PERFECTOID COVERS OF ABELIAN VARIETIES

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Abstract. For an abelian variety A over an algebraically closed non-archimedean field of residue characteristic p, we show that there exists a perfectoid space which is the tilde-limit of $\varprojlim_{[p]} A$. Our proof also works for the larger class of abeloid varieties.

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1. Introduction

Let p be a prime and let K be an algebraically closed non-archimedean field of residue characteristic p. For an abelian variety A over K we consider the inverse system of A under the p-multiplication morphism:

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A.$$

Via the adic analytification functor, we may see this as an inverse system of analytic adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K. The primary goal of this article is to show that the "inverse limit" of this tower exists in some way and is a perfectoid space: Since inverse limits rarely exist in the category of adic spaces, in [7] Huber introduced the weaker notion of tilde-limits to remedy this problem. This is the notion of "limits" we are going to use. More precisely, we prove the following slightly more general result:

Theorem 1. *Let A be an abeloid variety over K, for instance an abelian variety seen as a rigid space. Then there is a unique perfectoid space A_∞ over K such that $A_\infty \sim \varprojlim_{[p]} A$ is a tilde-limit.*

The possibility of results in this direction is mentioned in §7 and §13 of [17], and in the case of abelian varieties with good reduction, this theorem was proven already in [12, Lemme A.16]. We recall the argument in Lemma 2.15 below.

In general, A has semi-stable reduction by the assumption that K is algebraically closed. Consequently, the theory of Raynaud extensions provides us with a short exact sequence

$$0 \to T \to E \to B \to 0$$

of rigid groups, where $T = (\mathbb{G}_m^{\mathrm{an}})^d$ is a split rigid torus and B is the analytification of an abelian variety with good reduction, such that $A = E/M$ for a discrete lattice $M \subset E$. This short exact sequence is split locally on B, allowing us to locally write E as a product of T and an open subspace of B. Our strategy for the proof of Theorem 1, which more generally applies to any abeloid variety over K, is now similar to the good reduction case:

(1) Construct a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$. This is easy.
(2) Use T_∞ and B_∞ to construct a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

(3) Study the quotient map $E \to A$ in the limit over $[p]$ to construct the desired space A_∞.

More precisely, this article is organised as follows: In §2 we recall the definition of tilde-limits and collect some useful lemmas about tilde-limits and perfectoid spaces. In particular, we construct the perfectoid tilde-limit T_∞. In §3 we use the language of fibre bundles to construct a perfectoid tilde-limit E_∞: The Raynaud extension of A mentioned earlier arises from a short exact sequence of formal group schemes over $\mathcal{O}_K$

$$0 \to \overline{T} \to \overline{E} \to \overline{B} \to 0$$

by taking generic fibres and forming the pushout with respect to the open immersion $\overline{T}_\eta \to T$. Since the sequence is locally split, we can see $\overline{E} \to \overline{B}$ as a principal $\overline{T}$-bundle and formation of E amounts to a change of fibre from $\overline{T}_\eta$ to T. We get the desired tilde-limit by tracing the local splitting through the tower of multiplication by $[p]$. This will also show that there is a short exact sequence of perfectoid groups

$$0 \to T_\infty \to E_\infty \to B_\infty \to 0.$$

In §4 we finish the proof of Theorem 1 by constructing A_∞ from E_∞ as follows: After choosing lattices $M \subset M_n \subset E$ that map isomorphically to M under $[p^n]\colon E \to E$, the $[p]$-multiplication tower of $A = E/M$ naturally factors into two separate towers: One is the tower of maps $E/M_{n+1} \to E/M_n$ induced from $[p]$-multiplication of E, and the other is induced from the projection maps $v^n\colon E/M \to E/M_n$. Using local splittings, one can construct a perfectoid tilde-limit $A'_\infty \sim \varprojlim_n E/M_n$ of the first tower from E_∞. It fits into a short exact sequence

$$0 \to M \to E_\infty \to A'_\infty \to 0.$$

The existence of $A_\infty \sim \varprojlim_{[p]} A$ then follows as the quotient maps $v^n\colon E/M \to E/M_n$ are étale. In fact, they are locally split in the analytic topology, from which one can deduce the following analogue of Raynaud uniformisation for A_∞: Write D_n for the kernel of v^n. Then there is a profinite perfectoid tilde-limit $D_\infty \sim \varprojlim_{[p]} D_n$ and a short exact sequence of perfectoid groups

$$0 \to M \to D_\infty \times E_\infty \to A_\infty \to 0,$$

which we regard as an analogue of the sequence $0 \to M \to E \to A \to 0$.

We give three applications of Theorem 1 in §5: As observed by Hansen, one can deduce from Theorem 1 the existence of certain universal covers of curves by embedding them into their Jacobian:

Corollary 1.1 (Hansen, [6])**.** *Let C be a connected smooth projective curve of genus $g \geq 1$ over K. Fix a geometric point $x\colon \operatorname{Spec}(K) \to C$ and for each open subgroup H of $\pi_1(C, x)$, let C_H denote the finite étale cover of C corresponding to H. We regard C and C_H as analytic adic spaces.*

(1) There is a perfectoid tilde-limit $\tilde{C} \sim \varprojlim_H C_H$ where H ranges over the open subgroups of $\pi_1(C, x)$.

(2) The morphism $\tilde{C} \to C$ is a pro-étale $\pi_1(C, x)$-torsor. It is universal with this property in the sense that it represents the fibre functor sending pro-finite-étale perfectoid covers $X \to C$ to the $\pi_1(C, x)$-module $F(X) = \operatorname{Hom}_C(x, X)$.

(3) For any pro-finite-étale morphism $X \to C$, there is a natural isomorphism

$$X = \underline{F(X)} \times^{\pi_1(C,x)} \tilde{C} := (\underline{F(X)} \times \tilde{C})/\pi_1(C, x).$$

Here the right hand side is the categorical quotient in adic spaces for the antidiagonal action.

Second, we note that the analogue of this corollary also works for C replaced by an abelian variety, in which case the pro-étale fundamental group is isomorphic to the absolute Tate module

$TA := \varprojlim_{N\in\mathbb{N}} A[N]$. In particular, one obtains from this two different natural ways to uniformise the diamond $A^\diamond$ attached to A: On the one hand, as a consequence of Theorem 1, we can write

$$A^\diamond = A_\infty/T_pA.$$

On the other hand, one can deduce from Theorem 1 that there is also a perfectoid tilde-limit $\tilde{A} \sim \varprojlim_{[N]} A$ which gives rise to a natural isomorphism

$$A^\diamond = \tilde{A}/TA.$$

Here the second equation describes A in terms of the universal connected pro-finite-étale cover $\tilde{A} \to A$, whereas the first uses the universal connected pro-finite-étale pro-p-cover. Either may be seen as a sort of analogue of Riemann uniformisation of abelian varieties over $\mathbb{C}$.

Our third application of Theorem 1 states that in line with this analogy to the complex case, the cohomology of constructible $\mathbb{F}_p$-sheaves on A_∞ behaves like that of a Stein space: This follows in combination with a result of Reinecke:

Corollary 1.2 (Reinecke)**.** *Let L be a constructible sheaf of $\mathbb{F}_p$-modules on $A_{\text{ét}}$. Then for $i > \dim A$,*

$$\varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^*L) = 0.$$

The paper ends with an appendix on fibre bundles and associated fibre bundle constructions in the context of adic spaces, which develops some language that we need in the construction of A_∞.

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Notation

Let K be an algebraically closed non-archimedean field, let $\mathcal{O}_K$ be the ring of integers of K and fix a pseudo-uniformiser $\varpi \in \mathcal{O}_K$ such that $p \in \varpi\mathcal{O}_K$.

We will use adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Huber, and perfectoid spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ in the sense of Scholze [14]. We denote by $X \mapsto X^{\mathrm{an}}$ the analytification functor from schemes of finite type over X to analytic adic spaces over $(K, \mathcal{O}_K)$.

By a rigid space, we shall always mean an analytic adic space of topologically finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$. In particular, by an open cover of a rigid space we shall always mean a cover of the associated adic space, so that we do not need the notion of admissible covers.

For a ϖ-adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$, we denote by $\mathfrak{X}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O}_K,\mathcal{O}_K)} \mathrm{Spa}(K,\mathcal{O}_K)$ its adic generic fibre. Here $\mathfrak{X}^{\mathrm{ad}}$ is the adification in the sense of [20].

2. Tilde-limits of rigid groups

2.1. **Tilde-limits.** We begin with some lemmas on tilde-limits that we will need throughout.

Inverse limits often do not exist in the category of adic spaces, and neither do they in rigid spaces. Instead we use the notion of tilde-limits from [7, Definition 2.4.2]:

Definition 2.1. Let $(X_i)_{i\in I}$ be a filtered inverse system of adic spaces with quasi-compact and quasi-separated transition maps, and let X be an adic space with a compatible system of morphisms $f_i\colon X \to X_i$. We write $X \sim \varprojlim X_i$ and say that X is a **tilde-limit** of the inverse system $(X_i)_{i\in I}$ if

(1) the map of underlying topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and
(2) there exists an open cover of X by affinoids $\mathrm{Spa}(A, A^+) \subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(A_i,A_i^+)\subset X_i} A_i \to A$$

has dense image, where the direct limit runs over all $i \in I$ and all open affinoid subspaces $\mathrm{Spa}(A_i, A_i^+) \subset X_i$ through which the morphism $\mathrm{Spa}(A, A^+) \subseteq X \to X_i$ factors.

Definition 2.2. Suppose that there is an adic space S with compatible maps $f_i : X_i \to S$, giving rise to a map $f : X \to S$, and a cover $\mathfrak{U}$ of open subspaces of S such that for each $U \in \mathfrak{U}$, the $f_i^{-1}(U)$ and $f^{-1}(U)$ are all affine. Then if condition (2) above is satisfied with respect to the cover of X by the $f^{-1}(U)$, we shall say that $X \sim_{\mathfrak{U}} \varprojlim X_i$ is a tilde-limit with respect ot $\mathfrak{U}$. We say that it is a perfectoid tilde-limit with respect to $\mathfrak{U}$ if the $f^{-1}(U)$ are even affinoid perfectoid.

Remark 2.3. As pointed out after Proposition 2.4.4 of [20], tilde-limits (if they exist) are in general not unique. However, Corollary 2.7 below says that perfectoid tilde-limits are unique.

We recall a few results from [20], §2.4 on tilde-limits that we will use frequently throughout:

Proposition 2.4 ([20], Proposition 2.4.2). *Let (A_i, A_i^+) be a direct system of affinoids over $(K, \mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ carrying the ϖ-adic topology. Let $(A, A^+) = (\varinjlim A_i, \varinjlim A_i^+)$ be the affinoid algebra equipped with the topology making $\varinjlim A_{i,0}$ with the ϖ-adic topology a ring of definition. Then*

$$\mathrm{Spa}(A, A^+) \sim \varprojlim \mathrm{Spa}(A_i, A_i^+).$$

Proposition 2.5 ([20], Proposition 2.4.3). *Let $X \sim \varprojlim_{i\in I} X_i$ be a tilde-limit and let $U_i \hookrightarrow X_i$ be an open immersion for some $i \in I$. Set $U_j := U_i \times_{X_i} X_j$ for $j \geq i$ and $U := U_i \times_{X_i} X$. Then*

$$U \sim \varprojlim_{j\geq i} U_j.$$

Proposition 2.6 ([20], Proposition 2.4.5). *Let $(X_i)_{i\in I}$ be an inverse system of adic spaces over $(K, \mathcal{O}_K)$ and assume that there is a perfectoid space X such that $X \sim \varprojlim_{i\in I} X_i$. Then for any perfectoid K-algebra (B, B^+),*

$$X(B, B^+) = \varprojlim_{i\in I} X_i(B, B^+).$$

Corollary 2.7. *Any two perfectoid spaces that are tilde-limits of the same inverse system of adic spaces over $(K, \mathcal{O}_K)$ are canonically isomorphic.*

In the situation of the corollary, we will also refer to such a perfectoid space as *the* perfectoid tilde-limit of the inverse system. Of course perfectoid tilde-limits do not always exist. An example of a basic situation in which they do is the following:

Lemma 2.8. *Let $(S_i)_{i\in I}$ be an inverse system of finite sets. Let $S=\varprojlim_{i\in I} S_i$. Then the system of constant groups $\underline{S_i}=\mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S_i,K),\mathrm{Map}_{\mathrm{cts}}(S_i,\mathcal{O}_K))$ has a perfectoid tilde-limit*

$$\underline{S}:=\mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(S,K),\mathrm{Map}_{\mathrm{cts}}(S,\mathcal{O}_K))\sim\varprojlim_{i\in I}\underline{S_i}.$$

Proof. Since S is compact, $\mathrm{Map}_{\mathrm{cts}}(S,K)=\mathrm{Map}_{\mathrm{cts}}(S,\mathcal{O}_K)[\frac{1}{\varpi}]$. Perfectoidness now follows from $\mathrm{Map}_{\mathrm{cts}}(S,\mathcal{O}_K)/\varpi=\mathrm{Map}_{\mathrm{lc}}(S,\mathcal{O}_K/\varpi)$. The tilde-limit property follows from Proposition 2.4. $\square$

We will need the following basic lemma later on.

Lemma 2.9. *Let (A_i,A_i^+) and (B_i,B_i^+) be direct systems of affinoids over $(K,\mathcal{O}_K)$ with compatible rings of definition $A_{i,0}$ and $B_{i,0}$ carrying the ϖ-adic topology. Assume that there are perfectoid tilde-limits $\mathrm{Spa}(A,A^+)\sim\varprojlim\mathrm{Spa}(A_i,A_i^+)$ and $\mathrm{Spa}(B,B^+)\sim\varprojlim\mathrm{Spa}(B_i,B_i^+)$. Then*

$$\mathrm{Spa}(A,A^+)\times_{\mathrm{Spa}(K,\mathcal{O}_K)}\mathrm{Spa}(B,B^+)\sim\varprojlim(\mathrm{Spa}(A_i,A_i^+)\times_{\mathrm{Spa}(K,\mathcal{O}_K)}\mathrm{Spa}(B_i,B_i^+))$$

is also a perfectoid tilde-limit.

Proof. The fibre product $\mathrm{Spa}(A,A^+)\times_{\mathrm{Spa}(K,\mathcal{O}_K)}\mathrm{Spa}(B,B^+)$ exists and is perfectoid by [14, Prop 6.18]. In fact, it is represented by $\mathrm{Spa}(C,C^+)$, where $C=A\widehat{\otimes}_K B$ and C^+ is the ϖ-adic completion of the integral closure of the image of $A^+\otimes_{\mathcal{O}_K}B^+$.

We first check the condition on topological spaces: Since fibre products commute with limits in the category of sheaves, it follows from Proposition 2.6 that for any perfectoid field (D,D^+) over $(K,\mathcal{O}_K)$, we have

$$(\mathrm{Spa}(A,A^+)\times_{\mathrm{Spa}(K,\mathcal{O}_K)}\mathrm{Spa}(B,B^+))(D,D^+)=\varprojlim(\mathrm{Spa}(A_i,A_i^+)\times_{\mathrm{Spa}(K,\mathcal{O}_K)}\mathrm{Spa}(B_i,B_i^+))(D,D^+).$$

Since the topological space can be reconstructed from this data, it follows that the underlying topological spaces of both sides coincide.

It remains to check that if $\varinjlim A_i\to A$ has dense image and $\varinjlim B_i\to B$ has dense image, then $\varinjlim(A_i\otimes B_i)\to A\otimes B$ has dense image. As direct limits commute with tensor products, we have $\varinjlim(A_i\otimes B_i)=(\varinjlim A_i)\otimes(\varinjlim B_i)$. Now density can be checked directly on elements. $\square$

Lemma 2.10. *Let $(X_{i,j})_{i\le j\in\mathbb{N}}$ be an inverse system of adic spaces of the form*

$$\begin{tikzcd}
{} \arrow[dr, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] \\
 & X_{3,3} \arrow[r] \arrow[dr] & X_{2,3} \arrow[r] \arrow[d] & X_{1,3} \arrow[d] \\
 & & X_{2,2} \arrow[r] \arrow[dr] & X_{1,2} \arrow[d] \\
 & & & X_{1,1}.
\end{tikzcd}$$

Suppose that there exists a cover $\mathfrak{U}$ of $X_{1,1}$ for which there are tilde-limits $X_{i,\infty}\sim_{\mathfrak{U}}\varprojlim_j X_{i,j}$ with respect to $\mathfrak{U}$. Suppose moreover that there is a tilde-limit $X_\infty\sim_{\mathfrak{U}}\varprojlim_i X_{i,\infty}$ with respect to $\mathfrak{U}$. Then this is also a tilde-limit for the diagonal tower,

$$X_{\infty,\infty}\sim_{\mathfrak{U}}\varprojlim_{n\in\mathbb{N}}X_{n,n}.$$

Proof. By the assumption that the tilde-limits exist with respect to $\mathfrak{U}$, we may without loss of generality assume that each $X_{i,j} = \mathrm{Spa}(A_{i,j})$ is affinoid and that $\mathfrak{U} = \{X_{1,1}\}$. By assumption, we have a system

$$\cdots \to \mathrm{Spa}(A_{2,\infty}) \to \mathrm{Spa}(A_{2,\infty}) \to \mathrm{Spa}(A_{1,\infty}).$$

With tilde-limit $X_{\infty,\infty} = \mathrm{Spa}(A_\infty, A_\infty) \sim \mathrm{Spa}(A_{i,\infty})$ with respect to $\mathfrak{U}$. To see that $X_{\infty,\infty} \sim \varprojlim X_{n,n}$, we note that the first condition follows from $\varprojlim_n |X_{n,n}| = \varprojlim_i \varprojlim_j |X_{i,j}|$. To check the second, it suffices to prove that the image of

$$\varinjlim_n A_{n,n} \to A_{\infty,\infty}$$

is dense. This follows from a straight-forward pointwise approximation argument using that all tilde-limit properties hold with respect to $\mathfrak{U}$. □

2.2. **Perfectoid tilde-limits for rigid groups.** One reason why perfectoid tilde-limits along group homomorphisms are particularly interesting is that these again have a group structure:

Definition 2.11. A **perfectoid group** is a group object in the category of perfectoid spaces.

The category of perfectoid spaces over K has finite products, so this is a well-defined notion.

Lemma 2.12. *Let $(G_i)_{i\in I}$ be an inverse system of adic groups such that the transition maps are homomorphisms of adic groups. Assune that there is a perfectoid tilde-limit $G_\infty \sim \varprojlim_{i\in I} G_i$.*

(1) There is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms
(2) Given a morphism of inverse systems of adic groups $(G_i)_{i\in I} \to (H_j)_{j\in J}$ and a perfectoid tilde-limit $H_\infty \sim \varprojlim_{j\in J} H_j$, there is a unique morphism of perfectoid groups $G_\infty \to H_\infty$ commuting with all projection maps.

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, Proposition 2.6, which shows that one can argue like in the case of categorical limits. □

Let G be an adic group locally of finite type over $(K, \mathcal{O}_K)$, that is, a group object in the category of adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. Throughout we will always consider commutative groups. The main topic of study of this work is the $[p]$-multiplication tower

$$\cdots \xrightarrow{[p]} G \xrightarrow{[p]} G.$$

We will usually assume that G is p-divisible, i.e. that $[p]\colon G \to G$ is surjective.

Question 2.13. When is there a perfectoid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$ is a tilde-limit?

We are primarily interested in the following examples:

(1) Analytifications over $\mathrm{Spa}(K, \mathcal{O}_K)$ of finite type group schemes over K. Examples include analytifications of abelian varieties A over K and of tori T over K.
(2) Generic fibres of locally topologically finite type formal group schemes over $\mathcal{O}_K$.
(3) Raynaud's covering space E of an abelian variety with semi-stable reduction.

Remark 2.14. More generally, one could ask Question 2.13 for families of abelian varieties over $\mathrm{Spec}(R)$ where R is any perfectoid ring. Considering the fibers of such a family in any point of $\mathrm{Spa}(R, R^\circ)$ motivates to also study analytifications over $\mathrm{Spa}(K, K^+)$ where K^+ is any open bounded integrally closed subring of $\mathcal{O}_K$. However, one can reduce this case to the one of $K^+ = \mathcal{O}_K$.

Indeed, this follows from the following technical observation: Let $(X_i)_{i\in I}$ be an inverse system of adic spaces X_i of finite type over (K,K^+) with finite transition maps. Let $X_{i,\eta} := X_i \times_{\operatorname{Spa}(K,K^+)} \operatorname{Spa}(K,\mathcal{O}_K)$. Then the following are equivalent:

(1) There is a perfectoid tilde-limit $X_\infty \sim \varprojlim_{i\in I} X_i$.
(2) There is a perfectoid tilde-limit $X_{\infty,\eta} \sim \varprojlim_{i\in I} X_{i,\eta}$.

We will therefore restrict attention to the case of $K^+ = \mathcal{O}_K$ without loss of generality.

As we have already mentioned in the introduction, Question 2.13 has an affirmative answer in the case of abelian varieties of good reduction by [12, Lemme A.16]. More generally:

Lemma 2.15. *Let $\mathfrak{G}$ be a flat commutative formal group scheme over $\mathcal{O}_K$ such that $[p] : \mathfrak{G} \to \mathfrak{G}$ is affine. Let $G = \mathfrak{G}_\eta^{\mathrm{ad}}$ be the adic generic fibre. Then $G_\infty := (\varprojlim_{[p]} \mathfrak{G})_\eta^{\mathrm{ad}}$ is a perfectoid tilde-limit*

$$G_\infty \sim \varprojlim_{[p]} G.$$

In particular, if B is an abelian variety of good reduction over K, there is a perfectoid tilde-limit $B_\infty \sim \varprojlim B$.

Proof. This holds by the same proof as in [12, Lemme A.16], (see also Exercise 4 – 6 in [1]): Let $\varpi \in \mathcal{O}_K$ be a pseudo-uniformiser such that $p \in \varpi\mathcal{O}_K$. The assumption that $[p] : \mathfrak{G} \to \mathfrak{G}$ is affine ensures that the limit $\mathfrak{G}_\infty := \varprojlim_{[p]} \mathfrak{G}$ exists.

The mod ϖ special fibre $\tilde{G} = \mathfrak{G} \times \operatorname{Spec}(\mathcal{O}_K/\varpi)$ is a group scheme over $\mathcal{O}_K/\varpi$, so the map $[p]\colon \tilde{G} \to \tilde{G}$ factors through the relative Frobenius map. Consequently, the mod ϖ special fibre $\varprojlim_{[p]} \tilde{G}$ of $\mathfrak{G}_\infty$ is relatively perfect over $\mathcal{O}_K/\varpi$. This implies that the adic generic fibre of $\mathfrak{G}_\infty$ is perfectoid by [14, Theorem 5.2]. $\square$

Lemma 2.16. *Let T be a torus over K. Then there is a perfectoid tilde-limit $T_\infty \sim \varprojlim_{[p]} T$.*

Proof. Since we assume K algebraically closed, we may choose a splitting $T \cong (\mathbb{G}_m^{\mathrm{an}})^d$ for some $d \in \mathbb{N}$. By Lemma 2.9, it suffices to consider the case of $d = 1$. For this, we may use the open embedding $\mathbb{G}_m^{\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}} \setminus \{0,\infty\} \subseteq \mathbb{P}^{1,\mathrm{an}}$. Sending $(x : y) \mapsto (x^p : y^p)$ defines a morphism $\varphi : \mathbb{P}^{1,\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$. The pullback of φ to $\mathbb{G}_m^{\mathrm{an}}$ is precisely $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$. We can therefore apply Proposition 2.5 to the perfectoid tilde-limit $\mathbb{P}_1^{\mathrm{perf}} \sim \varprojlim_\varphi \mathbb{P}^{1,\mathrm{an}}$ introduced in [14]. $\square$

Example 2.17. Regarding Question 2.13, we note that if G is not p-divisible, $\varprojlim_{[p]} G$ might have a tilde-limit for trivial reasons: For example, let $\mathfrak{G}_a$ be the p-adic completion of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Then the trivial group $\operatorname{Spa}(K,\mathcal{O}_K) \sim \varprojlim_{[p]} (\mathfrak{G}_a)_\eta^{\mathrm{ad}}$ is a perfectoid tilde-limit.

3. Perfectoid tilde-limits of Raynaud extensions

In this section we study the p-multiplication tower of the Raynaud extensions associated to abeloid varieties over an algebraically closed perfectoid field K. The main result of this section is Proposition 3.8, which shows that the existence of perfectoid tilde-limits is closed under analytic-locally split extensions, and thus there exists a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

Remark 3.1. Everything in this section also works with minor modifications over a general perfectoid field. But we opt to work over an algebraically closed field to simplify the exposition.

3.1. **Raynaud extensions.** We briefly sketch the theory of Raynaud extensions here, and refer the readers to [2, 10, 11] for more details on the setup.

Let A be an abelian variety over K. There exists a unique open rigid analytic subgroup $\overline{A}$ of A such that $\overline{A}$ admits a formal model $\overline{E}$ that is a connected smooth $\mathcal{O}_K$-group scheme fitting into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0, \tag{1}$$

where $\overline{B}$ is a formal abelian scheme over $\mathcal{O}_K$ with rigid generic fibre $B := \overline{B}_\eta$, and $\overline{T}$ is the completion of a torus $T_{\mathcal{O}_K}$ of rank r over $\mathcal{O}_K$. We set $T := T_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K$ and denote its analytification also by T. Then the rigid generic fibre $\overline{T}_\eta$ of the formal torus $\overline{T}$ canonically embeds into T. This induces a pushout exact sequence in the category of rigid groups: More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout:

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0. \end{tikzcd} \tag{2}$$

The abelian variety A can be uniformized in terms of E as follows:

Definition 3.2. A subset M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}^r}$.

Theorem 3.3. *There exists a lattice $M \subset E$ of rank equal to the rank r of the torus such that the quotient E/M exists as a rigid space and such that there is a natural isomorphism*

$$A = E/M$$

making $E \to E/M = A$ a rigid group homomorphism.

The data of the extension (1) together with the lattice $M \subset E$ is what we refer to as a Raynaud uniformisation of A. This will be the only input we need to construct the perfectoid tilde-limit A_∞. Consequently, our method generalises to the class of rigid groups which admit Raynaud uniformisation, namely to abeloid varieties:

Theorem 3.4 (Lütkebohmert, [11], Theorem 7.6.4)**.** *Let A be an abeloid variety, that is, a connected smooth proper commutative rigid group over K. Then A admits a Raynaud uniformisation.*

In the situation of Raynaud uniformisation, since M is discrete, the local geometry of A is determined by the local geometry of E. We will therefore first study the $[p]$-multiplication tower of E in the rest of this section and will then deduce properties of the $[p]$-multiplication tower of A in the next section.

Our strategy is to study the local geometry of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that we would like to work locally on B, but the notion of short exact sequences does not make sense if we replace B by an open $U \subseteq B$ which might not itself have a group structure. Instead, we have the following crucial lemma, which says that one may regard Raynaud extensions as T-torsors of formal schemes.

Lemma 3.5. *The short exact sequence (1) admits local sections, that is there is a cover of $\overline{B}$ by formal open subschemes $\overline{U}_i$ such that there exist local sections $s : \overline{U}_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times \overline{U}_i \hookrightarrow \overline{E}$.*

Proof. This is proved in Proposition A.2.5 in [11], where it is fomulated in terms of the group $\mathrm{Ext}^1(B,T)$. Also see [3], §1. □

Remark 3.6. In the following, we will freely work with fibre bundles of formal schemes and rigid and perfectoid spaces. For some background material on these we refer to Appendix A.

The sequence (1) gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}_\eta$ via push-out along $\overline{T}_\eta \to T$ can be expressed in terms of the associated fibre bundle by saying that $E = T \times^{\overline{T}_\eta} \overline{E}_\eta$ in the sense of Definition A.7.

Definition 3.7. We call a sequence of adic groups $0 \to T \to E \xrightarrow{\pi} B \to 0$ an analytic-locally split short exact sequence if T is the kernel of π and $\pi : E \to B$ is a principal T-torsor in the analytic topology. This implies that B is the categorial quotient in adic groups of $T \to E$.

In particular, any Raynaud extension is an analytic-locally split short exact sequence. The main goal of this section is to use this to deduce from the following from the existence of perfectoid tilde-limits $B_\infty \sim \varprojlim_{[p]} B$ and $T_\infty \sim \varprojlim_{[p]} T$:

Proposition 3.8. *Let $0 \to T \to E \to B \to 0$ be a rigid Raynaud extension. Then there is a perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$. It fits into an analytic-locally split short exact sequence of perfectoid groups*

$$0 \to T_\infty \to E_\infty \to B_\infty \to 0.$$

Our strategy of proof is as follows: Recall that $E = T \times^{\overline{T}_\eta} \overline{E}_\eta$. We would first like to *define*

$$E_\infty := T_\infty \times^{\overline{T}_{\eta,\infty}} \overline{E}_{\eta,\infty}.$$

We then use local splitting to argue that locally over affinoid opens $U_\infty \subseteq B_\infty$ this of the form $T_\infty \times U_\infty$ and thus perfectoid. However, note that for E_∞ to be well-defined, we need to know that $\overline{E} \to \overline{B}_\infty$ is a fibre bundle. This is guaranteed by the following Proposition:

Proposition 3.9. *Let $0 \to \overline{T} \to \overline{E} \to \overline{B} \to 0$ be any formal Raynaud extension. Then applying $\varprojlim_{[p]}$ on the generic fibre gives an analytic-locally split short exact sequence*

$$0 \to \overline{T}_{\eta,\infty} \to \overline{E}_{\eta,\infty} \to B_\infty \to 0.$$

Proof. We first consider the case that K is of characteristic p, and deduce the case of characteristic 0 by untilting. If K is of characteristic p, then $[p] : \overline{B} \to \overline{B}$ decomposes into the Verschiebung $V : \overline{B} \to \overline{B}^{(p^{-1})}$ and the Frobenius morphism $F : \overline{B}^{(p^{-1})} \to \overline{B}$, and we can thus write

$$\varprojlim_{[p]} \overline{B} = \varprojlim_{V} \varprojlim_{F} \overline{B}^{(p^{-n})}.$$

and similarly for $\overline{E}$. Here the limit $\varprojlim_F$ amounts to the perfection functor, which is defined more generally on formal schemes over $\mathcal{O}_K$. For this reason, we see that by functoriality, the sequence

$$0 \to \overline{T}^{\mathrm{perf}} \to \overline{E}^{\mathrm{perf}} \to \overline{B}^{\mathrm{perf}} \to 0$$

is again locally split exact. Note that $\overline{T}^{\mathrm{perf}} = \varprojlim_{[p]} \overline{T} = \overline{T}_\infty$. Consequently, we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{T}_\infty \arrow[r] \arrow[d, equal] & \overline{E}_\infty \arrow[r] \arrow[d] & \overline{B}_\infty \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{T}^{\mathrm{perf}} \arrow[r] \arrow[d] & \overline{E}^{\mathrm{perf}} \arrow[r] \arrow[d] & \overline{B}^{\mathrm{perf}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{T} \arrow[r] & \overline{E} \arrow[r] & \overline{B} \arrow[r] & 0
\end{tikzcd}$$

in which the top morphism of sequences is a pull-back along $\overline{B}_\infty \to \overline{B}^{\mathrm{perf}}$. In particular, the top morphism is again Zariski-locally split. The result follows by passing to generic fibres.

To deduce the case of characteristic 0, it suffices to prove that the desired sequence is the untilt of a sequence in characteristic p: Recall that we have a canonical identification $\mathcal{O}_{K^\flat}/t = \mathcal{O}_K/p$. Via this identification, one can always lift the reduction

$$0 \to \overline{T}/p \to \overline{E}/p \to \overline{B}/p \to 0$$

over $\mathcal{O}_K/p$ to a Raynaud extension $0 \to \overline{T}' \to \overline{E}' \to \overline{B}' \to 0$ over $\mathcal{O}_{K^\flat}$: For this, one first lifts the abelian scheme $\overline{B}/p$ along $\mathcal{O}_{K^\flat} \to \mathcal{O}_K/p$, and then the fibre bundle $\overline{E}'$: To see that this is possible, one reduces to the case of line bundles by considering characters $T \to \mathbb{G}_m$. But such line-bundles are parametrised by the Picard group $\mathrm{Pic}^0(\overline{B'})$ which is a smooth $\mathcal{O}_K$-scheme, and thus translation-invariant line bundles can always be lifted.

By applying $\varprojlim_{[p]}$, we see that in particular, we have an identification

$$\begin{tikzcd}
0 \arrow[r] & \overline{T}_\infty/p \arrow[r] \arrow[d, equal] & \overline{E}_\infty/p \arrow[r] \arrow[d, equal] & \overline{B}_\infty/p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overline{T}'_\infty/t \arrow[r] & \overline{E}'_\infty/t \arrow[r] & \overline{B}'_\infty/t \arrow[r] & 0
\end{tikzcd}$$

Arguing like in the proof of [18, Corollary III.2.19], it now follows from the tilting equivalence that the sequence stated in the Proposition is analytic-locally split exact, as desired. □

Proof of Proposition 3.8. We can split $[p^n] : E \to E$ into two morphisms as follows:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "{[p^n]}"] & E \arrow[r, "\pi"] \arrow[d, "{[p^n]\times\pi}"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & E \times_{B,[p^n]} B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "{[p^n]}"] & 0 \\
0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0
\end{tikzcd} \tag{3}$$

Here the bottom morphism of short exact sequences is the base-change of the T-bundle $E \to B$ along $[p^n] : B \to B$, whereas the top morphism of short exact sequences is the change of fibre along $[p^n] : T \to T$. It is clear that the vertical maps in the middle compose to $[p^n] : E \to E$.

The basic idea is now to trace local splittings of π through the diagram: Let $\mathfrak{U}$ be a cover of B of opens U over which $E \to B$ is split, and let U_n be the pullback of U along $[p^n]$. Then by the universal property of the fibre product, the section of $E|_U = T \times U$ induces a section of $E \times_{B,[p^n]} B|_{U_n} \to U_n$. The pullback of the above diagram to U is thus of the form

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & E_{|U_n} \arrow[r] \arrow[d] & U_n \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & T \times U_n \arrow[r] \arrow[d, "\mathrm{id}\times[p]"] & U_n \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & T \arrow[r] & T \times U \arrow[r] & U \arrow[r] & 0.
\end{tikzcd}$$

If we knew that the morphism on the top right was again split over U_n, this would prove the Proposition using Lemma 2.9. However, while we know that this map is locally split, it is not clear that a splitting exists over U_n. Of course one could refine $\mathfrak{U}$, but it is not a priori clear that this refinement stabilises for $n \to \infty$. To circumvent this problem, we will instead use that by Proposition 3.9, such a splitting always exists in the limit.

For varying n, we can split the $[p]$-multiplication tower into two towers

$$\begin{tikzcd}
E \arrow[r, "{[p]\times\mathrm{id}}"] \arrow[dr, "{[p]}"'] & E \times_{B,[p]} B \arrow[r, "{[p]\times\mathrm{id}}"] \arrow[d, "\mathrm{id}\times[p]"] & E \times_{B,[p^2]} B \arrow[d, "\mathrm{id}\times[p]"] \\
& E \arrow[r, "{[p]\times\mathrm{id}}"] \arrow[dr, "{[p]}"'] & E \times_{B,[p]} B \arrow[d, "\mathrm{id}\times[p]"] \\
& & E
\end{tikzcd}$$

We now first take the limit over the vertical tower on the right: We conclude that in the tilde-limit, we obtain an analytic locally split short exact sequence

$$0 \to T \to E \times_B B_\infty \to B_\infty \to 0$$

of sousperfectoid adic spaces. Since over U, we have

$$E \times_B B_\infty|_{U_\infty} = T \times U_\infty \sim \varprojlim_n T \times U_n,$$

we moreover have $E \times_B B_\infty \sim \varprojlim E \times_{B,[p^n]} B$ with respect to $\mathfrak{U}$.

At this point we have constructed tilde-limits for the vertical towers in the above diagram. These fit into a horizontal tower

$$\dots \xrightarrow{[p]\times\mathrm{id}} E \times_B B_\infty \xrightarrow{[p]\times\mathrm{id}} E \times_B B_\infty.$$

Recall that by definition, we have $E_\infty := T_\infty \times^{\overline{T}_\infty} \overline{E}_\infty$. This fits into an inverse system of analytic-locally split exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_\infty \arrow[r] \arrow[d] & E_\infty \arrow[r] \arrow[d] & B_\infty \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T \arrow[r] \arrow[d, "{[p]}"] & E \times_B B_\infty \arrow[r] \arrow[d, "{[p]\times\mathrm{id}}"] & B_\infty \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T \arrow[r] & E \times_B B_\infty \arrow[r] & B_\infty \arrow[r] & 0.
\end{tikzcd}$$

which we may think of being the limiting horizontal tower of the above diagonal diagram. These sequences are now again locally compatibly split, but for a different reason: They are all push-outs for the various projection maps $T_\infty = \varprojlim_{[p]} T \to T$. Since we know that the sequence on top is analytic-locally split, we get compatible splittings at each level. After refining $\mathfrak{U}$, we can assume that these splitting exist over the opens U_∞ from above. Arguing as in the first part of the proof, this give a tilde-limit relation

$$E_\infty \sim \varprojlim_{[p]\times\mathrm{id}} E \times_B B_\infty$$

with respect to $\mathfrak{U}$. We now conclude by applying Lemma 2.10 to the above diagonal diagram. □

Remark 3.10. There is an alternative proof of the tilde-limit property that also constructs a formal model $\mathfrak{E}_\infty$ of E_∞, like in Lemma 2.15. For this, one first takes a sequence of formal models

$$\cdots \to \mathfrak{T}_2 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{T}_1 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{T}_0$$

of $\cdots \xrightarrow{[p]} T \xrightarrow{[p]} T$. This can be done in such a way that each $[\mathfrak{p}]_i$ reduces to the relative Frobenius mod p. Then $\mathfrak{T}_\infty := \varprojlim_{[\mathfrak{p}]_i} \mathfrak{T}_i$ is a formal model of the perfectoid space T_∞ (giving an alternative proof that T_∞ is a perfectoid tilde-limit). When we set $\mathfrak{E}_i := \mathfrak{T}_i \times^{\overline{T}} \overline{E}$, we get an inverse system

$$\cdots \to \mathfrak{E}_2 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{E}_1 \xrightarrow{[\mathfrak{p}]_1} \mathfrak{E}_0$$

with transition maps that factor through the relative Frobenius map mod p. Thus the generic fibre of $\mathfrak{E}_\infty := \varprojlim_{[\mathfrak{p}]_i} \mathfrak{E}_i$ is a perfectoid tilde-limit of $\cdots \xrightarrow{[p]} E \xrightarrow{[p]} E$.

However, this construction does not give the local splittings in Proposition 3.8.

Remark 3.11. With some work, the arguments in this section can be extended to any perfectoid base field. For instance, the Raynaud uniformisation of Theorem 3.3 might only be defined over a finite extension L of K. Our argument then gives a perfectoid space over the (necessarily perfectoid) field L. We can then use Galois descent to get a perfectoid space over our original field K. This uses that the quotient of a perfectoid space by a finite group often remains perfectoid: see Theorem 1.4 of [5] for details. Finally, one checks that this Galois descent commutes with tilde-limits.

4. The case of abeloid varieties

We now prove Theorem 1, building on the preceding sections. Recall our setup: Let A be an abeloid variety over K. Let E be the Raynaud extension associated to A from Proposition 3.4, which is an extension of an abeloid variety B of good reduction by a split rigid torus T of rank r, and $M \subset E$ is a lattice of rank r such that $A = E/M$.

By Proposition 3.4, the quotient map $\pi\colon E \to A$ is locally split in the analytic topology on A: As the action of M on E is totally discontinuous, for every point $x \in A$ there is an open neighbourhood U' of E such that π maps isomorphically onto an open $U := \pi(U')$ containing x. Here we are careful to distinguish $U' \subset E$ and $U \subset A$, even though the two are isomorphic via π.

We fix from now on a cover $\mathfrak{U}$ of A by opens U of this form.

The pullback of U' along $[p]\colon A \to A$ will in general be bigger than the pullback of U along $[p] : E \to E$: e.g. in characteristic 0, the first is an étale $A[p]$-torsor, whereas the latter is an étale $E[p]$-torsor, and by the Snake Lemma we have a short exact sequence

$$0 \to E[p] \to A[p] \to M/pM \to 0$$

To relate the pullbacks, we subdivide the tower

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$$

into two partial towers. For this we make some auxiliary choices: Since K is algebraically closed, we can choose lattices $M_n \subseteq E$ such that $M_0 = M$ and $[p]\colon E \to E$ restricts to isomorphisms $M_{n+1} \to M_n$ for all n.

Remark 4.1. Such a choice is equivalent to the choice of subgroups $D_n \subseteq A[p^n]$ of order p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$. Namely, given the lattices M_n, we obtain the desired torsion subgroups by setting $D_n := M_n/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/p^nM \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ with properties as above, we recover M_n as the kernel of $E \to A \to A/D_n$.

One might call the choice of D_n for all n a partial anticanonical $\Gamma_0(p^\infty)$-structure, because if B admits a canonical subgroup (that is, if it satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A always has a partial anticanonical subgroup even if B does not have a canonical subgroup.

Following the remark, denote by D_n the torsion subgroup $M_n/M \subset A$. The quotient $A_n := A/D_n = E/M_n$ is then another abeloid variety over K and the quotient map $v^n\colon A = E/M \to A_n = E/M_n$ is an isogeny of degree p^{rn} through which $[p^n]\colon A \to A$ factors. The $[p]$-multiplication tower now splits into two towers, one written vertically, the other horizontally:

(4)
$$\begin{tikzcd}
{} \arrow[dr, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] \\
& A \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[r, "v"] \arrow[d, "{[p]_E}"] & A_2 \arrow[d, "{[p]_E}"] \\
& & A_1 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & A_1 \arrow[d, "{[p]_E}"] \\
& & & A.
\end{tikzcd}$$

Since each $D_n = M_n/M$ is finite étale, all horizontal maps are finite étale. The vertical tower on the other hand fits into a commutative diagram which compares it to the $[p]$-tower of E:

(5)
$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & A_1 \arrow[r] \arrow[d, "{[p]_E}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Definition 4.2. Let $M_\infty := \varprojlim_{n\in\mathbb{N}} M_n$ be the limit of the left vertical tower.

We note that M_∞ is an actual limit, not just a tilde-limit, because the transition maps are isomorphisms. In particular, the projection $M_\infty \to M$ is an isomorphism as well. By Proposition 2.6, we get a natural map $M_\infty \to E_\infty$.

Proposition 4.3. *There is a perfectoid tilde-limit $A'_\infty \sim \varprojlim_{n\in\mathbb{N}} A_n$. It fits into an analytic-locally split short exact sequence of perfectoid groups*

$$0 \to M_\infty \to E_\infty \to A'_\infty \to 0.$$

Proof. We work locally on opens $U' \subset E$ mapping isomorphically to U in our cover $\mathfrak{U}$ of A. Write $\pi_n\colon E \to A_n$ for the quotient map. Since the rows in (5) are exact, and the transition maps on the left are isomorphisms, it follows that for each $n \in \mathbb{N}$, the quotient map π_n sends the pullback $U'_n := [p^n]^{-1}(U')$ isomorphically onto $U_n := \pi_n(U'_n) \subseteq A_n$. Thus (5) is locally of the form

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cong"] & M_1 \times U'_1 \arrow[r] \arrow[d, "{[p]}"] & U_1 \arrow[r] \arrow[d, "{[p]_E}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \times U' \arrow[r] & U \arrow[r] & 0. \end{tikzcd} \tag{6}$$

Let U_∞ be the pullback of U' along $E_\infty \to E$. We have $U_\infty \sim \varprojlim U'_n \cong \varprojlim U_n$. The system $(U_n)_{n\in\mathbb{N}}$ thus has a perfectoid tilde-limit. This shows that $\varprojlim A_n$ has a perfectoid tilde-limit. We can therefore apply Proposition 2.6 to get a morphism $E_\infty \to A'_\infty$, obtaining the desired short exact sequence in the limit over diagram (5) since the transition maps in (6) respect the splitting. $\square$

We will keep the notation introduced in the above proof: U' is an open of E mapping isomorphically to $U \subset A$. The open $U'_n := [p^n]^{-1}(U') \subset E$ maps isomorphically to its image $U_n \subset A_n$ and we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_n \arrow[r] \arrow[d, equal] & M_n \times U'_n \arrow[r] \arrow[d, hook] & U_n \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r, "\pi_n"] & A_n \arrow[r] & 0. \end{tikzcd}$$

To construct a tilde-limit for $\varprojlim A$, we use the fact that the horizontal maps in diagram (4) are all finite étale. They are even finite covering maps, in the following sense:

Lemma 4.4. *For any $n \geq 0$, the preimage of $U_n \subset A_n$ under the horizontal map $v^n \colon A \to A_n$ is isomorphic to p^{rn} disjoint copies of U_n. More canonically, it is isomorphic to $D_n \times U_n$, where $D_n = M_n/M$ (see Remark 4.1).*

Proof. For the first part, we observe that the map v^n fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & E \arrow[r] & A_n \arrow[r] & 0 \end{tikzcd} \tag{7}$$

where the map on the left is the natural inclusion. Upon restriction to $U_n \subset A_n$, this becomes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook] & M_n \times U'_n \arrow[r] \arrow[d, equal] & (v^n)^{-1}(U_n) \arrow[r] \arrow[d, "v^n"] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & M_n \times U'_n \arrow[r] & U_n \arrow[r] & 0 \end{tikzcd} \tag{8}$$

and the claim follows the fact that M is a discrete lattice of rank r, and from $U'_n \cong U_n$. $\square$

Definition 4.5. The $[p]$-multiplication on E maps M_{n+1} onto M_n and therefore the $[p]$-multiplication tower of A induces a tower

$$\cdots \xrightarrow{[p]} D_{n+1} = M_{n+1}/M \xrightarrow{[p]} D_n = M_n/M \to \cdots .$$

Since K is algebraically closed, the finite étale groups D_n are already constant. By Lemma 2.8, there is a profinite perfectoid group D_∞ such that

$$D_\infty \sim \varprojlim_n D_n.$$

The quotient maps $M_n \to D_n = M_n \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ in the limit give rise to a closed immersion of perfectoid groups $M_\infty \hookrightarrow D_\infty = M_\infty \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Theorem 1 is now part of the following theorem:

Theorem 4.6. *(1) There is a perfectoid space A_∞ which is a tilde-limit of $\varprojlim_{[p]} A$.*

(2) It is independent up to canonical isomorphism of the auxiliary choice of lattices M_n with $D_n = M_n/M$, but it remembers the choice as a pro-finite étale closed subgroup $D_\infty \subseteq A_\infty$.

(3) The preimage of any $U \in \mathfrak{U}$ under the projection $A_\infty \to A$ is isomorphic to $D_\infty \times U_\infty$.

(4) There is a natural map of analytic-locally split short exact sequences of perfectoid groups

$$\begin{tikzcd}
0 \arrow[r] & M_\infty \arrow[r] \arrow[d, hook] & E_\infty \arrow[r] \arrow[d, hook] & A'_\infty \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D_\infty \arrow[r] & A_\infty \arrow[r] & A'_\infty \arrow[r] & 0.
\end{tikzcd}$$

(5) One can describe A_∞ as the associated fibre bundle

$$A_\infty = D_\infty \times^{M_\infty} E_\infty.$$

In particular, we have an analytic-locally split short exact sequence of perfectoid groups

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the antidiagonal embedding of M_∞ into $D_\infty \times E_\infty$.

Remark 4.7. We think of part (5) as the analogue of the Raynaud uniformisation

$$0 \to M \to E \to A \to 0.$$

Here we note that while the map $E \to A$ is a quotient, in the limit over $[p]$ it becomes an immersion $E_\infty \hookrightarrow A_\infty$: The reason is that the projective system $(M, [p])$ has vanishing lim but non-vanishing Rlim^1, for instance, when considered as abelian sheaves on perfectoid spaces for the pro-étale topology in the sense of [19] (assuming that K is of characteristic 0). A toy example of this phenomenon would be the inverse system over $[p]$ on the short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ whose limit yields an exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \varprojlim^1_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

We therefore think the quotient $D_\infty/M_\infty = M_\infty \otimes_{\mathbb{Z}} (\mathbb{Z}_p/\mathbb{Z}$ implicit in part (5) as being an incarnation of $\mathrm{Rlim}^1_{[p]} M_\infty$.

Proof of Theorem 4.6. We keep the notation from the proof of Proposition 4.3: We have a cover of A_n by open subsets U_n and a perfectoid open subspace $U_\infty \subseteq E_\infty$ for which $U_\infty \sim \varprojlim U_n$.

By Lemma 4.4, the restriction of diagram (4) to the open U of the bottom A becomes

$$\begin{tikzcd}
{} \arrow[dr, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] \\
& D_2 \times U_2 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & v^{-1}(U_2) \arrow[r, "v"] \arrow[d, "{[p]_E}"] & U_2 \arrow[d, "{[p]_E}"] \\
& & D_1 \times U_1 \arrow[r, "v"] \arrow[dr, "{[p]}"'] & U_1 \arrow[d, "{[p]_E}"] \\
& & & U.
\end{tikzcd}$$

Hence the restriction of the tower $\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$ to U becomes the inverse system

$$\cdots \to D_{n+1} \times U_{n+1} \to D_n \times U_n \to \cdots .$$

By Lemma 2.9 this inverse system has perfectoid tilde-limit $D_\infty \times U_\infty$. These local tilde-limits glue together to give the desired tilde-limit A_∞. This proves parts (1), (2) and (3), and shows that the second row of part (4) is locally split and in particular exact.

The first row in part (4) is from Proposition 4.3. Part (5) follows immediately from part (4). □

Remark 4.8. When working over a general perfectoid base field, the lattices M_n may no longer be defined over K. Instead, one can show that the natural map $A[p^n] \times U_n \to V_n$ is an étale $E[p^n]$-torsor for the diagonal action where V_n is the pullback of U along $[p^n]\colon A \to A$. The point is that this torsor is split when K is algebraically closed.

5. Applications

In this section, we give three applications of our main result. For all of these, we assume that K is of characteristic 0, i.e. K is an algebraically closed non-archimedean field extension of $\mathbb{Q}_p$.

5.1. **Uniformisation.** Our first application is a "p-adic uniformisation" of abelian varieties. Recall that any abelian variety A over $\mathbb{C}$ of dimension d has a uniformisation in terms of a complex torus $A \cong \mathbb{C}^d/\Lambda$ for some $2d$-dimensional lattice $\Lambda \subseteq \mathbb{C}^d$. More canonically, it admits the presentation

$$A \cong \operatorname{Lie} A/H_1(A, \mathbb{Z}).$$

We have the following analogue of this over K: Let A be an abeloid variety over K of dimension d, considered as a rigid space. Then in the limit over n, the short exact sequences

$$0 \to A[p^n] \to A \to A \to 0$$

give rise to a short exact sequence of sheaves on perfectoid K-algebras with the pro-étale topology

$$0 \to T_pA \to A_\infty \to A \to 0.$$

Using the language of diamonds from [19], we then have:

Corollary 5.1. *The diamond $A^\diamond$ associated to A has a natural presentation*

$$A^\diamond = A_\infty/T_pA$$

given by the perfectoid space A_∞ with its pro-étale subgroup T_pA.

Here we think of $T_pA = H_1^{\text{ét}}(A, \mathbb{Z}_p)$ as the analogue of $H_1(A, \mathbb{Z})$ in the complex setting.

Of course this p-adic uniformisation of A is very closely related to the uniformisation of the associated p-divisible group $A[p^\infty]$ described in [20] and [16, §4]: Indeed, in the language used there, we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_p(A[p^\infty]) \arrow[r] \arrow[d, equal] & \widetilde{A[p^\infty]} \arrow[r] \arrow[d, hook] & A[p^\infty] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_pA \arrow[r] & A_\infty \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Remark 5.2. We note that for two abelian varieties A and B of dimension d, the universal covers A_∞ and B_∞ are different in general, so that this is only a "uniformisation" in a rather weak sense. However, they are canonically isomorphic if A and B are abelian varieties of good reduction with the same special fibre, or if A and B are p-power isogeneous, so that in these cases we can really think of T_pA as a a $2d$-dimension $\mathbb{Z}_p$-lattice determining A.

There is a second closely related uniformisation which we discuss in §5.3 below.

5.2. **Stein property.** As a second application, we can combine our main theorem with work of Reinecke to deduce the following Artin vanishing result:

Corollary 5.3 (Reinecke)**.** *Let A be an abeloid variety over K. Let L be a constructible sheaf of $\mathbb{F}_p$-modules on $A_{\text{ét}}$. Then for any $i > \dim A$,*

$$\varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^*L) = 0.$$

Proof. Due to Theorem 1, we can apply [13, Theorem 3.3] to the system $\cdots \to A \xrightarrow{[p]} A$. $\square$

A theorem of Artin and Grothendieck states if X is an affine algebraic variety over K, then $H^i_{\text{ét}}(X, L) = 0$ for any constructible $\mathbb{F}_p$-module L and any $i > \dim A$. However, the rigid analogue of this statement is false in general. The point of the Corollary is that an analogue of this vanishing statement is true for the pullback of L to A_∞ in the following sense: Consider the morphism of sites $\nu\colon A_{\text{proét}} \to A_{\text{ét}}$. Then by regarding A_∞ as an object in $A_{\text{proét}}$ via the pro-étale morphism $A_\infty \to A$, one can show

$$H^i_{\text{proét}}(A_\infty, \nu^*L) = \varinjlim_{n\in\mathbb{N}} H^i_{\text{ét}}(A, [p^n]^*L).$$

Let us sketch a proof that the left hand side vanishes for $i > \dim A$: Since A is proper and smooth, one can use the Primitive Comparison Theorem [15, Theorem 5.1] [16, Theorem 3.13] to reduce to showing that $H^i_{\text{proét}}(A_\infty, \nu^*L \otimes_{\mathbb{F}_p} \mathcal{O}^+_{A_{\text{ét}}}/p)$ is almost zero for $i > d$. For this one can use that $H^j(V, \nu^*L \otimes_{\mathbb{F}_p} \mathcal{O}^+_{A_{\text{ét}}}/p)^a = 0$ for any affinoid perfectoid $V \subseteq A_\infty$ and any $j > 0$. This reduces the desired statement to a computation in Čech cohomology, which indeed vanishes in degree $> \dim A$.

5.3. **Universal perfectoid covers of curves.** As a third application, we describe how one can obtain universal perfectoid pro-étale covers of curves over K. This was first observed by Hansen [6].

We start by recalling some background in a more general setting: Let C be a connected smooth projective scheme over K, which we consider as an analytic adic space. By [9, Theorem 3.1], GAGA induces an equivalence of categories between finite étale covers of the scheme C and finite étale covers of the adic space C. We can therefore fix a geometric base point $x : \mathrm{Spa}(K, \mathcal{O}_K) \to C$ and study the usual étale fundamental group $\pi_1(C, x)$ using the language of adic spaces.

To prepare our discussion, we recall from [15, §3] a few facts on the pro-finite-étale site of C: This is the category $C_{\text{profét}} = \text{pro}{-}C_{\text{fét}}$ of small cofiltered inverse systems $(X_i)_{i\in I}$ in the finite étale site $C_{\text{fét}}$. An object in $C_{\text{profét}}$ is called perfectoid if there is a perfectoid tilde-limit $X_\infty \sim \varprojlim X_i$. Let $C^{\text{perf}}_{\text{profét}}$ be the full subcategory of perfectoid objects, and let Perf_C be the category of perfectoid spaces over C. Then the argument in the proof of [21, Lemma 8.2.3] shows:

Lemma 5.4. *Sending perfectoid pro-étale objects to their tilde-limits defines a fully faithful functor*

$$C^{\text{perf}}_{\text{profét}} \to \mathrm{Perf}_C, \quad (X_i)_{i\in I} \mapsto X_\infty \sim \varprojlim_{i\in I} X_i.$$

We call the objects $X_\infty \to C$ in the essential image the pro-finite-étale perfectoid covers of C.

Proposition 5.5 (Proposition 3.5, [15]). *There is an equivalence of categories*

$$F : C_{\text{profét}} \to \pi_1(C,x)\text{–pfSets}, \quad (Y_i)_{i\in I} \mapsto F(X) := \varprojlim_{i\in I} |Y_i \times_C x| = \varprojlim_{i\in I} \mathrm{Hom}_C(x, Y_i)$$

from the pro-finite-étale site of C to the category of profinite sets with continuous $\pi_1(C,x)$-action.

This restricts to the usual equivalence of finite étale covers to finite sets with continuous $\pi_1(C,x)$-action. In particular, for every open subgroup $H \subseteq \pi_1(C,x)$, there is a corresponding finite étale morphism $C_H \to C$ from a connected scheme C_H, considered as an analytic adic space. For any two open subgroups $H_1 \subseteq H_2 \subseteq \pi_1(C,x)$, there is a natural map $C_{H_1} \to C_{H_2}$. For varying H, one therefore has a filtered inverse system $(C_H)_{H\subseteq \pi_1(C,x)}$ which we may regard as an object in $C_{\text{proét}}$.

Corollary 5.6 (Hansen, [6]). *Let C be a connected smooth projective curve of genus $g \geq 1$ over K, considered as an analytic adic space.*

1. *There is a perfectoid tilde-limit $\tilde{C} \sim \varprojlim_H C_H$ where H ranges over the open subgroups of $\pi_1(C,x)$.*
2. *The morphism $\tilde{C} \to C$ is a pro-étale $\pi_1(C,x)$-torsor. It is universal with this property in the sense that it represents the functor sending pro-finite-étale perfectoid covers $X \to C$ to the $\pi_1(C,x)$-module $F(X) = \mathrm{Hom}_C(x, X)$.*
3. *For any $X \in C_{\text{profét}}$, for example for any finite étale $X \to C$, there is a natural isomorphism*
$$X = \underline{F(X)} \times^{\pi_1(C,x)} \tilde{C} := (\underline{F(X)} \times \tilde{C})/\pi_1(C,x).$$
Here the right hand side is the categorical quotient in adic spaces for the antidiagonal action.

Remark 5.7. Parts (2) and (3) say that we may reasonably regard $\tilde{C} \to C$ as the "universal cover" of C, in analogy with this notion in topology.

The proof also works in the case that C is an abelian variety. In this case, the étale fundamental group is simply the absolute Tate module $\pi_1(A,x) = TA := \varprojlim_{N\in\mathbb{N}} A[N](K)$. We then have:

Corollary 5.8. *Let A be an abelian variety over K, then there is a perfectoid tilde-limit*

$$\tilde{A} \sim \varprojlim_{[N]} A$$

and the analogous statements of Corollary 5.6.2 and 3 hold for the TA-torsor $\tilde{A} \to A$. In particular, there is a natural isomorphism

$$A^\diamond = \tilde{A}/TA = \tilde{A}/\pi_1(A,x).$$

Proof of Cor. 5.6 and Cor. 5.8. To ease notation, let us abbreviate $G := \pi_1(C,x)$.

We construct $\tilde{C}$ in two steps. The choice of the base point x gives an embedding $\iota\colon C \to A$ of C into its Jacobian. Let C_n be the pullback of C along the map $[p^n]\colon A \to A$. Combining our main theorem with [18, Lemma II.2.2], we can pull back perfectoid tilde-limits along closed immersions and hence get a perfectoid space $C_\infty \sim \varprojlim C_n$ with a Zariski closed embedding $C_\infty \to A_\infty$.

We now use the fact that pro-étale covers of perfectoid spaces are again perfectoid to construct a perfectoid cover $\tilde{C}$ of C_∞ that packages up the entire étale fundamental group of C. As we are assuming that K has characteristic 0, the maps $[p^n]\colon A \to A$ are finite étale, so the induced covers $C_n \to C$ are finite étale. The inverse system

$$\cdots \to C_n \to \cdots \to C_1 \to C$$

therefore corresponds to a chain of subgroups

$$\dots < G_n < \dots < G_1 < G = \pi_1(C, x).$$

For any open subgroup H of G corresponding to the finite étale cover $C_H \to C$, we have a decreasing sequence of positive integers

$$\dots \leq [G_n : G_n \cap H] \leq \dots \leq [G_1 : G_1 \cap H] \leq [G : G \cap H]$$

which stabilises for $n \gg 0$. So there is an integer d such that for all $n \gg 0$, we have $[G_n : G_n \cap H] = d$. Translating back to the language of finite étale covers, we see that for such n, the map

$$C_{G_{n+1}\cap H} \to C_{G_n \cap H} \times_{C_{G_n}} C_{G_{n+1}}$$

coming from the universal property of the fibre product is an isomorphism: Both spaces are finite étale covers of $C_{G_{n+1}}$ of degree d, so the map is a finite étale cover of degree 1. This implies that the natural morphism $\varprojlim C_{G_n\cap H} \to \varprojlim C_{G_n}$ of objects of $C_{\text{proét}}$ is finite étale in the sense of [15, Definition 3.9]. To simplify notation, we write this morphism as $C_{H,\infty} \to C_\infty$ (via Lemma 5.4, one can also think of this as the corresponding map of perfectoid spaces).

We can now rewrite in $C_{\text{proét}}$:

$$\varprojlim_{H\to 1} C_H = \varprojlim_{H\to 1} \varprojlim_{n\to\infty} C_{G_n\cap H} = \varprojlim_{H\to 1} C_{H,\infty}.$$

As the $C_{H,\infty}$ have compatible finite étale maps to C_∞, we obtain a morphism in $C_{\text{proét}}$

$$\varprojlim_{H\to 1} C_{H,\infty} \to C_\infty.$$

By [15, Lemma 4.6], pro-finite-étale covers of perfectoid objects are again perfectoid, giving us the desired perfectoid space

$$\tilde{C} \sim \varprojlim_{H\to 1} C_H.$$

This completes the construction of $\tilde{C}$, and thus proves part (1).

To see part (2), we note that we can write $G = \varprojlim_N G/N$ where N ranges through the normal open subgroups. These are precisely the subgroups for which $C_N \to C$ is already a finite étale G/N-torsor. Concretely, this means that the following natural morphism is already an isomorphism:

$$G/N \times_C C_N \to C_N \times_C C_N.$$

We note that we also have $\tilde{C} \sim \varprojlim_N C_N$, as normal open subgroups are cofinal in the inverse system of all open subgroups. In the limit, this shows that $\tilde{C}$ is a pro-finite-étale G-torsor.

To see that $F(X) = \mathrm{Hom}_C(\tilde{C}, X)$, we recall that for any Galois cover $C_N \to C$ with a finite Galois map $C_N \to X$, we have $F(X) = \mathrm{Hom}_C(C_N, X)$. It therefore suffices to see that

$$\mathrm{Hom}_C(\tilde{C}, X) = \varinjlim_N \mathrm{Hom}_C(C_N, X).$$

But this follows from Lemma 5.4.

For (3), write $S = F(X)$, then it suffices to prove that the natural morphism

$$\rho : \underline{S} \times \tilde{C} \to X$$

is a pro-finite-étale G-torsor for the antidiagonal action. Indeed, this implies that X is the categorical quotient by the action of G: This is because the torsor property implies $\mathcal{O}_X = (\rho_* \mathcal{O}_{\underline{S}\times\tilde{C}})^G$ by combining [4, Lemma 2.24] and [8, Theorem 8.2.3].

Since connected components of X correspond to G-orbits of S, we may reduce to the case where X is connected and G acts transitively on $\underline{S}$. By writing X as a system of finite étale covers, we may further reduce to the case that S is finite. Fix $s \in S$ and let $H \subseteq G$ be the stabiliser of s, then $X = C_H$. It now suffices to show that for any normal open subgroup $N \subseteq G$ with $N \subseteq H$, the natural morphism

$$G/H \times C_N \to C_H$$

is a G/N-torsor, as the desired result will follow in the limit $N \to 1$. But this follows by Galois descent from the diagram

$$\begin{tikzcd} G/H \times C_N \arrow[r] & C_H \\ G/N \times C_N \arrow[r] \arrow[u] & C_N \arrow[u] \end{tikzcd}$$

which is Cartesian as $C_N \to C_H$ is a finite étale H/N-torsor. $\square$

Appendix A. Fibre bundles of adic spaces

In this appendix we review the theory of fibre bundles in the setting of analytic adic spaces. We will always implicitly assume that finite products of the adic spaces we work with exist: This is for instance the case when we work with rigid or perfectoid spaces, the cases we are most interested in.

Notation A.1. In the following, if $\pi\colon E \to B$ is a morphism of adic spaces, then for an open subspace $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.2. Let T be an adic group. Throughout we will assume that T is commutative.

Let F be an adic space with an action $m\colon T \times F \to F$. A morphism $\pi\colon E \to B$ of adic spaces is called a **fibre bundle with fibre** F **and structure group** T if there is a cover $\mathfrak{U}$ of B of open subspaces $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl, "p_2"] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i\colon F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that ϕ_{ij} coincides with the composite

$$F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}.$$

Definition A.3. When $F = T$ with the action on itself by left multiplication, then a fibre bundle $\pi\colon E \to B$ with fibre T and structure group T is called a T**-torsor**.

Example A.4. The short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0$ from §3 yields a T-torsor $\overline{E} \xrightarrow{\pi} \overline{B}$ by Lemma 3.5. Moreover, for any open subspace $U \subseteq \overline{B}$, the map $E|_U \to U$ is a T-torsor.

The ϕ_{ij} from condition (b) are determined by the maps $\psi_{ij} \colon U_{ij} \to T$. By glueing, one sees:

Lemma A.5. *Suppose we are given adic spaces F and B and an adic group T with an action on F. Then fibre bundles $\pi \colon E \to B$ with fibre F and structure group T are equivalent to the data (up to refinement) of a cover $\mathfrak{U}$ of B by open subspaces and morphisms $\psi_{ij} \colon U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk}|_{U_{ijk}} \cdot \psi_{ij}|_{U_{ijk}} = \psi_{ik}|_{U_{ijk}}$ on the intersection $U_{ijk} := U_i \cap U_j \cap U_k$.*

Lemma A.6. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. Then the natural T-action on $F \times U_i$ for each i via the first factor glue to a natural T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.7. Let $\pi \colon E \to B$ be a T-torsor. Let F be an adic space with an action by T. Since the data in Lemma A.5 are completely independent of the fibre, the morphisms $\psi_{ij} \colon U_{ij} \to T$ by Lemma A.5 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel–Weil construction.

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