PERFECTOID LIMITS OF RIGID GROUPS VIA FORMAL MODELS

CLIFFORD BLAKESTAD, BEN HEUER, DAMIAN GVIRTZ, KOJI SHIMIZU, PETER WEAR, DARIA SCHEDRINA, AND ZIJIAN YAO

Abstract. For an abelian variety A over an algebraically closed non-archimedean field, we show that there is a perfectoid space A_∞ such that $A_\infty \sim \varprojlim_{[p]} A$.

Contents

1. Introduction

To do: Rewrite Introduction, summarize main results, outline, remarks about GL_n, etc.
Instead of an actual introduction, for now we have the following
Outline:

For an abelian variety A over an algebraically closed non-archimedean field we show that there is a perfectoid space A_∞ such that $A \sim \varprojlim_{[p]} A$.

We first more generally consider a rigid group G over a non-archimedean field K. While inverse limits usually don't exist in the rigid analytic category, limits are much better behaved in formal schemes over the ring of integers $\mathcal{O}_K$ of K. One can therefore give a simple criterion in terms of formal models that guarantees that a tilde-limit $G_\infty \sim \varprojlim_{[p]} G$ exists, namely that there is a well-behaved formal model of the $[p]$-multiplication tower. If K is perfectoid, we give a stronger criterion involving a Frobenius factorisation condition, which implies that G_∞ is perfectoid.

In the case of a rigid analytic split torus T, one can use a family of explicit covers by affinoids to construct formal models for which both of these conditions are satisfied.

Next we consider the case of the Raynaud extension E associated to a semistable abelian variety A over a perfectoid field K. One can construct E by extending the rigid fibre of a formal group scheme $\overline{E}$ by a rigid torus T. In order to construct a formal model of E one therefore just needs to extend $\overline{E}$ by a formal model of T. While this can be done explicitly using affinoid covers, the language of formal and rigid fibre bundles allows for a more conceptual treatment. Using the associated fibre construction we then show that there is a formal model of the $[p]$-multiplication tower of E which satisfies all the necessary criteria to show that E_∞ exists and is perfectoid.

We then construct a tilde-limit of $\varprojlim_{[p]} A$ from E_∞: By Raynaud uniformisation, A is naturally isomorphic to the rigid analytic quotient of E by a lattice M. After a choice of $\Gamma_0(p^\infty)$-structure, the $[p]$-multiplication tower of E/M factors in a "ramified" and an "étale" part. By a careful choice of charts of E/M in terms of subspaces of E that behave well under $[p^n] : E \to E$, one can explicitly construct first a perfectoid tilde-limit of the "ramified" tower, and then in a second step the space A_∞. This space is independent of the choice of $\Gamma_0(p^\infty)$-structure but remembers it as a pro-étale subgroup $D_\infty \subseteq A_\infty$. The construction shows that the perfectoid tilde-limit A_∞ still exists under the weaker assumption that K is a perfectoid field over which there exist lattices M^{1/p^n} for all n whose p^n-th multiple is M.

The approach via explicit covers finally gives an explicit description of A_∞ in terms of open subspaces of E_∞ which we use in the last section to study the induced map $E_\infty \to A_\infty$. We show that E_∞ is in fact an open subspace of A_∞ and thus obtain a second description of A_∞ as a quotient group $(D_\infty \times E_\infty)/M_\infty$.

Notation:

Throughout we will study rigid analytic spaces over K. If such a space is obtained from a K-scheme X via rigid-analytification $X \mapsto X^{\mathrm{an}}$, we will often denote both by the same symbol X. Also, we will make no distinction between rigid analytic spaces and their corresponding adic spaces.

Throughout we denote by $\mathfrak{X}$ a topologically finite type formal scheme over $\operatorname{Spf} R$ with the π-adic topology. Let $\mathfrak{X}_\eta \to \operatorname{Sp} K$ be its rigid generic fibre, and let $\tilde{X} = \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_K} \operatorname{Spf} \mathcal{O}_K/\pi$ be its special, considered as a scheme over $\operatorname{Spec} \mathcal{O}_K/\pi$. Add definition of tilde limits

Add acknowledgement

2. Tilde-limits of rigid groups

Let p be a prime. Let K be a complete non-archimedean field that is either an extension of $\mathbb{Q}_p$ or of characteristic p. Denote by $\mathcal{O}_K$ the ring of integers and let π denote a pseudo-uniformiser.

Let G be a rigid group, that is a group object in the category of rigid spaces. One way that rigid groups arise is by analytification of finite type group schemes over K: We will be most interested in the analytification of an abelian variety A, but other important cases are the analytifications $\mathbb{G}_a^{\mathrm{an}}$ of $\mathbb{G}_a$ and $\mathbb{G}_m^{\mathrm{an}}$ of $\mathbb{G}_m$, or more generally of tori T over K. A second source of rigid groups are generic fibres of topologically finite type formal group schemes over $\mathcal{O}_K$. A third important example is the covering space E in the sense of Raynaud of an abelian variety with semi-stable reduction.

This note is concerned with the following question:

Question. Given a rigid group G, when is there an adic space G_∞ such that

$$G_\infty \sim \varprojlim_{[p]} G$$

in the sense of [11]? If it exists, and K is perfectoid, when is G_∞ perfectoid?

Note that if a perfectoid tilde-limit exists, it is unique with this property by Proposition 2.4.5 in [11]. Our main goal will be to show that this is true for A an abelian variety over a perfectoid field. But before we give proves for examples of rigid groups G for which a perfectoid tilde-limit exists, we first note that the second question certainly doesn't have an affirmative answer for all rigid group varieties:

Example. For the additive group $\mathbb{G}_a^{\mathrm{an}}$, we know that $[p]$ is an isomorphism and therefore $\varprojlim_{[p]} \mathbb{G}_a = \mathbb{G}_a$ exists (even as an actual limit in the category of adic spaces) but is certainly not perfectoid.

2.1. A condition ensuring that the tilde-limit exists.

Inverse limits often don't exist in the category of adic spaces, and neither do they in rigid spaces. They do, however, often exist in the category of formal schemes:

Lemma 2.1. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps $\phi_{ij} : \mathfrak{X}_j \to \mathfrak{X}_i$. Then the inverse limit $\mathfrak{X} = \varprojlim \mathfrak{X}_i$ exists in the category of formal schemes over $\mathcal{O}_K$. If all the $\mathfrak{X}_i$ are flat formal schemes, so is $\mathfrak{X}$.*

Proof. In the affine case, if the inverse system is $\mathrm{Spf}\, A_i$, take A to be the p-adic completion of $\varinjlim A_i$, then $\mathrm{Spf}\, A$ is the inverse limit of the $\mathrm{Spf}\, A_i$. In general, we can use the fact that the transition maps are affine to reduce to the affine case. ☐

In the situation of the lemma, the transition maps are affine and hence quasi-compact and quasi-separated, so after passing to adic spaces, $\mathfrak{X}$ is also the tilde-limit $\mathfrak{X} \sim \varprojlim \mathfrak{X}_i$ in the sense of [11]. Even better, this remains true after passing to the generic fibre $\mathrm{Spa}(K, \mathcal{O}_K) \to \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$.

Lemma 2.2. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps ϕ_{ij} and let $\mathfrak{X} = \varprojlim_{\phi_j} \mathfrak{X}_i$ be the limit. Let $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$ and $\mathcal{X} = (\mathfrak{X})_\eta$ be the adic generic fibres. Then*

$$\mathcal{X} \sim \varprojlim \mathcal{X}_i.$$

Proof. This is a consequence of [11], Proposition 2.4.2: The transition maps in the system are affine, hence quasi-separated quasi-compact. In order to prove the Lemma, we can restrict to an affine open subset $\mathrm{Spf}(A)$ of $\mathfrak{X}$ that arises as the inverse limit of affine open subsets $\mathrm{Spf}(A_i) \subseteq \mathfrak{X}_i$. Here all formal schemes are considered with the π-adic topology and A is the π-adic completion of $\varinjlim A_i$. On the generic fibre, A_i with ideal of definition $I_i = \pi A_i$ is an open subring of definition of $A_i[1/\pi]$. We then clearly have $I_i A_j = A_j$ for any $j \geq i$. The inverse system therefore satisfies the conditions of [11], Proposition 2.4.2, and we conclude that $\mathrm{Spf}(A)_\eta \sim \varprojlim \mathrm{Spf}(A_i)_\eta$ as desired. □

Remark 2.3. This means that one can always construct the limit of an inverse system of rigid spaces $\mathcal{X}_i$ if it arises from an inverse system of formal schemes $\mathfrak{X}_i$ with affine transition maps. This is precisely what Scholze uses in [10] in order to construct the space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ (see the proof of Corollary III.2.19 in [10]).

If one starts with an inverse system of rigid spaces $\mathcal{X}_i$, a straightforward strategy to construct "the" tilde limit $\varprojlim \mathcal{X}_i$ is thus to look for formal models $\mathfrak{X}_i$, that is formal schemes over $\mathrm{Spf}\,\mathcal{O}_K$ such that $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$, as well as affine formal models $\phi_{ji} : \mathfrak{X}_j \to \mathfrak{X}_i$ of the transition maps. If such data exists, Lemma 2.2 produces a tilde-limit $\mathcal{X} \sim \varprojlim \mathcal{X}_i$. Here we follow the following standard terminology:

Definition 2.4.
(1) Let $\mathcal{X}$ be a rigid space over K. Then a **formal model** of $\mathcal{X}$ is an admissible topologically finite type formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ together with an isomorphism of its generic fibre $\mathfrak{X}_\eta \xrightarrow{\sim} \mathcal{X}$ (which is often suppressed from notation).
(2) Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a morphism of rigid spaces over K. Let $\mathfrak{X}, \mathfrak{Y}$ be formal models of $\mathcal{X}, \mathcal{Y}$ respectively. Then a morphism of formal schemes $\Phi : \mathfrak{X} \to \mathfrak{Y}$ is a **formal model** of ϕ if the following diagram commutes:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\phi"] & \mathcal{Y} \\ \mathfrak{X}_\eta \arrow[u, "\cong"] \arrow[r, "\Phi"] & \mathfrak{Y}_\eta \arrow[u, "\cong"'] \end{tikzcd}$$

The theory of Raynaud's formal models explains under which circumstances formal models of rigid spaces and their maps exist. We need the following definition:

Definition 2.5 ([1], Def 8.2.12)**.** A topological (resp G-topological) space X is called **quasi-paracompact** if there exists an open (resp admissible open) cover $\mathfrak{U}$ of X such that

- each $U \in \mathfrak{U}$ is quasi-compact and
- the cover $\mathfrak{U}$ is of finite type, that is for each $U_i \in \mathfrak{U}$ there are only finitely many $U_j \in \mathfrak{U}$ such that $U_i \cap U_j \neq \emptyset$.

For instance, the spaces $\mathbb{G}_a^{\mathrm{an}}$ and $\mathbb{G}_m^{\mathrm{an}}$ are not quasi-compact, but they are quasi-paracompact since they can be covered using families of annuli that are admissible covers of finite type. Similarly, one should be able to show (replacing annuli by intersections of Laurent domains and Weierstrass domains) that if X is any quasi-compact space and $S \subseteq X$ is a Zariski-closed subset, then $X \setminus S$ is quasi-paracompact.

The main result of Raynaud's theory of formal models is then:

Theorem 2.6 ([1], section 8.4)**.**
(1) Let X be a quasi-separated quasi-paracompact rigid space over K. Then there exist an admissible quasi-paracompact formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ such that $X = \mathfrak{X}_\eta$.

(2) *If $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blow-up of admissible formal schemes, then its generic fibre is an isomorphism $\mathfrak{X}'_\eta \xrightarrow{\sim} \mathfrak{X}_\eta$.*
(3) *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be admissible quasi-paracompact formal schemes over $\mathcal{O}_K$ and let $f : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ be a morphism of their associated rigid spaces. Then there exist an admissible blow-up $\pi : \mathfrak{X}' \to \mathfrak{X}$ and a map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{Y}$ such that $\mathfrak{f}_\eta = f \circ \pi_\eta$.*

$$\begin{tikzcd} \mathfrak{X}' \arrow[d, "\pi"] \arrow[dr, "\mathfrak{f}"] & \\ \mathfrak{X} & \mathfrak{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{X}'_\eta \arrow[d, "\pi_\eta"', "\cong"] \arrow[dr, "\mathfrak{f}_\eta"] & \\ \mathfrak{X}_\eta \arrow[r, "f"] & \mathfrak{Y}_\eta \end{tikzcd}$$

The theorem implies that given an inverse system $(\mathcal{X}_i, \phi_{ij})$ of rigid spaces, one can always choose formal models $\mathfrak{X}_i$ and by successive admissible blow-ups while going along the inverse system one can also find models for the ϕ_{ij}. If it is possible to do this in such a way that transition maps are affine, this way one always obtains a construction of G_∞. More precisely, we can formalise this as follows:

Definition 2.7. For a rigid analytic group G, we call **$[p]$-model tower** the data of:
(1) a family of formal models $\mathfrak{G}_n$ of G for $n \in \mathbb{N}$,
(2) morphisms of formal schemes $[p] : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$ satisfying the following conditions:
  (a) $[p] : \mathfrak{G}_n \to \mathfrak{G}_{n+1}$ is a formal model of $[p] : G \to G$.
  (b) $[p]$ is an affine morphism

More generally, by a $[p]$-model tower of an admissible open subset $U \subseteq G$ we mean the same formal model data for the tower of pull-backs over U

$$\begin{tikzcd} \dots \arrow[r] & {[p^2]}^{-1}(U) \arrow[r, "{[p]}"] \arrow[d, hook] & {[p]}^{-1}(U) \arrow[r, "{[p]}"] \arrow[d, hook] & U \arrow[d, hook] \\ \dots \arrow[r, "{[p]}"] & G \arrow[r, "{[p]}"] & G \arrow[r, "{[p]}"] & G \end{tikzcd}$$

We can summarise our discussion in this chapter by the following Proposition:

Proposition 2.8. *Let G be a rigid analytic group. Then if G has a $[p]$-model tower, there exists a space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$.*

2.2. **A condition ensuring that the tilde-limit is perfectoid.** First we consider the case where $G = A$ is an abelian variety with good reduction. Slightly more generally, we have

Proposition 2.9. *Assume that K is perfectoid. Let $\mathfrak{G}$ be a flat commutative formal group scheme of topologically finite type over $\mathcal{O}_K$ for which the p-multiplication map $[p] : \mathfrak{G} \to \mathfrak{G}$ is affine. Let $G = \mathfrak{G}_\eta$ be the rigid group obtained on the generic fibre. Then G_∞ exists and is perfectoid.*

Proof. Following notation from the previous subsection, the map $[p] : \mathfrak{G} \to \mathfrak{G}$ is a formal model of the map $[p]$ on its rigid generic fiber. By Lemma 2.2 we therefore have

$$G_\infty = (\varprojlim_{[p]} \mathfrak{G})_\eta \sim \varprojlim_{[p]} G.$$

To see that G_∞ is perfectoid, we proceed exactly like in the proof of [10], Corollary III.2.19. It suffices to prove that $\mathfrak{G}_\infty = \varprojlim_{[p]} \mathfrak{G}$ can be covered by formal schemes of the form $\mathrm{Spf}(S)$ where S

is a flat $\mathcal{O}_K$-algebra such that the Frobenius map

$$S/p^{1/p} \to S/p$$

is an isomorphism. Lemma 5.6 of [7] then shows that $S[1/p]$ is perfectoid.

The key observation here is that upon reduction mod p, the p-multiplication factors through relative Frobenius. More precisely, denote by $\tilde{G}$ the reduction of $\mathfrak{G}$ mod p. Then $[p] : \tilde{G} \to \tilde{G}$ factors as

$$\begin{tikzcd} & \tilde{G} \arrow[dr, dashed] & \\ \tilde{G}^{(p)} \arrow[ur, "F_{rel}"] \arrow[rr, "{[p]}"] & & \tilde{G} \end{tikzcd}$$

This has the following consequence: Let $\mathrm{Spf}(S_1)$ be any affine open subspace of $\mathfrak{G}$ and let $\mathrm{Spf}\, S_n$ be the pullback via $[p^n] : \mathfrak{G} \to \mathfrak{G}$. Then we have a commutative diagram:

$$\begin{tikzcd} & & \tilde{S}_n^{(p)} \arrow[dr, "F_{rel}"] & & \tilde{S}_{n+1}^{(p)} \arrow[dr, "F_{rel}"] & & \\ \dots \arrow[r] & \tilde{S}_{n-1} \arrow[ur, dashed, "V"] \arrow[rr] & & \tilde{S}_n \arrow[ur, dashed, "V"] \arrow[rr, "{[p]}"] & & \tilde{S}_{n+1} \arrow[r] & \dots \end{tikzcd}$$

From this we can check on elements that relative Frobenius is an isomorphism on $\tilde{S}_\infty := \varinjlim_n \tilde{S}_n$. Since K is perfectoid, we moreover have an isomorphism $\mathcal{O}_K/p^{1/p} \to \mathcal{O}_K/p$ from the absolute Frobenius on $\mathcal{O}_K/p$. Therefore absolute Frobenius on S_∞/p induces an isomorphism

$$S_\infty/p^{1/p} \xrightarrow{\sim} S_\infty/p.$$

Since $\mathfrak{G}$ is flat, so are the S_n and thus so is S_∞. Thus $S_\infty[1/p]$ is a perfectoid K-algebra. Since G_∞ is covered by affinoids of the form $\mathrm{Spf}(S_\infty)_\eta$, this shows that G_∞ is perfectoid. $\square$

Corollary 2.10. *Let A be an abelian variety of good reduction over a perfectoid field K. Then A_∞ exists and is perfectoid.*

Before we proceed, we would like to mention two illustrative examples:

Example. Let $\mathfrak{G}$ be the p-adic completion of the affine group scheme $\mathbb{G}_m$ over $\mathcal{O}_K$, that is the formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, S$ where $S = \mathcal{O}_K\langle X^{\pm 1}\rangle$. It is clear that $\mathbb{G}$ satisfies the conditions of Proposition 2.9, so for the generic fibre $G = \mathfrak{G}_\eta$ we obtain a perfectoid tilde-limit $G_\infty = \varprojlim_{[p]} G$. More precisely, the $[p]$-multiplication map corresponds to the homomorphism

$$[p] : \mathcal{O}_K\langle X^{\pm 1}\rangle \to \mathcal{O}_K\langle X^{\pm 1}\rangle, \quad X \to X^p.$$

In the direct limit, we obtain the algebra $S_\infty = (\varinjlim_{[p]} S)^\wedge = \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle$. On the generic fibre we thus obtain

$$G_\infty = \mathrm{Spa}(K\langle X^{\pm 1/p^\infty}\rangle, \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle)$$

and one can verify by hand that we indeed have $G_\infty \sim \varprojlim_{[p]} G$.

Example. An example of a very different flavour is $\mathfrak{G}$ the p-adic completion of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Note that $G = \mathfrak{G}_\eta$ is not equal to $\mathbb{G}_a^{an}$, but is the closed unit disc in the latter.

While the underlying formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, S$ where $S = \mathcal{O}_K\langle X\rangle$ as before, the $[p]$-multiplication is now given by

$$[p] : \mathcal{O}_K\langle X\rangle \to \mathcal{O}_K\langle X\rangle, \quad X \to pX.$$

In the direct limit, we first obtain the algebra $S'_\infty = \varinjlim_{[p]} S = \mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle$ of power series $f = \sum_{n=0}^\infty a_n X^n \in \mathcal{O}_K[[X]]$ for which there is $m \in \mathbb{Z}_{\geq 0}$ such that $|p^{nm} a_n| \to 0$. In order to form S_∞, we need to p-adically complete S'_∞. But we have

$$p^n \mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle = p^n \mathcal{O}_K + X\mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle$$

and therefore $S'_\infty/p^n = \mathcal{O}_K/p^n\mathcal{O}_K$. Consequently, the completion is $S_\infty = \mathcal{O}_K$ and thus $G_\infty = \mathrm{Spa}(K, \mathcal{O}_K)$ is perfectoid, but just one point!

Geometrically, this makes sense: On the level of K-points, the formal scheme G is the closed unit disc and $[p]$ is scaling points by p. A K-point in $\varprojlim_{[p]} G(K)$ therefore corresponds to a sequence of K-points of the closed unit disc of the form $x, \frac{1}{p}x, \frac{1}{p^2}x, \dots$. But for this to be contained in the closed unit disc, we must have $x = 0$. Thus $\varprojlim_{[p]} G(K) = 0$.

Looking closely at Proposition 2.9, all we need is in fact a model for the p-multiplication morphism $[p] : G \to G$, and we never use that all the $\mathfrak{G}$ in the tower are copies of the same formal scheme. Weakening these two conditions, we arrive at the following definition:

Definition 2.11. For a rigid analytic group G, we call **$[p]$-F-model tower** the data of:

1. a family of flat formal models $\mathfrak{G}_n$ of G for $n \in \mathbb{N}$,
2. morphisms of formal schemes $[\mathfrak{p}] : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$ satisfying the following conditions:
   - (a) the generic fibre of $[\mathfrak{p}] : \mathfrak{G}_n \to \mathfrak{G}_{n+1}$ coincides with $[p] : G \to G$.
   - (b) $[\mathfrak{p}]$ is an affine morphism
   - (c) Denote by $\tilde{G}_n$ the reduction of $\mathfrak{G}_n$ mod p. Then $[\mathfrak{p}]$ factors through the relative Frobenius morphism:

$$\begin{tikzcd} & \tilde{G}_n^{(p)} \arrow[dr, dashed] & \\ \tilde{G}_n \arrow[ur, "F_{rel}"] \arrow[rr, "{[p]}"] & & \tilde{G}_{n-1} \end{tikzcd}$$

More generally, as before, we can consider $[p]$-F-model towers of admissible open subset $U \subseteq G$ by which we mean the same formal model data for the tower of pull-backs over U.

Example. If $\mathfrak{G}$ is a flat commutative formal group scheme such that p-multiplication is affine, then setting $\mathfrak{G}_n = \mathfrak{G}$ and taking for $[\mathfrak{p}]$ the actual p-multiplication maps $[p]$ defines a $[p]$-F-model tower for the rigid analytic group $G = \mathfrak{G}_\eta$.

The same proof for Proposition 2.9 works with the weakened conditions:

Proposition 2.12. *Let G be a rigid analytic group over a perfectoid field K. If G admits a $[p]$-F-model, then G_∞ exists and is perfectoid.*

What we aim to prove in the rest of this write-up is that for a Raynaud extension $0 \to T \to E \to B \to 0$, there is a $[p]$-F-model for T which induces a $[p]$-F-model for E. This will prove that tilde-limits T_∞ and E_∞ exist and are perfectoid if K is perfectoid.

2.3. **A few consequences.** One reason why perfectoid limits along group morphisms are particularly interesting is that the perfectoidness ensures that the limit has again a group structure:

Definition 2.13. By a **perfectoid group** we mean a group object in the category of perfectoid spaces. Note that the category of perfectoid spaces over K has finite products, so the notion of a group object makes sense.

Proposition 2.14. *Let G be a rigid group and assume that there is a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$. Then*

(1) there is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms
(2) given a rigid group H with perfectoid tilde-limit $H_\infty \sim \varprojlim_{[p]} H$ and a group homomorphism $H \to G$, there is a unique group homomorphism $H_\infty \to G_\infty$ commuting with all projection maps. In particular, formation of $\varprojlim_{[p]}$ is functorial.

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, cf Proposition 2.4.5 of [11], which shows that one can argue like in the case of usual limits. □

3. Formal models for tori

Let K be perfectoid. In this section we want to show that for a split rigid torus T over K, a tilde-limit T_∞ exists and is perfectoid. We do this by exhibiting a $[p]$-F-model of T.

As a preparation, we consider the torus $\mathbb{G}_m^{\mathrm{an}}$ over K. Recall that it arises from rigid analytification of the affine torus $\mathbb{G}_m$ over K. Note however that $\mathbb{G}_m^{\mathrm{an}}$ is not affinoid (and not even quasi-compact). It contains the generic fibre of the p-adic completion of $\mathbb{G}_m$ as an open subspace. If we see $\mathbb{G}_m^{\mathrm{an}}$ as the rigid affine line with origin removed, this subspace $\widehat{\mathbb{G}}_m$ can be identified with the open annulus of radius 1. In other words, on the level of points it corresponds to $\mathcal{O}_K^\times \subseteq K^\times$.

Finally, recall that for every $x \in K^\times$ we have a translation map

$$\mathbb{G}_m^{\mathrm{an}} \xrightarrow{x\cdot} \mathbb{G}_m^{\mathrm{an}}$$

that is an isomorphism of rigid spaces sending the point 1 to x.

3.1. A family of explicit covers.

We briefly recall how $\mathbb{G}_m^{\mathrm{an}}$ is constructed: The following is inspired by [1], §9.2, although we choose slightly different constructions. Throughout we use the following shorthand notation: for any $a \in K$ we write

$$K\langle X, a/X\rangle = K\langle X, Z\rangle/(X \cdot Z - a).$$

Let $q \in K^\times$ with $|q| \le 1$. Consider the annulus $\mathcal{B}(q,1)$ of radii $|q|$ and 1 inside $\mathbb{A}_K^{\mathrm{an}}$:

$$\mathcal{B}(q,1) = \mathrm{Sp}(L_q), \quad \text{where } L_q = K\langle X, q/X\rangle/a.$$

Similarly, for $q \in K^\times$ with $|q| \ge 1$ one constructs the annulus $\mathcal{B}(1,q)$ by

$$\mathcal{B}(1,q) = \mathrm{Sp}(L_q), \quad \text{where } L_q = K\langle X/q, 1/X\rangle$$

where $K\langle X/q\rangle$ denotes the ring of those power series $f = \sum c_n X^m \in K[[X]]$ for which $|c_n/q| \to 0$ for $n \to \infty$. In particular, we have isomorphisms

$$K\langle X', q^{-1}/X'\rangle \cong K\langle X/q, 1/X\rangle, \quad X' \mapsto q^{-1}X.$$

One can now construct $\mathbb{G}_m$ as follows: Choose sequences $a_n, b_n \in K^\times$ with $a_0 = 1 = b_0$ such that $|a_n| < |a_{n-1}| < ... < 1$ and $|a_n| \to 0$ and similarly $|b_n| > |b_{n-1}| > ... > 1$ and $|b_n| \to \infty$. Then one can glue the annuli $\mathcal{B}(a_n, 1)$ and $\mathcal{B}(1, b_n)$ using the following maps:

$$\begin{aligned} \mathcal{B}(a_n,1) &\hookleftarrow \mathcal{B}(a_{n-1},1) \\ L_{a_n} = K\langle X, a_n/X\rangle &\to K\langle X, a_{n-1}/X\rangle = L_{a_{n-1}} \\ X, a_n/X &\mapsto X, \frac{a_n}{a_{n-1}} a_{n-1}/X \end{aligned} \tag{1}$$

and similarly

$$\begin{aligned} \mathcal{B}(1,b_n) &\hookleftarrow \mathcal{B}(1,b_{n-1}) \\ L_{b_n} = K\langle X/b_n, 1/X\rangle &\to K\langle X/b_{n-1}, 1/X\rangle = L_{b_{n-1}} \\ X/b_n, 1/X &\mapsto \frac{b_{n-1}}{b_n} X/b_{n-1}, 1/X. \end{aligned} \tag{2}$$

Also, via the above maps, the annuli $\mathcal{B}(a_n, 1)$ and $\mathcal{B}(1, b_m)$ are glued along $\mathcal{B}(a_0, 1) = \mathcal{B}(1, 1) = \mathcal{B}(1, b_0)$. This gives the desired space $\mathbb{G}_m^{\mathrm{an}}$.

Since we are mainly interested in the p-multiplication map, we will more precisely use the following cover on which $[p]$ can be seen directly: Choose $q \in K^\times$ with $|q| < 1$. Then for the sequences a_n and b_n from above we take $a_n = q^n$, $b_n = q^{-n}$. We call this cover $\mathfrak{U}_q$.

Assume now that q has a p-th root $q^{1/p}$ in K. The above then gives a finer cover $\mathfrak{U}_{q^{1/p}}$ of $\mathbb{G}_m^{\mathrm{an}}$. Using both covers $\mathfrak{U}_q$ and $\mathfrak{U}_{q^{1/p}}$, we can easily see the $[p]$-multiplication $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ as follows: Consider the affinoid open subsets $\mathcal{B}(q^{1/p}, 1)$ of the source and $\mathcal{B}(q^{1/p}, 1)$ of the target. Then $[p]$ restricts to

$$\begin{aligned} \mathcal{B}(q,1) &\xleftarrow{[p]} \mathcal{B}(q^{1/p},1) \\ K\langle X, q/X\rangle &\to K\langle X, q^{1/p}/X\rangle \\ X, q/X &\mapsto X^p, (q^{1/p}/X)^p \end{aligned} \tag{3}$$

and similarly, on $\mathcal{B}(1, q^{-1/p})$ and $\mathcal{B}(1, q^{-1})$ the map is

$$\begin{aligned} \mathcal{B}(1,q^{-1}) &\xleftarrow{[p]} \mathcal{B}(1,q^{-1/p}) \\ K\langle X/q, 1/X\rangle &\to K\langle X/q^{1/p}, 1/X\rangle \\ X/q, 1/X &\mapsto (X/q^{1/p})^p, (1/X)^p. \end{aligned} \tag{4}$$

The same works for the other affinoid open subspaces $\mathcal{B}(q^n, 1) \xleftarrow{[p]} \mathcal{B}(q^{n/p}, 1)$ and for $\mathcal{B}(1, q^{-n}) \xleftarrow{[p]} \mathcal{B}(1, q^{-n/p})$. One can then show that the maps (3) and (4) are compatible with the glue maps (1) and (2). In the case of (3) this is basically because $a_n/a_{n-1} = q$ or $a_n/a_{n-1} = q^{1/p}$ depending on whether we work with $\mathfrak{U}_q$ or $\mathfrak{U}_{q^{1/p}}$ respectively, and the only thing to check is that the following diagram commutes:

$$\begin{tikzcd} \mathcal{B}(q^n,1) & \mathcal{B}(q^{n-1},1) \arrow[l, hook'] & & q^n/X \arrow[r, mapsto] \arrow[d, mapsto] & q\cdot q^{n-1}/X \arrow[d, mapsto] \\ \mathcal{B}(q^{n/p},1) \arrow[u, "{[p]}"] & \mathcal{B}(q^{(n-1)/p},1) \arrow[l, hook'] \arrow[u, "{[p]}"'] & & (q^{n/p}/X)^p \arrow[r, mapsto] & q\cdot (q^{(n-1)/p}/X)^p. \end{tikzcd} \tag{5}$$

The case of (4) is very similar.

3.2. **A family of formal models.** Recall that we have constructed a cover $\mathfrak{U}_q$ of $\mathbb{G}_m^{\mathrm{an}}$ depending on a choice of $q \in K^\times$ with $|q| < 1$. The affinoid subspaces $\mathcal{B}(q^n, 1)$ that we have used for this admit natural formal models: Namely, consider the $\mathcal{O}_K$-algebra

$$L_q^\circ := \mathcal{O}_K\langle X, Z\rangle/(XZ - q).$$

This is clearly of topologically finite type over $\mathcal{O}_K$. It is moreover flat as an $\mathcal{O}_K$-algebra (this should follow from Lemma 8.2.1 in [1]). For the same reason (or by $L_{q^{-1}} \cong L_q$) we see that

$$L_{q^{-1}}^\circ := \mathcal{O}_K\langle X', Z\rangle/(X'Z - q)$$

is a flat topologically finite type $\mathcal{O}_K$-algebra. Consequently, we have flat formal models

$$\begin{aligned} \mathfrak{B}(q,1) &:= \mathrm{Spf}(L_q^\circ), & \mathfrak{B}(q,1)_\eta &= \mathcal{B}(q,1) \\ \mathfrak{B}(1,q) &:= \mathrm{Spf}(L_{q^{-1}}^\circ), & \mathfrak{B}(1,q)_\eta &= \mathcal{B}(1,q) \end{aligned}$$

For the glueing maps (1) and (2) it is clear from $a_n/a_{n-1} = b_{n-1}/b_n = q$ that these extend to glueing maps $\mathfrak{B}(q^n, 1) \hookleftarrow \mathfrak{B}(q^{n-1}, 1)$ and $\mathfrak{B}(1, q^{-n}) \hookleftarrow \mathfrak{B}(1, q^{-(n-1)})$. We conclude:

Lemma 3.1. *The affine formal schemes $\mathfrak{B}(q^n, 1)$ and $\mathfrak{B}(1, q^n)$ glue together to a flat formal scheme $\mathfrak{G}_q$ such that $(\mathfrak{G}_q)_\eta = \mathbb{G}_m^{\mathrm{an}}$. In other words, $\mathfrak{G}_q$ is a formal model for $\mathbb{G}_m^{\mathrm{an}}$.*

3.3. **A family of formal models for p-multiplication.** As before choose $q \in K^\times$ such that $|q| < 1$ and such that there exists a p-th root $q^{1/p} \in K$. A closer look at the maps (3) and (4) shows that the $[p]$-multiplication extends to a morphism of formal schemes

$$\mathfrak{B}(q, 1) \xleftarrow{[p]} \mathfrak{B}(q^{1/p}, 1) : [\mathfrak{p}]$$

and similarly for $\mathfrak{B}(1, q^{-1})$. The diagram (5) shows that these maps glue to a morphism

$$[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q.$$

By construction, after tensoring $- \otimes_{\mathcal{O}_K} K$ all morphisms on algebras coincide with those defined in (1), (2), (3), (4) respectively. We conclude:

Proposition 3.2. *The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ is a formal model of $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$.*

We moreover see directly from the construction:

Proposition 3.3. *The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ reduces mod p to the relative Frobenius map.*

We now have everything together to finish our proof that $(\mathbb{G}_m^{\mathrm{an}})_\infty$ is perfectoid:

Proposition 3.4. *The space $\mathbb{G}_m^{\mathrm{an}}$ has a $[p]$-F-model tower. In particular, there exists a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ such that $(\mathbb{G}_m^{\mathrm{an}})_\infty \sim \varprojlim_{[p]} \mathbb{G}_m^{\mathrm{an}}$.*

Proof. Since K is perfectoid, we can find $q \in K^\times$ such that $|q| < 1$ for which there exist arbitrary p^n-th roots. We choose such a q and roots q^{1/p^n} for all n. Then the two Propositions above combine to show that

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_q$$

is a $[p]$-F-model tower. Proposition 2.12 then gives the desired space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ and shows that it is perfectoid. $\square$

3.4. **The action of $\overline{T}$.** The multiplication $\mathbb{G}_m^{\mathrm{an}} \times \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ can locally be described in terms of the rigid analytic cover that we have defined above as follows : Let $a, b \in K^\times$ such that $|a|, |b| \leq 1$, then the multiplication map restricts to

$$\begin{aligned} \mathcal{B}(a,1) \times \mathcal{B}(b,1) &\xrightarrow{m} \mathcal{B}(ab,1) \\ K\langle X, ab/X\rangle &\to K\langle X, a/X\rangle \widehat{\otimes} K\langle X, b/X\rangle \\ X &\mapsto X \otimes X \\ ab/X &\mapsto b/X \otimes a/X \end{aligned} \tag{6}$$

and similarly on the $\mathcal{B}(1,a) \times \mathcal{B}(1,b)$ for $|a|, |b| \geq 1$. Multiplication on the $\mathcal{B}(a,1) \times \mathcal{B}(1,b)$ for $|a| < 1 < |b|$ is more difficult to see on the cover that we have chosen.

The same arguments as in the last section show that the map described in (6) has a flat formal model

$$\mathfrak{B}(a,1) \times \mathfrak{B}(b,1) \to \mathfrak{B}(ab,1).$$

This does *not* mean that multiplication has a formal model $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$. Indeed, the chosen description has different covers on source and target which in the formal case give rise to different formal schemes (the inversion map $i : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ on the other hand does have a formal model). Nevertheless, if we take $a = 1$ in the above , we see that we do have an action of the torus $\overline{T} := \mathfrak{B}(1,1)$ on each of $\mathfrak{B}(b,1)$ and $\mathfrak{B}(1,b)$. Using the formal models from the last section, we conclude:

Proposition 3.5. *For any $q \in K^\times$ with $|q| < 1$, the formal torus $\overline{T} := \mathfrak{B}(1,1)$ has a natural action on $\mathfrak{G}_q$ via a map*

$$\mathfrak{m} : \overline{T} \times \mathfrak{G}_q \to \mathfrak{G}_q.$$

This map is a formal model of the action of the annulus $\mathcal{B}(1,1)$ on $\mathbb{G}_m^{\mathrm{an}}$. Furthermore, this action is compatible with the models for $[p]$ in the sense that if there is a p-th root $q^{1/p} \in K$, then the following diagram commutes.

$$\begin{array}{ccc} \overline{T} \times \mathfrak{G}_{q^{1/p}} & \xrightarrow{\mathfrak{m}} & \mathfrak{G}_{q^{1/p}} \\ \big\downarrow{\scriptstyle [p]\times[\mathfrak{p}]} & & \big\downarrow{\scriptstyle [\mathfrak{p}]} \\ \overline{T} \times \mathfrak{G}_q & \xrightarrow{\mathfrak{m}} & \mathfrak{G}_q. \end{array}$$

Proof. The existence of $\mathfrak{m}$ follows from the above consideration concerning the map (6). The rest is clear from the construction: All adic rings we have used in the construction are $\mathcal{O}_K$-subalgebras of the affinoid K-algebras used to define $\mathbb{G}_m^{\mathrm{an}}$, so the equalities hold because they hold for $\mathbb{G}_m^{\mathrm{an}}$. $\square$

3.5. **The case of general tori.** By taking products everywhere, all of the statements in this section immediately generalises to split tori:

Corollary 3.6. *Let T be a split torus over K of the form $T = (\mathbb{G}_m^{\mathrm{an}})^d$. Then for any $q \in K^\times$ with $|q| < 1$ the formal scheme $\mathfrak{T}_q := (\mathfrak{G}_q)^d$ is a formal model of T. If there is a p-th root $q^{1/p} \in K$, the p-multiplication map has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ that locally on polyannuli is of the form $[\mathfrak{p}] : \mathfrak{B}(q^{1/p},1)^d \to \mathfrak{B}(q,1)^d$. Moreover this map reduces mod p to the relative Frobenius morphism.*

Corollary 3.7. *Let T be a split torus over K, considered as a rigid space. Then T has a $[p]$-F-model tower. In particular, there exists a perfectoid space T_∞ such that $T_\infty \sim \varprojlim_{[p]} T$.*

Corollary 3.8. *Let T be any split torus over K. For any $q \in K^\times$ with $|q| < 1$, the formal completion $\overline{T}$ has a natural action on $\mathfrak{T}_q$ via a map*

$$\mathfrak{m} : \overline{T} \times \mathfrak{T}_q \to \mathfrak{T}_q.$$

This map is a formal model of the action of the annulus $\overline{T}$ on T. Furthermore, this action is compatible with the models for $[p]$ in the sense that if there is a p-th root $q^{1/p} \in K$, then the map $[\mathfrak{p}] : \mathfrak{T}_q^{1/p} \to \mathfrak{T}_q$ is semi-linear with respect to $[p] : \overline{T} \to \overline{T}$.

4. Raynaud extensions as principal bundles of formal and rigid spaces

In the following discussion let A be an abelian variety over K of semi-stable reduction. We denote by N the identity component of the Néron-model and by $\overline{E}$ its completion along the special fibre. Then by the theory of Raynaud, $\overline{E}$ is a formal group that fits into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0 \tag{7}$$

where $\overline{B}$ is the completion of an abelian variety B over K of good reduction (we also denote by B the rigid space associated to it), and $\overline{T}$ is the completion of a torus of rank r over K. After passing to a finite extension of K, we can always assume that the torus is split. The rigid generic fibre $\overline{T}_\eta$ of the torus $\overline{T}$ canonically embeds into the torus T^{an} which again we simply denote by T. One can show that this induces a pushout exact sequence in the category of rigid groups, see §1 of [3]. More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout.

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{8}$$

The abelian variety A we started with can then be uniformized in terms of E as follows:

Definition 4.1. A subset M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete torsion-free subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}}^r$.

Proposition 4.2. *There exists a lattice $M \subseteq E$ of rank equal to the rank r of the torus for which E/M exists as a rigid space, has a group structure such that $E \to E/M$ is a rigid group homomorphism, and for which there is a natural isomorphism*

$$A = E/M.$$

Since M is discrete, the local geometry of A is thus determined by the local geometry of E. More precisely, we will first study the $[p]$-multiplication tower of E and deduce properties of the $[p]$-multiplication tower of A later.

In order to do so, we would like to study the local geometry of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that one cannot direcly study short exact sequences locally on T, E or B. An important tool is therefore the following Lemma:

Lemma 4.3 ([3], §1)**.** *The short exact sequence (7) admits local sections, that is there is a cover of B by formal open subschemes U_i such that there exist sections $s : U_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times U_i \hookrightarrow E$.*

Proof. This is proved in Proposition A.2.5 in [5], in terms of the group $\mathrm{Ext}(B, T)$. □

The last Lemma suggest that instead of considering Raynaud extensions from the abelian category viewpoint, one should consider them as fibre bundles of formal schemes with structure group T, or more precisely as principal T-bundles of formal schemes, which are also called torsors. This is the language we want to use in the following: We will work with fibre bundles of formal schemes, rigid spaces and schemes. The main technical tool we will need is the associated fibre construction in these settings. For a rigorous treatment of these we refer to the Appendix which should be replaced by a link to the relevant literature.

First of all, we note that the sequence (7) from the last section gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}_\eta$ via push-out from $\overline{T}_\eta \to T$ can now conveniently be expressed in terms of the associated fibre bundle by saying that $E_\eta = T \times^{\overline{T}_\eta} \overline{E}_\eta$ in the sense of Definition A.8. We have the following description of $[p]$:

Lemma 4.4. *The map $[p] : E \to E$ coincides with the morphism*

$$[p] \times^{[p]} [p] : T \times^{\overline{T}_\eta} \overline{E}_\eta \to T \times^{\overline{T}_\eta} \overline{E}_\eta$$

induced by the different $[p]$-multiplication maps by Proposition A.17.

Proof. Lemma A.18 in light of Remark A.19 applied to the maps $g = [p] : \overline{T}_\eta \to \overline{T}_\eta$, $h = [p] : T \to T$ and $f = [p] : \overline{E}_\eta \to \overline{E}_\eta$ says that there is a unique morphism of fibre bundles $E \to E$ making the following diagram commute:

(9)

$$\begin{tikzcd}
& T \arrow[rr] & & E \\
T \arrow[ur, "{[p]}"] \arrow[rr] & & E \arrow[ur, dotted, "\exists!"] & \\
& \overline{T}_\eta \arrow[uu, hook] \arrow[rr] & & \overline{E}_\eta \arrow[uu] \\
\overline{T}_\eta \arrow[uu, hook] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E}_\eta \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd}$$

Since $[p] : E \to E$ is such a map, the Lemma follows. □

5. Formal models for E

In this subsection we prove that E admits a $[p]$-F-tower model. The first step is to construct a family of formal models for E. We do this by using the formal models $\mathfrak{T}_q$.

Proposition 5.1. *Let $q \in K^\times$ with $|q| < 1$. Let $\mathfrak{T}_q$ be the formal model from Corollary 3.6. Then there is a formal scheme $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E}$ that is a formal model of the rigid space E. Furthermore, there exists a morphism*

$$\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$$

which is a fibre bundle and a formal model of $E \to B$.

Proof. Recall from Proposition 3.8 that $\mathfrak{T}_q$ has a $\overline{T}$-action that is a model of the $\overline{T}_\eta$-action on T. In particular, the associated fibre construction for the principal $\overline{T}$-bundle $\overline{E}$ gives a fibre bundle $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$. Since $\mathfrak{T}_q$ is a formal model of T, one sees by Lemma A.20 that this is a formal model of $T \times^{\overline{T}_\eta} \overline{E}_\eta$ which by definition is equal to E. □

Next we want to construct a model for the $[p]$-multiplication map. Here we can use again that $[p]$ exists on $\overline{E}$ and on $\mathfrak{T}_{q^{1/p}}$.

Proposition 5.2. *Let $q \in K^\times$ be such that $|q| < 1$ and assume there exists a p-th root $q^{1/p} \in K$. Then there is an affine morphism*

$$[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$$

which is a formal model of $[p] : E \to E$.

Proof. Recall that the multiplication map $[p] : T \to T$ has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ by Corollary 3.6. This fits into a commutative diagram

$$\begin{tikzcd}
\mathfrak{T}_{q^{1/p}} \arrow[r, "{[\mathfrak{p}]}"] & \mathfrak{T}_q \\
\overline{T} \arrow[u, hook] \arrow[r, "{[p]}"] & \overline{T}. \arrow[u, hook]
\end{tikzcd}$$

Functoriality of the associated fibre construction in the general case, Proposition A.17, applied to the diagram below then gives a natural map $\mathfrak{E}_{q^{1/p}} \to \mathfrak{E}$ making the diagram commute:

$$\begin{tikzcd}
& \mathfrak{T}_q \arrow[rr] & & \mathfrak{E}_q \\
\mathfrak{T}_{q^{1/p}} \arrow[ur, "{[\mathfrak{p}]}"] \arrow[rr] & & \mathfrak{T}_{q^{1/p}} \times^{\overline{T}} \overline{E} \arrow[ur, dotted, "\exists"] & \\
& \overline{T} \arrow[uu] \arrow[rr] & & \overline{E} \arrow[uu] \\
\overline{T} \arrow[uu] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E} \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd} \tag{10}$$

By Lemma 4.4 this diagram equals diagram (9) on the generic fibre.

To see that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ is affine, first note that $[p] : \overline{B} \to \overline{B}$ is an affine morphism. The map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ is affine by construction, namely by Corollary 3.6 it is locally on $\mathfrak{T}_q$ of the form $[\mathfrak{p}] : \mathfrak{B}(q^{1/p}, 1)^d \to \mathfrak{B}(q, 1)^d$. Note that both of these affine open subsets are fixed by the action of $\overline{T}$. We conclude from the construction in the proof of Proposition A.17 that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ locally on the target is of the form

$$[\mathfrak{p}] : \mathfrak{B}(q^{1/p}, 1)^d \times U' \to \mathfrak{B}(q, 1)^d \times U$$

for an affine open formal subscheme $U \subseteq \overline{B}$ with affine preimage U' under $[p] : \overline{B} \to \overline{B}$. This shows that the morphism is affine locally on the target, and therefore is affine. □

We have thus proved the first part of what we want to show about tilde-limits of E:

Proposition 5.3. *Let K be perfectoid. Then E has a $[p]$-model tower of the form*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

for some $q \in K^\times$. In particular, there exists a space E_∞ such that $E_\infty \sim \varprojlim_{[p]} E$.

Proof. By Proposition 5.2, any choice of $q \in K^\times$ with $|q| < 1$ for which there exists a compatible system of p^n-th roots $q^{1/p^n} \in K^\times$ gives a tower

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

that on the generic fibre equals $\dots \xrightarrow{[p]} E \xrightarrow{[p]} E$. This is the desired $[p]$-model tower. $\square$

We are now ready to prove the main result of this note, namely that E_∞ is perfectoid.

Theorem 5.4. *Let K be perfectoid. Then the $[p]$-model tower from Proposition 5.3*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

is already a $[p]$-F-model tower. In particular, the corresponding space E_∞ is perfectoid.

Proof. It suffices to prove that for any $q \in K^\times$ with $|q| < 1$ and a p-th root $q^{1/p}$, the map $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ upon reduction mod p factors through relative Frobenius.

In the following we denote reduction of a formal scheme by a $\sim$ over the formal scheme, for example the reductions of $\overline{T}$, $\overline{E}$ and $\mathfrak{T}$ are denoted by $\tilde{T}$, $\tilde{E}$ and $\tilde{\mathfrak{T}}$.

Recall that $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ was constructed using the $[p]$-multiplication cube in diagram (10) and functoriality of the associated bundle. Also recall that all statements we have used about fibre bundles also hold when we replace formal schemes over $\mathcal{O}_K$ by schemes over $\mathcal{O}_K/p$, and formation of the associated bundle commutes with this reduction. In particular,

$$\tilde{\mathfrak{E}}_q = \tilde{\mathfrak{T}}_q \times^{\tilde{T}} \tilde{E}.$$

By Corollary 3.6, the model of the multiplication map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ reduces to relative Frobenius over p. In particular, we have a natural isomorphism

$$\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \cong \tilde{\mathfrak{T}}_q$$

and we can identify $\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} = \tilde{\mathfrak{T}}_q$ in the following. The same is true for $\tilde{T}^{(p)} = \tilde{T}$.

Since $\tilde{E}$ and $\tilde{T}$ are group schemes, the reduction of $[p]$ on them factors through the relative Frobenius maps F_E and F_E respectively. By functoriality of relative Frobenius ("Frobenius commutes with any map") we have a commutative diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "F_{\tilde{E}}"] & \tilde{E}^{(p)} \\ \tilde{T} \arrow[u, hook] \arrow[r, "F_{\tilde{T}}"] & \tilde{T}^{(p)}. \arrow[u, hook] \end{tikzcd}$$

In other words, $F_{\tilde{E}}$ is a $F_{\tilde{T}}$-linear morphism of fibre bundles. Again by functoriality of Frobenius we also have a commutative diagram

$$\begin{tikzcd}
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[r, "F_{\tilde{\mathfrak{T}}}"] & \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \\
\tilde{T} \arrow[u, hook] \arrow[r, "F_{\tilde{T}}"] & \tilde{T}^{(p)}. \arrow[u, hook]
\end{tikzcd}$$

By Proposition A.17, we thus obtain a natural morphism

$$F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}} : \tilde{\mathfrak{T}}_{q^{1/p}} \times^{\tilde{T}} \tilde{E} \to \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \times^{\tilde{T}^{(p)}} \tilde{E}^{(p)}.$$

Using the explicit description of $F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}}$ in the proof of Proposition A.17, we easily check that this morphism is just the relative Frobenius of $\tilde{\mathfrak{E}}_{q^{1/p}}$: This is a consequence of the fact that relative Frobenius on the fibre product $\tilde{T} \times \tilde{U}$ for any $\tilde{U} \subseteq \tilde{B}$ is just the product of the relative Frobenius morphisms of $\tilde{T}$ and $\tilde{U}$, and thus the morphisms θ_i from Lemma A.7 are all trivial.

But this means that again by Proposition A.17, the reduction of the formal model of the p-multiplication cube in diagram 10 admits the following factorisation:

$$\begin{tikzcd}
 & & \tilde{\mathfrak{T}}_q \arrow[rr] & & \tilde{\mathfrak{E}}_q \\
 & \tilde{\mathfrak{T}}^{(p)}_q \arrow[ur, equal] \arrow[rr] & & \tilde{\mathfrak{E}}^{(p)}_{q^{1/p}} \arrow[ur, dotted] & \\
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[ur, "F"'] \arrow[rr] & & \tilde{\mathfrak{E}}_{q^{1/p}} \arrow[ur, "F"'] & & \\
 & & \tilde{T} \arrow[uuu] \arrow[rr] & & \tilde{E} \arrow[uuu] \\
 & \tilde{T}^{(p)} \arrow[uuu] \arrow[ur, equal] \arrow[rr] & & \tilde{E}^{(p)} \arrow[uuu] \arrow[ur] & \\
\tilde{T} \arrow[uuu] \arrow[ur, "F"'] \arrow[rr] & & \tilde{E} \arrow[uuu] \arrow[ur, "F"'] & &
\end{tikzcd}$$

Since the composed maps $\tilde{E} \to \tilde{E}$ on the bottom right, $\tilde{T} \to \tilde{T}$ on the bottom left and $\tilde{\mathfrak{T}}_{q^{1/p}} \to \tilde{\mathfrak{T}}_q$ on the upper left by construction are the reductions of the respective p-multiplication maps $[p]$, the functoriality of the associated bundle construction in Proposition A.17 implies that the two maps on the upper right compose to the reduction of $[\mathfrak{p}] \times^{[p]} [p]$. But $[\mathfrak{p}] \times^{[p]} [p]$ is equal to $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ by definition of the latter. This completes the proof that the reduction of $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through the relative Frobenius on $\tilde{\mathfrak{E}}_{q^{1/p}}$.

The conclusion that E_∞ exists and is perfectoid then follows from Proposition 2.12. □

6. The case of abelian varieties with semi-stable reduction

Let K be a complete non-archimedean field with ring of integers $\mathcal{O}_K$. Let A be an abelian variety over K of dimension g. Recall from Proposition 4.2 that associated to A we have a Raynaud extension E which is an extension of a rigid torus T of rank r which we assume to be split by an abelian variety B of good reduction. Moreover there is a lattice $M \subseteq E$ of rank r such that $A = E/M$. In this chapter, we want to prove the main result of this write-up:

Theorem 6.1. *Assume that A admits a constant partial anticanonical $\Gamma_0(p^\infty)$-structure (this will be defined later, but we remark already that this condition is always satisfied if K is algebraically closed). Then there is a perfectoid space A_∞ such that*

$$A_\infty \sim \varprojlim_{[p]} A.$$

The proof of Theorem 6.1 will be completed in several steps over the following sections. Our strategy is to describe the $[p]$-multiplication tower on E/M locally in terms of the $[p]$-multiplication tower of E. As a first step towards this goal, in the following section, we want to give a cover of E/M by subspaces of E that behaves well under $[p]$-multiplication.

6.1. **Covering A by subspaces of E.** As a first step we recall how to relate the lattice M to a Euclidean lattice in $\mathbb{R}^r$, cf §2.7 and §6.2 in [5]. On the level of points, $\mathbb{G}_m^{\mathrm{an}}$ has an absolute value map

$$|-|: \mathbb{G}_m^{\mathrm{an}}(K) = K^\times \to \mathbb{R}^\times, \quad x \mapsto |x|$$

which induces the following group homomorphism from the torus T:

$$|-|: T(K) = (K^\times)^r \to (\mathbb{R}^\times)^r, \quad (x_1, \dots, x_n) \mapsto (|x_1|, \dots, |x_n|)$$

Since when working with lattices we prefer additive notation, we also consider the map

$$\ell : T(K) = (K^\times)^r \to \mathbb{R}^r, \quad x_1, \dots, x_n \mapsto (-\log|x_1|, \dots, -\log|x_n|).$$

Note that this map has dense image by our assumption that K is perfectoid.

The formal torus $\overline{T}$ on K-points corresponds to $\overline{T}_\eta(K) = (\mathcal{O}_K^\times)^r$ and is thus in the kernel of $-\log$. We can therefore extend ℓ to $E(K)$ as follows: Locally over an open subspace $U \subseteq B$ we have $E|_U = T \times^{\overline{T}_\eta} \overline{E}_\eta|_U$ and we define ℓ by projection from the first factor. The different $E|_U$ are then glued on intersections using the $\overline{T}_\eta$-action on T. But since ℓ on T is invariant under the $\overline{T}_\eta$-action, the maps glue together to a group homomorphism

$$\ell : E(K) \to \mathbb{R}^r.$$

Since $A = E/M$ is proper, the lattice M is sent by $-\log$ to an Euclidean lattice $\Lambda \subset \mathbb{R}^r$ of full rank r (see Proposition 6.1.4 in [5]). In particular, this induces an isomorphism of discrete torsionfree groups

$$\ell : M \xrightarrow{\sim} \Lambda \subseteq \mathbb{R}^r.$$

The idea is now that one can understand the quotient E/M in terms of the quotient $\mathbb{R}^r/\Lambda$. We are going to make this precise in the following:

In the space $\mathbb{R}^r$, we can now choose a $d \in \mathbb{R}^r_{>0}$ and a cuboid with center at the origin

$$S(d) = \{(x_1, \dots, x_r) \in \mathbb{R}^r | |x_i| \leq d_i\}$$

that intersects Λ only in $0 \in \Lambda$ and $q_1, \dots, q_r \in K$ such that $|q_i| = \exp(-d_i)$. We denote by $\mathcal{B}(q, q^{-1})$ the affinoid open multi-annulus T centered at 1 of radii $|q_i| < 1 < |q_i|^{-1}$ in every direction.

Lemma 6.2. *The inverse image $\ell^{-1}(S(d)) \subseteq E(K)$ is the underlying set of the admissible open subset $E(q) := \mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$.*

Proof. One shows this first for the map $T \to \mathbb{R}^d$, where it is clear that the preimage is $\mathcal{B}(q, q^{-1})$. This is also described in §6.4 of [4]. The statement for $\mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ follows by direct inspection on local trivialisations $\mathcal{B}(q, q^{-1}) \times U$ for $U \subseteq B$. □

Note that the map $\mathbb{R}^d \to \mathbb{R}^d/\Lambda$ maps $S(d)$ bijectively onto its image. Lemma 6.2 says that we can use $\mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ as a chart for E/M around the origin.

In order to obtain charts around other points of E/M, we simply need to consider translations: Recall that for every $c \in T(K)$, the translation map

$$T \xrightarrow{\cdot c} T$$

is an isomorphism of rigid spaces that sends the unit 1 to c. We denote the image of any admissible open set U under translation by $c \cdot U$.

Lemma 6.3. *With notation as before, let $c \in T(K)$ be any point such that $l(c) = s$. Then the inverse image $\ell^{-1}(s + S(d)) \subseteq E(K)$ of the translation of $S(d)$ by s is the underlying set of the admissible open subset $E(c, q) := (c\cdot\mathcal{B}(q, q^{-1}))\times^{\overline{T}_\eta}\overline{E}_\eta \subseteq E$. We can choose $c \in T(K)$ and $q \in T(K)$ in such a way that they admit arbitrary p^n-th roots.*

Proof. Since ℓ commutes with the translations

$$\begin{array}{ccc} T(K) & \xrightarrow{\ell} & \mathbb{R}^r \\ {\scriptstyle \cdot c}\downarrow & & \downarrow{\scriptstyle +s} \\ T(K) & \xrightarrow{\ell} & \mathbb{R}^r, \end{array}$$

the first part is an immediate consequence of Lemma 6.2. For the second, note that for any $c' \in T(K)$ with $l(c') = l(c)$ and $q \in T(K)$ with $l(q') = l(q)$ we have $c \cdot \mathcal{B}(q, q^{-1}) = c' \cdot \mathcal{B}(q, q^{-1})$, and thus $E(c, q) = E(c', q')$. The statement therefore follows from K being perfectoid, for instance using $\mathcal{O}_K^\flat = \varprojlim_{x\mapsto x^p} \mathcal{O}_K$. □

Definition 6.4. We call spaces of the form $E(c, q) \subseteq E$ **cuboids** centered at c. More precisely they are locally a cuboid $c \cdot \mathcal{B}(q, q^{-1}) \subseteq T$ times an admissible open subset of the abelian variety B.

Lemma 6.5. *There exist finitely many admissible open cuboids $E_1, \dots, E_k \subseteq E$ which map isomorphically to $A = E/M$ and which cover A admissibly.*

One can reconstruct A from any such cover by glueing $E_1, \dots, E_k$ as follows: By construction, for any E_i the translates $q \cdot E_i$ by $q \in M$ are pairwise disjoint and we thus have a canonical projection π from the union $\cup_{m\in M}(m \cdot E_i) \subseteq E$ to E_i. Let $E_{ij} := (\bigcup_{m\in M} m \cdot E_i) \cap E_j \subseteq E$. Then we glue E_j to E_i via the map

$$E_{ij} \to \bigcup_{m\in M} q \cdot E_i \xrightarrow{\pi} E_i.$$

Proof. Since $\mathbb{R}^r/\Lambda$ is compact, we can find finitely many $s_1, \dots, s_k$ and $d_1, \dots, d_k \in \mathbb{R}^r_{>0}$ such that $\mathbb{R}^r/\Lambda$ is covered by the $s_i + S(d_i)$. When we choose corresponding $c_1, \dots, c_k \in T(K)$ and $q_1, \dots, q_k$ as in Lemma 6.3, then the corresponding $E_i := E(c_i, q_i)$ are an atlas of $A = E/M$ by admissible open subsets of E.

In order to reconstruct A, note that $\cup_{m\in M} q \cdot E_i$ is precisely the preimage of E_i under the projection $E \to E/M$. In particular, the subspace E_{ij} is precisely the preimage of E_i under the composition $E_j \hookrightarrow E \to E/M$. In other words, the subspace $E_{ij} \subseteq E_j$ is the intersection of E_i and E_j when considered as subspaces of A. This shows that as charts of A, the spaces E_i and E_j are glued via E_{ij} as described. □

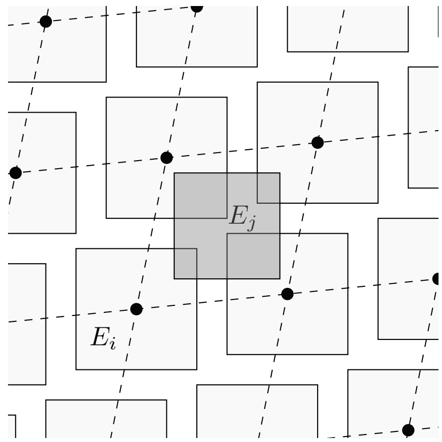

FIGURE 1. Given two charts E_i and E_j, the chart E_j is glued to E_i along intersections with all translates of E_i by $q \in M$.

Finally, we need some control about what happens to the cubes under $[p]$-multiplication. Recall from Lemma 6.3 that we can always assume that c admits p-th roots.

Lemma 6.6. *Let $c^{1/p}$ be a p-th root of c and let $q^{1/p}$ be a p-th root of q in $(K^\times)^r$. Then under $[p] : E \to E$, the admissible open $E(c_i, q)$ pulls back to the admissible open $E(c_i^{1/p}, q^{1/p})$.*

Proof. It is clear that under $[p] : T \to T$, the admissible open cuboid $c \cdot \mathcal{B}(q, q^{-1})$ centered at c pulls back to $c^{1/p} \cdot \mathcal{B}(q^{1/p}, q^{-1/p})$. Note that this is independent of the choices of $c^{1/p}$ and $q^{1/p}$. Now recall that in terms of fibre bundles, multiplication $[p] : E \to E$ is

$$[p] \times^{[p]} [p] : T \times^{\overline{T}_\eta} \overline{E}_\eta \to T \times^{\overline{T}_\eta} \overline{E}_\eta$$

by Lemma 4.4. Thus $(c \cdot \mathcal{B}(q, q^{-1})) \times^{\overline{T}_\eta} \overline{E}_\eta$ pulls back to $(c^{1/p} \cdot \mathcal{B}(q^{1/p}, q^{-1/p})) \times^{\overline{T}_\eta} \overline{E_\eta}$. □

6.2. **The two towers.** In this section we want to separate the $[p]$-multiplication of A into two different towers, which we think of as being a "ramified" tower and an "étale" tower. Of course in characteristic 0 both towers will actually be étale, and these words are only meant to describe the behaviour compared to $[p] : E \to E$.

The "ramified" tower only exists under an additional condition, which we want to briefly discuss now:

Definition 6.7. By a **partial anticanonical $\Gamma_0(p^\infty)$-structure** on A we mean a choice of subgroups $D_n \subseteq A[p^n]$ of rank p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$.

Note that the conditions imply that D_n is necessarily finite étale, and after a finite extension of K is isomorphic to the constant group $\underline{(\mathbb{Z}/p^n\mathbb{Z})^r}$. We say that a partial anticanonical $\Gamma_0(p^\infty)$-structure is **constant** if such an isomorphism exists over K for all n.

The name is chosen because if B admits a canonical subgroup (that is, satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A may have a partial anticanonical subgroup even if B does not have a canonical subgroup. For instance, it is clear that A always has a constant partial anticanonical subgroup if K is algebraically closed.

We will use this definition only via the following equivalent description:

Lemma 6.8. *Existence and choice of a constant partial anticanonical $\Gamma_0(p^\infty)$-structure on A are equivalent to existence and choice of lattices $M^{1/p^n} \subseteq E$ defined over K such that $[p] : E \to E$ restricts to isomorphisms $M^{1/p^{n+1}} \to M^{1/p^n}$ for all n.*

Proof. Given lattices $M^{1/p^{n+1}}$ as in the lemma, we obtain a split partial anticanonical $\Gamma_0(p^\infty)$ structure by setting $D_n := M^{1/p^{n+1}}/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/M^p \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ that form a partial anticanonical $\Gamma_0(p^\infty)$ structure, we recover M^{1/p^n} as the kernel of $E \to A \to A/D_n$. The finite group $M^{1/p^n}/M$ is then constant if and only if M^{1/p^n} is a lattice. $\square$

Assumption 6.9. From now on we will assume that A admits a constant partial anticanonical $\Gamma_0(p^\infty)$-structure. Let us fix a choice of such a structure, that is we choose lattices M^{1/p^n} defined over K that map isomorphically to M under $[p^n]$.

The quotient $A/D_n = E/M^{1/p^n}$ is then another abelian variety over K and the quotient map $E/M \to E/M^{1/p^n}$ is an isogeny of degree p^{2gn} through which $[p^n] : A \to A$ factors:

$$\begin{tikzcd} & E/M^{1/p} \arrow[dr, "{[p]}"] & \\ E/M \arrow[ur, "v"] \arrow[rr, "{[p]}"] & & E/M. \end{tikzcd} \tag{11}$$

We think of these maps as being an analogue of Frobenius and Verschiebung, which is why we denote the left map by v. Putting everything together, the $[p]$-multiplication tower splits into two towers

$$\begin{tikzcd} \ddots \arrow[dr] & & \vdots \arrow[d] & \vdots \arrow[d] \\ & E/M \arrow[r, "v"] \arrow[dr, "{[p]}"'] & E/M^{1/p} \arrow[r, "v"] \arrow[d, "{[p]}"] & E/M^{1/p^2} \arrow[d, "{[p]}"] \\ & & E/M \arrow[r, "v"] \arrow[dr, "{[p]}"'] & E/M^{1/p} \arrow[d, "{[p]}"] \\ . & & & E/M \end{tikzcd} \tag{12}$$

Since each quotient $M^{1/p^n}/M$ is a finite étale group scheme, all horizontal maps are finite étale. The vertical tower on the other hand fits into a second commutative diagram of rigid groups which compares it to the $[p]$-tower of E:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M^{1/p^2} \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p^2} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M^{1/p} \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & E/M \arrow[r] & 0
\end{tikzcd}
\tag{13}
$$

6.3. **Constructing a limit of the vertical tower.** Our first step is to show that the tower on the right has a perfectoid tilde-limit. Recall from Lemma 6.5 that E/M can be covered by admissible open subspaces $E_1, \ldots, E_k \subseteq E$ which map isomorphically onto an admissible open via $E \to E/M$. Denote by $E_i^{1/p^n} \subseteq E$ the pullback along $[p^n] : E \to E$. Also denote by $E_{ij}^{1/p^n} \subseteq E$ the pullback of E_{ij}. Because of our assumption, we can then reconstruct the space $E/M^{1/p^n}$ from the E_i^{1/p^n} as follows:

Lemma 6.10.

(1) The restriction to $E_i^{1/p^n} \subseteq E$ of $E \to E/M^{1/p^n}$ is an isomorphism onto its image. In particular, we can view E_i^{1/p^n} as a chart of $E/M^{1/p^n}$, and this is the preimage of E_i under $E/M^{1/p^n} \to E/M$.

(2) The collection of E_i^{1/p^n} is an atlas for $E/M^{1/p^n}$.

(3) We can reconstruct $E/M^{1/p^n}$ from glueing the E_i^{1/p^n} along the E_{ij}^{1/p^n}.

(4) The map $[p^n] : E/M^{1/p^n} \to E/M$ can be glued from the restrictions of $[p^n] : E \to E$ to $E_i^{1/p^n} \to E_i$, that is these maps commute with the glueing maps on E_{ij}^{1/p^n}.

The situation is thus like in Figure 2.

Proof. The first part follows because the map on the left of diagram 13 is an isomorphism. The second follows from pulling back the E_i along $E/M^{1/p^n} \to E/M$, using that the diagram commutes. We thus obtain an admissible cover by cuboids $E_1^{1/p^n}, \ldots, E_k^{1/p^n}$ of $E/M^{1/p^n}$. The glueing maps that can be used to reconstruct $E/M^{1/p^n}$ by glueing along subspaces E_{ij}^{1/p^n} then exist by the second part of Lemma 6.5: Note that we can apply Lemma 6.5 also to $E/M^{1/p^n}$ because of Assumption 6.9.

Finally, in order to see that one can glue together the map $[p] : E/M^{1/p^n} \to E/M$ from the E_i^{1/p^n}, use that intersection of cuboids are again cuboids, and so E_{ij}^{1/p^n} is a disjoint union of cuboids. It then follows from Lemma 6.6 that E_{ij} pulls back under $[p]$ to the intersection $E_{ij}^{1/p^n} \subseteq E/M^{1/p^n}$. That $[p]$ commutes with the glueing maps is clear because we know from diagram (11) that $[p] : E \to E$ descends to a morphism $[p] : E/M^{1/p^n} \to E/M$. $\square$

We are now ready to prove:

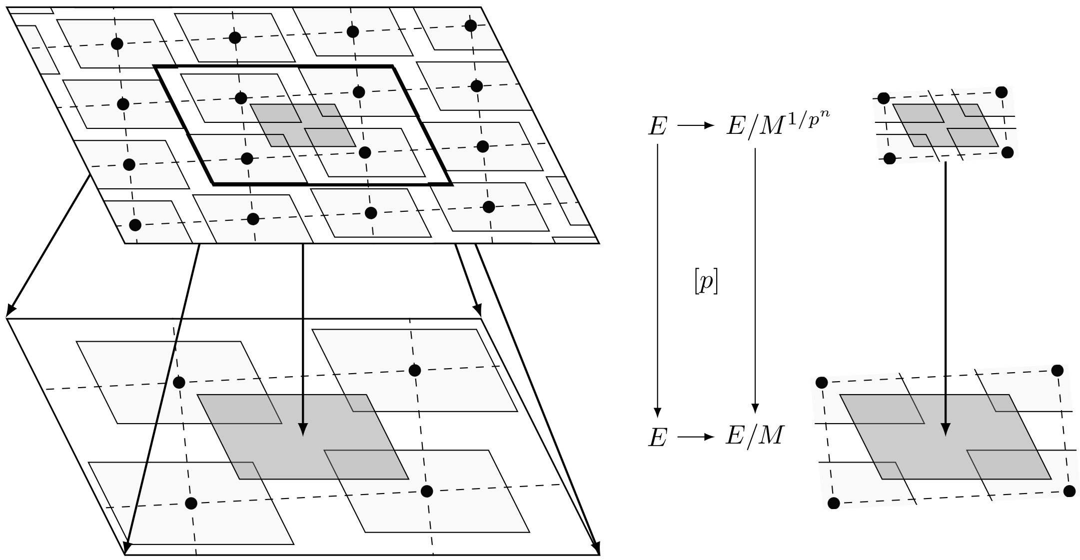

FIGURE 2. Illustration of how $[p] : E/M^{1/p^n} \to E/M$ can be glued from the maps $E_j^{1/p^n} \to E_j$. Here E_j on bottom and E_j^{1/p^n} on top are represented by the grey cuboids in the middle. On the left they are embedded into E whereas on the right they are considered as charts for E/M and $E/M^{1/p}$.

Proposition 6.11. *There is a perfectoid space $E/M^{1/p^\infty}$ such that*

$$E/M^{1/p^\infty} \sim \varprojlim_n E/M^{1/p^n}.$$

Proof. Denote by E_i^{1/p^∞} the pullback of $E_i \subseteq E$ to E_∞. This is an open subspace of a perfectoid space and hence perfectoid. Moreover, by Proposition 2.4.3 in [11] we have

$$E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}.$$

Given two different E_i, E_j, we know by Lemma 6.10 that at every step in the tower, the pullbacks E_i^{1/p^n} and E_j^{1/p^n} to $E/M^{1/p^n}$ intersect in E_{ij}^{1/p^n}. We can thus glue the E_i^{1/p^∞} along pullbacks E_{ij}^{1/p^∞} of the intersections $E_{ij} = E_i \cap E_j$ to E_∞ and thus obtain a perfectoid space $E/M^{1/p^\infty}$. This is a tilde-limit for $\varprojlim_{[p]} E/M^{1/p^n}$ because by construction it is so locally, and the definition of tilde-limits in Definition 2.4.1 of [11] is local on the source. $\square$

6.4. **Constructing a limit of the horizontal tower.** In order to construct a tilde-limit for $\varprojlim A$, we can now use that the horizontal maps in diagram (12) are all finite étale.

Lemma 6.12. *For any $0 \leq m \leq n$, the preimage of E_i^{1/p^n} from Lemma 6.10 under the horizontal map $v^m : E/M^{1/p^m} \to E/M^{1/p^n}$ is isomorphic to $p^{r(n-m)}$ disjoint copies of E_i^{1/p^n}. More canonically, it can be described as the isomorphic image of $D_n/D_m \times E_i^{1/p^n}$ under the multiplication on the abelian variety $E/M^{1/p^n}$.*

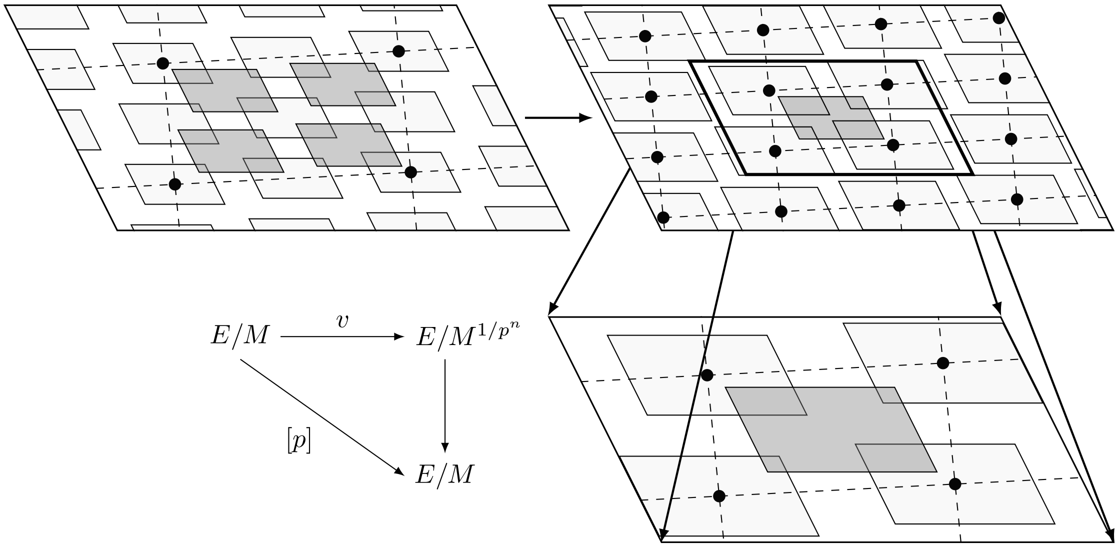

FIGURE 3. Illustration of how $[p] : E/M \to E/M$ factors in a part that is "ramified" (the vertical tower) and a part that is "étale" (the horizontal tower) with respect to our cover.

Proof. By the first part of Lemma 6.10, we know that the preimage of E_i^{1/p^n} under the projection $E \to E/M^{1/p^n}$ is a disjoint union of translates of E_i^{1/p^n} by M^{1/p^n}. The result then follows because $M^{1/p^n}/M^{1/p^m} = D_n/D_m = \underline{(\mathbb{Z}/p^{n-m}\mathbb{Z})^r}$. $\square$

We also record the following immediate consequence:

Lemma 6.13. *The preimage of E_i under $[p^n] : A \to A$ is isomorphic to p^{rn} disjoint copies of E_i^{1/p^n}. More canonically, we can describe the preimage as the isomorphic image of $D_n \times E_i^{1/p^n}$ under the multiplication $A \times A \to A$. The situation is thus as in figure 3.*

Proof. This follows from the first part of Lemma 6.10 combined with Lemma 6.12 in the case of $m = 0$. $\square$

Lemma 6.14. *The squares in diagram (12) are all pullback diagrams.*

$$\begin{tikzcd} E/M^{1/p^n} \arrow[r, "v"] \arrow[d, "{[p]}"] & E/M^{1/p^{n+1}} \arrow[d, "{[p]}"] \\ E/M^{1/p^{n-1}} \arrow[r, "v"] & E/M^{1/p^n} \end{tikzcd}$$

Proof. This can for instance be checked locally: The admissible open subset $E_i^{1/p^n} \subseteq E/M^{1/p^n}$ from Lemma 6.10 is pulled back to $E_i^{1/p^{n+1}}$ under the vertical map $[p] : E/M^{1/p^{n+1}} \to E/M^{1/p^n}$. The preimage of E_i^{1/p^n} under the horizontal map $E/M^{1/p^{n-1}} \to E/M^{1/p^n}$ is p^r disjoint copies of E_i^{1/p^n} by Lemma 6.12. The pullback of E_i^{1/p^n} to the upper right is thus p^r disjoint copies of $E_i^{1/p^{n+1}}$, which is clearly the fibre product. $\square$

Lemma 6.15. *The horizontal maps in diagram (12) induce natural finite étale morphisms* $v : E/M^{1/p^\infty} \to E/M^{1/p^\infty}$ *that fit into Cartesian diagrams*

$$\begin{array}{ccc} E/M^{1/p^\infty} & \xrightarrow{v^m} & E/M^{1/p^\infty} \\ \downarrow & & \downarrow \\ E/M^{1/p^{n-m}} & \xrightarrow{v^m} & E/M^{1/p^n} \end{array}$$

In particular, the preimage of E_i^{1/p^∞} *under* v^m *is isomorphic to* p^{rm} *copies of* E_i^{1/p^∞}*.*

Proof. Since $E/M \to E/M^{1/p}$ is finite étale, the fibre product with $E/M^{1/p^\infty} \to E/M^{1/p}$ exists and is perfectoid by Proposition 7.10 of [8].

The universal property of the fibre product then gives a unique map

$$E/M^{1/p^\infty} \to E/M \times_{E/M^{1/p}} E/M^{1/p^\infty}$$

making the natural diagrams commute. On the other hand, using Lemma 6.14 we see that the fibre product has compatible maps into the vertical inverse system over E/M. Since by Proposition 2.4.5 of [11] the perfectoid tilde-limit $E/M^{1/p^\infty}$ is universal for maps from perfectoid spaces to the inverse system, we obtain a unique map into the other direction. □

We thus obtain a pro-étale tower

$$\dots \xrightarrow{v} E/M^{1/p^\infty} \xrightarrow{v} E/M^{1/p^\infty} \xrightarrow{v} E/M^{1/p^\infty} \tag{14}$$

which we think of as being a kind of vertical "limit" of diagram 12. One can always take the limit of such a tower, as the following Lemma asserts:

Lemma 6.16. *Let* X *be a perfectoid space and* $(X_i)_{i\in I}$ *be a pro-étale cover of* X*. Then there is a perfectoid space* X_∞ *such that*

$$X_\infty \sim \varprojlim_{i\in I} X_i$$

Proof. This follows from Lemma 4.6 of [9] and the discussion in the paragraph before it. □

We conclude from this that the tower in equation 14 has a perfectoid limit $(E/M^{1/p^\infty})_\infty$.

6.5. **The diagonal tower: proof of the main theorem.** We want to show that this space is in fact a tilde-limit of the $[p]$-multiplication tower. In other words, this says that the horizontal tilde-limit of the vertical tilde-limits in diagram 12 is also a diagonal tilde-limit. However, while it follows immediately from Proposition 2.4.5 of [11] that $(E/M^{1/p^\infty})_\infty$ is universal for maps of perfectoid spaces Y into the inverse system $\varprojlim_{[p]} E/M$, it does not follow immediately from the definition that it is also the tilde-limit of this inverse system. The problem is that for a space X to be a tilde-limit $X \sim \varprojlim_{[p]} X_i$, it suffices for X to satisfy a property locally on some affinoid cover. But it is not clear a priori how such affinoid covers of $(E/M^{1/p^\infty})_\infty$ and $(E/M^{1/p^\infty})$ are related. In our situation, on the other hand, we can use that we have a good understanding of the local behaviour of the maps in the tower in terms of the cuboids E_i to solve this problem:

Proposition 6.17. *The perfectoid space* $A_\infty := (E/M^{1/p^\infty})_\infty$ *is a tilde-limit of* $\varprojlim_{[p]} A$*. It is independent up to unique isomorphism of the choice of partial anticanonical* $\Gamma_0(p^\infty)$*-structure, but it remembers the choice as a pro-finite étale closed subgroup* $D_\infty \subseteq A_\infty$*.*

Proof. It is clear from $(E/M^{1/p^\infty})_\infty \sim \varprojlim E/M^{1/p^\infty}$ and $(E/M^{1/p^\infty}) \sim \varprojlim E/M^{1/p^n}$ that the underlying topological space is indeed isomorphic to $\varprojlim_{[p]} |E/M|$.

In order to show that it is a tilde-limit of $\varprojlim_{[p]} E/M$, it thus suffices to give an explicit cover of $(E/M^{1/p^\infty})_\infty$ by open affinoids satisfying the tilde-limit property. To this end, first note that the pro-étale system $D_n = M^{1/p^n}/M$ over $M/M = \operatorname{Sp} K$ under multiplication maps $[p]$ has a perfectoid tilde-limit D_∞ by Lemma 6.16. Since the D_n are all constant perfectoid groups $\cong (\mathbb{Z}/p^r\mathbb{Z})^n$, it is clear that D_∞ is isomorphic to the constant perfectoid group $\underline{\mathbb{Z}_p^r}$.

Recall that by construction of $(E/M^{1/p^\infty})$ we have a cover of E/M by open subsets E_i that pull back to perfectoid open subspaces E_i^{1/p^∞} for which $E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}$. Moreover, by the second part of Proposition we know that the pullback of E_i^{1/p^∞} to $(E/M^{1/p^\infty})_\infty$ is the disjoint union of $|\mathbb{Z}_p^r|$ copies of E_i^{1/p^∞}, which more precisely can be described as $D_\infty \times E_i^{1/p^\infty}$. But it is clear from Lemma 6.13 that the image of any such isomorphic copy of $E_i^{1/p^\infty} \hookrightarrow (E/M^{1/p^\infty})_\infty$ in the inverse system $\dots \xrightarrow{[p]} A \xrightarrow{[p]} A$ is isomorphic to the inverse system $\dots \to E_i^{1/p} \to E_i$. Since $E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}$, this shows that $(E/M^{1/p^\infty})_\infty$ is locally the tilde-limit of the $[p]$-tower on A as desired.

That A_∞ is independent of the $\Gamma_0(p^\infty)$-structure up to unique isomorphism is a consequence of the universal property of the perfectoid tilde-limit. That D_∞ can be described as a closed subgroup of A_∞ is then clear from the local description of A_∞ as $D_\infty \times E_i^{1/p^\infty}$. □

This finishes the proof of Theorem 6.1.

Note that while the approach via cuboids E_i may look a bit technical on first glance, it has the advantage of giving an explicit description of $(E/M)_\infty$ as being glued from pieces of E_∞ by glueing data that is controlled by the lattices M^{1/p^n}. This might be interesting for applications, and in particular for computing the tilt.

7. Limits of the covering maps

Over the course of the proof, we have used three different towers: The tower $\dots \to E \xrightarrow{[p]} E$, the tower $\dots \to E/M \xrightarrow{[p]} E/M$ and the tower $\dots \to E/M^{1/p} \xrightarrow{[p]} E/M$ (for the latter we also needed the additional Assumption 6.9). The three are related by covering maps which fit into a commutative diagram of towers

$$\begin{array}{ccccc} E & \longrightarrow & E/M & \longrightarrow & E/M^{1/p^{n+1}} \\ \downarrow{\scriptstyle [p]} & & \downarrow{\scriptstyle [p]} & & \downarrow{\scriptstyle [p]} \\ E & \longrightarrow & E/M & \longrightarrow & E/M^{1/p^n} \end{array}$$

As we have seen in the last sections, all three towers have perfectoid tilde-limits, that we have denoted by E_∞, A_∞ and $E/M^{1/p^\infty}$.

By Proposition 2.14 the map $\pi : E \to A = E/M$ in the limit induces a natural group homomorphism $\iota : E_\infty \to A_\infty$. A similar universal property argument shows that we obtain a natural group homomorphism $A_\infty \to E/M^{1/p^\infty}$. In this section we want to study these morphisms more closely. Throughout we are going to retain Assumption 6.9.

We start with the case of $E \to E/M^{1/p^\infty}$:

Proposition 7.1. *Denote by $M_\infty \cong M$ the perfectoid tilde-limit of the tower*

$$\longrightarrow M^{1/p^2} \xrightarrow[{[p]}]{\sim} M^{1/p} \xrightarrow[{[p]}]{\sim} M.$$

There is a natural map $M_\infty \to E_\infty$ with respect to which we can interpret M_∞ as a lattice of rank r in E_∞. The map fits into a short exact sequence of perfectoid groups

$$0 \to M_\infty \to E_\infty \to E/M^{1/p^\infty} \to 0.$$

Proof. The map $M_\infty \to E_\infty$ is induced by the universal property of the perfectoid tilde-limit as usual. In order to see that the sequence is exact, we need to see that the first map is a kernel of the second, and the second map is a categorical quotient of the first. To this end, we first analyse the morphism locally: The projections to the inverse system fit into a commutative diagram

$$\begin{tikzcd}
M_\infty \arrow[r] \arrow[d, dotted] & E_\infty \arrow[r] \arrow[d, dotted] & E/M^{1/p^\infty} \arrow[d, dotted] \\
M^{1/p} \arrow[r] \arrow[d, "{[p]}"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p} \arrow[d, "{[p]}"] \\
M \arrow[r] & E \arrow[r] & E/M
\end{tikzcd}$$

Let us consider the preimages of $E_i \subseteq E/M$ under these morphisms: By Lemma 6.5 we see that the pullback to E is $\bigsqcup_{q \in M} qE_i$. We can also see this as the isomorphic image of $M \times E_i$ under the multiplication map $E \times E \to E$.

The pullback of E_i along $[p^n] : E/M^{1/p^n} \to E/M$ is E_i^{1/p^n} as we have seen in Lemma 6.10. The same Lemma shows that the pullback of this to E is $\bigsqcup_{q \in M^{1/p^n}} qE_i^{1/p^n} = M^{1/p^n} \times E_i^{1/p^n}$.

We then see that the pullback to E_∞ is $M_\infty \times E_i^{1/p^\infty}$. By construction of $E/M^{1/p^\infty}$ in the proof of Proposition 6.11, the pullback of E_i to $E/M^{1/p^\infty}$ is E_i^{1/p^∞}. All in all, we obtain a pullback diagram

$$\begin{tikzcd}
& E_\infty \arrow[rr] \arrow[dd] & & E/M^{1/p^\infty} \arrow[dd] \\
M_\infty \times E_i^{1/p^\infty} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & E_i^{1/p^\infty} \arrow[ur, hook] \arrow[dd] & \\
& E \arrow[rr] & & E/M \\
M \times E_i \arrow[ur, hook] \arrow[rr] & & E_i \arrow[ur, hook] &
\end{tikzcd} \tag{15}$$

We conclude that $E_\infty \to E/M^{1/p^\infty}$ is a principal M_∞-torsor of perfectoid groups. It is then clear that M_∞ is the preimage of $0 \in E/M^{1/p^\infty}$, from which one easily verifies that $M_\infty \hookrightarrow E_\infty$ has the universal property of the kernel. Similarly, $E_\infty \to E/M^{1/p^\infty}$ has the universal property of the cokernel: Given any perfectoid group H and a group homomorphism $E_\infty \to H$ that is trivial on M_∞, the restriction $M_\infty \times E_i^{1/p^\infty} \to H$ gives a natural map $E_i^{1/p^\infty} \to H$. Since by construction of $E/M^{1/p^\infty}$ the spaces E_i^{1/p^∞} and E_j^{1/p^∞} are glued on E_{ij}^{1/p^∞} using translation by M, these glue together to a morphism of $E/M^{1/p^\infty}$. □

The case of $\iota : A_\infty \to E/M^{1/p^\infty}$ is similar:

Proposition 7.2. *The subgroups $D_n = M^{1/p^n}/M \subseteq A$ in the limit give rise to a perfectoid group $D_\infty \sim \varprojlim_{[p]} D_n$ which is a subgroup $D_\infty \subseteq A_\infty$ in a natural way. If we assume Assumption 6.9, this fits into a short exact sequence of perfectoid groups*

$$0 \to D_\infty \to A_\infty \to E/M^{1/p^\infty} \to 0.$$

Proof. The tilde-limit D_∞ of the pro-étale tower $\varprojlim_{[p]} D_n$ exists by Lemma 6.16. The group structure and the map to A_∞ follow from the universal property as usual. Under assumption 6.9 is is easy to see that D_∞ is in fact non-canonically isomorphic to the constant group $\underline{\mathbb{Z}_p^r}$.

We can now argue similarly as in the proof of the last Proposition: at finite level we obtain short exact sequences

$$\begin{tikzcd}
D_{n+1} \arrow[r, equal] \arrow[d] & M^{1/p^{n+1}}/M \arrow[r] \arrow[d, "{[p]}"] & E/M \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p^{n+1}} \arrow[d, "{[p]}"] \\
D_n \arrow[r, equal] & M^{1/p^n}/M \arrow[r] & E/M \arrow[r] & E/M^{1/p^n}
\end{tikzcd}$$

The preimages of E_i under these maps are

$$D_n \times E_i^{1/p^n} \to E_i^{1/p^n}$$

by Lemma 6.13. In the limit the pullback is then

$$D_\infty \times E_i^{1/p^\infty} \to E_i^{1/p^\infty}$$

which shows that $A_\infty \to E/M^{1/p^\infty}$ is a D_∞-torsor. As in the last proof, this implies that the sequence in the Proposition is a short exact sequence. □

Finally, we consider the case of $\iota : E \to A = E/M$. While the limits of the last two towers were fibre bundles again, the map ι shows quite a different behaviour and on the opposite is an injective group homomorphism. While this may seem strange at first, it is actually what one might expect following the intuition of the following example:

Remark 7.3. To illustrate why this phenomenon occurs, consider the following inverse system of abstract groups:

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "{[p]}"] & \mathbb{R} \arrow[r] \arrow[d, "{[p]}"] & \mathbb{R}/\mathbb{Z} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

While at infinite level the maps on the right are all covering maps, in the inverse limit the homological algebra of $\varprojlim$ produces a long exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \varprojlim^1_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

So in the limit the covering map becomes the kernel of a map to $\mathbb{Z}_p/\mathbb{Z}$.

For perfectoid groups the homological algebra argument of course doesn't apply in the same way. Nevertheless, if we retain Assumption 6.9, we can again use the explicit covers of the last section to show that the situation is very similar as in the remark:

Proposition 7.4. *The homomorphism $\iota : E_\infty \to A_\infty$ is an open and closed immersion. Moreover, there is a natural map $\rho : A_\infty \to D_\infty/M_\infty$ which is a quotient of ι in the category of perfectoid groups. Here D_∞/M_∞ is a constant group which is uncanonically isomorphic to the constant perfectoid group $\underline{(\mathbb{Z}_p/\mathbb{Z})^r}$. We thus obtain a short exact sequence of perfectoid groups*

$$0 \to E_\infty \to A_\infty \to D_\infty/M_\infty \to 0$$

which is uncanonically split on the level of the underlying perfectoid spaces.

Proof. To see that ι is open and closed, it suffices to prove this locally. So it suffices to consider the induced map over the preimages of $E_i \subseteq A$ under the projection $A_\infty \to A$.

Recall that the map $\iota : E_\infty \to A_\infty$ arises by a universal property from the inverse system

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E \arrow[r] \arrow[d, "{[p]}"'] & A \arrow[r, "v"] \arrow[d, "{[p]}"'] & E/M^{1/p} \arrow[d, "f"'] \\
E \arrow[r] & A \arrow[r, equal] & E/M.
\end{tikzcd}$$

Using Lemmas 6.10 and 6.12 we see that the pullback of this diagram to $E_i \subseteq E/M$ is

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\bigsqcup_{q \in M^{1/p}} q \cdot E_i^{1/p^n} \arrow[r] \arrow[d] & \bigsqcup_{q \in D_n} E_i^{1/p^n} \arrow[r] \arrow[d] & E_i^{1/p^n} \arrow[d] \\
\bigsqcup_{q \in M} q \cdot E_i \arrow[r] & E_i \arrow[r, equal] & E_i
\end{tikzcd}$$

where recall that we write D_n for the group $M^{1/p^n}/M \subseteq A[p^n]$. In particular, at every level in the tower over E_i the maps are trivial covering maps. This means that in order to understand them we only need to keep track of the maps on index sets, which are

$$\begin{tikzcd}
{} \arrow[d, dotted, "\cong"'] & {} \arrow[d, dotted, "{[p]}"] & {} \arrow[d, dotted] \\
M^{1/p} \arrow[r] \arrow[d, "\cong"', "{[p]}"] & M^{1/p}/M \arrow[r] \arrow[d, "{[p]}"] & 1 \arrow[d] \\
M \arrow[r] & 1 \arrow[r] & 1
\end{tikzcd}$$

Recall that we have defined $D_\infty = \varprojlim_{[p]} D_n = \varprojlim_{[p]} M^{1/p^n}/M$ and $M_\infty \cong \varprojlim_{[p]} M^{1/p^n} \hookrightarrow E_\infty$. In the limit, the diagram gives rise to a natural map $M_\infty \hookrightarrow D_\infty$. Since $M \cong \underline{\mathbb{Z}^r}$ and $D_\infty \cong \underline{\mathbb{Z}_p^r}$ uncanonically, the quotient D_∞/M_∞ exists and is uncanonically isomorphic to

$$D_\infty/M_\infty \cong \underline{(\mathbb{Z}_p/\mathbb{Z})^r}.$$

Using the map $M_\infty \hookrightarrow D_\infty$ on indices, we see that the pullback to $E_i \subseteq E/M$ of the morphisms $\iota : E_\infty \to A_\infty \to E/M^{1/p^\infty}$ is given by the diagram

$$\begin{tikzcd}
 & E_\infty \arrow[rr, "\iota"] \arrow[dd] & & A_\infty \arrow[rr] \arrow[dd] & & E/M^{1/p^\infty} \arrow[dd] \\
\bigsqcup\limits_{M_\infty} E_i^{1/p^\infty} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & \bigsqcup\limits_{D_\infty} E_i^{1/p^\infty} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & E_i^{1/p^\infty} \arrow[ur, hook] \arrow[dd] & \\
 & E \arrow[rr] & & A \arrow[rr] & & E/M \\
\bigsqcup_M E_i \arrow[ur, hook] \arrow[rr] & & E_i \arrow[ur, hook] \arrow[rr] & & E_i \arrow[ur, hook] & .
\end{tikzcd}$$

But since $M_\infty \hookrightarrow D_\infty$ is injective, the map $\iota : \bigsqcup_{M_\infty} E_i^{1/p^\infty} \to \bigsqcup_{D_\infty} E_i^{1/p^\infty}$ is clearly open and closed. This shows that $\iota : E_\infty \to A_\infty$ is open and closed.

In order to see that ι has a quotient, note that the last diagram implies that the morphism

$$D_\infty \times E_\infty \to A_\infty$$

induced from multiplication on A_∞ is surjective. More precisely, for any point of $x \in D_\infty$, the translate $x \cdot E_\infty$ inside A_∞ is *equal* to E_∞ if $x \in M_\infty \hookrightarrow E_\infty$ and is *disjoint* from E_∞ otherwise. This shows that any choice of a set of coset representatives (x_i) of D_∞/M_∞ or equivalently a set-theoretic section $x : D_\infty/M_\infty \to D_\infty$ gives a morphism of perfectoid spaces

$$\bigsqcup_{i \in D_\infty/M_\infty} x_i \cdot E_\infty \xrightarrow{\sim} A_\infty \tag{16}$$

which is an isomorphism because it is bijective and each $x_i E_\infty \hookrightarrow A_\infty$ is an open immersion. Moreover, since $\iota : E_\infty \to A_\infty$ is a group homomorphism, the multiplication $A_\infty \times A_\infty \to A_\infty$ locally on the left hand side of this isomorphism restricts to a morphism

$$\begin{tikzcd}
E_\infty \times E_\infty \arrow[r, "(\cdot x_i, \cdot x_j)", "\sim"'] \arrow[d, "m"] & x_i E_\infty \times x_j E_\infty \arrow[r] \arrow[d] & A_\infty \times A_\infty \arrow[d, "m"] \\
E_\infty \arrow[r, "\cdot x_i x_j", "\sim"'] & x_i x_j E_\infty \arrow[r] & A_\infty
\end{tikzcd}$$

commuting with the translation maps.

Now since D_∞/M_∞ is a constant group, there is a natural locally constant morphism

$$\bigsqcup_{i \in D_\infty/M_\infty} x_i \cdot E_\infty \to D_\infty/M_\infty$$

which is independent of the choice of the x_i. Commutativity of the diagram shows that it moreover composes with equation 16 to a group homomorphism $\rho : A_\infty \to D_\infty/M_\infty$. It is then clear from 16 that this map has the universal property of a quotient for $\iota : E_\infty \to A_\infty$: Given any perfectoid group H and a homomorphism $g : A_\infty \to H$ such that $g \circ \iota = 0$, the map g must be constant on any $x_i E_\infty$ and thus factors uniquely through the morphism $D_\infty/M_\infty \to H$, $i \mapsto g(x_i)$.

That ι is the kernel of ρ follows from the fact that E_∞ is the pullback of $0 \in D_\infty/M_\infty$ under ρ. This shows that the sequence in the Proposition is exact.

That the short exact sequence is split on the level of perfectoid spaces follows from the projection maps $\bigsqcup_{i\in D_\infty/M_\infty} x_i \cdot E_\infty \to E_\infty$, which depend on the uncanonical choice of a set-theoretic section of the map of abstract groups underlying $D_\infty \to D_\infty/M_\infty$. □

We summarise all this in the following:

Theorem 7.5. *We have a commutative diagram with short exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_\infty \arrow[r] \arrow[d] & E_\infty \arrow[r] \arrow[d] & E/M^{1/p^\infty} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D_\infty \arrow[r] \arrow[d] & A_\infty \arrow[r] \arrow[d] & E/M^{1/p^\infty} \arrow[r] & 0 \\
 & D_\infty/M_\infty \arrow[r, equal] \arrow[d] & D_\infty/M_\infty \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Moreover, the square on the upper right is a pullback as well as a pushout square. In particular, we have a natural short exact sequence of perfectoid groups

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the diagonal embedding of M_∞ into $D_\infty \times E_\infty$. In particular, we can describe A_∞ as the quotient $(D_\infty \times E_\infty)/M_\infty$.

Proof. The diagram of short exact sequences follow from Proposition 7.1, Proposition 7.2 and Proposition 7.4. That the square is Cartesian is an immediate consequence of the universal property of the kernel. That the square satisfies the pushout universal property can be verified using the isomorphism in equation 16.

The universal property of the pushout then implies that A_∞ has the universal property of the quotient of the diagonal map $M \to D_\infty \times E_\infty$. □

Appendix A. Fibre bundles of formal and rigid spaces

In this chapter we review the theory of fibre bundles with structure group T and in particular of principal T-bundles in the setting of formal and rigid geometry.

In this chapter we denote by T a commutative formal group scheme over $\mathcal{O}_K$. We denote the multiplication map by $m : T \times T \to T$. By a T-action on a formal scheme X we mean a morphism $m_X : T \times X \to X$ such that the usual associativity diagram commutes.

Definition A.1. By a T**-linear map** of schemes X and Y with T-actions we mean a morphism $\phi : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} T \times X \arrow[r, "\mathrm{id}_T \times \phi"] \arrow[d, "m_X"] & T \times Y \arrow[d, "m_Y"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

We denote by $\mathbf{FormAct}_T$ the category of formal schemes with action by T.

The definition of a principal T-bundle is just what we get when we take the definition of a principal G-bundle and replace the category of topological spaces by the category of formal schemes.

Notation. In the following, if $\pi : E \to B$ is a morphism of formal schemes, then for a formal open subscheme $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.2. Let T be a formal group scheme. Let F be a formal scheme with an action $m : T \times F \to F$. A morphism $\pi : E \to B$ of formal schemes is called a **fibre bundle with fibre** F **and structure group** T if there is a cover $\mathfrak{U}$ of B of open formal subschemes $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i : F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that

$$\phi_{ij} = F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}$$

Definition A.3. When we take F equal to the formal scheme T with the action on itself by left multiplication, then a fibre bundle $\pi : E \to B$ with fibre T and structure group T is called a **principal** T**-bundle**. This is also called a T-torsor.

Example. For the short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} B \to 0$ from the last section, $\overline{E} \xrightarrow{\pi} B$ defines a principal $\overline{T}$-bundle by Lemma 4.3. Moreover, for any formal open subscheme $U \subseteq B$, the map $E|_U \to U$ is still a principal $\overline{T}$-bundle. This is what we mean when we say that the notion of principal $\overline{T}$-bundles is better suited for studying E locally on B than the notion of short exact sequences is.

The morphism ϕ_{ij} from condition (b) is fully determined by the morphism $\psi_{ij} : U_{ij} \to T$. By a glueing argument, one shows:

Lemma A.4. *Suppose we are given formal schemes F and B and a formal group scheme T with an action on F. Then fibre bundles $\pi : E \to B$ with fibre F and structure group T are equivalent to* *the data (up to refinement) of a cover $\mathfrak{U}$ by formal open subschemes and morphisms $\psi_{ij} : U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk} \cdot \psi_{ij} = \psi_{ik}$, by which we mean that the following diagram commutes:*

$$\begin{tikzcd} U_{ijk} \arrow[r, "\psi_{ij}\times\psi_{jk}"] \arrow[d, equal] & T \times T \arrow[d, "m"] \\ U_{ijk} \arrow[r, "\psi_{ik}"] & T. \end{tikzcd} \tag{17}$$

In order to define the category of fibre bundles, we also need the following:

Lemma A.5. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. With notations like in Definition A.2 we have a natural T-actions on $F \times U_i$ when we let T act trivially on U_i. These actions glue together to a T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.6. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T. Then a **morphism of fibre bundles** $f : (E', B', \pi') \to (E, B, \pi)$ is a commutative diagram of formal schemes

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

in which the morphism f_E is also T-linear (we often abbreviate this by writing $f : E' \to E$). We thus obtain the category of fibre bundles over T that we denote by $\mathbf{FormFibBun}_T$ and the full subcategory of principle T-bundles, that we denote by $\mathbf{FormPrinBun}_T$.

In the case of principal T-bundles, this data can be given as follows: Let $\mathfrak{U}$ be a cover over which E is trivialised. Then we can always refine U in such a way that for all $U \in \mathfrak{U}$ the fibre bundle E' is trivial over $U' := f_B^{-1}(U)$. The induced map $f_E : T \times U \to T \times U'$ is then T-linear and thus can be reconstructed from the induced map

$$\theta : U' = 1 \times U' \hookrightarrow T \times U \xrightarrow{f_E} T \times U \xrightarrow{p_1} T.$$

Lemma A.7. *Given a morphism $f_B : B' \to B$, and using notation as above, the data of a morphism $f = (f_E, f_B)$ of principal T-bundles is equivalent to the data of morphisms $\theta_i : U'_i \to T$ for all $U_i \in \mathfrak{U}$ such that for all i, j th following diagram commutes:*

$$\begin{tikzcd} T \times U'_{ij} \arrow[r, "\phi'_{ij}"] \arrow[d, "f_E"] & T \times U'_{ij} \arrow[d, "f_E"] \\ T \times U_{ij} \arrow[r, "\phi_{ij}"] & T \times U_{ij}. \end{tikzcd}$$

Moreover, commutativity of the above diagram is equivalent to commutativity of

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij}\times\theta_j"] \arrow[d, "(\psi_{ij}\circ f)\times\theta_i"'] & T \times T \arrow[d, "m"] \\ T \times T \arrow[r, "m"] & T. \end{tikzcd}$$

Or in short hand notation,

$$\psi'_{ij}(u)\theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u)$$

Proof. One direction is clear. For the other, the first part follows from glueing. The second part is a consequence of all maps in the first diagram being T-linear. □

Definition A.8. Let $\pi : E \to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Since the data in the equivalent characterisation of Lemma A.4 is completely independent of the fibre, the morphisms $\psi_{ij} : U_{ij} \to T$ by Lemma A.4 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel-Weil construction.

Note that in many authors in differential geometry and topology denote the associated bundle by "$F \times^T E$" instead of $F \times^T E$. In our setting, however, this is slightly confusing since we often have natural maps from T to F and E, but $F \times^T E$ is usually *not* their fibre product. In fact it behaves more like a pushout, for instance in the case that E comes from a short exact sequence.

Proposition A.9. *The associated bundle construction is a bifunctor*

$$- \times^T - : \mathbf{FormAct}_T \times \mathbf{FormPrinBun}_T \to \mathbf{FormFibBun}_T$$

from the categories of formal schemes with T-action $\times$ the category of principal T-bundles to the category of fibre bundles with structure group T.

Proof. Let E and E' be principal T-bundles and let $f : E' \to E$ be a morphism of T-bundles. Let F and F' be formal schemes with T-action and let $\gamma : F' \to F$ be a T-equivariant morphism. Then we can find compatible covers $\mathfrak{U}'$ of E' and $\mathfrak{U}$ of E such that locally we have diagrams like in Lemma A.7. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U'_i$ such that we obtain a natural map

$$F' \times U'_i \xrightarrow{(\lambda \times^T \pi)} F \times U_i, \quad (f, u) \mapsto (\lambda(f)\theta_i(u), \pi(u))$$

(of course this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersection Lemma A.7 implies that we have a commutative diagram

$$\begin{array}{ccc} F' \times U'_{ij} & \xrightarrow{\lambda \times^T \pi} & F \times U_{ij} \\ {\scriptstyle \psi'_{ij} \times \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \psi_{ij} \times \mathrm{id}} \\ F' \times U'_{ij} & \xrightarrow{\lambda \times^T \pi} & F \times U_{ij}. \end{array}$$

One easily checks that this is functorial in both components. □

Lemma A.10. *Let S be another formal group scheme that receives an action of T from a group homomorphism $g : T \to S$. Then for any principal T-bundle E, the Borel construction $S \times^T E$ is a principal S-bundle.*

Proof. This follows from Lemma A.4. The only thing we need to check is that the cocycle condition from diagram (17) also holds with respect to S. But g is a homomorphism and therefore the following diagram commutes:

$$\begin{array}{ccc} T \times T & \xrightarrow{g\times g} & S \times S \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ T & \xrightarrow{g} & S. \end{array}$$

□

Lemma A.11. *The Borel construction is a functor $S \times^T -$ from principal T-bundles to principal S-bundles.*

Proof. This is a consequence of Lemma A.7. One obtains the necessary data by composing the morphisms $\theta' : U_i' \to T$ with the morphism $T \to S$. These morphisms glue together because the second diagram of Lemma A.7 commutes, as one easily sees from the fact that $T \to S$ is a morphism of formal groups. □

The Borel construction satisfies the following universal property:

Lemma A.12. *Let $g : T \to S$ be a homomorphism of formal group schemes and let $E \to B$ be a principal T-bundle. Let X be any principle S-bundle. Note that X receives a T-action from g. Then there is a functorial one-to-one correspondence between T-linear morphisms $E \to X$ and morphisms of principal S-bundles $S \times^T E \to X$.*

A.1. **The semi-linear case.** We later want to consider morphisms of fibre bundles that are induces from morphisms of short exact sequences. In this situation. in order to describe the morphism of the kernels, we need to incorporate morphisms of the structure group into the notion of morphisms of fibre bundles. For this we need semi-linear group actions.

Definition A.13. Let T and S be formal group schemes and let $g : T \to S$ be a homomorphism. Let X and Y be formal schemes with actions $m : T \times X \to X$ and $m : S \times Y \to Y$ respectively. Then by a g-linear morphism $f : X \to Y$ we mean a morphism of formal schemes such that the following diagram commutes

$$\begin{array}{ccc} T \times X & \xrightarrow{g\times f} & S \times Y \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{f} & Y. \end{array}$$

Definition A.14. We denote by **FormAct** the category of pairs (T, X) where T is a formal group scheme and X is a formal scheme with T action, and morphisms are pairs of (g, f) where g is a group homomorphism and f is a g-linear morphism. It has a natural forgetful functor to **FormGrp**, the category of formal group schemes.

Definition A.15. Let $g : T' \to T$ be a homomorphisms of formal group schemes. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T' . Then a g-linear morphism of principal bundles is a diagram

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & B' \\ \downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f_B} \\ E & \xrightarrow{\pi} & B \end{array}$$

such that f_E is g-linear. We denote by **FormPrinBun** the category of fibre bundles (E, B, π, T, F) with arrows being the morphisms of principal bundles. It has a natural forgetful functor $(E, B, \pi, T, F) \mapsto T$ to the category **FormGrp** of formal group schemes

We get the natural analogue of Lemma A.7:

Lemma A.16. *With the notations from Lemma A.7, a g-linear morphism of a principal T'-bundle to a principal T-bundle is equivalent to the data of morphisms $\theta : U'_i \to T$ such that the following diagram commutes on intersections:*

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij}\times\theta_j"] \arrow[d, "(\psi_{ij}\circ f)\times\theta_i"'] & T'\times T \arrow[r, "g\times\mathrm{id}"] & T\times T \arrow[d, "m"] \\ T\times T \arrow[rr, "m"] & & T \end{tikzcd}$$

Or in short hand notation,

$$g(\psi'_{ij}(u))\cdot\theta_j(u) = \psi_{ij}(f(u))\cdot\theta_i(u). \tag{18}$$

Similarly as in Proposition A.9 one can conclude from this that change of fibre is functorial in the following sense:

Proposition A.17. *Given any homomorphism of group schemes $g : T' \to T$ and a g-linear homomorphism $h : F' \to F$ of formal schemes with T' and T-actions respectively, and a homomorphism $f : E' \to E$ of principal T' and T-bundles over g, one obtains a morphism*

$$h \times^g f : F' \times^{T'} E' \to F \times^T E$$

of fibre bundles over g, in a way that is functorial in h, g, f. More precisely, the associated bundle construction is a fibered bifunctor

$$-\times^- - : \mathbf{FormAct} \times_{\mathbf{FormGrp}} \mathbf{FormPrinBun} \to \mathbf{FormBun}.$$

Proof. Let (E,B,π,T) and (E',B',π',T') be principal bundles. Let F and F' be formal schemes with T-action and T' action respectively. Let $g : T \to T'$ be a group homomorphism and let $h : F' \to F$ be a g-equivariant morphism. Let $f : E' \to E$ be a morphism of principle fibre bundles over g. Then we can find compatible covers $\mathfrak{U}'$ of E' and $\mathfrak{U}$ of E such that locally we have diagrams like in Lemma A.7. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U'_i$ such that we obtain a natural map

$$F' \times U'_i \xrightarrow{(h\times^T\pi)} F \times U_i, \quad (x,u) \mapsto (h(x)\theta_i(u), f_\pi(u))$$

(as before this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersection Lemma A.7 implies that we have a commutative diagram

$$\begin{tikzcd} F'\times U'_{ij} \arrow[r, "h\times^T\pi"] & F\times U_{ij} \\ F'\times U'_{ij} \arrow[r, "h\times^T\pi"] \arrow[u, "\psi'_{ij}\times\mathrm{id}"] & F\times U_{ij}. \arrow[u, "\psi_{ij}\times\mathrm{id}"'] \end{tikzcd}$$

More precisely, by g-linearity of h one has

$$h(x\cdot\psi'_{ij}(u))\cdot\theta_j(u) = h(x)\cdot g(\psi'_{ij}(u))\cdot\theta_j(u) \overset{(18)}{=} h(x)\cdot\psi_{ij}(f(u))\cdot\theta_i(u).$$

This shows that the maps glue to a morphism $h \times^g f$ as desired.

By refining covers, one easily checks that this is functorial in both components. $\square$

We obtain a variant of Lemma A.12 in the semilinear case:

Lemma A.18. *Let E' be a principal T' bundle and let E be a principal T-bundle. Let H' and H be formal group schemes and assume there is a commutative diagram of group homomorphisms*

$$\begin{array}{ccc} H' & \xrightarrow{h} & H \\ \uparrow & & \uparrow \\ T' & \xrightarrow{g} & T. \end{array}$$

Let moreover $f : E' \to E$ be a g-linear morphism of fibre bundles. Then the map $h \times^g f$ from Proposition A.17 is the unique h-linear morphism of fibre bundles making the following diagram commute:

$$\begin{array}{ccc} H' \times^{T'} E' & \xrightarrow{h \times^f g} & H \times^T E \\ \uparrow & & \uparrow \\ E' & \xrightarrow{f} & E. \end{array}$$

Proof. The morphism exists by Proposition A.17. The vertical maps in the commutative diagram exist by functoriality via $E = T \times^T E \to H \times^T E$.

On the other hand, on any compatible trivialisation $T' \times U' \to T \times U$ of $f : E' \to E$ there is clearly only one way to extend this to $H' \times U' \to H \times U$ in a h-linear way. □

Remark A.19. All that we have done in this chapter can be done in completely the same way with formal schemes replaced by rigid spaces (covers being replaced by admissible covers) and also for schemes, or in fact for any site. We have preferred to use formal schemes to make things more explicit. The different categories of fibre bundles are well-behaved with respect to the usual functors between the different categories: For instance, by functoriality of fibre products there are natural rigidification and reduction functors from formal principal T-bundles over $\mathcal{O}_K$ to rigid principal T_η-bundles over K on the generic fibre, and to principal $\overline{T}$-bundles on the reduction $\mathcal{O}_K/p$. Moreover, these generic fibre and reduction functors commute with the associated fibre construction:

Lemma A.20. *Let T be a formal group scheme and let $\pi : E \to B$ be a principial T-bundle. Let F be a formal scheme with an action by T. Then*

$$(E \times^T B)_\eta = E_\eta \times^{T_\eta} B_\eta$$

Proof. This can be checked locally on any trivialising cover, where it is clear. □

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