PERFECTOID COVERS OF ABELIAN VARIETIES

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Abstract. For an abelian variety A over an algebraically closed non-archimedean field which is either p-adic or of characteristic p, we show that there exists a perfectoid space which is the tilde-limit of $\varprojlim_{[p]} A$.

1. Introduction

Let p be a prime and K a non-archimedean field which is either p-adic or characteristic p. For simplicity we assume that K is algebraically closed in this introduction. The primary goal of this article is to show that the "inverse limit" of an abelian variety A over K under the multiplication by p map carries a natural structure of perfectoid spaces. More precisely, we prove

Theorem 1. *Let A be an abelian variety over K. Then there exists a unique perfectoid space A_∞ over K such that $A_\infty \sim \varprojlim_{[p]} A$.*

Remark 2. We will recall the definition of the tilde-limit, namely the meaning of $A_\infty \sim \varprojlim_{[p]} A$, in Section 2 of the paper. This notion appears in [8] to remedy the problem that inverse limits rarely exist in the category of adic spaces in the sense of *ibid* (even if transition maps are affine).

Remark 3. char p case

To prove the main theorem, we need to

(1) construct a tilde-limit A_∞ for the system $\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$;
(2) show that the adic space A_∞ is perfectoid.

In fact, we will achieve (1) and (2) simultaneously.

Remark 4. Let us consider the motivating example of the case of good reduction. This example is generalized from Exercise 4 – 6 in [?] Cite Pilloni-Stroh instead (or both? I prefer to also cite Bhatt, the outline there is clear enough) and spelt out in details in Subsection 2.3 [Corollary 2.12].

Let $\mathcal{A}$ be a formal abelian scheme over $\mathcal{O}_K$ with generic fiber A. Then on the mod π special fiber $\mathcal{A}_s = \mathcal{A} \times \mathcal{O}_K/\pi$, multiplication by p factors through the relative Frobenius map, forcing the inverse limit $\varprojlim_{[p]} \mathcal{A}_s$ (which exists in the category of schemes) to be relatively perfect over $\mathcal{O}_K/\pi$. Let $\mathcal{A}_\infty = \varprojlim_{[p]} \mathcal{A}$ (which exists by Lemma 2.3), then its adic generic fiber A_∞ is perfectoid and a tilde-limit of $\varprojlim_{[p]} A$.

Now we explain the outline of the proof of Theorem 1. The main strategy is, roughly speaking, to study abelian varieties with good reduction and totally degenerate ones "separately", and use the theory of Raynaud extensions to combine the two situations. More precisely, since A is semi-stable (K being algebraically closed), Raynaud's theory provides us with a short exact sequence

$$0 \to T \to E \to B \to 0$$

of rigid group schemes, where $T = (\mathbb{G}_m^{\mathrm{an}})^d$ is a split rigid tori and B the analytification of an abelian variety with good reduction, such that $A = E/M$ for a discrete lattice $M \subset E$. Our strategy is

(1) Show that there exists a perfectoid tilde-limit E_∞ of $\varprojlim_{[p]} E$;
(2) Use part (1) above to construct the desired perfectoid space A_∞.

Our approach to prove part (1) of in the outline (namely that E_∞ is perfectoid) relies on constructing various formal models of relevant objects and take their (adic) generic fibres. More precisely, the construction of the perfectoid space E_∞ is realized in several mini-steps:

- Develop a notion of a $[p]$-F-formal tower for a rigid group G over K such that, if G admits such a tower, then there exists a unique perfectoid tilde-limit $G_\infty \sim \varprojlim_{[p]} G$;
- Construct a $[p]$-F-formal tower $\mathfrak{T}$ for T;
- Use the tower $\mathfrak{T}$ from above, apply the associated fibre bundle construction (developed in the Appendix) to construct a $[p]$-F-formal tower for E. From this we will obtain E_∞

Finally, to construct $\varprojlim_{[p]} A$, we will "separate" the $[p]$-multiplication tower of $A = E/M$ factors into a "ramified" and an "étale" part. By carefully choosing charts of A in terms of certain subspaces of E that behave well under $[p^n] : E \to E$, we will

- explicitly construct a perfectoid tilde-limit of the "ramified" tower;
- use this "ramified tilde-limit" to construct A_∞.

Notation

We fix the prime p. Let K be a perfectoid field (either p-adic or of characteristic p), with the ring of integers $\mathcal{O}_K$ and a fixed pseudo-uniformiser $\pi \in \mathcal{O}_K$.

By adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, we mean adic spaces in the sense of [8], and we adopt the notion of perfectoid spaces defined in §2 *ibid.* In their language, adic spaces in the sense of Huber are referred to as *honest* adic spaces. Throughout the article, we make no distinction between rigid spaces (resp. formal schemes) and their corresponding honest adic spaces (resp. adic spaces).

If a rigid space is obtained from a K-scheme X via rigid-analytification $X \mapsto X^{\mathrm{an}}$, we will often denote both by the same symbol X.

For a formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$ with the π-adic topology, we denote by $\mathfrak{X}_\eta$ its adic generic fibre. We denote by $\tilde{X} = \mathfrak{X} \times_{\mathrm{Spf}\, \mathcal{O}_K} \mathrm{Spf}\, \mathcal{O}_K/\pi$ its mod π special fibre, considered as a scheme over $\mathrm{Spec}\, \mathcal{O}_K/\pi$.

Finally, recall the following standard terminology:

(1) Let X be a rigid space over K. A **formal model** of X is an admissible topologically finite type formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ together with an isomorphism of its generic fibre $\mathfrak{X}_\eta \xrightarrow{\sim} X$ (which is often suppressed from notation).
(2) Let $\phi : X \to Y$ be a morphism of rigid spaces over K, with formal models $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. A morphism of formal schemes $\Phi : \mathfrak{X} \to \mathfrak{Y}$ is a **formal model** of ϕ if it agrees with ϕ on the adic generic fiber.

Acknowledgements

This work started as a group project at the 2017 Arizona Winter School. We would like to thank Bhargav Bhatt for proposing the project, for his guidance and for his constant encouragement. In addition we would like to thank the organizers of the Arizona Winter School for setting up a great environment for us to participate in this project.

During this work Damián Gvirtz and Ben Heuer were supported by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. Peter Wear was supported by NSF grant DMS-1502651 and UCSD and would like to thank Kiran Kedlaya for helpful discussions. Other grants?

2. Tilde-limits of rigid groups

2.1. **Tilde-limits and formal models.** As we have seen in Remark 2 in the introduction, inverse limits often do not exist in the category of adic spaces (and neither in in rigid spaces). Instead we use the notion of tilde-limits from [8]:

Definition 2.1. Let X_i be a filtered inverse system of adic spaces with quasi-compact and quasi-separated transition maps, let X be an adic space with a compatible system of morphisms $f_i : X \to X_i$. We write $X \sim \varprojlim X_i$ (X is a **tilde-limit** of X_i) if the map of underlying topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism, and there exists an open cover of X by affinoids $\mathrm{Spa}(A, A^+) \subset X$ such that the map

$$\varinjlim_{\mathrm{Spa}(A_i, A_i^+) \subset X_i} A_i \to A$$

has dense image, where the direct limit runs over all open affinoids $\mathrm{Spa}(A_i, A_i^+) \subset X_i$ containing the image of $\mathrm{Spa}(A, A^+) \subset X$.

Remark 2.2. As pointed out after Proposition 2.4.4. of [8], tilde-limits (if they exist) are in general not unique. For example, consider the inverse system consisting of a single affinoid pre-perfectoid space $X = \mathrm{Spa}(A, A^+)$, then its strong completion $\hat{X} = \mathrm{Spa}(\hat{A}, \hat{A}^+)$ is also a tilde-limit of X.

One way to construct tilde-limits is by constructing certain formal models. First we observe that inverse limits are much better behaved in the category of formal schemes.

Lemma 2.3. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps $\phi_{ij} : \mathfrak{X}_j \to \mathfrak{X}_i$. Then the inverse limit $\mathfrak{X} = \varprojlim \mathfrak{X}_i$ exists in the category of formal schemes over $\mathcal{O}_K$. If all the $\mathfrak{X}_i$ are flat formal schemes, so is $\mathfrak{X}$.*

Proof. In the affine case, if the inverse system is $\mathrm{Spf}\, A_i$, take A to be the π-adic completion of $\varinjlim A_i$, then $\mathrm{Spf}\, A$ is the inverse limit of the $\mathrm{Spf}\, A_i$. In general, we can use the fact that the transition maps are affine to reduce to the affine case. □

In the lemma above, $\mathfrak{X}$ is also a tilde-limit $\mathfrak{X} \sim \varprojlim \mathfrak{X}_i$. This remains true after passing to the adic generic fibre after base-change $\mathrm{Spa}(K, \mathcal{O}_K) \hookrightarrow \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$.

Lemma 2.4. *Let $(\mathfrak{X}_i, \phi_{ij})_{i\in I}$ be an inverse system of formal schemes $\mathfrak{X}_i$ over $\mathcal{O}_K$ with affine transition maps ϕ_{ij} and let $\mathfrak{X} = \varprojlim_{\phi_j} \mathfrak{X}_i$ be the limit. Let $\mathcal{X}_i = (\mathfrak{X}_i)_\eta$ and $\mathcal{X} = (\mathfrak{X})_\eta$ be the adic generic fibres. Then*

$$\mathcal{X} \sim \varprojlim \mathcal{X}_i.$$

Proof. This is a consequence of [8], Proposition 2.4.2: The transition maps in the system are affine, hence quasi-separated quasi-compact. In order to prove the Lemma, we can restrict to an affine open subset $\mathrm{Spf}(A)$ of $\mathfrak{X}$ that arises as the inverse limit of affine open subsets $\mathrm{Spf}(A_i) \subseteq \mathfrak{X}_i$. Here all formal schemes are considered with the π-adic topology and A is the π-adic completion of $\varinjlim A_i$. On the generic fibre, A_i with ideal of definition $I_i = \pi A_i$ is an open subring of definition of $A_i[1/\pi]$. We then clearly have $I_i A_j = A_j$ for any $j \geq i$. The inverse system therefore satisfies the conditions of [8], Proposition 2.4.2, and we conclude that $\mathrm{Spf}(A)_\eta \sim \varprojlim \mathrm{Spf}(A_i)_\eta$ as desired. □

Remark 2.5. This lemma essentially says that one may construct a tilde-limit of an inverse system of rigid spaces $\mathcal{X}_i$ if it arises from an inverse system of formal schemes $\mathfrak{X}_i$ with affine transition maps. This is precisely what Scholze uses in [7] to construct the space $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ (see Corollary III.2.19 in [7] and its proof).

Remark 2.6. Let us recall Raynaud's theory of formal models: under mild assumptions, one can always find formal models of rigid spaces, and (possibly after formal blow ups) of morphisms between them. More precisely, Raynaud's theorem [1, section 8.4] states that

(1) Let X be a quasi-separated quasi-paracompact rigid space over K. Then there exist an admissible quasi-paracompact formal model $\mathfrak{X}$ for X.
(2) If $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blow-up of admissible formal schemes, then it induces an isomorphism on the generic fibre $\mathfrak{X}'_\eta \xrightarrow{\sim} \mathfrak{X}_\eta$.
(3) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be admissible quasi-paracompact formal schemes over $\mathcal{O}_K$ and let $f : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ be a morphism of their associated rigid spaces. Then there exist an admissible blow-up $\pi : \mathfrak{X}' \to \mathfrak{X}$ and a map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{Y}$ such that $\mathfrak{f}_\eta = f \circ \pi_\eta$.

In particular, given an inverse system $(\mathcal{X}_i, \phi_{ij})$ of rigid spaces, one can always choose formal models $\mathfrak{X}_i$ for the $\mathcal{X}_i$, and by successive admissible blow-ups one can also find models for the transition maps ϕ_{ij} (*which might not be affine morphisms*).

2.2. **Tilde-limits for rigid groups.** Let G be a rigid group over K, that is, a group object in the category of rigid spaces over K. We are primarily interested in the following examples

(1) Analytifications of finite type group schemes over K. Examples include the analytification of an abelian variety A over K, of tori T over K, *etc.*
(2) Generic fibres of topologically finite type formal group schemes over $\mathcal{O}_K$.
(3) Raynaud's covering space E of an abelian variety with semi-stable reduction.

Lemma 2.3 and 2.4 motivate the following definition.

Definition 2.7.

(1) Let G be a rigid group. A $[p]$**-formal tower** for G is a family of formal models $\{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ of G, together with affine transition maps $[\mathfrak{p}]_{n+1} : \mathfrak{G}_{n+1} \to \mathfrak{G}_n$ which are formal models of $[p] : G \to G$. Sometimes we suppress notation and write $[\mathfrak{p}]$ for $[\mathfrak{p}]_n$ on $\mathfrak{G}_n$.
(2) More generally, let $U \subseteq G$ be an admissible open subset, a $[p]$-formal tower for $U \subseteq G$ is a tower of formal models for

$$\cdots \xrightarrow{[p]} [p^2]^{-1}(U) \xrightarrow{[p]} [p]^{-1}(U) \xrightarrow{[p]} U.$$

In particular, a $[p]$-formal tower for G gives rise to a $[p]$-formal tower for U for each admissible open $U \subseteq G$ by pullback.

We now summarise our discussion in the previous subsection by the following

Corollary 2.8. *Let G be a rigid group. If G admits a $[p]$-formal tower, then there exists a rigid space G_∞ such that $G_\infty \sim \varprojlim_{[p]} G$.*

For example, for any formal group $\mathfrak{G}$ where $[p]$ is affine, the $[p]$-multiplication tower of $\mathfrak{G}$ gives rise to a $[p]$-formal tower for its generic fibre $\mathfrak{G}_\eta$.

2.3. **Perfectoidness.** In this subsection we define a formalism to address part (2) of Question **??** in the introduction. In light of Remark 4, we first make the following definition

Definition 2.9. Let G be a rigid analytic group, a **$[p]$-F-formal tower** for G is a $[p]$-formal tower

$$(\{\mathfrak{G}_n\}_{n\in\mathbb{N}}, [\mathfrak{p}]_{n+1} : \mathfrak{G}_{n+1} \to \mathfrak{G}_n)$$

such that each $\mathfrak{G}_n$ is flat over $\mathcal{O}_K$, and on the mod π special fibre $\tilde{G}_n$, each $[\mathfrak{p}]_{n+1}$ factors through the relative Frobenius:

$$\begin{tikzcd} & \tilde{G}_{n+1}^{(p)} \arrow[dr, dashed] & \\ \tilde{G}_{n+1} \arrow[ur, "F_{rel}"] \arrow[rr, "{[p]}"] & & \tilde{G}_n \end{tikzcd}$$

Proposition 2.10. *Let G be a rigid analytic group over a perfectoid field K. If G admits a $[p]$-F-formal tower, then G_∞ exists and is perfectoid.*

Remark 2.11. In contrast to Remark 2.2, by Proposition 2.4.5 of [8], if $X \sim \varprojlim X_i$ and X is perfectoid, then X is the unique perfectoid tilde-limit of $\varprojlim X_i$ up to unique isomorphism. In such situations we will refer to X by *the* tilde-limit of $\varprojlim X_i$.

Proof. Let $(\{\mathfrak{G}_n\}_{n\in\mathbb{N}}, [\mathfrak{p}]_{n+1} : \mathfrak{G}_{n+1} \to \mathfrak{G}_n)$ be a $[p]$-F-formal tower for G. By Lemma 2.4 we therefore have

$$G_\infty := (\varprojlim_{[\mathfrak{p}]} \mathfrak{G})_\eta \sim \varprojlim_{[p]} G.$$

To see that G_∞ is perfectoid, we proceed as the proof of [7], Corollary III.2.19. It suffices to prove that $\mathfrak{G}_\infty = \varprojlim_{[\mathfrak{p}]} \mathfrak{G}$ can be covered by formal schemes of the form $\mathrm{Spf}(S)$ where S is a flat $\mathcal{O}_K$-algebra such that the Frobenius map

$$S/\pi^{1/p} \to S/\pi$$

is an isomorphism. Lemma 5.6 of [5] then shows that $S[1/\pi]$ is perfectoid.

By assumption, on the mod π special fibre $\tilde{G}_n$, $[\mathfrak{p}]_{n+1}$ factors through the relative Frobenius. Now take any affine open subspace $\mathrm{Spf}(S_0) \subseteq \mathfrak{G}_0$. Let $[\mathfrak{p}^i] : \mathfrak{G}_i \to \mathfrak{G}_0$ be the composition $[\mathfrak{p}]_i \circ \cdots \circ [\mathfrak{p}]_1$, and let $\mathrm{Spf}\, S_i \subset \mathfrak{G}_i$ be the pullback of $\mathrm{Spf}\, S_0$ via $[p^i]$. Then we have a commutative diagram:

$$\begin{tikzcd} & & \tilde{S}_i^{(p)} \arrow[dr, "F_{rel}"] & & \tilde{S}_{i+1}^{(p)} \arrow[dr, "F_{rel}"] & & \\ \dots \arrow[r] & \tilde{S}_{i-1} \arrow[ur, dashed, "V"] \arrow[rr] & & \tilde{S}_i \arrow[ur, dashed, "V"] \arrow[rr] & & \tilde{S}_{i+1} \arrow[r] & \dots \end{tikzcd}$$

where the horizontal maps are induced from $[\mathfrak{p}] \mod \pi$.

From this we can check on elements that relative Frobenius is an isomorphism on $\tilde{S}_\infty := \varinjlim_i \tilde{S}_i$. Since K is perfectoid, we moreover have an isomorphism $\mathcal{O}_K/\pi^{1/p} \to \mathcal{O}_K/\pi$ from the absolute Frobenius on $\mathcal{O}_K/\pi$. Therefore absolute Frobenius on S_∞/π induces an isomorphism

$$S_\infty/\pi^{1/p} \xrightarrow{\sim} S_\infty/\pi.$$

Since each $\mathfrak{G}_i$ is flat, so are the S_i and thus so is S_∞. Thus $S_\infty[1/\pi]$ is a perfectoid K-algebra. Since G_∞ is covered by affinoids of the form $\mathrm{Spf}(S_\infty)_\eta$, this shows that G_∞ is perfectoid. □

This in particular implies what we promised in Remark 4:

Corollary 2.12. *Let A be an abelian variety of good reduction over a perfectoid field K. Then A_∞ exists and is perfectoid.*

More generally, If $\mathfrak{G}$ is a flat commutative formal group scheme such that p-multiplication is affine, then multiplication by p on $\mathfrak{G}$ defines a $[p]$-F-formal tower for the rigid analytic group $G = \mathfrak{G}_\eta$, and $G_\infty := (\varprojlim_{[p]} \mathfrak{G})_\eta$ is the perfectoid tilde-limit of $\varprojlim_{[p]} G$.

2.4. **Examples.**

Example. Let $\mathfrak{G}$ be the p-adic completion of the affine group scheme $\mathbb{G}_m$ over $\mathcal{O}_K$. The underlying formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, \mathcal{O}_K\langle X^{\pm 1}\rangle$. Multiplication by p on $\mathbb{G}_m$ gives a $[p]$-F-formal tower for G, so for the generic fibre $G = \mathfrak{G}_\eta$ we obtain the perfectoid tilde-limit $G_\infty := (\mathfrak{G}_\infty)_\eta \sim \varprojlim_{[p]} G$. More precisely, multiplication by p corresponds to the homomorphism

$$[p] : \mathcal{O}_K\langle X^{\pm 1}\rangle \to \mathcal{O}_K\langle X^{\pm 1}\rangle, \quad X \to X^p.$$

In the direct limit, we obtain $(\varinjlim_{[p]} \mathcal{O}_K\langle X^{\pm 1}\rangle)^\wedge = \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle$. Therefore, taking the generic fiber we get

$$G_\infty = \mathrm{Spa}(K\langle X^{\pm 1/p^\infty}\rangle, \mathcal{O}_K\langle X^{\pm 1/p^\infty}\rangle)$$

and one can verify directly that we indeed have $G_\infty \sim \varprojlim_{[p]} G$.

Example. An example of a very different flavour is the p-adic completion $\mathfrak{G}$ of the affine group scheme $\mathbb{G}_a$ over $\mathcal{O}_K$. Note that $G = \mathfrak{G}_\eta$ is not equal to $\mathbb{G}_a^{an}$, but is the closed unit disc in the latter.

The underlying formal scheme of $\mathfrak{G}$ is $\mathrm{Spf}\, S$ with $S = \mathcal{O}_K\langle X\rangle$, and the $[p]$-multiplication is now given by

$$[p] : \mathcal{O}_K\langle X\rangle \to \mathcal{O}_K\langle X\rangle, \quad X \to pX.$$

In the direct limit, we first obtain the algebra $S'_\infty = \mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle$, consisting of power series $f = \sum_{n=0}^\infty a_n X^n \in \mathcal{O}_K[[X]]$ for which there exists an $m \in \mathbb{Z}_{\geq 0}$ so that $|p^{nm} a_n| \to 0$. Next we need to take the p-adic completion to form S_∞. But we have

$$p^n \mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle = p^n \mathcal{O}_K + X\mathcal{O}_K\langle \frac{1}{p^\infty} X\rangle$$

and therefore $S'_\infty/\pi^n = \mathcal{O}_K/\pi^n\mathcal{O}_K$. Consequently, the completion is just $S_\infty = \mathcal{O}_K$ and thus its adic generic fiber is $G_\infty = \mathrm{Spa}(K, \mathcal{O}_K)$. This is still perfectoid but is just one point!

Let us explain the heuristic geometrically: on the level of K-points, the formal scheme G is the closed unit disc and $[p]$ is scaling by p. A K-point in $\varprojlim_{[p]} G(K)$ therefore corresponds to a sequence of K-points of the closed unit disc of the form

$$(x, \frac{1}{p}x, \frac{1}{p^2}x, \dots).$$

But for this to be contained in the closed unit disc, we must have $x = 0$.

2.5. **A few consequences.** One reason why perfectoid limits along group morphisms are particularly interesting is that the perfectoidness ensures that the limit has again a group structure:

Definition 2.13. A **perfectoid group** is a group object in the category of perfectoid spaces.

Note that the category of perfectoid spaces over K has finite products, so the notion of a group object makes sense.

Proposition 2.14. *Let G be a rigid group with a perfectoid tilde-limit G_∞. Then*

(1) there is a unique way to endow G_∞ with the structure of a perfectoid group in such a way that all projections $G_\infty \to G$ are group homomorphisms
(2) given a rigid group H with perfectoid tilde-limit $H_\infty \sim \varprojlim_{[p]} H$ and a group homomorphism $H \to G$, there is a unique group homomorphism $H_\infty \to G_\infty$ commuting with all projection maps. In particular, formation of $\varprojlim_{[p]}$ is functorial.

Proof. These are all consequences of the universal property of the perfectoid tilde-limit, cf Proposition 2.4.5 of [8], which shows that one can argue like in the case of usual limits. □

3. Formal models for tori

In this section we construct a $[p]$-F-formal tower for a split rigid torus T over K, therefore proving the existence of the perfectoid tilde-limit T_∞ of $\varprojlim_{[p]} T$.

As a preparation, we consider the torus $\mathbb{G}_m^{\mathrm{an}}$ over K. Recall that it arises from rigid analytification of the affine torus $\mathbb{G}_m$ over K. Note however that $\mathbb{G}_m^{\mathrm{an}}$ is not affinoid (and not even quasi-compact). It contains the generic fibre of the p-adic completion of $\mathbb{G}_m$ as an open subspace. If we see $\mathbb{G}_m^{\mathrm{an}}$ as the rigid affine line with the origin removed, this subspace $(\widehat{\mathbb{G}}_m)_\eta$ can be identified with the open annulus of radius 1.

Finally, recall that for every $x \in K^\times$ we have a translation map

$$\mathbb{G}_m^{\mathrm{an}} \xrightarrow{x\cdot} \mathbb{G}_m^{\mathrm{an}}$$

which is the isomorphism of rigid spaces sending the point 1 to x.

3.1. **A family of explicit covers.** We briefly recall how $\mathbb{G}_m^{\mathrm{an}}$ is constructed: The following is inspired by [1], §9.2, although we choose slightly different constructions which is suitable for constructing a $[p]$-F-formal tower for $\mathbb{G}_m^{\mathrm{an}}$.

Notation 3.1. (1) Throughout we use the following notation: For any $a \in K$ we write

$$K\langle X, a/X\rangle = K\langle X, Z\rangle/(X \cdot Z - a).$$

(2) Let $a \in K^\times$ with $|a| \leq 1$. Denote by $\mathcal{B}(a, 1) \subset \mathbb{A}_K^{\mathrm{an}}$ the annulus of radii between $|a|$ and 1:

$$\mathcal{B}(a, 1) = \operatorname{Sp} K\langle X, a/X\rangle.$$

(3) Similarly, for $b \in K^\times$ with $|b| \geq 1$, the annulus $\mathcal{B}(1, b)$ of radii 1 and $|b|$ is:

$$\mathcal{B}(1, b) = \operatorname{Sp} K\langle X/b, 1/X\rangle$$

Note that we have isomorphisms

$$K\langle X', b^{-1}/X'\rangle \cong K\langle X/b, 1/X\rangle, \quad X' \mapsto b^{-1}X.$$

Construction 3.2. We review the construction of $\mathbb{G}_m^{\mathrm{an}}$ by constructing a cover $\mathfrak{U}_q$ of affinoids, for some (fixed) $q \in K^\times$ with $|q| < 1$. First, let

$$a_n = a_{n,q} := q^n, \quad b_n = b_{n,q} := q^{-n}.$$

Then we glue the annuli $\mathcal{B}(a_n, 1)$ using maps:

$$\mathcal{B}(a_{n-1}, 1) \hookrightarrow \mathcal{B}(a_n, 1) \tag{1}$$

induced from $K\langle X, a_n/X\rangle \to K\langle X, a_{n-1}/X\rangle$ by sending X to X (and $a_n/X \mapsto \frac{a_n}{a_{n-1}} a_{n-1}/X$). Likewise, we glue $\mathcal{B}(1, b_n)$ by

$$\mathcal{B}(1, b_{n-1}) \hookrightarrow \mathcal{B}(1, b_n) \tag{2}$$

Now, via the above maps, the annuli $\mathcal{B}(a_n, 1)$ and $\mathcal{B}(1, b_m)$ are glued along $\mathcal{B}(a_0, 1) = \mathcal{B}(1, 1) = \mathcal{B}(1, b_0)$ to form $\mathbb{G}_m^{\mathrm{an}}$. We denote by this particular cover constructed above $\mathfrak{U}_q$, which of course depends on the choice of q.

Assume now that q has a p-th root $q^{1/p}$ in K (for example we may choose $q = \pi$ a perfectoid pseudo-uniformizer which admits all p-power roots in K). The same procedure above (replacing q by $q^{1/p}$) then gives a finer cover $\mathfrak{U}_{q^{1/p}}$ of $\mathbb{G}_m^{\mathrm{an}}$.

Using both covers $\mathfrak{U}_q$ and $\mathfrak{U}_{q^{1/p}}$, we can easily see the $[p]$-multiplication $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ as follows: on the affinoid open subsets $\mathcal{B}(q^{1/p}, 1)$, $[p]$ restricts to

$$\mathcal{B}(q^{1/p}, 1) \xrightarrow{[p]} \mathcal{B}(q, 1) \tag{3}$$

where the map on $K\langle X, q/X\rangle \to K\langle X, q^{1/p}/X\rangle$ is given by $X \mapsto X^p$. Similarly, on $\mathcal{B}(1, q^{-1/p})$ the map is given by

$$\mathcal{B}(1, q^{-1/p}) \xrightarrow{[p]} \mathcal{B}(1, q^{-1}) \tag{4}$$

again given by $X \mapsto X$ on the corresponding affinoid algebras.

The same in fact works for the other affinoid open subspaces $\mathcal{B}(q^{n/p}, 1) \xrightarrow{[p]} \mathcal{B}(q^n, 1)$ and for $\mathcal{B}(1, q^{-n/p}) \xrightarrow{[p]} \mathcal{B}(1, q^{-n})$.

Finally, one can show that the maps (3) and (4) are compatible with the glue maps (1) and (2). For maps (3), this is saying that the following diagram commutes:

$$\begin{tikzcd} \mathcal{B}(q^n, 1) & \mathcal{B}(q^{n-1}, 1) \arrow[l, hook'] & & q^n/X \arrow[r, mapsto] \arrow[d, mapsto] & q \cdot q^{n-1}/X \arrow[d, mapsto] \\ \mathcal{B}(q^{n/p}, 1) \arrow[u, "{[p]}"] & \mathcal{B}(q^{(n-1)/p}, 1) \arrow[l, hook'] \arrow[u, "{[p]}"'] & & (q^{n/p}/X)^p \arrow[r, mapsto] & q \cdot (q^{(n-1)/p}/X)^p. \end{tikzcd} \tag{5}$$

The case of (4) is very similar.

3.2. A $[p]$-F-formal tower for $\mathbb{G}_m^{\mathrm{an}}$.

At this point we have constructed a cover $\mathfrak{U}_q$ of $\mathbb{G}_m^{\mathrm{an}}$ depending on a choice of $q \in K^\times$ with $|q| < 1$.

Construction 3.3. The affinoid subspaces $\mathcal{B}(q^n, 1) \in \mathfrak{U}_q$ admit natural formal models:

$$\mathfrak{B}(q^n, 1) := \mathrm{Spf}\, \mathcal{O}_K\langle X, Z\rangle/(XZ - q^n).$$

This is clearly of topologically finite type over $\mathcal{O}_K$. It is moreover flat over $\mathcal{O}_K$ by Lemma 8.2.1 in [1]. For the same reason we see that

$$\mathfrak{B}(1,q^n) := \operatorname{Spf} \mathcal{O}_K\langle X', Z\rangle/(X'Z - q^n)$$

is a flat topologically finite type $\mathcal{O}_K$-formal scheme.

Moreover, observe that the gluing maps (1) and (2) in Construction 3.2 extend to glueing maps

$$\mathfrak{B}(q^{n-1},1) \hookrightarrow \mathfrak{B}(q^n,1), \qquad \mathfrak{B}(1,q^{-n+1}) \hookrightarrow \mathfrak{B}(1,q^{-n})$$

for each $n \geq 1$. We therefore conclude:

Lemma 3.4. *Let $q \in K^\times$ be an element which has $|q| < 1$ (and admits all p-power roots in K). The affine formal schemes $\mathfrak{B}(q^n,1)$ and $\mathfrak{B}(1,q^n)$ glue together to form a flat formal scheme $\mathfrak{G}_q$ over $\operatorname{Spf} \mathcal{O}_K$ such that $(\mathfrak{G}_q)_\eta = \mathbb{G}_m^{\mathrm{an}}$. In other words, $\mathfrak{G}_q$ is a formal model for $\mathbb{G}_m^{\mathrm{an}}$.*

Construction 3.5. A closer look at the maps (3) and (4) shows that the $[p]$-multiplication map extends to a morphism of formal schemes

$$[\mathfrak{p}] : \mathfrak{B}(q^{n/p},1) \xrightarrow{[p]} \mathfrak{B}(q^n,1)$$

and similarly for $\mathfrak{B}(1,q^{-1})$. The diagram (5) shows that these maps glue to a morphism

$$[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q.$$

It is clear from definition that, after tensoring $-\otimes_{\mathcal{O}_K} K$ all morphisms on algebras coincide with those defined in (1), (2), (3), (4) respectively; moreover, after reducing mod π it reduces to the relative Frobenius map.

Lemma 3.6.

(1) The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ described above is a formal model of $[p] : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$.
(2) The map $[\mathfrak{p}] : \mathfrak{G}_{q^{1/p}} \to \mathfrak{G}_q$ reduces mod π to the relative Frobenius map.

We now have everything together to finish our proof that $(\mathbb{G}_m^{\mathrm{an}})_\infty$ is perfectoid:

Proposition 3.7. *The space $\mathbb{G}_m^{\mathrm{an}}$ admits a $[p]$-F-formal tower. In particular, there exists a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ such that $(\mathbb{G}_m^{\mathrm{an}})_\infty \sim \varprojlim_{[p]} \mathbb{G}_m^{\mathrm{an}}$.*

Proof. Since K is perfectoid, we can find $q \in K^\times$ such that $|q| < 1$ for which there exist arbitrary p^n-th roots. We choose such a q and roots q^{1/p^n} for all n. Then the two Propositions above combine to show that

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{G}_q$$

is a $[p]$-F-formal tower. By Proposition 2.10 we obtain a perfectoid space $(\mathbb{G}_m^{\mathrm{an}})_\infty$ as desired. ☐

3.3. **The action of $\overline{T}$.** The multiplication $\mathbb{G}_m^{\mathrm{an}} \times \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ of the rigid group can locally be described in terms of the rigid analytic cover that we have defined above as follows : Let $a, b \in K^\times$ such that $|a|, |b| \leq 1$, then the multiplication map restricts to

$$\mathcal{B}(a,1) \times \mathcal{B}(b,1) \xrightarrow{m} \mathcal{B}(ab,1) \tag{6}$$

which is given on the level of algebra by the map $K\langle X, ab/X\rangle \to K\langle X, a/X\rangle \widehat{\otimes} K\langle X, b/X\rangle$ sending $X \mapsto X \otimes X$ and $ab/X \mapsto a/X \otimes b/X$. We have a similar description of the multiplication map on the $\mathcal{B}(1,a) \times \mathcal{B}(1,b)$ for $|a|, |b| \geq 1$.

Remark 3.8. The multiplication on the $\mathcal{B}(a,1)\times\mathcal{B}(1,b)$ for $|a|<1<|b|$ is more difficult to see on the cover that we have chosen.

Remark 3.9. The map (6) has a flat formal model

$$\mathfrak{B}(a,1)\times\mathfrak{B}(b,1)\to\mathfrak{B}(ab,1).$$

However, this does *not* mean that multiplication has a formal model $\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$. Indeed, the chosen description use different covers on source and target which gives rise to different formal schemes (the inversion map $i:\mathbb{G}_m^{\mathrm{an}}\to\mathbb{G}_m^{\mathrm{an}}$, on the other hand, does admit a formal model).

Nevertheless, if we take $a=1$ in the above, we see that we do have an action of the torus $\overline{T}:=\mathfrak{B}(1,1)$ on each of $\mathfrak{B}(b,1)$ and $\mathfrak{B}(1,b)$. Using the formal models from the last section, we conclude:

Proposition 3.10. *For any $q\in K^\times$ with $|q|<1$, the formal torus $\overline{T}:=\mathfrak{B}(1,1)$ has a natural action on $\mathfrak{G}_q$ via a map*

$$\mathfrak{m}:\overline{T}\times\mathfrak{G}_q\to\mathfrak{G}_q.$$

This map is a formal model of the action of the annulus $\mathcal{B}(1,1)$ on $\mathbb{G}_m^{\mathrm{an}}$. Furthermore, this action is compatible with the models for $[p]$ in the sense that, for a p-th root $q^{1/p}\in K$, the following diagram commutes.

$$\begin{tikzcd}
\overline{T}\times\mathfrak{G}_{q^{1/p}} \arrow[r, "\mathfrak{m}"] \arrow[d, "{[p]\times[\mathfrak{p}]}"'] & \mathfrak{G}_{q^{1/p}} \arrow[d, "{[\mathfrak{p}]}"] \\
\overline{T}\times\mathfrak{G}_q \arrow[r, "\mathfrak{m}"] & \mathfrak{G}_q.
\end{tikzcd}$$

Proof. The existence of $\mathfrak{m}$ follows from the above consideration concerning the map (6). The rest is clear from the construction: All adic rings we have used in the construction are $\mathcal{O}_K$-subalgebras of the affinoid K-algebras used to define $\mathbb{G}_m^{\mathrm{an}}$, so the equalities hold because they hold for $\mathbb{G}_m^{\mathrm{an}}$. $\square$

3.4. **The case of general tori.** By taking products everywhere, all of the statements in this section immediately generalise to split tori:

Corollary 3.11. *Let T be a split torus over K of the form $T=(\mathbb{G}_m^{\mathrm{an}})^d$. Then for any $q\in K^\times$ with $|q|<1$ the formal scheme $\mathfrak{T}_q:=(\mathfrak{G}_q)^d$ is a formal model of T. For a p-th root $q^{1/p}\in K$, the p-multiplication map has a formal model $[\mathfrak{p}]:\mathfrak{T}_{q^{1/p}}\to\mathfrak{T}_q$ that locally has the form $[\mathfrak{p}]:\mathfrak{B}(q^{1/p},1)^d\to\mathfrak{B}(q,1)^d$. Moreover this map reduces mod p to the relative Frobenius morphism.*

Corollary 3.12. *Let T be a split torus over K, considered as a rigid space. Then T admits a $[p]$-F-formal tower. In particular, there exists a perfectoid space T_∞ such that $T_\infty\sim\varprojlim_{[p]}T$.*

Corollary 3.13. *Let T and $q\in K^\times$ be as above, then the formal completion $\overline{T}$ has a natural action on $\mathfrak{T}_q$ via a map*

$$\mathfrak{m}:\overline{T}\times\mathfrak{T}_q\to\mathfrak{T}_q.$$

This map is a formal model of the action of the annulus $\overline{T}$ on T. Furthermore, this action is compatible with the models for $[p]$, namely, the map $[\mathfrak{p}]:\mathfrak{T}_q^{1/p}\to\mathfrak{T}_q$ is semi-linear with respect to $[p]:\overline{T}\to\overline{T}$.

4. A $[p]$-F-formal tower for Raynaud extensions

In this section we study the p-multiplication tower of the Raynaud extensions associated to abelian varieties over an algebraically closed perfectoid field K. The main result of this section is Theorem 4.11, namely the Raynaud extension E of an abelian variety A over K admits a $[p]$-F-formal tower, hence we obtain the perfectoid tilde-limit $E_\infty \sim \varprojlim_{[p]} E$.

Remark 4.1. While we expect that our main theorem still holds over any perfectoid field, it is easier to work in the algebraically closed case since this simplifies the theory of Raynaud uniformisation, in particular Lemma 4.5 below.

4.1. **Raynaud extensions.** We briefly sketch the theory of Raynaud extensions here, and refer the readers to [4] for more details on the setup.

Let now A be an abelian variety over K. There exists a unique connected open rigid analytic subgroup $\overline{A}$ of A which extends to a formal smooth $\mathcal{O}_K$-group scheme $\overline{E}$ with semi-abelian reduction. Then $\overline{E}$ fits into a short exact sequence of formal group schemes

$$0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0 \tag{7}$$

where $\overline{B}$ is the completion of an abelian variety B over K of good reduction (we also denote by B the rigid space associated to it), and $\overline{T}$ is the completion of a torus of rank r over K. The rigid generic fibre $\overline{T}_\eta$ of the torus $\overline{T}$ canonically embeds into the torus T^{an} which again we simply denote by T. One can show that this induces a pushout exact sequence in the category of rigid groups. More precisely, there exists a rigid group variety E such that the following diagram commutes and the left square is a pushout.

$$\begin{tikzcd} 0 \arrow[r] & \overline{T}_\eta \arrow[r] \arrow[d, hook] & \overline{E}_\eta \arrow[r] \arrow[d, hook] & \overline{B}_\eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{8}$$

The abelian variety A we started with can then be uniformized in terms of E as follows:

Definition 4.2. A subset M of a rigid space G is called **discrete** if the intersection of M with any affinoid open subset of G is a finite set of points. Let G be a rigid group, then a **lattice** in G of rank r is a discrete subgroup M of G which is isomorphic to the constant rigid group $\underline{\mathbb{Z}}^r$.

Proposition 4.3. *There exists a lattice $M \subseteq E$ of rank equal to the rank r of the torus for which the quotient E/M exists as a rigid space and has a group structure such that $E \to E/M$ is a rigid group homomorphism. Moreover, there is a natural isomorphism*

$$A = E/M.$$

Since M is discrete, the local geometry of A is thus determined by the local geometry of E. More precisely, we will first study the $[p]$-multiplication tower of E and will then deduce properties of the $[p]$-multiplication tower of A.

Remark 4.4. Our strategy is to study the local geometry of E and $\overline{E}$ via T and B. An obstacle in doing this is that the categories of formal or rigid groups are not abelian, which makes working with short exact sequences a subtle issue. Another issue is that one cannot direcly study short exact sequences locally on T, E or B.

An important tool is therefore the following Lemma:

Lemma 4.5. *The short exact sequence (7) admits local sections, that is there is a cover of B by formal open subschemes U_i such that there exist sections $s : U_i \to \overline{E}$ of π. In particular, one can cover $\overline{E}$ by formal open subschemes of the form $\overline{T} \times U_i \hookrightarrow E$.*

Proof. This is proved in Proposition A.2.5 in [4], where it is fomulated in terms of the group $\mathrm{Ext}(B, T)$. Also see [2], §1. ☐

This Lemma suggests that instead of considering Raynaud extensions from the abelian category viewpoint, one should consider them as principal T-bundles of formal schemes (which are also referred to as T-torsors in the article).

Remark 4.6. This is the language we use in the rest of the paper: We will work with fibre bundles of formal schemes, rigid spaces and schemes. The main technical tool we need is the associated fibre construction in these settings. For a rigorous treatment of these we refer to the Appendix.

First of all, we note that the sequence (7) from the last section gives rise to a principal $\overline{T}$-bundle $\overline{E} \to \overline{B}$. The fact that E is obtained from $\overline{E}_\eta$ via push-out from $\overline{T}_\eta \to T$ can now conveniently be expressed in terms of the associated fibre bundle by saying that $E_\eta = T \times^{\overline{T}_\eta} \overline{E}_\eta$ in the sense of Definition A.9. We have the following description of $[p]$:

Lemma 4.7. *The map $[p] : E \to E$ coincides with the morphism*

$$[p] \times^{[p]} [p] : T \times^{\overline{T}_\eta} \overline{E}_\eta \to T \times^{\overline{T}_\eta} \overline{E}_\eta$$

induced by the different $[p]$-multiplication maps by Proposition A.18.

Proof. Lemma A.19 in light of Remark A.20 applied to the maps $g = [p] : \overline{T}_\eta \to \overline{T}_\eta$, $h = [p] : T \to T$ and $f = [p] : \overline{E}_\eta \to \overline{E}_\eta$ says that there is a unique morphism of fibre bundles $E \to E$ making the following diagram commute:

$$\begin{tikzcd}
& T \arrow[rr] & & E \\
T \arrow[ur, "{[p]}"] \arrow[rr] & & E \arrow[ur, dotted, "\exists!"] & \\
& \overline{T}_\eta \arrow[uu, hook] \arrow[rr] & & \overline{E}_\eta \arrow[uu] \\
\overline{T}_\eta \arrow[uu, hook] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E}_\eta \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd} \tag{9}$$

Since $[p] : E \to E$ is such a map, the Lemma follows. ☐

4.2. **A $[p]$-F-formal tower for E.** In this subsection we prove that E admits a $[p]$-F-formal tower. The first step is to construct a family of formal models for E.

Lemma 4.8. *Let $q \in K^\times$ with $|q| < 1$. Let $\mathfrak{T}_q$ be the formal model from Corollary 3.11. Then there is a formal scheme $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E}$ that is a formal model of the rigid space E. Furthermore, there exists a morphism*

$$\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$$

which is a fibre bundle and a formal model of $E \to B$.

Proof. Recall from Proposition 3.13 that $\mathfrak{T}_q$ has a $\overline{T}$-action that is a model of the $\overline{T}_\eta$-action on T. In particular, the associated fibre construction for the principal $\overline{T}$-bundle $\overline{E}$ gives a fibre bundle $\mathfrak{E}_q := \mathfrak{T}_q \times^{\overline{T}} \overline{E} \to \overline{B}$. Since $\mathfrak{T}_q$ is a formal model of T, one sees by Lemma A.21 that this is a formal model of $T \times^{\overline{T}_\eta} \overline{E}_\eta$ which by definition is equal to E. □

Next we construct a model for the $[p]$-multiplication map. Here we can use again that $[p]$ exists on $\overline{E}$ and on $\mathfrak{T}_{q^{1/p}}$.

Lemma 4.9. *Let $q \in K^\times$ be such that $|q| < 1$ and let $q^{1/p} \in K$ be a p-th root of q. Then there is an affine morphism*

$$[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$$

which is a formal model of $[p] : E \to E$.

Proof. Recall that the multiplication map $[p] : T \to T$ has a formal model $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ by Corollary 3.11. This fits into a commutative diagram

$$\begin{tikzcd}
\mathfrak{T}_{q^{1/p}} \arrow[r, "{[\mathfrak{p}]}"] & \mathfrak{T}_q \\
\overline{T} \arrow[u, hook] \arrow[r, "{[p]}"] & \overline{T}. \arrow[u, hook]
\end{tikzcd}$$

Functoriality of the associated fibre construction in the general case, Proposition A.18, applied to the diagram below then gives a natural map $\mathfrak{E}_{q^{1/p}} \to \mathfrak{E}$ making the diagram commute:

(10)
$$\begin{tikzcd}
& \mathfrak{T}_q \arrow[rr] & & \mathfrak{E}_q \\
\mathfrak{T}_{q^{1/p}} \arrow[ur, "{[\mathfrak{p}]}"] \arrow[rr] & & \mathfrak{T}_{q^{1/p}} \times^{\overline{T}} \overline{E} \arrow[ur, dotted, "\exists"] & \\
& \overline{T} \arrow[uu] \arrow[rr] & & \overline{E} \arrow[uu] \\
\overline{T} \arrow[uu] \arrow[ur, "{[p]}"] \arrow[rr] & & \overline{E} \arrow[uu] \arrow[ur, "{[p]}"] &
\end{tikzcd}$$

By Lemma 4.7 this diagram equals diagram (9) on the generic fibre.

To see that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ is affine, first note that $[p] : \overline{B} \to \overline{B}$ is an affine morphism. The map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ is affine by construction, namely by Corollary 3.11 it is locally on $\mathfrak{T}_q$ of the form $[\mathfrak{p}] : \mathfrak{B}(q^{1/p}, 1)^d \to \mathfrak{B}(q, 1)^d$. Note that both of these affine open subsets are fixed by the action of $\overline{T}$. We conclude from the construction in the proof of Proposition A.18 that the morphism $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ locally on the target is of the form

$$[\mathfrak{p}] : \mathfrak{B}(q^{1/p}, 1)^d \times U' \to \mathfrak{B}(q, 1)^d \times U$$

for an affine open formal subscheme $U \subseteq \overline{B}$ with affine preimage U' under $[p] : \overline{B} \to \overline{B}$. This shows that the morphism is affine locally on the target, and therefore is affine. □

We have thus proved the first part of what we want to show about tilde-limits of E:

Lemma 4.10. *Retain notations from above. E admits a $[p]$-formal tower of the form*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

for some $q \in K^\times$. In particular, there exists a space E_∞ such that $E_\infty \sim \varprojlim_{[p]} E$.

Proof. By Lemma 4.9, any choice of $q \in K^\times$ with $|q| < 1$ for which there exists a compatible system of p^n-th roots $q^{1/p^n} \in K^\times$ gives a tower

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

that on the generic fibre equals $\dots \xrightarrow{[p]} E \xrightarrow{[p]} E$. This is the desired $[p]$-formal tower. □

We are now ready to prove the main result of this section, namely that E_∞ is perfectoid.

Theorem 4.11. *Let K be perfectoid. Then the $[p]$-formal tower from Proposition 4.10*

$$\dots \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p^2}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_{q^{1/p}} \xrightarrow{[\mathfrak{p}]} \mathfrak{E}_q$$

is already a $[p]$-F-formal tower. In particular, the space E_∞ is perfectoid.

Proof. It suffices to prove that, the reduction mod π of the map $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through relative Frobenius.

In the following we denote reduction of a formal scheme by a $\sim$ over the formal scheme, for example the reductions of $\overline{T}$, $\overline{E}$ and $\mathfrak{T}$ are denoted by $\tilde{T}$, $\tilde{E}$ and $\tilde{\mathfrak{T}}$.

Recall that $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ was constructed using the $[p]$-multiplication cube in diagram (10) and functoriality of the associated bundle. Also recall that all statements we have used about fibre bundles also hold when we replace formal schemes over $\mathcal{O}_K$ by schemes over $\mathcal{O}_K/\pi$, and formation of the associated bundle commutes with this reduction. In particular,

$$\tilde{\mathfrak{E}}_q = \tilde{\mathfrak{T}}_q \times^{\tilde{T}} \tilde{E}.$$

By Corollary 3.11, the model of the multiplication map $[\mathfrak{p}] : \mathfrak{T}_{q^{1/p}} \to \mathfrak{T}_q$ reduces to relative Frobenius over p. In particular, we have a natural isomorphism

$$\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \cong \tilde{\mathfrak{T}}_q$$

and we can identify $\tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} = \tilde{\mathfrak{T}}_q$ in the following. The same is true for $\tilde{T}^{(p)} = \tilde{T}$.

Since $\tilde{E}$ and $\tilde{T}$ are group schemes, the reduction of $[p]$ on them factors through the relative Frobenius maps F_E and F_T respectively. By functoriality of relative Frobenius ("Frobenius commutes with any map") we have a commutative diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "F_{\tilde{E}}"] & \tilde{E}^{(p)} \\ \tilde{T} \arrow[u, hook] \arrow[r, "F_{\tilde{T}}"] & \tilde{T}^{(p)}. \arrow[u, hook] \end{tikzcd}$$

In other words, $F_{\tilde{E}}$ is an $F_{\tilde{T}}$-linear morphism of fibre bundles. Again by functoriality of Frobenius we also have a commutative diagram

$$\begin{tikzcd}
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[r, "F_{\tilde{\mathfrak{T}}}"] & \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \\
\tilde{T} \arrow[u, hook] \arrow[r, "F_{\tilde{T}}"] & \tilde{T}^{(p)}. \arrow[u, hook]
\end{tikzcd}$$

By Proposition A.18, we thus obtain a natural morphism

$$F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}} : \tilde{\mathfrak{T}}_{q^{1/p}} \times^{\tilde{T}} \tilde{E} \to \tilde{\mathfrak{T}}^{(p)}_{q^{1/p}} \times^{\tilde{T}^{(p)}} \tilde{E}^{(p)}.$$

Using the explicit description of $F_{\tilde{\mathfrak{T}}} \times^{F_{\tilde{T}}} F_{\tilde{E}}$ in the proof of Proposition A.18, we easily check that this morphism is just the relative Frobenius of $\tilde{\mathfrak{E}}_{q^{1/p}}$: This is a consequence of the fact that relative Frobenius on the fibre product $\tilde{T} \times \tilde{U}$ for any $\tilde{U} \subseteq \tilde{B}$ is just the product of the relative Frobenius morphisms of $\tilde{T}$ and $\tilde{U}$, and thus the morphisms θ_i from Lemma A.8 are all trivial.

But this means that again by Proposition A.18, the reduction of the formal model of the p-multiplication cube in diagram 10 admits the following factorisation:

$$\begin{tikzcd}
 & & \tilde{\mathfrak{T}}_q \arrow[rr] & & \tilde{\mathfrak{E}}_q \\
 & \tilde{\mathfrak{T}}^{(p)}_q \arrow[ur, equal] \arrow[rr] & & \tilde{\mathfrak{E}}^{(p)}_{q^{1/p}} \arrow[ur, dotted] & \\
\tilde{\mathfrak{T}}_{q^{1/p}} \arrow[ur, "F"'] \arrow[rr] & & \tilde{\mathfrak{E}}_{q^{1/p}} \arrow[ur, "F"'] & & \\
 & & & & \\
 & & \tilde{T} \arrow[uuuu] \arrow[rr] & & \tilde{E} \arrow[uuuu] \\
 & \tilde{T}^{(p)} \arrow[uuuu] \arrow[ur, equal] \arrow[rr] & & \tilde{E}^{(p)} \arrow[uuuu] \arrow[ur] & \\
\tilde{T} \arrow[uuuu] \arrow[ur, "F"'] \arrow[rr] & & \tilde{E} \arrow[uuuu] \arrow[ur, "F"'] & &
\end{tikzcd}$$

Since the composed maps $\tilde{E} \to \tilde{E}$ on the bottom right, $\tilde{T} \to \tilde{T}$ on the bottom left and $\tilde{\mathfrak{T}}_{q^{1/p}} \to \tilde{\mathfrak{T}}_q$ on the upper left by construction are the reductions of the respective p-multiplication maps $[p]$, the functoriality of the associated bundle construction in Proposition A.18 implies that the two maps on the upper right compose to the reduction of $[\mathfrak{p}] \times^{[p]} [p]$. But $[\mathfrak{p}] \times^{[p]} [p]$ is equal to $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ by definition of the latter. This completes the proof that the reduction of $[\mathfrak{p}] : \mathfrak{E}_{q^{1/p}} \to \mathfrak{E}_q$ factors through the relative Frobenius on $\tilde{\mathfrak{E}}_{q^{1/p}}$.

The conclusion that E_∞ is perfectoid then follows from Proposition 2.10. □

5. The case of abelian varieties with semi-stable reduction

We now want to prove the main result of this work, building on the preceding chapters:

Theorem 5.1. *Let A be an abelian variety over an algebraically closed perfectoid field K. Then there is a perfectoid space A_∞ such that*

$$A_\infty \sim \varprojlim_{[p]} A.$$

We have seen this already for abelian varieties with good reduction in Corollary 2.12. It thus suffices to deal with the bad reduction case. Since K is algebraically closed, this means that A has semi-stable reduction.

Before we proceed with the proof of the main theorem which occupies the rest of the section, let us recall some notation:

(1) g is the dimension of A.
(2) E is the Raynaud extension associated to A from Proposition 4.3, which is an extension of a split rigid torus T of rank r by an abelian variety B of good reduction.
(3) $M \subseteq E$ is a lattice of rank r such that $A = E/M$.

5.1. **Covering A by subspaces of E.** The first step towards towards the proof is to construct a cover of $A = E/M$ by subspaces of E that behaves well under $[p]$-multiplication.

As a first step we recall how to relate the lattice M to a Euclidean lattice in $\mathbb{R}^r$, cf §2.7 and §6.2 in [4]. On the level of points, $\mathbb{G}_m^{\mathrm{an}}$ has an absolute value map

$$|-|: \mathbb{G}_m^{\mathrm{an}}(K) = K^\times \to \mathbb{R}^\times, \quad x \mapsto |x|$$

which induces the following group homomorphism from the torus T:

$$|-|: T(K) = (K^\times)^r \to (\mathbb{R}^\times)^r, \quad (x_1, \dots, x_n) \mapsto (|x_1|, \dots, |x_n|)$$

Since when working with lattices we prefer additive notation, we also consider the map

$$\ell : T(K) = (K^\times)^r \to \mathbb{R}^r, \quad x_1, \dots, x_n \mapsto (-\log|x_1|, \dots, -\log|x_n|).$$

Note that this map has dense image due to our assumptions on K.

The formal torus $\overline{T}$ corresponds on K-points to $\overline{T}_\eta(K) = (\mathcal{O}_K^\times)^r$ and is thus in the kernel of ℓ. We can therefore extend ℓ to $E(K)$ as follows: Locally over an open subspace $U \subseteq B$ we have $E|_U = T \times^{\overline{T}_\eta} \overline{E}_\eta|_U$ and we define ℓ by projection from the first factor. The different $E|_U$ are then glued on intersections using the $\overline{T}_\eta$-action on T. But since ℓ on T is invariant under the $\overline{T}_\eta$-action, the maps glue together to a group homomorphism

$$\ell : E(K) \to \mathbb{R}^r.$$

Since $A = E/M$ is proper, the lattice M is sent by ℓ to a Euclidean lattice $\Lambda \subset \mathbb{R}^r$ of full rank r (see Proposition 6.1.4 in [4]). In particular, this induces an isomorphism of discrete torsionfree groups

$$\ell : M \xrightarrow{\sim} \Lambda \subseteq \mathbb{R}^r.$$

The idea is now that one can understand the quotient E/M in terms of the quotient $\mathbb{R}^r/\Lambda$. We are going to make this precise in the following:

For any $d \in \mathbb{R}^r_{>0}$, consider the cuboid with length $2d$ centered at the origin:

$$S(d) = \{(s_1, \dots, s_r) \in \mathbb{R}^r | |s_i| \leq d_i\}$$

Choose d small enough such that $S(d)$ intersects Λ only in $0 \in \Lambda$. Choose $q_1, \dots, q_r \in K$ such that $|q_i| = \exp(-d_i)$. We denote by $\mathcal{B}(q, q^{-1})$ the affinoid open rigid multi-annulus in T centered at 1 of radii $|q_i| < 1 < |q_i|^{-1}$ in every direction.

Lemma 5.2. *The inverse image $\ell^{-1}(S(d)) \subseteq E(K)$ is the underlying set of the admissible open subset $E(q) := \mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$.*

Proof. One shows this first for the map $T \to \mathbb{R}^r$, where it is clear that the preimage is $\mathcal{B}(q, q^{-1})$. This is also described in §6.4 of [3]. The statement for $\mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ follows by direct inspection on local trivialisations $\mathcal{B}(q, q^{-1}) \times U$ for $U \subseteq B$. □

Note that by our choice of d, the map $\mathbb{R}^r \to \mathbb{R}^r/\Lambda$ maps $S(d)$ bijectively onto its image. Lemma 5.2 says that we can use $\mathcal{B}(q, q^{-1}) \times^{\overline{T}_\eta} \overline{E}_\eta$ as a chart for E/M around the origin.

In order to obtain charts around other points of E/M, we simply need to consider translations: Recall that for every $c \in T(K)$, the translation map

$$T \xrightarrow{\cdot c} T$$

is an isomorphism of rigid spaces that sends the unit 1 to c. We denote the image of any admissible open set U under translation by $c \cdot U$.

Lemma 5.3. *With notation as before, let $c \in T(K)$ be any point and let $s = l(c)$. Then the inverse image $\ell^{-1}(s + S(d)) \subseteq E(K)$ of the translation of $S(d)$ by s is the underlying set of the admissible open subset $E(c, q) := (c \cdot \mathcal{B}(q, q^{-1})) \times^{\overline{T}_\eta} \overline{E}_\eta \subseteq E$. We can choose $c \in T(K)$ and $q \in T(K)$ in such a way that they admit arbitrary p^n-th roots.*

Proof. Since ℓ commutes with the translations

$$\begin{array}{ccc} T(K) & \xrightarrow{\ell} & \mathbb{R}^r \\ {\scriptstyle \cdot c}\downarrow & & \downarrow{\scriptstyle +s} \\ T(K) & \xrightarrow{\ell} & \mathbb{R}^r, \end{array}$$

the first part is an immediate consequence of Lemma 5.2. For the second, note that for any $c' \in T(K)$ with $l(c') = l(c)$ and $q \in T(K)$ with $l(q') = l(q)$ we have $c \cdot \mathcal{B}(q, q^{-1}) = c' \cdot \mathcal{B}(q, q^{-1})$, and thus $E(c, q) = E(c', q')$. The statement therefore follows from K being perfectoid, for instance using $\mathcal{O}_K^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_K$. □

Definition 5.4. We call spaces of the form $E(c, q) \subseteq E$ **cuboids** centered at c. More precisely they are locally a cuboid $c \cdot \mathcal{B}(q, q^{-1}) \subseteq T$ times an admissible open subset of the abelian variety B.

Lemma 5.5. *There exist finitely many admissible open cuboids $E_1, \dots, E_k \subseteq E$ which map isomorphically onto their images in $A = E/M$ and which cover A admissibly.*

One can reconstruct A from any such cover by glueing $E_1, \dots, E_k$ as follows: By construction, for any E_i the translates $m \cdot E_i$ by $m \in M$ are pairwise disjoint and we thus have a canonical projection π from the union $\cup_{m \in M}(m \cdot E_i) \subseteq E$ to E_i. Let $E_{ij} := (\bigcup_{m \in M} m \cdot E_i) \cap E_j \subseteq E$. Then we glue E_j to E_i via the map

$$E_{ij} \to \bigcup_{m \in M} m \cdot E_i \xrightarrow{\pi} E_i.$$

Proof. Since $\mathbb{R}^r/\Lambda$ is compact, we can find finitely many $s_1, \dots, s_k$ and $d_1, \dots, d_k \in \mathbb{R}^r_{>0}$ such that $\mathbb{R}^r/\Lambda$ is covered by the $s_i + S(d_i)$. When we choose corresponding $c_1, \dots, c_k \in T(K)$ and $q_1, \dots, q_k$ as in Lemma 5.3, then the corresponding $E_i := E(c_i, q_i)$ are an atlas of $A = E/M$ by admissible open subsets of E.

In order to reconstruct A, note that $\cup_{m \in M} m \cdot E_i$ is precisely the preimage of E_i under the projection $E \to E/M$. In particular, the subspace E_{ij} is precisely the preimage of E_i under the

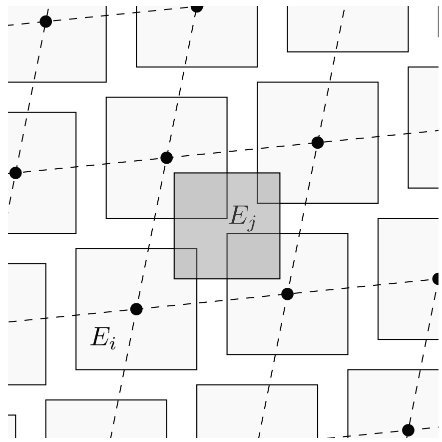

FIGURE 1. Given two charts E_i and E_j, the chart E_j is glued to E_i along intersections with all translates of E_i by $q \in M$.

composition $E_j \hookrightarrow E \to E/M$. In other words, the subspace $E_{ij} \subseteq E_j$ is the intersection of E_i and E_j when considered as subspaces of A. This shows that as charts of A, the spaces E_i and E_j are glued via E_{ij} as described. □

Finally, we need some control about what happens to the cubes under $[p]$-multiplication. Recall from Lemma 5.3 that we can always assume that c admits p-th roots.

Lemma 5.6. *Let $c^{1/p}$ be a p-th root of c and let $q^{1/p}$ be a p-th root of q in $(K^\times)^r$. Then under $[p] : E \to E$, the admissible open $E(c_i, q)$ pulls back to the admissible open $E(c_i^{1/p}, q^{1/p})$.*

Proof. It is clear that under $[p] : T \to T$, the admissible open cuboid $c \cdot \mathcal{B}(q, q^{-1})$ centered at c pulls back to $c^{1/p} \cdot \mathcal{B}(q^{1/p}, q^{-1/p})$. Note that this is independent of the choices of $c^{1/p}$ and $q^{1/p}$. Now recall that in terms of fibre bundles, multiplication $[p] : E \to E$ is

$$[p] \times^{[p]} [p] : T \times^{\overline{T}_\eta} \overline{E}_\eta \to T \times^{\overline{T}_\eta} \overline{E}_\eta$$

by Lemma 4.7. Thus $(c \cdot \mathcal{B}(q, q^{-1})) \times^{\overline{T}_\eta} \overline{E}_\eta$ pulls back to $(c^{1/p} \cdot \mathcal{B}(q^{1/p}, q^{-1/p})) \times^{\overline{T}_\eta} \overline{E_\eta}$. □

5.2. **The two towers.** In this section we want to separate the $[p]$-multiplication of A into two different towers, which we think of as being a “ramified” tower and an “étale” tower. Of course in characteristic 0 both towers will actually be étale, and these words are only meant to describe the behaviour of the maps relative to the lattice M.

For the ramified tower, we first make an auxiliary choice of certain torsion subgroups of A: Since K is algebraically closed, we can choose lattices $M^{1/p^n} \subseteq E$ such that $[p] : E \to E$ restricts to isomorphisms $M^{1/p^{n+1}} \to M^{1/p^n}$ for all n.

Remark 5.7. Such a choice is equivalent to the choice of subgroups $D_n \subseteq A[p^n]$ of rank p^{rn} for all n such that $pD_{n+1} = D_n$ and $D_n + E[p^n] = A[p^n]$. Namely, given the lattices $M^{1/p^{n+1}}$, we obtain torsion subgroups by setting $D_n := M^{1/p^{n+1}}/M$. This is because any such lattice gives a splitting of the short exact sequence $0 \to E[p^n] \to A[p^n] \to M/M^{p^n} \to 0$.

Conversely, given subgroups $D_n \subseteq A[p^n]$ that form a partial anticanonical $\Gamma_0(p^\infty)$ structure, we recover M^{1/p^n} as the kernel of $E \to A \to A/D_n$.

One might call the choice of D_n for all n a partial anticanonical $\Gamma_0(p^\infty)$-structure, because if B admits a canonical subgroup (that is, satisfies a condition on its Hasse invariant), the choice of a (full) anticanonical $\Gamma_0(p^\infty)$-structure on A is equivalent to the choice of a partial anticanonical $\Gamma_0(p^\infty)$-structure on A and an anticanonical $\Gamma_0(p^\infty)$-structure on B. Note however that A always has a partial anticanonical subgroup even if B does not have a canonical subgroup.

Note that in the case of K perfectoid but not necessarily algebraically closed, one can still carry out the constructions in the following using partial anticanonical $\Gamma_0(p^\infty)$-structures, whereas the lattices M^{1/p^n} might not be defined over K.

Following the remark, denote by D_n the torsion subgroup $M^{1/p^n}/M \subseteq A$. The quotient $A/D_n = E/M^{1/p^n}$ is then another abelian variety over K and the quotient map $v^n : E/M \to E/M^{1/p^n}$ is an isogeny of degree p^{2gn} through which $[p^n] : A \to A$ factors:

$$
\begin{tikzcd}
 & E/M^{1/p} \arrow[dr, "{[p^n]}"] & \\
E/M \arrow[ur, "v^n"] \arrow[rr, "{[p^n]}"] & & E/M.
\end{tikzcd}
\tag{11}
$$

We think of these maps as being an analogue of Frobenius and Verschiebung, which is why we denote the left map by v. Putting everything together, the $[p]$-multiplication tower splits into two towers

$$
\begin{tikzcd}
\ddots \arrow[dr] & & \vdots \arrow[d] & \vdots \arrow[d] \\
 & E/M \arrow[r, "v"] \arrow[dr, "{[p]}"'] & E/M^{1/p} \arrow[r, "v"] \arrow[d, "{[p]}"] & E/M^{1/p^2} \arrow[d, "{[p]}"] \\
 & & E/M \arrow[r, "v"] \arrow[dr, "{[p]}"'] & E/M^{1/p} \arrow[d, "{[p]}"] \\
. & & & E/M
\end{tikzcd}
\tag{12}
$$

Since each quotient $M^{1/p^n}/M$ is a finite étale group scheme, all horizontal maps are finite étale. The vertical tower on the other hand fits into a second commutative diagram of rigid groups which compares it to the $[p]$-tower of E:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M^{1/p^2} \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p^2} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M^{1/p} \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & E/M \arrow[r] & 0
\end{tikzcd}
\tag{13}
$$

5.3. **Constructing a limit of the vertical tower.** Our first step is to show that the tower on the right has a perfectoid tilde-limit. Recall from Lemma 5.5 that E/M can be covered by admissible open subspaces $E_1, \dots, E_k \subseteq E$ which map isomorphically onto an admissible open via $E \to E/M$. Denote by $E_i^{1/p^n} \subseteq E$ the pullback along $[p^n] : E \to E$. Also denote by $E_{ij}^{1/p^n} \subseteq E$ the pullback of E_{ij}. We can then reconstruct the space $E/M^{1/p^n}$ from the E_i^{1/p^n} as follows:

Lemma 5.8.

(1) The restriction to $E_i^{1/p^n} \subseteq E$ of $E \to E/M^{1/p^n}$ is an isomorphism onto its image. In particular, we can view E_i^{1/p^n} as a chart of $E/M^{1/p^n}$, and this is the preimage of E_i under $E/M^{1/p^n} \to E/M$.

(2) The collection of E_i^{1/p^n} is an atlas for $E/M^{1/p^n}$.

(3) We can reconstruct $E/M^{1/p^n}$ from glueing the E_i^{1/p^n} along the E_{ij}^{1/p^n}.

(4) The map $[p^n] : E/M^{1/p^n} \to E/M$ can be glued from the restrictions of $[p^n] : E \to E$ to $E_i^{1/p^n} \to E_i$, that is these maps commute with the glueing maps on E_{ij}^{1/p^n}.

The situation is thus like in Figure 2.

Proof. The first part follows because the map on the left of diagram 13 is an isomorphism. The second follows from the pullback of the E_i along $E/M^{1/p^n} \to E/M$, using that the diagram commutes. We thus obtain an admissible cover by cuboids $E_1^{1/p^n}, \dots, E_k^{1/p^n}$ of $E/M^{1/p^n}$. The second part of Lemma 5.5 applied to $E/M^{1/p^n}$ then shows that $E/M^{1/p^n}$ can be reconstructed by glueing along subspaces E_{ij}^{1/p^n}.

Finally, in order to see that one can glue together the map $[p^n] : E/M^{1/p^n} \to E/M$ from the E_i^{1/p^n}, use that intersection of cuboids are again cuboids, and so E_{ij}^{1/p^n} is a disjoint union of cuboids. It then follows from Lemma 5.6 that E_{ij} pulls back under $[p^n]$ to the intersection $E_{ij}^{1/p^n} \subseteq E/M^{1/p^n}$. That $[p^n]$ commutes with the glueing maps is clear because we know from diagram (11) that $[p^n] : E \to E$ induces a morphism $[p^n] : E/M^{1/p^n} \to E/M$. □

We are now ready to prove:

Proposition 5.9. *There is a perfectoid space $E/M^{1/p^\infty}$ such that*

$$E/M^{1/p^\infty} \sim \varprojlim_n E/M^{1/p^n}.$$

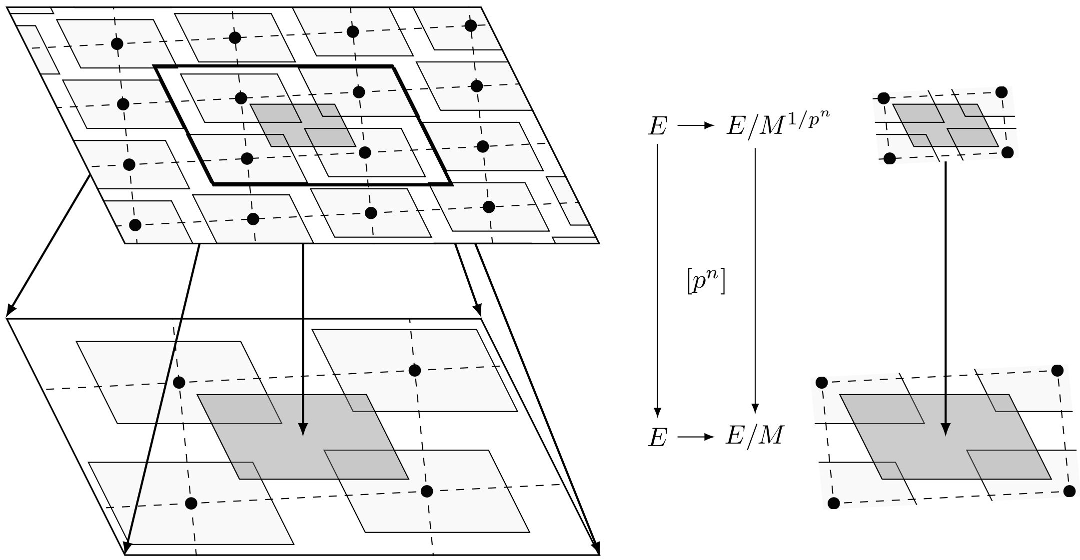

FIGURE 2. Illustration of how $[p^n] : E/M^{1/p^n} \to E/M$ can be glued from the maps $E_j^{1/p^n} \to E_j$. Here E_j on bottom and E_j^{1/p^n} on top are represented by the grey cuboids in the middle. On the left they are embedded into E whereas on the right they are considered as charts for E/M and $E/M^{1/p}$.

Proof. Denote by E_i^{1/p^∞} the pullback of $E_i \subseteq E$ to E_∞. This is an open subspace of a perfectoid space and hence perfectoid. Moreover, by Proposition 2.4.3 in [8] we have

$$E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}.$$

Given two different E_i, E_j, we know by Lemma 5.8 that at every step in the tower, the pullbacks E_i^{1/p^n} and E_j^{1/p^n} to $E/M^{1/p^n}$ intersect in E_{ij}^{1/p^n}. We can thus glue the E_i^{1/p^∞} along pullbacks E_{ij}^{1/p^∞} of the intersections $E_{ij} = E_i \cap E_j$ to E_∞ and thus obtain a perfectoid space $E/M^{1/p^\infty}$. This is a tilde-limit for $\varprojlim_{[p]} E/M^{1/p^n}$ because by construction it is so locally, and the definition of tilde-limits in Definition 2.4.1 of [8] is local on the source. $\square$

5.4. **Constructing a limit of the horizontal tower.** In order to construct a tilde-limit for $\varprojlim A$, we can now use that the horizontal maps in diagram (12) are all finite étale. They are even finite covering maps, in the following sense:

Lemma 5.10. *For any $0 \le m \le n$, the preimage of E_i^{1/p^n} from Lemma 5.8 under the horizontal map $v^{n-m} : E/M^{1/p^m} \to E/M^{1/p^n}$ is isomorphic to $p^{r(n-m)}$ disjoint copies of E_i^{1/p^n}. More canonically, it can be described as the isomorphic image of $M^{1/p^n}/M^{1/p^m} \times E_i^{1/p^m}$ under the multiplication map $E/M^{1/p^m} \times E/M^{1/p^m} \to E/M^{1/p^m}$.*

Proof. By the first part of Lemma 5.8, we know that the preimage of E_i^{1/p^n} under the projection $E \to E/M^{1/p^n}$ is a disjoint union of translates of E_i^{1/p^n} by M^{1/p^n}. The result then follows because $M^{1/p^n}/M^{1/p^m} = \underline{(\mathbb{Z}/p^{n-m}\mathbb{Z})^r}$. $\square$

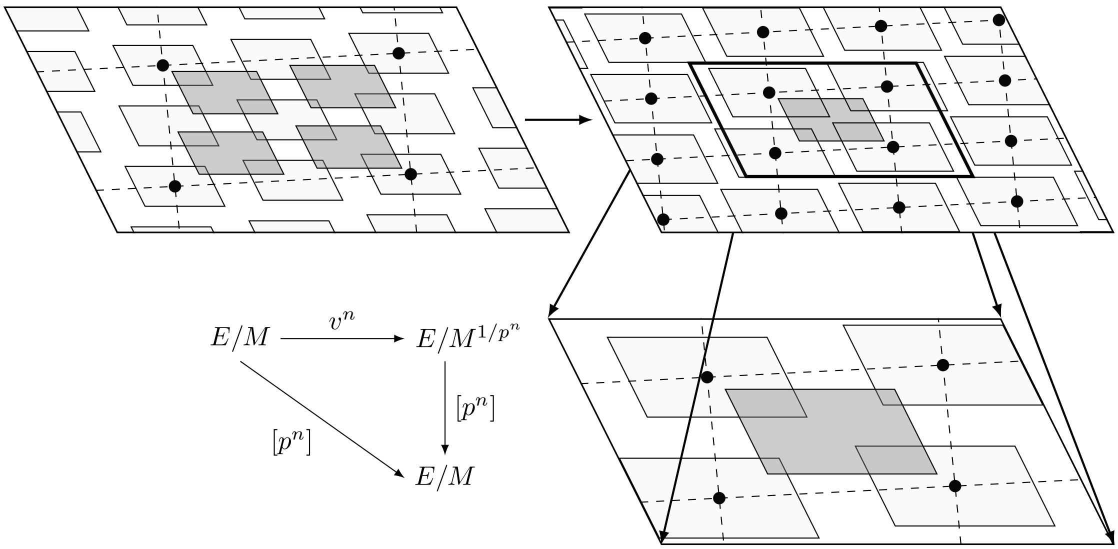

FIGURE 3. Illustration of how $[p] : E/M \to E/M$ factors in a part that is "ramified" (the vertical tower) and a part that is "étale" (the horizontal tower) with respect to our cover.

We also record the following immediate consequence:

Lemma 5.11. *The preimage of E_i under $[p^n] : A \to A$ is isomorphic to p^{rn} disjoint copies of E_i^{1/p^n}. More canonically, we can describe the preimage as the isomorphic image of $D_n \times E_i^{1/p^n}$ under the multiplication $A \times A \to A$. The situation is thus as in figure 3.*

Proof. This follows from the first part of Lemma 5.8 combined with Lemma 5.10 in the case of $m = 0$. □

Lemma 5.12. *The squares in diagram (12) are all pullback diagrams.*

$$\begin{array}{ccc} E/M^{1/p^n} & \xrightarrow{v} & E/M^{1/p^{n+1}} \\ \downarrow{\scriptstyle [p]} & & \downarrow{\scriptstyle [p]} \\ E/M^{1/p^{n-1}} & \xrightarrow{v} & E/M^{1/p^n} \end{array}$$

Proof. This can for instance be checked locally: The admissible open subset $E_i^{1/p^n} \subseteq E/M^{1/p^n}$ from Lemma 5.8 is pulled back to $E_i^{1/p^{n+1}}$ under the vertical map $[p] : E/M^{1/p^{n+1}} \to E/M^{1/p^n}$. The preimage of E_i^{1/p^n} under the horizontal map $E/M^{1/p^{n-1}} \to E/M^{1/p^n}$ is p^r disjoint copies of E_i^{1/p^n} by Lemma 5.10. The pullback of E_i^{1/p^n} to the upper right is thus p^r disjoint copies of $E_i^{1/p^{n+1}}$, which is clearly the fibre product. □

Lemma 5.13. *The horizontal maps in diagram (12) induce natural finite étale morphisms $v : E/M^{1/p^\infty} \to E/M^{1/p^\infty}$ that fit into Cartesian diagrams*

$$\begin{array}{ccc} E/M^{1/p^\infty} & \xrightarrow{v^m} & E/M^{1/p^\infty} \\ \downarrow & & \downarrow \\ E/M^{1/p^{n-m}} & \xrightarrow{v^m} & E/M^{1/p^n} \end{array}$$

In particular, the preimage of E_i^{1/p^∞} under v^m is isomorphic to p^{rm} copies of E_i^{1/p^∞}.

Proof. Since $E/M \to E/M^{1/p}$ is finite étale, the fibre product with $E/M^{1/p^\infty} \to E/M^{1/p}$ exists and is perfectoid by Proposition 7.10 of [5].

The universal property of the fibre product then gives a unique map

$$E/M^{1/p^\infty} \to E/M \times_{E/M^{1/p}} E/M^{1/p^\infty}$$

making the natural diagrams commute. On the other hand, using Lemma 5.12 we see that the fibre product has compatible maps into the vertical inverse system over E/M. Since by Proposition 2.4.5 of [8] the perfectoid tilde-limit $E/M^{1/p^\infty}$ is universal for maps from perfectoid spaces to the inverse system, we obtain a unique map into the other direction. □

We thus obtain a pro-étale tower

$$\dots \xrightarrow{v} E/M^{1/p^\infty} \xrightarrow{v} E/M^{1/p^\infty} \xrightarrow{v} E/M^{1/p^\infty} \tag{14}$$

which we think of as being a kind of vertical "limit" of diagram 12.

Remark 5.14. It is clear at this point that the limit of this tower exists, even as a categorical limit of honest adic spaces, since limits of affinoid perfectoid spaces exist in honest adic spaces and are affinoid perfectoid:

If $(\mathrm{Spa}(A_i, A_i^+))_{i\in I}$ is an inverse system of affinoid perfectoid spaces, then it is easy to check that for A^+ the π-adic completion of $\varinjlim A_i$ and for $A = A^+[1/\pi]$, the affinoid algebra (A, A^+) is again perfectoid. It then follows from topological algebra that $\mathrm{Spa}(A, A^+)$ has the desired universal property for maps from honest adic spaces into the inverse system $(\mathrm{Spa}(A_i, A_i^+))_{i\in I}$.

From the construction it follows that this is a tilde-limit (the condition on the underlying sets can be checked using the argument of Lemma 6.13.(ii) in [5]). The affinoid case implies the result for the above tower by a glueing argument.

Despite the remark, we now want to give a slightly more explicit construction of this limit that also says something about its local geometry. We first looks at the tower

$$\dots \xrightarrow{[p]} D_{n+1} \xrightarrow{[p]} D_n \to \dots .$$

Lemma 5.15. *There is a perfectoid space D_∞ such that $D_\infty = \varprojlim_{[p]} D_n$ as a limit of adic spaces.*

Proof. The D_n are étale over $\mathrm{Spa}(K)$ and thus perfectoid. Like in Remark 5.14, we then obtain a perfectoid space D_∞. More explicitly, since the D_n are isomorphic to $\underline{(\mathbb{Z}_p/p^n\mathbb{Z})^r}$, we can write

$$D_\infty \cong \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^r, K), \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^r, \mathcal{O}_K)).$$

□

Proposition 5.16. *There exists a perfectoid space $(E/M^{1/p^\infty})_\infty$ such that*

$$(E/M^{1/p^\infty})_\infty = \varprojlim_v (E/M^{1/p^\infty}).$$

Moreover, the projection map $\pi : (E/M^{1/p^\infty})_\infty \to E/M^{1/p^\infty}$ *is a profinite covering map, that is every point of* $E/M^{1/p^\infty}$ *has an open neighbourhood* U *such that* $\pi^{-1}(U)$ *is isomorphic to* $D_\infty \times U$.

Proof. By Lemma 5.5, the preimage of E_i^{1/p^∞} under v^m of $E/M^{1/p^\infty}$ is isomorphic to $D_m \times E_i^{1/p^\infty}$. Since projective limits commute with products, the restriction of the tower to E_i^{1/p^∞} therefore has limit $D_\infty \times E_i^{1/p^\infty}$, which is perfectoid as a product of perfectoid spaces.

One can then glue using the same arguments as in the proof of Proposition 5.9. ☐

5.5. **The diagonal tower: proof of the main theorem.** We now want to show that $(E/M^{1/p^\infty})_\infty$ is in fact a tilde-limit of the $[p]$-multiplication tower. In other words, this says that the horizontal limit of the vertical tilde-limits in diagram 12 is also a diagonal tilde-limit. This isn't just a formal consequence since tilde-limits aren't limits. But using the local geometry of the maps in the tower in terms of cuboids, it is still easy to see:

Proposition 5.17. *The perfectoid space* $A_\infty := (E/M^{1/p^\infty})_\infty$ *is a tilde-limit of* $\varprojlim_{[p]} A$. *It is independent up to unique isomorphism of the choice of partial anticanonical* $\Gamma_0(p^\infty)$*-structure, but it remembers the choice as a pro-finite étale closed subgroup* $D_\infty \subseteq A_\infty$.

The preimage of $E_i \subseteq A$ *under the projection* $A_\infty \to A$ *is isomorphic to* $D_\infty \times E_i^{1/p^\infty}$.

Proof. It is clear from $(E/M^{1/p^\infty})_\infty \sim \varprojlim_v E/M^{1/p^\infty}$ and $E/M^{1/p^\infty} \sim \varprojlim_{[p]} E/M^{1/p^n}$ that the underlying topological space of $(E/M^{1/p^\infty})_\infty$ is indeed isomorphic to $\varprojlim_{[p]} |E/M|$.

In order to show that it is a tilde-limit of $\varprojlim_{[p]} E/M$, it thus suffices to give an explicit cover of $(E/M^{1/p^\infty})_\infty$ by open affinoids satisfying the tilde-limit property.

Recall that by construction of $(E/M^{1/p^\infty})$ we have a cover of E/M by open subsets E_i that pull back to perfectoid open subspaces E_i^{1/p^∞} for which $E_i^{1/p^\infty} \sim \varprojlim E_i^{1/p^n}$. Moreover, by the second part of Proposition we know that the pullback of E_i^{1/p^∞} to $(E/M^{1/p^\infty})_\infty$ is $D_\infty \times E_i^{1/p^\infty}$.

On the other hand, when we go along the diagonal tower, we obtain the inverse system

$$\cdots \to D_{n+1} \times E_i^{1/p^{n+1}} \to D_n \times E_i^{1/p^n} \to \dots.$$

More explicitly, after choosing compatible isomorphisms $D_n \cong \underline{(\mathbb{Z}/p^n\mathbb{Z})^r}$ for all n, we can for any affinoid open $V = \mathrm{Spa}(A_n, A_n^+) \subseteq E_i^{1/p^n}$ write the affinoid open subset $D_n \times V \subseteq D_n \times E_i^{1/p^n}$ as

$$D_n \times V \cong \mathrm{Spa}(\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_n), \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_n^+))$$

We claim that this tower has tilde-limit $D_\infty \times E_i^{1/p^\infty} \sim \varprojlim D_n \times E_i^{1/p^n}$. To see this, cover E_i^{1/p^∞} by open affinoids $U = \mathrm{Spa}(A, A^+)$ such that

$$\varinjlim_{\mathrm{Spa}(A_j, A_j^+) \subseteq E_i^{1/p^n}} A_j \to A$$

has dense image. Here the direct limit runs through all affinoid open subspaces $\mathrm{Spa}(A_j, A_j^+) \subseteq E_i^{1/p^n}$ for all n through which $U \subseteq E_i^{1/p^\infty} \to E_i^{1/p^n}$ factors, as in the definition of tilde-limits. Then by the construction of products of affinoid perfectoid spaces, the pullback of U to $(E/M^{1/p^\infty})_\infty$ is

$$D_\infty \times U \cong \mathrm{Spa}(\mathrm{Map}_{cts}(\mathbb{Z}_p^r, A), \mathrm{Map}_{cts}(\mathbb{Z}_p^r, A^+))$$

It thus suffices to show that $\varinjlim \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j)$ is dense in $\mathrm{Map}_{cts}(\mathbb{Z}_p^r, A)$. As a first step, even though in the above limit n depends on j, we may write this as two separate limits,

$$\varinjlim \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j) = \varinjlim_j \varinjlim_n \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j).$$

For fixed j, we then see that

$$\varinjlim_n \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^r, A_j) = \mathrm{Map}_{lc}(\mathbb{Z}_p^r, A_j)$$

where the right hand side denotes locally constant morphisms. But it then follows from a pointwise approximation argument that $\varinjlim \mathrm{Map}_{lc}(\mathbb{Z}_p^r, A_j)$ has dense image in $\mathrm{Map}_{lc}(\mathbb{Z}_p^r, A)$. It is then clear that the latter has dense image in $\mathrm{Map}_{cts}(\mathbb{Z}_p^r, A)$. We conclude that $D_\infty \times E_i^{1/p^\infty} \sim \varprojlim D_n \times E_i^{1/p^n}$.

That A_∞ is independent of the $\Gamma_0(p^\infty)$-structure up to unique isomorphism is a consequence of the universal property of the perfectoid tilde-limit. To see that D_∞ is a closed subgroup of A_∞, choose i such that the unit section of E/M lies in E_i. Then the unit section $\mathrm{Spa}(K, \mathcal{O}_K) \to E_i^{1/p^\infty}$ induces a closed immersion $D_\infty \hookrightarrow D_\infty \times E_i^{1/p^\infty} \hookrightarrow A_\infty$. □

This finishes the proof of Theorem 5.1.

Note that while the approach via cuboids E_i may look a bit technical on first glance, it has the advantage of giving an explicit description of $(E/M)_\infty$ as being glued from pieces of E_∞ by glueing data that is controlled by the lattices M^{1/p^n}. This might be interesting for applications, and in particular for computing the tilt.

6. Limits of the covering maps

In this section we use the explicit constructions of the space A_∞ to study its geometry more closely. We retain notation and assumptions from the last chapter.

Over the course of the proof of Theorem 5.1, we have used three different towers: The tower $\cdots \to E \xrightarrow{[p]} E$, the tower $\cdots \to E/M \xrightarrow{[p]} E/M$ and the tower $\cdots \to E/M^{1/p} \xrightarrow{[p]} E/M$. The three are related by covering maps which fit into a commutative diagram of towers

$$\begin{tikzcd} E \arrow[r] \arrow[d, "{[p]}"] & E/M \arrow[r] \arrow[d, "{[p]}"] & E/M^{1/p^{n+1}} \arrow[d, "{[p]}"] \\ E \arrow[r] & E/M \arrow[r] & E/M^{1/p^n} \end{tikzcd}$$

As we have seen in the last sections, all three towers have perfectoid tilde-limits, that we have denoted by E_∞, A_∞ and $E/M^{1/p^\infty}$.

By Proposition 2.14 the map $\pi : E \to A = E/M$ in the limit induces a natural group homomorphism $\iota : E_\infty \to A_\infty$. A similar universal property argument shows that we obtain a natural group homomorphism $A_\infty \to E/M^{1/p^\infty}$. The composition of these two maps is the morphism $E_\infty \to E/M^{1/p^\infty}$, which is the limit of the maps $E \to E/M^{1/p^n}$ in the above diagram. We now want to look at these morphisms more closely one after the other.

We start with the case of $E_\infty \to E/M^{1/p^\infty}$:

Proposition 6.1. *Denote by $M_\infty \cong M$ the perfectoid tilde-limit of the tower*

$$\cdots \xrightarrow[{[p]}]{\sim} M^{1/p^2} \xrightarrow[{[p]}]{\sim} M^{1/p} \xrightarrow[{[p]}]{\sim} M.$$

There is a natural map $M_\infty \to E_\infty$ with respect to which we can interpret M_∞ as a lattice of rank r in E_∞. The map fits into a short exact sequence of perfectoid groups

$$0 \to M_\infty \to E_\infty \to E/M^{1/p^\infty} \to 0.$$

that is locally split. In particular, we can view E_∞ as an M_∞-torsor over $E/M^{1/p^\infty}$.

Proof. The map $M_\infty \to E_\infty$ is induced by the universal property of the perfectoid tilde-limit as usual. In order to see that the sequence is exact, we need to see that the first map is a kernel of the second, and the second map is a categorical quotient of the first. To this end, we first analyse the morphism locally: The projections to the inverse system fit into a commutative diagram

$$\begin{tikzcd}
M_\infty \arrow[r] \arrow[d, dotted] & E_\infty \arrow[r] \arrow[d, dotted] & E/M^{1/p^\infty} \arrow[d, dotted] \\
M^{1/p^n} \arrow[r] \arrow[d, "{[p^n]}"] & E \arrow[r] \arrow[d, "{[p^n]}"] & E/M^{1/p^n} \arrow[d, "{[p^n]}"] \\
M \arrow[r] & E \arrow[r] & E/M
\end{tikzcd}$$

Let us consider the preimages of $E_i \subseteq E/M$ under these morphisms: By Lemma 5.5 we see that the pullback to E is $\bigsqcup_{q\in M} qE_i$. We can also see this as the isomorphic image of $M \times E_i$ under the multiplication map $E \times E \to E$. The pullback of E_i along $[p^n] : E/M^{1/p^n} \to E/M$ is E_i^{1/p^n} as we have seen in Lemma 5.8. The same Lemma shows that the pullback of this along $E \to E/M^{1/p^n}$ is $\bigsqcup_{q\in M^{1/p^n}} qE_i^{1/p^n} = M^{1/p^n} \times E_i^{1/p^n}$. We conclude that the pullback to E_∞ is $M_\infty \times E_i^{1/p^\infty}$. By construction of $E/M^{1/p^\infty}$ in the proof of Proposition 5.9, the pullback of E_i to $E/M^{1/p^\infty}$ is E_i^{1/p^∞}. All in all, we obtain a pullback diagram

$$\begin{tikzcd}
 & E_\infty \arrow[rr] \arrow[dd] & & E/M^{1/p^\infty} \arrow[dd] \\
M_\infty \times E_i^{1/p^\infty} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & E_i^{1/p^\infty} \arrow[ur, hook] \arrow[dd] & \\
 & E \arrow[rr] & & E/M \\
M \times E_i \arrow[ur, hook] \arrow[rr] & & E_i \arrow[ur, hook] &
\end{tikzcd}$$

We conclude that $E_\infty \to E/M^{1/p^\infty}$ is a principal M_∞-torsor of perfectoid groups. It is then clear that M_∞ is the preimage of $0 \in E/M^{1/p^\infty}$, from which one easily verifies that $M_\infty \hookrightarrow E_\infty$ has the universal property of the kernel. It remains to see that $E_\infty \to E/M^{1/p^\infty}$ has the universal property of the cokernel: Given any perfectoid group H and a group homomorphism $E_\infty \to H$ that is trivial on M_∞, the restriction $M_\infty \times E_i^{1/p^\infty} \to H$ gives a natural map $E_i^{1/p^\infty} \to H$. Since by construction of $E/M^{1/p^\infty}$ the spaces E_i^{1/p^∞} and E_j^{1/p^∞} are glued on E_{ij}^{1/p^∞} using translation by M_∞, these glue together to the desired morphism of $E/M^{1/p^\infty}$. $\square$

The case of $\iota : A_\infty \to E/M^{1/p^\infty}$ is similar:

Proposition 6.2. *The subgroup $D_\infty \subseteq A_\infty$ gives rise to a short exact sequence of perfectoid groups*

$$0 \to D_\infty \to A_\infty \to E/M^{1/p^\infty} \to 0.$$

that is locally split. In particular, we can view A_∞ as a D_∞-torsor over $E/M^{1/p^\infty}$.

Proof. By Proposition 5.16 the pullback of E_i^{1/p^∞} under $A_\infty \to E/M^{1/p^\infty}$ is

$$D_\infty \times E_i^{1/p^\infty} \to E_i^{1/p^\infty}$$

which shows that $A_\infty \to E/M^{1/p^\infty}$ is a D_∞-torsor. As in the last proof, this implies that the sequence in the Proposition is a short exact sequence. $\square$

Finally, we consider the case of $\iota : E \to A = E/M$. While the limits of the last two towers were fibre bundles again, the map ι shows quite a different behaviour and on the opposite is an injective group homomorphism. This may seem strange at first, but it is actually what one might expect following the intuition of the following example:

Remark 6.3. Consider the following inverse system of abstract groups:

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "{[p]}"] & \mathbb{R} \arrow[r] \arrow[d, "{[p]}"] & \mathbb{R}/\mathbb{Z} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

While at finite level the maps on the right are all covering maps, in the inverse limit the homological algebra of $\varprojlim$ produces a long exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \varprojlim_{[p]} \mathbb{R}/\mathbb{Z} \longrightarrow \varprojlim^1_{[p]} \mathbb{Z} = \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

So in the limit the covering map becomes the kernel of a map to $\mathbb{Z}_p/\mathbb{Z}$.

For perfectoid groups the homological algebra argument of course doesn't apply. Nevertheless, we can again use the explicit covers of the last section to show that the situation is very similar as in the remark. In the following, we use the notion of injective morphism from [**?**], Definition 5.1.

Theorem 6.4. *The map $\iota : E_\infty \to A_\infty$ is an injective group homomorphism. It fits into the following commutative diagram of locally split short exact sequences of perfectoid groups:*

$$\begin{tikzcd}
0 \arrow[r] & M_\infty \arrow[r] \arrow[d, hook] & E_\infty \arrow[r] \arrow[d, hook] & E/M^{1/p^\infty} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D_\infty \arrow[r] & A_\infty \arrow[r] & E/M^{1/p^\infty} \arrow[r] & 0
\end{tikzcd}$$

The morphism ι is compatible with the splittings: Locally on open subspaces $U \subseteq E/M^{1/p^\infty}$ the morphism $E_\infty \hookrightarrow A_\infty$ is of the form $M_\infty \times U \to D_\infty \times U$. In particular, one can describe A_∞ as the associated fibre bundle

$$A_\infty = D_\infty \times^{M_\infty} E_\infty.$$

Proof. Recall that the map $\iota : E_\infty \to A_\infty$ arises by a universal property from the inverse system

$$\begin{tikzcd}
\vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] \\
E \arrow[r] \arrow[d, "{[p]}"'] & A \arrow[r, "v"] \arrow[d, "{[p]}"'] & E/M^{1/p} \arrow[d, "f"'] \\
E \arrow[r] & A \arrow[r, equal] & E/M.
\end{tikzcd}$$

Using Lemmas 5.8 and 5.10 we see that the pullback of this diagram to $E_i \subseteq E/M$ is

$$\begin{tikzcd}
\vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] \\
\bigsqcup_{q \in M^{1/p}} q \cdot E_i^{1/p^n} \arrow[r] \arrow[d] & \bigsqcup_{q \in D_n} E_i^{1/p^n} \arrow[r] \arrow[d] & E_i^{1/p^n} \arrow[d] \\
\bigsqcup_{q \in M} q \cdot E_i \arrow[r] & E_i \arrow[r, equal] & E_i
\end{tikzcd}$$

We see from this description and from the last part of Proposition 5.17 that the pullback to infinite level is the sequence

$$M_\infty \times E_i^{1/p^\infty} \to D_\infty \times E_i^{1/p^\infty} \to E_i^{1/p^\infty}. \tag{15}$$

This shows that the diagram of short exact sequences commutes. Since the E_i^{1/p^∞} cover $E/M^{1/p^\infty}$ by construction, it also shows the description of A_∞ in terms of the associated fibre bundle.

It remains to prove that ι is injective: Choose a basis of M, and thus a trivialisation of all D_n. We then see that the map $M_\infty \to D_\infty$ on the level of the underlying topological spaces can be described as the inclusion $\mathbb{Z}^r \hookrightarrow \mathbb{Z}_p^r$. This shows that $M_\infty \hookrightarrow D_\infty$ is an injective group homomorphism of perfectoid spaces. Thus by equation (15), the morphism $E_\infty \to A_\infty$ is injective as well. □

Corollary 6.5. *The injection $E_\infty \to A_\infty$ induces a short exact sequence of perfectoid groups*

$$0 \to M_\infty \to D_\infty \times E_\infty \to A_\infty \to 0$$

where the map on the left is the diagonal embedding of M_∞ into $D_\infty \times E_\infty$. In particular, we can describe A_∞ as the quotient $(D_\infty \times E_\infty)/M_\infty$ in the category of perfectoid groups.

Proof. The map $D_\infty \times E_\infty \to A_\infty$ is just the composition of $D_\infty \times E_\infty \hookrightarrow A_\infty \times A_\infty$ with the multiplication of A_∞. It is then clear from the short exact sequences of Theorem 6.4 that M_∞ is the kernel of this map. That A_∞ has the universal property of the cokernel is a consequence of the universal property of the associated fibre bundle construction: Explicitly, this follows from the fact that locally over $U \subseteq E/M^{1/p^\infty}$, the map on the right is the projection

$$D_\infty \times M_\infty \times U \to D_\infty \times U.$$

This gives a local splitting, and thus the necessary map in the universal property of the cokernel. □

We can see the short exact sequence of 6.5 as the analogue at infinity of the short exact sequence of rigid groups

$$0 \to M \to E \to A \to 0.$$

In particular, while as a rigid analytic space A is locally isomorphic to open subspaces of E, the perfectoid space A_∞ is locally isomorphic to open subspaces of $D_\infty \times E_\infty$.

Appendix A. Fibre bundles of formal and rigid spaces

In this chapter we review the theory of fibre bundles with structure group T and in particular of principal T-bundles in the setting of formal and rigid geometry.

In this chapter we denote by T a commutative formal group scheme over $\mathcal{O}_K$. We denote the multiplication map by $m : T \times T \to T$. By a T-action on a formal scheme X we mean a morphism $m_X : T \times X \to X$ such that the usual associativity diagram commutes.

Definition A.1. By a **T-linear map** of schemes X and Y with T-actions we mean a morphism $\phi : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} T \times X \arrow[r, "\mathrm{id}_T \times \phi"] \arrow[d, "m_X"] & T \times Y \arrow[d, "m_Y"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

We denote by $\mathbf{FormAct}_T$ the category of formal schemes with action by T.

The definition of a principal T-bundle is just what we get when we take the definition of a principal G-bundle and replace the category of topological spaces by the category of formal schemes.

Notation A.2. In the following, if $\pi : E \to B$ is a morphism of formal schemes, then for a formal open subscheme $U \subseteq B$ we denote $E|_U := \pi^{-1}(U) \subseteq E$.

Definition A.3. Let T be a formal group scheme. Let F be a formal scheme with an action $m : T \times F \to F$. A morphism $\pi : E \to B$ of formal schemes is called a **fibre bundle with fibre F and structure group T** if there is a cover $\mathfrak{U}$ of B of open formal subschemes $U_i \subseteq B$ with isomorphisms $\varphi_i : F \times U_i \xrightarrow{\sim} E|_{U_i}$ which satisfy the following conditions:

(a) For every $U_i \in \mathfrak{U}$, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_i} \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

(b) For every two $U_i, U_j \in \mathfrak{U}$ with intersection U_{ij}, the commutative diagram

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "p_2"'] & E|_{U_{ij}} \arrow[d, "\pi"] & F \times U_{ij} \arrow[l, "\varphi_j"'] \arrow[dl, "p_2"] \\ & U_{ij} & \end{tikzcd}$$

produces an isomorphism $\phi_{ij} := \varphi_j^{-1} \circ \varphi_i : F \times U_{ij} \to F \times U_{ij}$ with the following property: There exists a morphism $\psi_{ij} : U_{ij} \to T$ such that

$$\phi_{ij} = F \times U_{ij} \xrightarrow{\psi_{ij} \times \mathrm{id} \times \mathrm{id}} T \times F \times U_{ij} \xrightarrow{m \times \mathrm{id}} F \times U_{ij}$$

Definition A.4. When we take F equal to the formal scheme T with the action on itself by left multiplication, then a fibre bundle $\pi : E \to B$ with fibre T and structure group T is called a **principal T-bundle**. This is also called a T-torsor.

Example. For the short exact sequence $0 \to \overline{T} \to \overline{E} \xrightarrow{\pi} \overline{B} \to 0$ from Section 4, $\overline{E} \xrightarrow{\pi} \overline{B}$ defines a principal $\overline{T}$-bundle by Lemma 4.5. Moreover, for any formal open subscheme $U \subseteq \overline{B}$, the map $E|_U \to U$ is still a principal $\overline{T}$-bundle. This is what we mean when we say that the notion of principal $\overline{T}$-bundles is better suited for studying $\overline{E}$ locally on $\overline{B}$ than the notion of short exact sequences.

The morphism ϕ_{ij} from condition (b) is fully determined by the morphism $\psi_{ij} : U_{ij} \to T$. By a glueing argument, one shows:

Lemma A.5. *Suppose we are given formal schemes F and B and a formal group scheme T with an action on F. Then fibre bundles $\pi : E \to B$ with fibre F and structure group T are equivalent to the data (up to refinement) of a cover $\mathfrak{U}$ of B by formal open subschemes and morphisms $\psi_{ij} : U_{ij} \to T$ for all $U_i, U_j \in \mathfrak{U}$ that satisfy the cocycle condition $\psi_{jk} \cdot \psi_{ij} = \psi_{ik}$, by which we mean that the following diagram commutes:*

$$\begin{tikzcd} U_{ijk} \arrow[r, "\psi_{ij}\times\psi_{jk}"] \arrow[d, equal] & T \times T \arrow[d, "m"] \\ U_{ijk} \arrow[r, "\psi_{ik}"] & T. \end{tikzcd} \tag{16}$$

In order to define the category of fibre bundles, we also need the following:

Lemma A.6. *Let $E \to B$ be a fibre bundle with fibre F and structure group T. With notations like in Definition A.3 we have a natural T-action on $F \times U_i$ for each i when we let T act trivially on U_i. These actions glue together to a T-action on E.*

Proof. This is immediate from condition (b). □

Definition A.7. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T. Then a **morphism of fibre bundles** $f : (E', B', \pi') \to (E, B, \pi)$ is a commutative diagram of formal schemes

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

in which the morphism f_E is also T-linear (we often abbreviate this by writing $f : E' \to E$). We thus obtain the category of fibre bundles over T that we denote by $\mathbf{FormFibBun}_T$ and the full subcategory of principal T-bundles, that we denote by $\mathbf{FormPrinBun}_T$.

In the case of principal T-bundles, this data can be given as follows: Let $\mathfrak{U}$ be of B a cover over which E is trivialised. Then we can always refine U in such a way that for all $U \in \mathfrak{U}$ the fibre bundle E' is trivial over $U' := f_B^{-1}(U)$. The induced map $f_E : T \times U' \to T \times U$ is then T-linear and thus can be reconstructed from the induced map

$$\theta : U' = 1 \times U' \hookrightarrow T \times U' \xrightarrow{f_E} T \times U \xrightarrow{p_1} T.$$

Lemma A.8. *Fix a morphism $f_B : B' \to B$ and fix notation as above. Then the data of a morphism $f = (f_E, f_B)$ of principal T-bundles is equivalent to the data of morphisms $\theta_i : U'_i = f_B^{-1}(U_i) \to T$ for some cover $\mathfrak{U}$ of B by formal open subschemes U_i such that each θ_i induces a map $f_{E,i}$ such that for all i, j the following diagram commutes:*

$$\begin{tikzcd}
T \times U'_{ij} \arrow[r, "\phi'_{ij}"] \arrow[d, "f_{E,i}"] & T \times U'_{ij} \arrow[d, "f_{E,j}"] \\
T \times U_{ij} \arrow[r, "\phi_{ij}"] & T \times U_{ij}.
\end{tikzcd}$$

Moreover, commutativity of the above diagram is equivalent to commutativity of

$$\begin{tikzcd}
U'_{ij} \arrow[r, "\psi'_{ij} \times \theta_j"] \arrow[d, "(\psi_{ij} \circ f) \times \theta_i"'] & T \times T \arrow[d, "m"] \\
T \times T \arrow[r, "m"] & T.
\end{tikzcd}$$

Or in short hand notation,

$$\psi'_{ij}(u)\theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u)$$

Proof. One direction is clear. For the other, the first part follows from glueing. The second part is a consequence of all maps in the first diagram being T-linear. □

Definition A.9. Let $\pi : E \to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Since the data in the equivalent characterisation of Lemma A.5 is completely independent of the fibre, the morphisms $\psi_{ij} : U_{ij} \to T$ by Lemma A.5 define a fibre bundle with fibre F and structure group T that we denote by $F \times^T E$. This is called the **associated bundle** or Borel-Weil construction.

Note that in many authors in differential geometry and topology denote the associated bundle by "$F \times_T E$" instead of $F \times^T E$. In our setting, however, this is slightly confusing since we often have natural maps from T to F and E, but $F \times^T E$ is usually *not* their fibre product. In fact it behaves more like a pushout, for instance in the case that E comes from a short exact sequence.

Proposition A.10. *The associated bundle construction is a bifunctor*

$$- \times^T - : \mathbf{FormAct}_T \times \mathbf{FormPrinBun}_T \to \mathbf{FormFibBun}_T$$

from the categories of formal schemes with T-action $\times$ the category of principal T-bundles to the category of fibre bundles with structure group T.

Proof. Let E and E' be principal T-bundles and let $f : E' \to E$ be a morphism of T-bundles. Let F and F' be formal schemes with T-action and let $h : F' \to F$ be a T-equivariant morphism. Then we can find compatible covers $\mathfrak{U}'$ of B' and $\mathfrak{U}$ of B such that locally we have diagrams like in Lemma A.8. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U'_i$ such that we obtain a natural map

$$F' \times U'_i \xrightarrow{h \times^T f} F \times U_i, \quad (x, u) \mapsto (h(x)\theta_i(u), f_B(u))$$

(of course this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersections Lemma A.8 implies that we have a commutative diagram

$$\begin{tikzcd}
F' \times U'_{ij} \arrow[r, "h \times^T f"] & F \times U_{ij} \\
F' \times U'_{ij} \arrow[r, "h \times^T f"] \arrow[u, "\psi'_{ij} \times \mathrm{id}"] & F \times U_{ij}. \arrow[u, "\psi_{ij} \times \mathrm{id}"']
\end{tikzcd}$$

One easily checks that this is functorial in both components. □

Lemma A.11. *Let S be another formal group scheme that receives an action of T from a group homomorphism $g : T \to S$. Then for any principal T-bundle E, the Borel construction $S \times^T E$ is a principal S-bundle.*

Proof. This follows from Lemma A.5. The only thing we need to check is that the cocycle condition from diagram (16) also holds with respect to S. But g is a homomorphism and therefore the following diagram commutes:

$$\begin{array}{ccc} T \times T & \xrightarrow{g \times g} & S \times S \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ T & \xrightarrow{g} & S. \end{array}$$

□

Lemma A.12. *The Borel construction is a functor $S \times^T -$ from principal T-bundles to principal S-bundles.*

Proof. This is a consequence of Lemma A.8. One obtains the necessary data by composing the morphisms $\theta' : U_i' \to T$ with the morphism $T \to S$. These morphisms glue together because the second diagram of Lemma A.8 commutes, as one easily sees from the fact that $T \to S$ is a morphism of formal groups. □

The Borel construction satisfies the following universal property:

Lemma A.13. *Let $g : T \to S$ be a homomorphism of formal group schemes and let $E \to B$ be a principal T-bundle. Let X be any principal S-bundle. Note that X receives a T-action from g. Then there is a functorial one-to-one correspondence between T-linear morphisms $E \to X$ and morphisms of principal S-bundles $S \times^T E \to X$.*

A.1. **The semi-linear case.** We later want to consider morphisms of fibre bundles that are induced from morphisms of short exact sequences. In this situation, in order to describe the morphism of the kernels, we need to incorporate morphisms of the structure group into the notion of morphisms of fibre bundles. For this we need semi-linear group actions.

Definition A.14. Let T and S be formal group schemes and let $g : T \to S$ be a homomorphism. Let X and Y be formal schemes with actions $m : T \times X \to X$ and $m : S \times Y \to Y$ respectively. Then by a g-linear morphism $f : X \to Y$ we mean a morphism of formal schemes such that the following diagram commutes

$$\begin{array}{ccc} T \times X & \xrightarrow{g \times f} & S \times Y \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{f} & Y. \end{array}$$

Definition A.15. We denote by **FormAct** the category of pairs (T, X) where T is a formal group scheme and X is a formal scheme with T-action, and morphisms are pairs of (g, f) where g is a group homomorphism and f is a g-linear morphism. It has a natural forgetful functor to **FormGrp**, the category of formal group schemes.

Definition A.16. Let $g : T' \to T$ be a homomorphism of formal group schemes. Let $\pi : E \to B$ be a fibre bundle with fibre F and structure group T and let $\pi' : E' \to B'$ be a fibre bundle with fibre F' and structure group T' . Then a g-linear morphism of principal bundles is a diagram

$$\begin{tikzcd} E' \arrow[r, "\pi'"] \arrow[d, "f_E"] & B' \arrow[d, "f_B"] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

such that f_E is g-linear. We denote by **FormFibBun** the category of fibre bundles (E, B, π, T, F) with arrows being the morphisms of principal bundles. It has a natural forgetful functor

$$(E, B, \pi, T, F) \mapsto T$$

to the category **FormGrp**. We denote by **FormPrinBun** the full subcategory of principal bundles.

We get the natural analogue of Lemma A.8:

Lemma A.17. *With the notations from Lemma A.8, a g-linear morphism of a principal T'-bundle to a principal T-bundle is equivalent to the data of morphisms $\theta : U'_i \to T$ such that the following diagram commutes on intersections:*

$$\begin{tikzcd} U'_{ij} \arrow[r, "\psi'_{ij}\times\theta_j"] \arrow[d, "(\psi_{ij}\circ f)\times\theta_i"'] & T' \times T \arrow[r, "g\times\mathrm{id}"] & T \times T \arrow[d, "m"] \\ T \times T \arrow[rr, "m"] & & T \end{tikzcd}$$

Or in short hand notation,

$$g(\psi'_{ij}(u)) \cdot \theta_j(u) = \psi_{ij}(f(u)) \cdot \theta_i(u). \tag{17}$$

Similarly as in Proposition A.10 one can conclude from this that change of fibre is functorial in the following sense:

Proposition A.18. *Given any homomorphism of group schemes $g : T' \to T$ and a g-linear homomorphism $h : F' \to F$ of formal schemes with T' and T-actions respectively, and a homomorphism $f : E' \to E$ of principal T' and T-bundles over g, one obtains a morphism*

$$h \times^g f : F' \times^{T'} E' \to F \times^T E$$

of fibre bundles over g, in a way that is functorial in h, g, f. More precisely, the associated bundle construction is a fibered bifunctor

$$- \times^- - : \mathbf{FormAct} \times_{\mathbf{FormGrp}} \mathbf{FormPrinBun} \to \mathbf{FormFibBun}.$$

Proof. Let (E,B,π,T) and (E',B',π',T') be principal bundles. Let F and F' be formal schemes with T-action and T'-action respectively. Let $g : T \to T'$ be a group homomorphism and let $h : F' \to F$ be a g-equivariant morphism. Let $f : E' \to E$ be a morphism of principal fibre bundles over g. Then we can find compatible covers $\mathfrak{U}'$ of B' and $\mathfrak{U}$ of B such that locally we have diagrams like in Lemma A.8. Then locally $F \times^T E$ and $F' \times^T E'$ are of the form $F \times U_i$ and $F' \times U'_i$ such that we obtain a natural map

$$F' \times U'_i \xrightarrow{(h\times^g f)} F \times U_i, \quad (x, u) \mapsto (h(x)\theta_i(u), f_B(u))$$

(as before this description is just a short hand for a diagram of maps, and not a description in terms of "points"). These maps glue together over the cover, since on intersection Lemma A.8 implies that we have a commutative diagram

$$\begin{tikzcd} F' \times U'_{ij} \arrow[r, "h\times^g f"] & F \times U_{ij} \\ F' \times U'_{ij} \arrow[u, "\psi'_{ij}\times\mathrm{id}"] \arrow[r, "h\times^g f"] & F \times U_{ij}. \arrow[u, "\psi_{ij}\times\mathrm{id}"'] \end{tikzcd}$$

More precisely, by g-linearity of h one has

$$h(x\cdot\psi'_{ij}(u))\cdot\theta_j(u) = h(x)\cdot g(\psi'_{ij}(u))\cdot\theta_j(u) \overset{(17)}{=} h(x)\cdot\psi_{ij}(f(u))\cdot\theta_i(u).$$

This shows that the maps glue to a morphism $h\times^g f$ as desired.

By refining covers, one easily checks that this is functorial in both components. □

We obtain a variant of Lemma A.13 in the semilinear case:

Lemma A.19. *Let E' be a principal T' bundle and let E be a principal T-bundle. Let H' and H be formal group schemes and assume there is a commutative diagram of group homomorphisms*

$$\begin{tikzcd} H' \arrow[r, "h"] & H \\ T' \arrow[u] \arrow[r, "g"] & T. \arrow[u] \end{tikzcd}$$

Let moreover $f: E'\to E$ be a g-linear morphism of fibre bundles. Then the map $h\times^g f$ from Proposition A.18 is the unique h-linear morphism of fibre bundles making the following diagram commute:

$$\begin{tikzcd} H'\times^{T'} E' \arrow[r, "h\times^f g"] & H\times^T E \\ E' \arrow[u] \arrow[r, "f"] & E. \arrow[u] \end{tikzcd}$$

Proof. The morphism exists by Proposition A.18. The vertical maps in the commutative diagram exist by functoriality via $E = T\times^T E \to H\times^T E$.

On the other hand, on any compatible trivialisation $T'\times U'\to T\times U$ of $f: E'\to E$ there is clearly only one way to extend this to $H'\times U'\to H\times U$ in a h-linear way. □

Remark A.20. All that we have done in this chapter can be done in completely the same way with formal schemes replaced by rigid spaces (covers being replaced by admissible covers) and also for schemes, or in fact for any site. We have preferred to use formal schemes to make things more explicit. The different categories of fibre bundles are well-behaved with respect to the usual functors between the different categories: For instance, by functoriality of fibre products there are natural rigidification and reduction functors from formal principal T-bundles over $\mathcal{O}_K$ to rigid principal T_η-bundles over K on the generic fibre, and to principal $\overline{T}$-bundles on the reduction $\mathcal{O}_K/\pi$. Moreover, these generic fibre and reduction functors commute with the associated fibre construction:

Lemma A.21. *Let T be a formal group scheme and let $\pi: E\to B$ be a principal T-bundle. Let F be a formal scheme with an action by T. Then*

$$(F\times^T E)_\eta = F_\eta \times^{T_\eta} E_\eta$$

Proof. This can be checked locally on any trivialising cover, where it is clear. □

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