work with fibre bundles of formal schemes, rigid spaces and schemes. The main technical tool we will need is the associated fibre construction in these settings. For a rigorous treatment of these we refer to the Appendix which should be replaced by a link to the relevant literature.

First of all, we note that the sequence (7) from the last section gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}$ via push-out from $\overline{T} \to T$ can now conveniently be expressed in terms of the associated fibre bundle by saying that $E = T \times^{\overline{T}} \overline{E}$ in the sense of Definition A.8. We moreover have the following description of $[p]$:

Lemma 3.2. *The map $[p] : E \to E$ coincides with the morphism $T \times^{\overline{T}} \overline{E} \to T \times^{\overline{T}} \overline{E}$ induced by Proposition A.17.*

Proof. The universal property of the associated bundle in the principal case, Lemma A.12 applied to the maps $g : \overline{T} \to T$ and $\overline{E} \xrightarrow{[p]} \overline{E} \to E$ says that there is a unique morphism of fibre bundles $E \to E$ making the following diagram commute:

$$
\begin{tikzcd}
& T \arrow[rr] & & E \\
T \arrow[ur, "{[p]}"] \arrow[rr] & & E \arrow[ur, dotted, "\exists!"] & \\
& \overline{T} \arrow[uu, hook] \arrow[rr] & & \overline{E} \arrow[uu] \\
\overline{T} \arrow[uu, hook] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E} \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd}
\tag{9}
$$

Since $[p] : E \to E$ is such a map, the Lemma follows. ☐

4. Formal models for E

We now want to prove step-by-step that E admits a $[p]$-F-tower model, which implies that there is a perfectoid tilde-limit E_∞ of the p-multiplication tower on E. The first step is to construct a family of formal models for E. We do this by using the formal models $\mathfrak{T}_q$.

Proposition 4.1. *Let $q \in K^\times$ with $|q| < 1$. Let $\mathfrak{T}_q$ be the formal model from Corollary 2.6. Then there is a formal scheme $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E}$ that is a formal model of the rigid space E. Furthermore, there exists a morphism*

$$\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to B$$

which is a fibre bundle and a formal model of $E \to B$.

Proof. Recall from Proposition 2.8 that $\mathfrak{T}_q$ has a $\overline{T}$-action that is a model of the $\overline{T}$-action on T. In particular, the associated fibre construction for the principal $\overline{T}$-bundle $\overline{E}$ gives a fibre bundle $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to B$. Since $\mathfrak{T}_q$ is a formal model of T, this is a formal model of $T \times^{\overline{T}} \overline{E}$ which by definition is equal to E. ☐

Next we want to construct a model for the $[p]$-multiplication map. Here we can use again that $[p]$ exists on $\overline{E}$ and on $\mathfrak{T}_{q^{1/p}}$.

Proposition 4.2. *Let $q \in K^\times$ be such that $|q| < 1$ and assume there exists a p-th root $q^{1/p} \in K$. Then there is an affine morphism*

$$[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$$

which is a formal model of $[p] : E \to E$.

Proof. Recall that the multiplication map $[p] : T \to T$ has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ by Corollary 2.6. This fits into a commutative diagram

$$
\begin{tikzcd}
\mathfrak{T}_{q^{1/p}} \arrow[r, "{[\mathfrak{p}]}"] & \mathfrak{T}_q \\
\overline{T} \arrow[u, hook] \arrow[r, "{[p]}"] & \overline{T}. \arrow[u, hook]
\end{tikzcd}
$$

Functoriality of the associated fibre construction in the general case, Proposition A.17, applied to the diagram below then gives a natural map $\mathfrak{E}_{q^{1/p}} \to \mathfrak{E}$ making the diagram commute:

(10)
$$
\begin{tikzcd}
 & \mathfrak{T}_q \arrow[rr] & & \mathfrak{E}_q \\
\mathfrak{T}_{q^{1/p}} \arrow[ur, "{[\mathfrak{p}]}"] \arrow[rr] & & \mathfrak{T}_{q^{1/p}} \times^{\overline{T}} \overline{E} \arrow[ur, dotted, "\exists"] & \\
 & \overline{T} \arrow[uu] \arrow[rr] & & \overline{E} \arrow[uu] \\
\overline{T} \arrow[uu] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E} \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd}
$$

By Lemma 3.2 this diagram equals diagram (9) on the generic fibre.

The morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ is affine because $[p] : \overline{E} \to \overline{E}$ is affine, the map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ is affine by construction, and the resulting map $\mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ is glued from covers which since $[p] : B \to B$ is affine we can without loss of generality assume to be affine. $\square$

We have thus proved the first part of what we want to show about tilde-limits of E:

Proposition 4.3. *Let K be perfectoid. Then E has a $[p]$-model tower of the form*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

for some $q \in K^\times$. In particular, there exists a space E_∞ such that $E_\infty \sim \varprojlim_{[p]} E$.

Proof. By Proposition 4.2, any choice of $q \in K^\times$ with $|q| < 1$ for which there exist arbitrary p^n-th roots $q^{1/p^n} \in K^\times$ gives a tower

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

that on the generic fibre equals $\dots \xrightarrow{[p]} E \xrightarrow{[p]} E$. This is the desired $[p]$-model tower. $\square$

We are now ready to prove the main result of this note, namely that E_∞ is perfectoid.

Theorem 4.4. *Let K be perfectoid. Then the $[p]$-model tower from Proposition 4.3*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

is already a $[p]$-F-model tower. In particular, the corresponding space E_∞ is perfectoid.

Proof. It suffices to prove that for any $q \in K^\times$ with $|q| < 1$ and a p-th root $q^{1/p}$, the map $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ upon reduction mod p factors through relative Frobenius.

In the following we denote reduction of formal scheme by a $\sim$ over the formal scheme, for example the reductions of $\overline{T}$, $\overline{E}$ and $\mathfrak{T}$ are denoted by $\tilde{T}$, $\tilde{E}$ and $\tilde{\mathfrak{T}}$.

Recall that $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ was constructed using the $[p]$-multiplication cube in diagram (10) and functoriality of the associated bundle. Also recall that all statements we have used about fibre bundles also hold when we replace formal schemes over R by schemes over R/p, and formation of the associated bundle commutes with this reduction. In particular,

$$\tilde{\mathfrak{E}}_q = \tilde{\mathfrak{T}}_q \times^{\tilde{T}} \tilde{E}.$$

By Corollary 2.6, the multiplication map $[\mathfrak{p}] : \tilde{\mathfrak{T}}_{q^{1/p}} \to \tilde{\mathfrak{T}}_q$ reduces to relative Frobenius over p. In particular, we have

$$\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} = \tilde{\mathfrak{T}}_q.$$

Since $\tilde{E}$ and $\tilde{T}$ are group schemes, the reduction of $[p]$ on them factors through the relative Frobenius maps F_E and F_E respectively. In particular, we have a commutative diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{F_{\tilde{E}}} & \tilde{E}^{(p)} \\ \uparrow & & \uparrow \\ \tilde{T} & \xrightarrow{F_{\tilde{T}}} & \tilde{T}^{(p)}. \end{array}$$

In other words, $F_{\tilde{E}}$ is a $F_{\tilde{T}}$-linear morphism of fibre bundles. By functoriality of relative Frobenius ("Frobenius commutes with any map") we also have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{T}}_{q^{1/p}} & \xrightarrow{F_{\tilde{\mathfrak{T}}}} & \tilde{\mathfrak{T}}_q \\ \uparrow & & \uparrow \\ \tilde{T} & \xrightarrow{F_{\tilde{T}}} & \tilde{T}^{(p)}. \end{array}$$

By Proposition A.17 we thus obtain a natural morphism

$$F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}} : \tilde{\mathfrak{T}}_{q^{1/p}} \times^{\tilde{T}} \tilde{E} \to \tilde{\mathfrak{T}}_q \times^{\tilde{T}^{(p)}} \tilde{E}^{(p)}.$$

Using the explicit description of $F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}}$ in the proof of Proposition A.17, we easily check that this morphism is just the relative Frobenius of $\tilde{\mathfrak{E}}_{q^{1/p}}$: This is a consequence of the fact that relative Frobenius on the fibre product $\tilde{T} \times \tilde{U}$ for any $\tilde{U} \subseteq \tilde{B}$ is just the product of the relative Frobenius morphisms of $\tilde{T}$ and $\tilde{U}$, and thus the morphisms θ_i from Lemma A.7 are all trivial.

But this means that again by Proposition A.17, the reduction of the formal model of the p-multiplication cube in diagram 10 admits the following factorisation:

$$\begin{tikzcd}
 & & \tilde{\mathfrak{T}}_q \arrow[rrr] & & & \tilde{\mathfrak{E}}_q \\
 & \tilde{\mathfrak{T}}_q \arrow[rrr] \arrow[ur, equal] & & & \tilde{\mathfrak{E}}^{(p)}_{q^{1/p}} \arrow[ur, dotted] & \\
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[rrr] \arrow[ur, "F"] & & & \tilde{\mathfrak{E}}_{q^{1/p}} \arrow[ur, "F"] & & \\
 & & \tilde{T} \arrow[rrr] \arrow[uuu] & & & \tilde{E} \arrow[uuu] \\
 & \tilde{T} \arrow[rrr] \arrow[ur, equal] \arrow[uuu] & & & \tilde{E}^{(p)} \arrow[ur] \arrow[uuu] & \\
\tilde{T} \arrow[rrr] \arrow[ur, "F"] \arrow[uuu] & & & \tilde{E} \arrow[ur, "F"] \arrow[uuu] & &
\end{tikzcd}$$

Since the composed maps $\tilde{E} \to \tilde{E}$ on the bottom left, $\tilde{T} \to \tilde{T}$ on the bottom right and $\tilde{\mathfrak{T}}_{q^{1/p}} \to \tilde{\mathfrak{T}}_q$ on the upper right by construction are the reductions of the respective p-multiplication maps $[p]$, the functoriality of the associated bundle construction in Proposition A.17 implies that the two maps on the upper left compose to the reduction of $[\mathfrak{p}] \times^{[p]} [p]$. But $[\mathfrak{p}] \times^{[p]} [p]$ is equal to $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ by definition of the latter. This completes the proof that the reduction of $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through the relative Frobenius on $\tilde{\mathfrak{E}}_{q^{1/p}}$.

The conclusion that E_∞ exists and is perfectoid then follows from Proposition 1.11. $\square$

Appendix A. Fibre bundles of formal and rigid spaces

In this chapter we review the theory of fibre bundles with structure group T and in particular of principal T-bundles in the setting of formal and rigid geometry. This should all be completely standard, but I didn't find a reference in the context of formal or rigid geometry. We should just find one and delete this whole appendix. I have just written this up in order to convince myself it's true. All of this are just translation from results that you find in any geometry or topology book on fibre bundles, except for Proposition A.17 for which I somehow couldn't find a reference (but it's straightforward).

Also this is all closely related to what Peter and Darya have written up about the tilt.

In this chapter we denote by T a commutative formal group scheme over R. We denote the multiplication map by $m : T \times T \to T$. By a T-action on a formal scheme X we mean a morphism $m_X : T \times X \to X$ such that the usual associativity diagram commutes.

Definition A.1. By a T-linear map of schemes X and Y with T-actions we mean a morphism $\phi : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} T \times X \arrow[r, "\mathrm{id}_T \times \phi"] \arrow[d, "m_X"] & T \times Y \arrow[d, "m_Y"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

We denote by $\mathbf{FormAct}_T$ the category of formal schemes with action by T.

The definition of a principal T-bundle is just what we get when we take the definition of a principal G-bundle and replace the category of topological spaces by the category of formal schemes.

Notation. In the following, if $\pi : E \to B$ is a morphism of formal schemes, then for a formal open subscheme $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.2. Let T be a formal group scheme. Let F be a formal scheme with an action $m : T \times F \to F$. A morphism $\pi : E \to B$ of formal schemes is called a **fibre bundle with fibre F and structure group T** if there is a cover $\mathfrak{U}$ of B of open formal subschemes $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i : F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that

$$\phi_{ij} = F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}$$

Definition A.3. When we take F equal to the formal scheme T with the action on itself by left multiplication, then a fibre bundle $\pi : E \to B$ with fibre T and structure group T is called a **principal T-bundle**.

Example. For the short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} B \to 0$ from the last section, $\overline{E} \xrightarrow{\pi} B$ defines a principal $\overline{T}$-bundle by Lemma 3.1. Moreover, for any formal open subscheme $U \subseteq B$, the map $E|_U \to U$ is still a principal $\overline{T}$-bundle. This is what we mean when we say that the notion of principal $\overline{T}$-bundles is better suited for studying E locally on B than the notion of short exact sequences is.

The morphism ϕ_{ij} from condition (b) is fully determined by the morphism $\psi_{ij} : U_{ij} \to T$. By a glueing argument, one shows:

Lemma A.4. *Suppose we are given formal schemes F and B and a formal group scheme T with an action on F. Then fibre bundles $\pi : E \to B$ with fibre F and structure group T are equivalent to the data (up to refinement) of a cover $\mathfrak{U}$ by formal open subschemes and morphisms $\psi_{ij} : U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk} \cdot \psi_{ij} = \psi_{ik}$, by which we mean that the following diagram commutes:*[2]

(11)
$$\begin{array}{ccc} U_{ijk} & \xrightarrow{\psi_{ij} \times \psi_{jk}} & T \times T \\ \| & & \downarrow m \\ U_{ijk} & \xrightarrow{\psi_{ik}} & T. \end{array}$$

In order to define the category of fibre bundles, we also need the following:

Lemma A.5. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. With notations like in Definition A.2 we have a natural T-actions on $F \times U_i$ when we let T act trivially on U_i. These actions glue together to a T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.6. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T. Then a morphism of fibre bundles $f : (E', B', \pi') \to (E, B, \pi)$ is a commutative diagram of formal schemes

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & B' \\ \downarrow f_E & & \downarrow f_B \\ E & \xrightarrow{\pi} & B \end{array}$$

in which the morphism f_E is also T-linear (we often abbreviate this by writing $f : E' \to E$). We thus obtain the category of fibre bundles over T that we denote by $\mathbf{FormFibBun}_T$ and the full subcategory of principle T-bundles, that we denote by $\mathbf{FormPrinBun}_T$.

In the case of principal T-bundles, this data can be given as follows: Let $\mathfrak{U}$ be a cover over which E is trivialised. Then we can always refine U in such a way that for all $U \in \mathfrak{U}$ the fibre bundle E' is trivial over $U' := f_B^{-1}(U)$. The induced map $f_E : T \times U \to T \times U'$ is then T-linear and thus can be reconstructed from the induced map

$$\theta : U' = 1 \times U' \hookrightarrow T \times U \xrightarrow{f_E} T \times U \xrightarrow{p_1} T.$$

Lemma A.7. *Given a morphism $f_B : B' \to B$, and using notation as above, the data of a morphism $f = (f_E, f_B)$ of principal T-bundles is equivalent to the data of morphisms $\theta_i : U_i' \to T$ for all $U_i \in \mathfrak{U}$ such that for all i, j th following diagram commutes:*

[2] One can probably add a remark about $H^1(B, T)$ here, but right now I am not sure how exactly this group is defined in this context.

$$\begin{tikzcd} T \times U'_{ij} \arrow[r, "\phi'_{ij}"] \arrow[d, "f_E"] & T \times U'_{ij} \arrow[d, "f_E"] \\ T \times U_{ij} \arrow[r, "\phi_{ij}"] & T \times U_{ij}. \end{tikzcd}$$

Moreover, commutativity of the above diagram is equivalent to commutativity of

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij}\times\theta_j"] \arrow[d, "(\psi_{ij}\circ f)\times\theta_i"'] & T \times T \arrow[d, "m"] \\ T \times T \arrow[r, "m"] & T. \end{tikzcd}$$

Or in short hand notation,

$$\psi'_{ij}(u)\theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u)$$

Proof. One direction is clear. For the other, the first part follows from glueing. The second part is a consequence of all maps in the first diagram being T-linear. □

Definition A.8. Let $\pi : E \to B$ be a principial T-bundle. Let F be a formal scheme with an action by T. Since the data in the equivalent characterisation of Lemma A.4 is completely independent of the fibre, the morphisms $\psi_{ij} : U_{ij} \to T$ by Lemma A.4 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel-Weil construction.

Note that in many authors in differential geometry and topology denote the associated bundle by "$F \times^T E$" instead of $F \times^T E$. In our setting, however, this is slightly confusing since we often have natural maps from T to F and E, but $F \times^T E$ is usually *not* their fibre product. In fact it behaves more like a pushout, for instance in the case that E comes from a short exact sequence.

Proposition A.9. *The associated bundle construction is a bifunctor*

$$- \times^T - : \mathbf{FormAct}_T \times \mathbf{FormPrinBun}_T \to \mathbf{FormFibBun}_T$$

from the categories of formal schemes with T-action $\times$ the category of principal T-bundles to the category of fibre bundles with structure group T.

Proof. Let E and E' be principal T-bundles and let $f : E' \to E$ be a morphism of T-bundles. Let F and F' be formal schemes with T-action and let $\gamma : F' \to F$ be a T-equivariant morphism. Then we can find compatible covers $\mathfrak{U}'$ of E' and $\mathfrak{U}$ of E such that locally we have diagrams like in Lemma A.7. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U'_i$ such that we obtain a natural map

$$F' \times U'_i \xrightarrow{(\lambda \times^T \pi)} F \times U_i, \quad (f, u) \mapsto (\lambda(f)\theta_i(u), \pi(u))$$

(of course this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersection Lemma A.7 implies that we have a commutative diagram

$$\begin{array}{ccc} F' \times U'_{ij} & \xrightarrow{\lambda\times^T\pi} & F \times U_{ij} \\ {\scriptstyle \psi'_{ij}\times\mathrm{id}}\uparrow & & \uparrow{\scriptstyle \psi_{ij}\times\mathrm{id}} \\ F' \times U'_{ij} & \xrightarrow{\lambda\times^T\pi} & F \times U_{ij}. \end{array}$$

One easily checks that this is functorial in both components. □

Lemma A.10. *Let S be another formal group scheme that receives an action of T from a group homomorphism $g : T \to S$. Then for any principal T-bundle E, the Borel construction $S \times^T E$ is a principal S-bundle.*

Proof. This follows from Lemma A.4. The only thing we need to check is that the cocycle condition from diagram (11) also holds with respect to S. But g is a homomorphism and therefore the following diagram commutes:

$$\begin{array}{ccc} T \times T & \xrightarrow{g\times g} & S \times S \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ T & \xrightarrow{g} & S. \end{array}$$

□

Lemma A.11. *The Borel construction is a functor $S \times^T -$ from principal T-bundles to principal S-bundles.*

Proof. This is a consequence of Lemma A.7. One obtains the necessary data by composing the morphisms $\theta' : U'_i \to T$ with the morphism $T \to S$. These morphisms glue together because the second diagram of Lemma A.7 commutes, as one easily sees from the fact that $T \to S$ is a morphism of formal groups. □

The Borel construction satisfies the following universal property:

Lemma A.12. *Let $g : T \to S$ be a homomorphism of formal group schemes and let $E \to B$ be a principal T-bundle. Let X be any principle S-bundle. Note that X receives a T-action from g. Then there is a functorial one-to-one correspondence between T-linear morphisms $E \to X$ and morphism of principal S-bundles $S \times^T E \to X$.*

A.1. **The semi-linear case.** We later want to consider morphisms of fibre bundles that are induces from morphisms of short exact sequences. In this situation. in order to describe the morphism of the kernels, we need to incorporate morphisms of the structure group into the notion of morphisms of fibre bundles. For this we need semi-linear group actions.

Definition A.13. Let T and S be formal group schemes and let $g : T \to S$ be a homomorphism. Let X and Y be formal schemes with actions $m : T \times X \to X$ and $m : S \times Y \to Y$ respectively. Then by a g-linear morphism $f : X \to Y$ we mean a morphism of formal schemes such that the following diagram commutes

$$\begin{tikzcd} T \times X \arrow[r, "g\times f"] \arrow[d, "m"'] & S \times Y \arrow[d, "m"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Definition A.14. We denote by **FormAct** the category of pairs (T, X) where T is a formal group scheme and X is a formal scheme with T action, and morphisms are pairs of (g, f) where g is a group homomorphism and f is a g-linear morphism. It has a natural forgetful functor to **FormGrp**, the category of formal group schemes.

Definition A.15. Let $g : T' \to T$ be a homomorphisms of formal group schemes. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T' . Then a g-linear morphism of principal bundles is a diagram

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

such that f_E is g-linear. We denote by **FormPrinBun** the category of fibre bundles (E, B, π, T, F) with arrows being the morphisms of principal bundles. It has a natural forgetful functor $(E, B, \pi, T, F) \mapsto T$ to the category **FormGrp** of formal group schemes

We get the natural analogue of Lemma A.7:

Lemma A.16. *With the notations from Lemma A.7, a g-linear morphism of a principal T'-bundle to a principal T-bundle is equivalent to the data of morphisms $\theta : U'_i \to T$ such that the following diagram commutes on intersections:*

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij}\times\theta_j"] \arrow[d, "(\psi_{ij}\circ f)\times\theta_i"'] & T' \times T \arrow[r, "g\times\mathrm{id}"] & T \times T \arrow[d, "m"] \\ T \times T \arrow[rr, "m"] & & T \end{tikzcd}$$

Or in short hand notation,

$$g(\psi'_{ij}(u)) \cdot \theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u). \tag{12}$$

Similarly as in Proposition A.9 one can conclude from this that change of fibre is functorial in the following sense:

Proposition A.17. *Given any homomorphism of group schemes $g : T' \to T$ and a g-linear homomorphism $h : F' \to F$ of formal schemes with T' and T-actions respectively, and a homomorphism $f : E' \to E$ of principal T' and T-bundles over g, one obtains a morphism*

$$h \times^g f : F' \times^{T'} E' \to F \times^T E$$

of fibre bundles over g, in a way that is functorial in h, g, f. More precisely, the associated bundle construction is a fibered bifunctor

$$- \times^- - : \mathbf{FormAct} \times_{\mathbf{FormGrp}} \mathbf{FormPrinBun} \to \mathbf{FormBun}.$$

Proof. Let (E,B,π,T) and (E',B',π',T') be principal bundles. Let F and F' be formal schemes with T-action and T' action respectively. Let $g : T \to T'$ be a group homomorphism and let $h : F' \to F$ be a g-equivariant morphism. Let $f : E' \to E$ be a morphism of principle fibre bundles over g. Then we can find compatible covers $\mathfrak{U}'$ of E' and $\mathfrak{U}$ of E such that locally we have diagrams like in Lemma A.7. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U_i'$ such that we obtain a natural map

$$F' \times U_i' \xrightarrow{(h \times^T \pi)} F \times U_i, \quad (f, u) \mapsto (h(f)\theta_i(u), \pi(u))$$

(as before this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersection Lemma A.7 implies that we have a commutative diagram

$$\begin{array}{ccc} F' \times U_{ij}' & \xrightarrow{h \times^T \pi} & F \times U_{ij} \\ {\scriptstyle \psi_{ij}' \times \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \psi_{ij} \times \mathrm{id}} \\ F' \times U_{ij}' & \xrightarrow{h \times^T \pi} & F \times U_{ij}. \end{array}$$

More precisely, by g-linearity of h one has

$$h(x \cdot \psi_{ij}'(u)) \cdot \theta_j(u) = h(x) \cdot g(\psi_{ij}'(u)) \cdot \theta_j(u) \stackrel{(12)}{=} h(x) \cdot \psi_{ij}(f(u)) \cdot \theta_i(u).$$

This shows that the maps glue to a morphism $h \times^g f$ as desired.

By refining covers, one easily checks that this is functorial in both components. □

All that we have done in this chapter can be done in completely the same way with formal schemes replaced by rigid spaces (covers being replaced by admissible covers) and also for schemes. In particular by functoriality of fibre products there are natural functors from formal principal T-bundles over R to rigid principal T_η-bundles over K on the generic fibre, and to principal $\overline{T}$-bundles on the reduction R/p. Moreover, these generic fibre and reduction functors commute with the associated fibre construction:

Lemma A.18. *Let T be a formal group scheme and let $\pi : E \to B$ be a principial T-bundle. Let F be a formal scheme with an action by T. Then*

$$(E \times^T B)_\eta = E_\eta \times^{T_\eta} B_\eta$$

Proof. This can be checked locally on any trivialising cover, where it is clear. □

References

[1] Siegfried Bosch *Lectures in formal and rigid geometry.*
[2] Siegfried Bosch *Zur Kohomologietheorie rigid analytischer Räume* .
[3] Bosch, Lütkebohmert *Degenerating abelian varieties.*
[4] Jean Fresnel, Marius van der Put *Rigid Analytic Geometry and its Applications.*
[5] Werner Lütkebohmert *Rigid Geometry of Curves and Their Jacobians.*
[6] Peter Scholze *Perfectoid spaces.*
[7] Peter Scholze *Torsion in cohomology of locally symmetric varieties.*
[8] Peter Scholze, Jared Weinstein *moduli spaces of p-divisible groups.*