Homework 5: Poisson matrix factorization

STAT 348, UChicago, Spring 2025

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Hours spent: 10

(Please let us know how many hours in total you spent on this assignment so we can calibrate for future assignments. Your feedback is always welcome!)

Open in Colab

Instructions

This homework focuses on themes in lectures 10-12 on coordinate ascent variational inference (CAVI), admixture models, and Poisson matrix factorization.

For reference, this homework is a close adaption of [HW5 for Scott Linderman's STATS 305C](#).

Assignment is due **Wednesday May 14, 11:59pm** on GradeScope.

Background

Poisson matrix factorization (PMF) is a mixed membership model like LDA, and it has close ties to non-negative factorization of count matrices. Let $\mathbf{X} \in \mathbb{N}^{N \times M}$ denote a count matrix with entries $x_{n,m}$. We model each entry as a Poisson random variable,

$$x_{n,m} \sim \mathrm{Po}\Big(\boldsymbol{\theta}_n^\top \boldsymbol{\phi}_m\Big) = \mathrm{Po}\Big(\sum_{k=1}^K \theta_{n,k}\phi_{m,k}\Big),$$

where $\boldsymbol{\theta}_n \in \mathbb{R}_+^K$ and $\boldsymbol{\phi}_m \in \mathbb{R}_+^K$ are *non-negative* feature vectors for row n and column m, respectively.

PMF has been used for recommender systems, aka collaborative filtering. In a recommender system, the rows correspond to users, the columns to items, and the entries $x_{n,m}$ to how much user n liked item m (on a scale of $0, 1, 2, \ldots$ stars, for example). The K feature dimensions capture different aspects of items that users may weight in their ratings.

Note that the Poisson rate must be non-negative. It is sufficient to ensure $\boldsymbol{\theta}_n$ and $\boldsymbol{\phi}_m$ are non-negative. To that end, PMF uses gamma priors,

$$\begin{aligned}\theta_{n,k} &\sim \mathrm{Ga}(\alpha_\theta, \beta_\theta)\\ \phi_{m,k} &\sim \mathrm{Ga}(\alpha_\phi, \beta_\phi),\end{aligned}$$

where $\alpha_\star$ and $\beta_\star$ are hyperparameters. When $\alpha_\star < 1$, the gamma distribution has a sharp peak at zero and the prior induces sparsity in the feature vectors.

Latent variable formulation

PMF can be rewritten in terms of a latent variable model. Note that,

$$\begin{aligned}x_{n,m} \sim \mathrm{Po}\Big(\sum_{k=1}^K \theta_{n,k}\phi_{m,k}\Big) \iff x_{n,m} &= \sum_{k=1}^K z_{n,m,k}\\ z_{n,m,k} &\sim \mathrm{Po}(\theta_{nk}\phi_{mk}) \quad \text{independently.}\end{aligned}$$

From this perspective, a user's rating of an item is a sum of ratings along each feature dimension, and each feature rating is an independent Poisson random variable.

The joint distribution is,

$$\begin{aligned}p(\mathbf{X}, \mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\Phi}) = &\Bigg[\prod_{n=1}^N \prod_{m=1}^M \mathbb{I}\Big[x_{n,m} = \sum_{k=1}^K z_{n,m,k}\Big] \prod_{k=1}^K \mathrm{Po}(z_{n,m,k} \mid \theta_{n,k}\phi_{m,k})\Bigg]\\ &\times \Bigg[\prod_{n=1}^N \prod_{k=1}^K \mathrm{Ga}(\theta_{n,k} \mid \alpha_\theta, \beta_\theta)\Bigg] \times \Bigg[\prod_{m=1}^M \prod_{k=1}^K \mathrm{Ga}(\phi_{m,k} \mid \alpha_\phi, \beta_\phi)\Bigg]\end{aligned}$$

where $\mathbf{Z} \in \mathbb{N}^{N \times M \times K}$ denotes the *tensor* of feature ratings, $\boldsymbol{\Theta} \in \mathbb{R}_+^{N \times K}$ is a matrix with rows $\boldsymbol{\theta}_n$, and $\boldsymbol{\Phi} \in \mathbb{R}_+^{M \times K}$ is a matrix with rows $\boldsymbol{\phi}_m$.

Setup

```
# from getpass import getpass
# token = getpass('Your token:')
```

```
# !git clone https://{token}@github.com/Zijiang-Yang/STAT_34800.git
# %cd /content/STAT34800/assignments/hw5
!git pull
```

```
Your token:··········
Cloning into 'STAT_34800'...
remote: Enumerating objects: 714, done.
remote: Counting objects: 100% (1/1), done.
remote: Total 714 (delta 0), reused 0 (delta 0), pack-reused 713 (from 2)
Receiving objects: 100% (714/714), 99.76 MiB | 21.74 MiB/s, done.
Resolving deltas: 100% (223/223), done.
[Errno 2] No such file or directory: '/content/STAT34800/assignments/hw5'
/content
fatal: not a git repository (or any of the parent directories): .git
```

```
import torch
from torch.distributions import Distribution, Gamma, Poisson, Multinomial
from torch.distributions.kl import kl_divergence

from tqdm.auto import trange

import matplotlib.pyplot as plt
import seaborn as sns
sns.set_context("notebook")
```

Problem 1: Conditional distributions [math]

Since this model is constructed from conjugate exponential family distributions, the conditionals are available in closed form. We will let $\mathbf{z}_{n,m} = (z_{n,m,1}, \ldots, z_{n,m,K})$.

Problem 1a: Derive the conditional for $\mathbf{z}_{n,m}$

Find the conditional density $p(\mathbf{z}_{n,m} \mid x_{n,m}, \boldsymbol{\theta}_n, \boldsymbol{\phi}_m)$.

双击（或按回车键）即可修改

Your answer here.

$$\begin{aligned} p(z_{n,m}|x_{n,m},\theta_n,\phi_m) &\propto p(x_{n,m}|z_{n,m},\theta_n,\phi_m)\cdot p(z_{n,m}|\theta_n,\phi_m) \\ &\propto 1(x_{n,m}=\sum_{k=1}^K z_{n,m,k})\prod_{k=1}^K Pois(z_{n,m,k}|\theta_{n,m},\phi_{m,k}) \\ &\propto 1(x_{n,m}=\sum_{k=1}^K z_{n,m,k})\prod_{k=1}^K \frac{(\theta_{n,k}\phi_{m,k})^{z_{n,m,k}}e^{-\theta_{n,k}\phi_{m,k}}}{z_{n,m,k}!} \end{aligned}$$

Denote $\pi_{n,m,k} = \frac{\theta_{n,k}\phi_{m,k}}{\sum_{k=1}^K \theta_{n,k}\phi_{m,k}}$

$$\begin{aligned} p(z_{n,m}|x_{n,m},\theta_n,\phi_m) &= Multinomial(z_{n,m};x_{n,m},\pi_{n,m}) \\ &= 1(x_{n,m}=\sum_{k=1}^K z_{n,m,k})\frac{x_{n,m}!}{\prod_{k=1}^K z_{n,m,k}}\prod_{k=1}^K \pi_{n,m,k}^{z_{n,m,k}} \end{aligned}$$

Problem 1b: Derive the conditional for $\theta_{n,k}$

Find the conditional density $p(\theta_{n,k} \mid \mathbf{Z}, \boldsymbol{\Phi})$.

Your answer here.

$$\begin{aligned} p(\theta_{n,k}|Z,\Phi) &\propto p(z_{n,1:M,k}|\theta_{n,k},\Phi)\cdot p(\theta_{n,k}) \\ &\propto \prod_{m=1}^M p(z_{n,m,k}|\theta_{n,k},\phi_{m,k})\cdot p(\theta_{n,k}) \\ &\propto \prod_{m=1}^M \frac{(\theta_{n,k}\phi_{m,k})^{z_{n,m,k}}e^{-\theta_{n,k}\phi_{m,k}}}{z_{n,m,k}!}\cdot \theta_{n,k}^{\alpha_\theta-1}e^{-\beta_\theta\theta_{n,k}} \\ &\propto \theta_{n,k}^{\sum_{m=1}^M z_{n,m,k}+\alpha_\theta-1}e^{-(\sum_{m=1}^M \phi_{m,k}+\beta_\theta)\theta_{n,k}} \\ &\propto Gamma(\theta_{n,k};\sum_{m=1}^M z_{n,m,k}+\alpha_\theta,\sum_{m=1}^M \phi_{m,k}+\beta_\theta) \end{aligned}$$

Denote $\alpha_m = \sum_{m=1}^M z_{n,m,k}+\alpha_\theta, \beta_m = \sum_{m=1}^M \phi_{m,k}+\beta_\theta$

$$p(\theta_{n,k}|Z,\Phi) = \frac{\beta_m^{\alpha_m}}{\Gamma(\alpha_m)}\theta_{n,k}^{\alpha_m-1}e^{-\beta_m\theta_{n,k}}$$

Problem 1c: Derive the conditional for $\phi_{m,k}$

Find the conditional density $p(\phi_{m,k} \mid \mathbf{Z}, \mathbf{\Theta})$.

Your answer here.

$$
\begin{aligned}
p(\phi_{m,k}|Z,\Phi) &\propto p(z_{1:N,m,k}|\phi_{m,k},\Theta)\cdot p(\phi_{m,k}) \\
&\propto \prod_{n=1}^{N} p(z_{n,m,k}|\theta_{n,k},\phi_{m,k})\cdot p(\phi_{m,k}) \\
&\propto \prod_{n=1}^{N} \frac{(\theta_{n,k}\phi_{m,k})^{z_{n,m,k}} e^{-\theta_{n,k}\phi_{m,k}}}{z_{n,m,k}!} \cdot \phi_{m,k}^{\alpha_\phi - 1} e^{-\beta_\phi \phi_{m,k}} \\
&\propto \phi_{m,k}^{\sum_{n=1}^{N} z_{n,m,k}+\alpha_\phi - 1} e^{-(\sum_{n=1}^{N}\theta_{n,k}+\beta_\theta)\phi_{m,k}} \\
&\propto Gamma(\sum_{n=1}^{N} z_{n,m,k}+\alpha_\phi, \sum_{n=1}^{N}\theta_{n,k}+\beta_\phi)
\end{aligned}
$$

Denote $\alpha_n = \sum_{n=1}^{N} z_{n,m,k}+\alpha_\phi, \beta_n = \sum_{n=1}^{N}\theta_{n,k}+\beta_\phi$

$$
p(\phi_{m,k}|Z,\Theta) = \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)}\phi_{m,k}^{\alpha_n-1}e^{-\beta_n\phi_{m,k}}
$$

Problem 2: Coordinate ascent variational inference [math]

We will perform inference in this model using a mean-field variational posterior which factorizes according to:

$$
\begin{aligned}
q(\mathbf{Z},\mathbf{\Phi},\mathbf{\Theta}) &= q(\mathbf{Z})q(\mathbf{\Phi})q(\mathbf{\Theta}) \\
&= \left[\prod_{n=1}^{N}\prod_{m=1}^{M} q(\mathbf{z}_{n,m})\right]\left[\prod_{n=1}^{N}\prod_{k=1}^{K} q(\theta_{n,k})\right]\left[\prod_{m=1}^{M}\prod_{k=1}^{K} q(\phi_{m,k})\right]
\end{aligned}
$$

The optimal mean field factors will have the same forms as the conditional distributions above.

Problem 2a: Derive the CAVI update for $q(\mathbf{z}_{n,m})$

Show that, fixing $q(\mathbf{\Phi})$ and $q(\mathbf{\Theta})$, the optimal $q(\mathbf{z}_{n,m})$ is given by:

$$
\begin{aligned}
q(\mathbf{z}_{n,m};\boldsymbol{\lambda}_{n,m}^{(z)}) &= \mathrm{Mult}(\mathbf{z}_{n,m}; x_{n,m}, \boldsymbol{\lambda}_{n,m}^{(z)}) \\
\log \lambda_{n,m,k}^{(z)} &= \mathbb{E}_q[\log\theta_{n,k} + \log\phi_{m,k}] + c
\end{aligned}
$$

Your answer here.

$$
\begin{aligned}
q(z_{n,m}) &\propto exp(E_{q_{\setminus z_{n,m}}}[logp(z_{n,m}|\Theta,\Phi,X)]) \\
&\propto 1(x_{n,m} = \sum_{k=1}^{K} z_{n,m,k}) exp(E_{q_{\setminus z_{n,m}}}[\sum_{k=1}^{K} z_{n,m,k} log\pi_{n,m,k} - \sum_{k=1}^{K} log z_{n,m,k}]) \\
&\propto 1(x_{n,m} = \sum_{k=1}^{K} z_{n,m,k}) exp(E_{q_{\setminus z_{n,m}}}[\sum_{k=1}^{K} z_{n,m,k}(log\theta_{n,k} + log\phi_{m,k} - log\sum\theta_{n,k}\phi_{m,k}) - \sum_{k=1}^{K} log z_{n,m,k}]) \\
&\propto 1(x_{n,m} = \sum_{k=1}^{K} z_{n,m,k}) exp(\sum_{k=1}^{K} z_{n,m,k} log\lambda_{n,m,k}^{(z)} - \sum_{k=1}^{K} log z_{n,m,k}) \\
&\propto Multinomial(z_{n,m}; x_{n,m}, \lambda_{n,m}^{(z)})
\end{aligned}
$$

Problem 2b: Derive the CAVI update for $q(\theta_{n,k})$

Show that, fixing $q(\mathbf{Z})$ and $q(\mathbf{\Phi})$, the optimal $q(\theta_{n,k})$ is given by:

$$
\begin{aligned}
q(\theta_{n,k};\lambda_{n,k,1}^{(\theta)},\lambda_{n,k,2}^{(\theta)}) &= \mathrm{Ga}(\theta_{n,k};\lambda_{n,k,1}^{(\theta)},\lambda_{n,k,2}^{(\theta)}) \\
\lambda_{n,k,1}^{(\theta)} &= \alpha_\theta + \sum_{m=1}^{M}\mathbb{E}_q[z_{n,m,k}] \\
\lambda_{n,k,2}^{(\theta)} &= \beta_\theta + \sum_{m=1}^{M}\mathbb{E}_q[\phi_{m,k}]
\end{aligned}
$$

Your answer here.

$$
\begin{aligned}
q(\theta_{n,k}) &= exp(E_{q_{\backslash \theta_{n,k}}} logp(\theta_{n,k}|Z,\Phi)) \\
&\propto exp(E_{q_{\backslash \theta_{n,k}}}(\sum_{m=1}^{M} z_{n,m,k} + \alpha_\theta - 1)log\theta_{n,k} - (\sum_{m=1}^{M} \phi_{m,k} + \beta_\theta)\theta_{n,k}) \\
&\propto exp((\lambda^{(\theta)}_{n,k,1} - 1)log\theta_{n,k} - \lambda^{(\theta)}_{n,k,2}\theta_{n,k}) \\
&= Gamma(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2})
\end{aligned}
$$

Problem 2c: Derive the CAVI update for $q(\phi_{m,k})$

Show that, fixing $q(\mathbf{Z})$ and $q(\mathbf{\Theta})$, the optimal $q(\phi_{m,k})$ is given by:

$$
\begin{aligned}
q(\phi_{m,k}; \lambda^{(\phi)}_{m,k,1}, \lambda^{(\phi)}_{m,k,2}) &= \mathrm{Ga}(\phi_{m,k}; \lambda^{(\phi)}_{m,k,1}, \lambda^{(\phi)}_{m,k,2}) \\
\lambda^{(\phi)}_{m,k,1} &= \alpha_\phi + \sum_{n=1}^{N} \mathbb{E}_q[z_{n,m,k}] \\
\lambda^{(\phi)}_{m,k,2} &= \beta_\phi + \sum_{n=1}^{N} \mathbb{E}_q[\theta_{n,k}]
\end{aligned}
$$

Your answer here.

$$
\begin{aligned}
q(\phi_{m,k}) &= exp(E_{q_{\backslash \phi_{m,k}}} logp(\phi_{m,k}|Z,\Theta)) \\
&\propto exp(E_{q_{\backslash \phi_{m,k}}}(\sum_{n=1}^{N} z_{n,m,k} + \alpha_\phi - 1)log\phi_{m,k} - (\sum_{n=1}^{N} \theta_{n,k} + \beta_\phi)\phi_{m,k}) \\
&\propto exp((\lambda^{(\phi)}_{m,k,1} - 1)log\phi_{m,k} - \lambda^{(\phi)}_{m,k,2}\phi_{m,k}) \\
&= Gamma(\phi_{m,k}; \lambda^{(\phi)}_{m,k,1}, \lambda^{(\phi)}_{m,k,2})
\end{aligned}
$$

Problem 2d: Find the expected sufficient statistics

To update the variational factors, we need the expectations $\mathbb{E}_q[z_{n,m,k}]$, $\mathbb{E}_q[\log\theta_{n,k} + \log\phi_{m,k}]$, $\mathbb{E}_q[\theta_{n,k}]$, and $\mathbb{E}_q[\phi_{m,k}]$. Assume that each factor follows the forms derived above. That is, assume $q(\mathbf{z}_{n,m})$ is multinomial with parameters $\lambda^{(z)}_{n,m}$ while $q(\theta_{n,k})$ and $q(\phi_{mk})$ are gamma with parameters $\left(\lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}\right)$ and $\left(\lambda^{(\phi)}_{m,k,1}, \lambda^{(\phi)}_{m,k,2}\right)$, respectively. Derive what each of these expectations are in closed form.

Your answer here.

$$
\begin{aligned}
E_q[z_{n,m,k}] &= x_{n,m}\lambda^{(z)}_{n,m,k} \\
E_q[log\theta_{n,k}] &= \psi(\lambda^{(\theta)}_{n,k,1}) - log\lambda^{(\theta)}_{n,k,2} \\
E_q[log\phi m,k] &= \psi(\lambda^{(\phi)}_{m,k,1}) - log\lambda^{(\phi)}_{m,k,2} \\
E_q[log\theta_{n,k} + log\phi m,k] &= \psi(\lambda^{(\theta)}_{n,k,1}) - log\lambda^{(\theta)}_{n,k,2} + \psi(\lambda^{(\phi)}_{m,k,1}) - log\lambda^{(\phi)}_{m,k,2} \\
E_q[\theta_{n,k}] &= \frac{\lambda^{(\theta)}_{n,k,1}}{\lambda^{(\theta)}_{n,k,2}} \\
E_q[\phi_{m,k}] &= \frac{\lambda^{(\phi)}_{m,k,1}}{\lambda^{(\phi)}_{m,k,2}}
\end{aligned}
$$

Here $\psi(x) = \frac{d}{dx}log\Gamma(x)$ is the digamma function.

Problem 3: Implement Coordinate Ascent Variational Inference [code]

First we'll give some helper functions and objects. Because PyTorch doesn't offer support for batched multinomial distributions in which the total counts differ (e.g. each $\mathbf{z}_{n,m}$ follows a multinomial distribution in which the total count is $x_{n,m}$), we have defined a `BatchedMultinomial` distribution for your convenience. This distribution doesn't support sampling, but will return the mean of each Multinomial variable in its batch. This is exactly what is needed for the CAVI updates.

```
def gamma_expected_log(gamma_distbn):
    """Helper function to compute the expectation of log(X) where X follows a
    gamma distribution.
    """
    return torch.digamma(gamma_distbn.concentration) - torch.log(gamma_distbn.rate)
```

```
class BatchedMultinomial(Multinomial):
    """
    Creates a Multinomial distribution parameterized by `total_count` and
    either `probs` or `logits` (but not both). The innermost dimension of
    `probs` indexes over categories. All other dimensions index over batches.

    The `probs` argument must be non-negative, finite and have a non-zero sum,
    and it will be normalized to sum to 1 along the last dimension. `probs` will
    return this normalized value. The `logits` argument will be interpreted as
    unnormalized log probabilities and can therefore be any real number. It will
    likewise be normalized so that the resulting probabilities sum to 1 along
    the last dimension. `logits` will return this normalized value.

    Args:
        total_count (Tensor): number of trials
        probs (Tensor): event probabilities
            Has shape total_count.shape + (num_categories,)
        logits (Tensor): event log probabilities (unnormalized)
            Has shape total_count.shape + (num_categories,)

    Note: this text is mostly from the PyTorch documentation for the
        Multinomial distribution
    """
    def __init__(self, total_count, probs=None, logits=None, validate_args=None):
        super().__init__(probs=probs, logits=logits, validate_args=validate_args)
        self.total_count = total_count

    @property
    def mean(self):
        return self.total_count[..., None] * self.probs
```

Problem 3a: Implement a CAVI update step

Using the update equations derived in Problem 2, complete the `cavi_step` function below.

Hint: Given a `Distribution` named `d`, `d.mean` returns the mean of that distribution.

```
def cavi_step(X, q_z, q_theta, q_phi, alpha_theta, beta_theta, alpha_phi, beta_phi):
    """One step of CAVI.

    Args:
        X: torch.tensor of shape (N, M)
        q_z: variational posterior over z, BatchedMultinomial distribution
        q_theta: variational posterior over theta, Gamma distribution
        q_phi: variational posterior over eta, Gamma distribution

    Returns:
        (q_z, q_theta, q_phi): Updated distributions after performing CAVI updates
    """
    ###
    # Your code here
    N, M = X.shape
    K = q_z.probs.shape[-1]

    E_log_theta = gamma_expected_log(q_theta)   # (N, K)
    E_log_phi = gamma_expected_log(q_phi)       # (M, K)

    # === Update q_z ===
    log_lambda_z = E_log_theta[:, None, :] + E_log_phi[None, :, :]   # (N, M, K)
    lambda_z = torch.softmax(log_lambda_z, dim=-1)                          # (N, M, K)
    q_z = BatchedMultinomial(total_count=X, logits=log_lambda_z)

    # === E_q[z] ===
    E_z = q_z.mean   # (N, M, K) = X[n,m] * lambda_z[n,m,k]

    # === Update q_theta ===
    lambda_theta_1 = alpha_theta + E_z.sum(dim=1)                         # (N, K)
    lambda_theta_2 = beta_theta + q_phi.mean.sum(dim=0)          # (N, K)
    q_theta = torch.distributions.Gamma(concentration=lambda_theta_1, rate=lambda_theta_2)

    # === Update q_phi ===
    lambda_phi_1 = alpha_phi + E_z.sum(dim=0)                                 # (M, K)
    lambda_phi_2 = beta_phi + q_theta.mean.sum(dim=0)              # (M, K)
    q_phi = torch.distributions.Gamma(concentration=lambda_phi_1, rate=lambda_phi_2)
    return q_z, q_theta, q_phi
```

Problem 3b: ELBO Calculation [math]

Recall that the evidence lower bound is defined as:

$$\mathcal{L}(q) = \mathbb{E}_q\left[\log p(\mathbf{X}, \mathbf{Z}, \mathbf{\Phi}, \mathbf{\Theta}) - \log q(\mathbf{Z}, \mathbf{\Phi}, \mathbf{\Theta})\right]$$

Assume that $q(\mathbf{Z})$ has support contained in $\{\mathbf{Z} : \sum_{k=1}^K z_{n,m,k} = x_{n,m} \text{ for all } n, m\}$. Show that we can rewrite $\mathcal{L}(q)$ as:

$$\mathcal{L}(q) = \mathbb{E}_q[\log p(\mathbf{Z} \mid \mathbf{\Theta}, \mathbf{\Phi}) - \log q(\mathbf{Z})] - \mathrm{KL}(q(\mathbf{\Theta})||p(\mathbf{\Theta})) - \mathrm{KL}(q(\mathbf{\Phi})||p(\mathbf{\Phi}))$$

Next, use that $q(\mathbf{z}_{n,m}; \boldsymbol{\lambda}^{(z)}_{n,m}) = \mathrm{Mult}(\mathbf{z}_{n,m}; x_{n,m}, \boldsymbol{\lambda}^{(z)}_{n,m})$ and by plug in the densities of the Poisson and Multinomial distributions to show that we have:

$$\begin{aligned}&\mathbb{E}_q[\log p(\mathbf{Z} \mid \mathbf{\Theta}, \mathbf{\Phi}) - \log q(\mathbf{Z})] = \\ &\quad \sum_{n=1}^N \sum_{m=1}^M \mathbb{E}_q\left[\sum_{k=1}^K -\theta_{n,k}\phi_{m,k} + z_{n,m,k}\log(\theta_{n,k}\phi_{m,k}) - z_{n,m,k}\log(\lambda^{(z)}_{n,m,k})\right] - \log(x_{n,m}!)\end{aligned}$$

Explain why we have:

$$\begin{aligned}&\mathbb{E}_q\left[-\theta_{n,k}\phi_{m,k} + z_{n,m,k}\log(\theta_{n,k}\phi_{m,k}) - z_{n,m,k}\log(\lambda^{(z)}_{n,m,k})\right] = \\ &\quad -\mathbb{E}_q\left[\theta_{n,k}\right]\mathbb{E}_q\left[\phi_{m,k}\right] + \mathbb{E}_q\left[z_{n,m,k}\right]\left(\mathbb{E}_q\left[\log(\theta_{n,k})\right] + \mathbb{E}_q\left[\log(\phi_{m,k})\right] - \log(\lambda^{(z)}_{n,m,k})\right)\end{aligned}$$

Your answer here.

1.

$$\begin{aligned}L(q) &= E_q[logp(X,Z,\Phi,\Theta) - logq(Z,\Phi,\Theta)] \\ &= E_q[logp(X|Z,\Phi,\Theta)p(Z,\Phi,\Theta) - logq(Z,\Phi,\Theta)] \\ &= E_q[logp(Z,\Phi,\Theta) - logq(Z,\Phi,\Theta)] \\ &= E_q[log(p(Z|\Phi,\Theta)p(\Phi)p(\Theta)) - log(q(Z)q(\Phi)q(\Theta))] \\ &= E_q[logp(Z|\Phi,\Theta) - logq(Z) + logp(\Phi) - logq(\Phi) + logp(\Theta) - logq(\Theta)] \\ &= E_q[logp(Z|\Phi,\Theta) - logq(Z) - KL(q(\Theta)||p(\Theta)) - KL(q(\Phi)||p(\Phi))]\end{aligned}$$

2.

$$logp(z_{n,m,k}|\theta_{n,k},\phi_{m,k}) = -\theta_{n,k}\phi_{m,k} + z_{n,m,k}log(\theta_{n,k}\phi_{m,k}) - logz_{n,m,k}!$$

$$logq(z_{n,m}) = logx_{n,m}! - \sum_{k=1}^K logz_{n,m,k}! + \sum_{k=1}^K z_{n,m,k}log\lambda^{(z)}_{n,m,k}$$

$$\begin{aligned}E_q[logp(Z|\Phi,\Theta) - logq(Z)] &= \sum_{n=1}^N\sum_{m=1}^M E_q[\sum_{k=1}^K(-\theta_{n,k}\phi_{m,k} + z_{n,m,k}log(\theta_{n,k}\phi_{m,k}) - logz_{n,m,k}!) - (logx_{n,m}! - \sum_{k=1}^K logz_{n,m,k}! + \sum_{k=1}^K z_{n,m,k}l \\ &= \sum_{n=1}^N\sum_{m=1}^M E_q[\sum_{k=1}^K(-\theta_{n,k}\phi_{m,k} + z_{n,m,k}log(\theta_{n,k}\phi_{m,k}) - z_{n,m,k}log\lambda^{(z)}_{n,m,k})] - log(x_{n,m})!\end{aligned}$$

3. Since $z_{n,m,k}$, $\theta_{n,k}$ and $\phi_{m,k}$ are independent under q, $\lambda^{(z)}_{n,m,k}$ is a constant, we have following equations:

$$\begin{aligned}E_q[\theta_{n,k}\phi_{m,k}] &= E_q[\theta_{n,k}]E_q[\phi_{m,k}] \\ E_q[z_{n,m,k}log\theta_{n,k}] &= E_q[z_{n,m,k}]E_q[log\theta_{n,k}] \\ E_q[z_{n,m,k}log\phi_{m,k}] &= E_q[z_{n,m,k}]E_q[log\phi_{m,k}] \\ E_q[z_{n,m,k}log\lambda^{(z)}_{n,m,k}] &= E_q[z_{n,m,k}]E_q[log\lambda^{(z)}_{n,m,k}]\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\mathbb{E}_q\left[-\theta_{n,k}\phi_{m,k} + z_{n,m,k}\log(\theta_{n,k}\phi_{m,k}) - z_{n,m,k}\log(\lambda^{(z)}_{n,m,k})\right] = \\ &\quad -\mathbb{E}_q\left[\theta_{n,k}\right]\mathbb{E}_q\left[\phi_{m,k}\right] + \mathbb{E}_q\left[z_{n,m,k}\right]\left(\mathbb{E}_q\left[\log(\theta_{n,k})\right] + \mathbb{E}_q\left[\log(\phi_{m,k})\right] - \log(\lambda^{(z)}_{n,m,k})\right)\end{aligned}$$

Problem 3c: Implement the ELBO [code]

Using our expression above, write a function which evaluates the evidence lower bound.

Hints:

- Use the `kl_divergence` function imported above to compute the KL divergence between two `Distributions` in the same family.
- Recall that for integers n, $\Gamma(n+1) = n!$ where Γ is the Gamma function. $\log\Gamma$ is implemented in PyTorch as `torch.lgamma`.

```
def elbo(X, q_z, q_theta, q_phi, p_theta, p_phi):
    """Compute the evidence lower bound.

    Args:
        X: torch.tensor of shape (N, M)
        q_z: variational posterior over z, BatchedMultinomial distribution
        q_theta: variational posterior over theta, Gamma distribution
        q_phi: variational posterior over eta, Gamma distribution
        p_theta: prior over theta, Gamma distribution
        p_phi: prior over eta, Gamma distribution

    Returns:
        elbo: torch.tensor of shape []
    """
```

```
    ###
    # Your code below

    N, M = X.shape
    K = q_z.probs.shape[-1]

    lambda_z = q_z.probs  # (N, M, K)
    E_z = q_z.mean  # (N, M, K)

    # Expectations of theta and phi
    E_theta = q_theta.mean  # (N, K)
    E_phi = q_phi.mean          # (M, K)
    E_log_theta = gamma_expected_log(q_theta)  # (N, K)
    E_log_phi = gamma_expected_log(q_phi)      # (M, K)              # (M, K)

    # Compute expected log p(z | theta, phi) - log q(z)
    E_theta_phi = E_theta.unsqueeze(1) * E_phi.unsqueeze(0)  # shape (N, M, K)
    E_log_theta_phi = E_log_theta.unsqueeze(1) + E_log_phi.unsqueeze(0)  # shape (N, M, K)
    log_lambda = lambda_z.log()  # shape (N, M, K)

    term1 = -E_theta_phi + E_z * (E_log_theta_phi - log_lambda)  # shape (N, M, K)
    term1_sum = term1.sum() - torch.lgamma(X + 1).sum()  # scalar

    # KL divergences
    kl_theta = kl_divergence(q_theta, p_theta).sum()
    kl_phi = kl_divergence(q_phi, p_phi).sum()

    elbo = term1_sum - kl_theta - kl_phi

    return elbo / torch.sum(X)
```

Implement CAVI loop [given]

Using your functions defined above, complete the function `cavi` below. `cavi` loops for some number of iterations, updating each of the variational factors in sequence and evaluating the ELBO at each step.

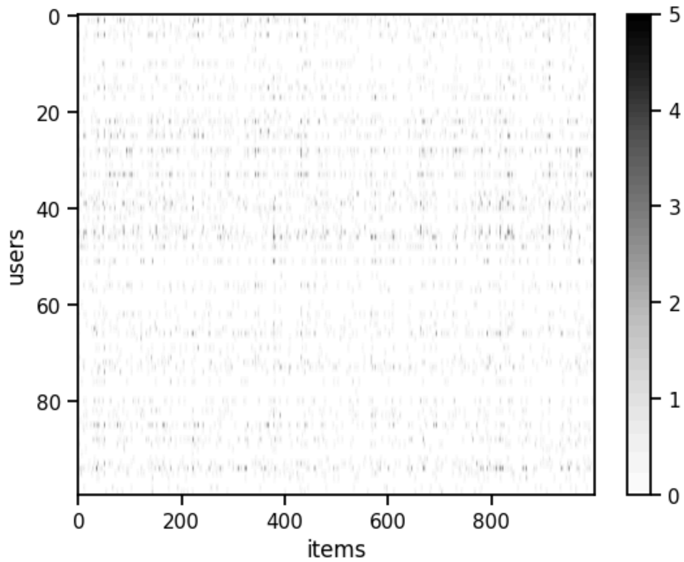
```
from torch.distributions import Uniform

def cavi(data,
         num_factors=10,
         num_iters=100,
         tol=1e-5,
         alpha_theta=0.1,
         beta_theta=1.0,
         alpha_phi=0.1,
         beta_phi=1.0,
         seed=0
        ):
    """Run coordinate ascent VI for Poisson matrix factorization.

    Args:

    Returns:
        elbos, (q_z, q_theta, q_phi):
    """
    data = data.float()
    N, M = data.shape
    K = num_factors          # short hand

    # Initialize the variational posteriors.
    q_phi = Gamma(Uniform(0.5 * alpha_phi, 1.5 * alpha_phi).sample((M, K)),
                  Uniform(0.5 * beta_phi, 1.5 * beta_phi).sample((M, K)))
    q_theta = Gamma(Uniform(0.5 * alpha_theta, 1.5 * alpha_theta).sample((N, K)),
                    Uniform(0.5 * beta_theta, 1.5 * beta_theta).sample((N, K)))
    q_z = BatchedMultinomial(data, logits=torch.zeros((N, M, K)))

    p_theta = Gamma(alpha_theta, beta_theta)
    p_phi = Gamma(alpha_phi, beta_phi)

    # Run CAVI
    elbos = [elbo(data, q_z, q_theta, q_phi, p_theta, p_phi)]
    for itr in trange(num_iters):
        q_z, q_theta, q_phi = cavi_step(data, q_z, q_theta, q_phi,
                                        alpha_theta, beta_theta,
                                        alpha_phi, beta_phi)

        elbos.append(elbo(data, q_z, q_theta, q_phi, p_theta, p_phi))
    return torch.tensor(elbos), (q_z, q_theta, q_phi)
```

Test your implementation on a toy dataset

To check your implementation is working properly, we will fit a mean-field variational posterior using data sampled from the true model.

```
# Constants
N = 100     # num "users"
M = 1000 # num "items"
K = 5        # number of latent factors

# Hyperparameters
alpha = 0.1   # sparse gamma prior with mean alpha/beta
beta = 1.0

# Sample data from the model
torch.manual_seed(305)
theta = Gamma(alpha, beta).sample(sample_shape=(N, K))
phi = Gamma(alpha, beta).sample(sample_shape=(M, K))
data = Poisson(theta @ phi.T).sample()

print(data.shape)
# Plot the data matrix
plt.imshow(data, aspect="auto", vmax=5, cmap="Greys")
plt.xlabel("items")
plt.ylabel("users")
plt.colorbar()

print("Max data:   ", data.max())
print("num zeros: ", torch.sum(data == 0))
```

```
torch.Size([100, 1000])
Max data:   tensor(14.)
num zeros:  tensor(95568)
```

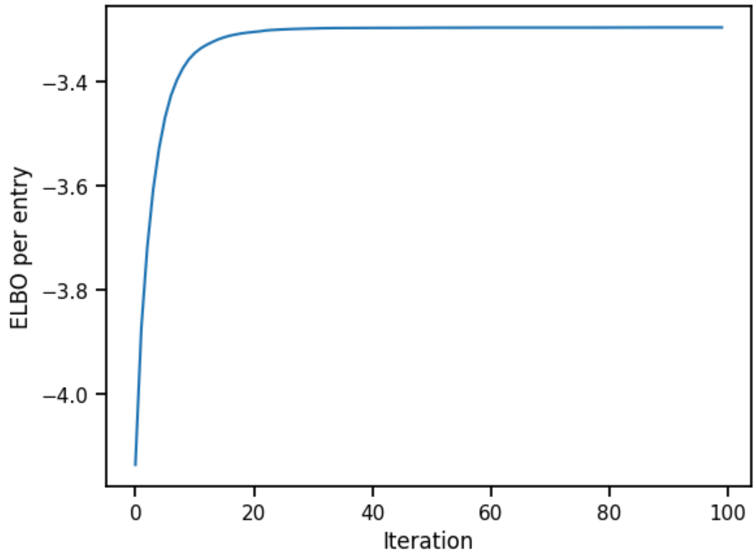



```
elbos, (q_z, q_theta, q_phi) = cavi(data)
```

100% 100/100 [00:13<00:00, 4.23it/s]

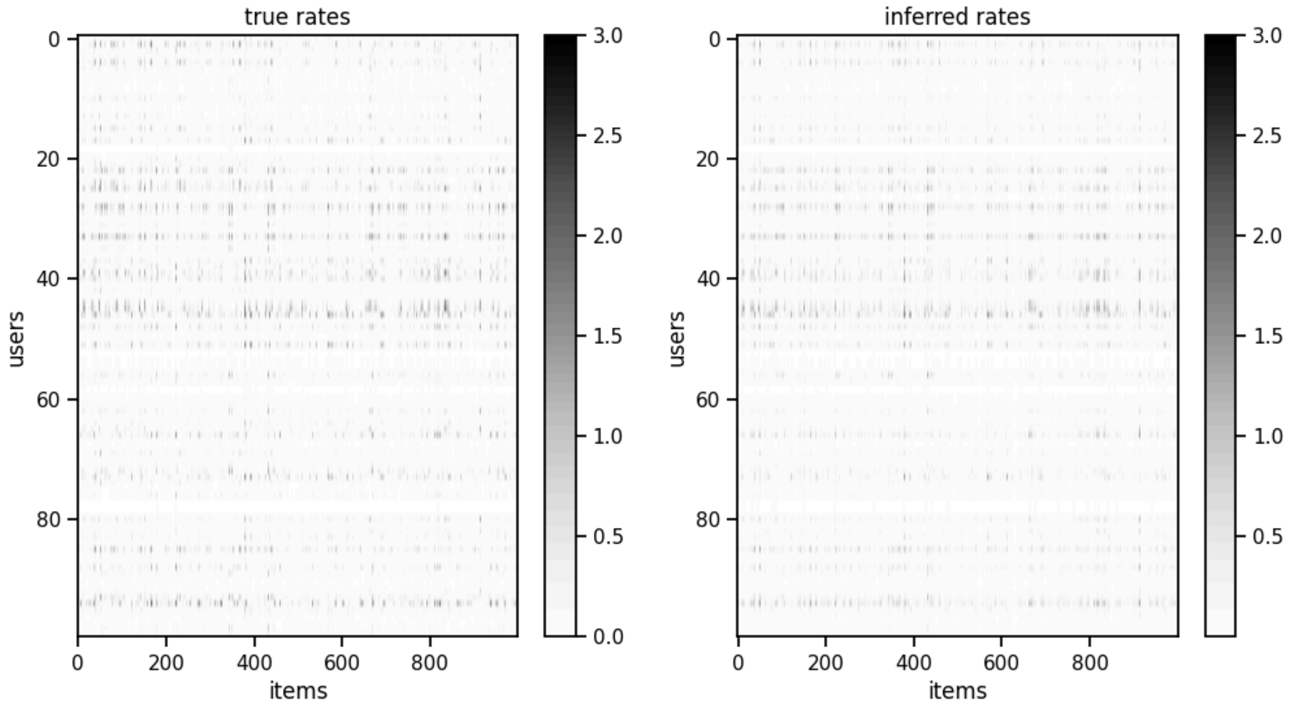
```
plt.plot(elbos[1:])
plt.xlabel("Iteration")
plt.ylabel("ELBO per entry")
```

```
Text(0, 0.5, 'ELBO per entry')
```




```
true_rates = theta @ phi.T
inf_rates = q_theta.mean @ q_phi.mean.T

# Plot the data matrix
plt.figure(figsize=(12, 6))
plt.subplot(1, 2, 1)
plt.imshow(true_rates, aspect="auto", vmax=3, cmap="Greys")
plt.xlabel("items")
plt.ylabel("users")
plt.title("true rates")
plt.colorbar()

plt.subplot(1, 2, 2)
plt.imshow(inf_rates, aspect="auto", vmax=3, cmap="Greys")
plt.xlabel("items")
plt.ylabel("users")
plt.title("inferred rates")
plt.colorbar()
```

```
<matplotlib.colorbar.Colorbar at 0x7c7c3a278690>
```



Problem 4: Run your code on a downsampled LastFM dataset

Next, we will use data gathered from [Last.FM](https://www.last.fm) users to fit a PMF model. We use a downsampled version of the [Last.FM-360K users](https://www.upf.edu) dataset. This dataset records how many times each user played an artist's songs. We downsample the data to include only the 2000 most popular artists, as measured by how many users listened to the artist at least once, and the 1000 most prolific users, as measured by how many artists they have listened to.

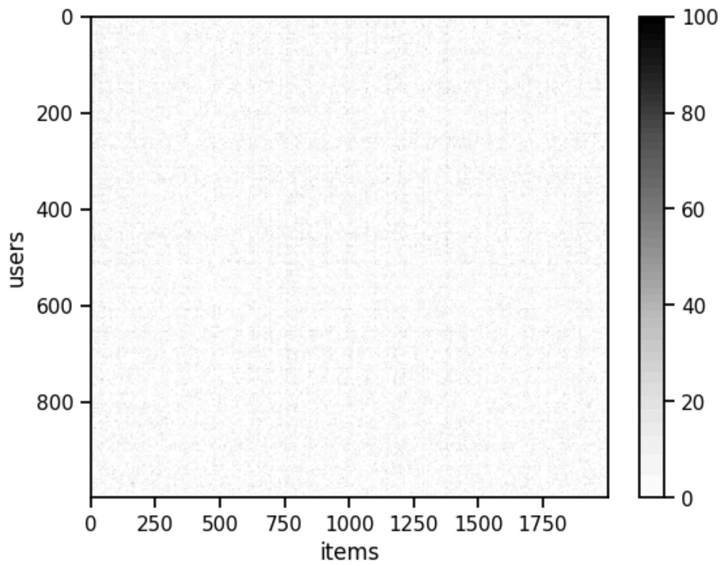
In the code below , we use `lfm` to represent the data matrix X in the model. That is, `lfm[n, d]` denotes how many times the `n`-th user played a song by the `d`-th artist.

```
import pandas as pd

lfm_df = pd.read_csv('/content/STAT_34800/assignments/hw5/subsampled_last_fm.csv')
lfm = lfm_df.pivot_table(index='UserID', columns='ItemID', values='Count', aggfunc=sum)\
        .fillna(0).astype(int).to_numpy()
lfm = torch.tensor(lfm, dtype=torch.int)
print(lfm.shape)
```

```
torch.Size([999, 2000])
<ipython-input-14-1b8bfd5340bc>:4: FutureWarning: The provided callable <built-in function sum> is currently using DataFrameGroupBy.sum. In a future
  lfm = lfm_df.pivot_table(index='UserID', columns='ItemID', values='Count', aggfunc=sum)\
```

```
plt.imshow(lfm, aspect="auto", vmax=100, cmap="Greys")
plt.xlabel("items")
plt.ylabel("users")
plt.colorbar()
```

```
<matplotlib.colorbar.Colorbar at 0x7c7c3e9f6250>
```



Using the code below, run coordinate ascent variational inference on this dataset. Our implementation takes around 10-15 minutes to finish, and achieves a rescaled ELBO of around -2.

```
elbos, (q_z, q_theta, q_phi) = cavi(lfm,
          num_factors=40, #40
          num_iters=200, #200
          alpha_theta=1.,
          beta_theta=0.5,
          alpha_phi=1.,
          beta_phi=0.5)
```

```
100%                                200/200 [24:06<00:00, 6.90s/it]
```

```
print(elbos[-1])
```

```
tensor(-1.9671)
```

Investigate "genres"

The columns of $\mathbf{H}$ correspond to weights on artists. Intuitively, each of the K columns should put weight on subsets of artists that are often played together. We might think of these columns as reflecting different "genres" of music. The code below the top 10 artists for a few of these columns.

```
# Find the 10 most used genres
genre_loading = q_theta.mean.sum(0)
genre_order = torch.argsort(genre_loading, descending=True)
```

```
# Print the top 10 artists for each of the top 10 genres
for genre in genre_order[:10]:
    print("genre ", genre)
    artist_idx = torch.argsort(q_phi.mean[:, genre],
                               descending=True)[:10].numpy()
    subset = lfm_df[lfm_df['ItemID'].isin(artist_idx)]
    print(subset[['ItemID', 'Artist']].drop_duplicates())
    print("")
```

```
1852     1279          snow patrol
29203     676            the stars

genre  tensor(32)
       ItemID             Artist
56       1377        the beatles
67         26              queen
106       730       eric clapton
116        63     frédéric chopin
183       911            madonna
280      1705               abba
324      1370  the rolling stones
330        13      elvis presley
565      1254                 u2
876       832       mushroomhead
18201    1377            beatles

genre  tensor(28)
      ItemID           Artist
170       37        daft punk
218     1007     depeche mode
222      574          blondie
865     1494        the knife
951     1930        fever ray
973     1379   crystal castles
992      657  yeah yeah yeahs
1303    1400         ladytron
1443    1313  franz ferdinand
1447    1947          justice

genre  tensor(5)
       ItemID                     Artist
59        645                miles davis
64       1672                johnny cash
69        857                  bob dylan
277      1043            various artists
293      1533                  tom waits
297       163  nick cave and the bad seeds
324      1370         the rolling stones
330        13              elvis presley
343       755               leonard cohen
443       163    nick cave & the bad seeds
913       848          bruce springsteen
18256    1043                       v.a.

genre  tensor(0)
       ItemID          Artist
328      1819      bad brains
452      1791      have heart
454       820    comeback kid
788      1692         ramones
821       277          rancid
860      1735            nofx
885      1281     against me!
894       142    bad religion
1364     1170     black flag
11243    1769   sick of it all
```

Problem 4a

Inspect the data either using the csv file or the pandas dataframe and choose a user who has listened to artists you recognize. If you are not familiar with any of the artists, use the listener with UserID 349, who mostly listens to hip-hop artists. For the particular user n you choose, find the 10 artists who are predicted to have the most plays by sorting the vector of mean song counts predicted by the model, i.e. the n^{th} row of $\mathbb{E}_q[\mathbf{\Theta}\mathbf{\Phi}^\top]$. Are these artists you would expect the user would enjoy? Are there any artists that the user has not listened to?

Hint: Use `torch.argsort(..., descending=True)` *to return the indices of the largest elements of a vector in descending order.*

```
###
user_id = 349

# 2. Compute expected play counts
expected_counts = q_theta.mean @ q_phi.mean.T   # shape (num_users, num_artists)

# 3. Get the predicted top 10 artists for the user
user_counts = expected_counts[user_id]   # shape (num_artists,)
top_artist_indices = torch.argsort(user_counts, descending=True)[:10]
```

```
top_artists = lfm_df[lfm_df['ItemID'].isin(top_artist_indices.numpy())]
print(top_artists[['ItemID', 'Artist']].drop_duplicates())

##
```

```
       ItemID                         Artist
336       442                      ghostface
817      1150             a tribe called quest
1419     1500                         common
1431      964                      the roots
2297     1639                            nas
2312     1213                        mos def
2693     1598                        j dilla
3613     1834                         madlib
3835       99                   wu-tang clan
4726      442                 ghostface killah
10000      99                   wu tang clan
15054    1319                   blessthefall
24609     964  the roots featuring d'angelo
```

Yes, I expect the user would enjoy their music. Yes, I believe there are some artists that he has not listened to.

Problem 5: Reflections

Problem 5a

Discuss one advantage and one disadvantage of fitting a posterior using variational inference vs. sampling from the posterior using MCMC.

Advantage: VI turns posterior approximation into an optimization problem, which is typically much faster than MCMC method.

Disadvantage: VI approximates the true posterior using a simplified family of distributions, which can lead to biased estimates—especially in multimodal or highly skewed posteriors. MCMC, by contrast, gives asymptotically exact samples from the true posterior.

Problem 5b

First, explain why the assumption that $\mathbf{Z}, \mathbf{\Phi}$ and $\mathbf{\Theta}$ are independent in the posterior will never hold.

Next, recall that maximizing the ELBO is equivalent to minimizing the KL divergence between the approximate posterior and the true posterior. In general, how will the approximate posterior differ from the true posterior, given that the variational family does not include the true posterior?

Your answer here.

1.

Since $z_{n,m,k} \sim \mathrm{Po}(\theta_{nk}\phi_{mk}), x_{n,m} = \sum_k z_{n,m,k}$, the posterior distribution of z with observations of X is still a function of Θ and Φ. Intuitively, if we observe high counts for some (n, m), this will constrain the likely combinations of θ_{nk} and ϕ_{mk} — they must co-adjust to explain the data. So they must not be independent in posterior.

2.

When the true posterior is not in the variational family, variational inference finds the closest match (via minimizing $\mathrm{KL}(q|p)$), which leads to:

Underestimated uncertainty: It avoids low-probability regions, focusing on one mode and shrinking variance.

Missing dependencies: Independence assumptions ignore correlations present in the true posterior.

Bias: The approximation may favor one mode, especially if the true posterior is multi-modal.

Problem 5c

Suppose we are using this model to recommend new items to users. Describe one improvement that could be made to the model which you think would lead to better recommendations.

Your answer here.

One possible improvement to the model is to incorporate side information about users or items, such as user demographics or item metadata.

This additional context can help the model better capture user preferences and item similarities, especially in sparse data settings or cold-start scenarios. For example, side information can be incorporated as priors or additional inputs in a context-aware Bayesian factor model, leading to more accurate and personalized recommendations.

Submission Instructions

Formatting: check that your code does not exceed 80 characters in line width. If you're working in Colab, you can set *Tools → Settings → Editor → Vertical ruler column* to 80 to see when you've exceeded the limit.

Download your notebook in .ipynb format and use the following commands to convert it to PDF:

```
jupyter nbconvert --to pdf hw5_yourname.ipynb
```

Dependencies:

- `nbconvert`: If you're using Anaconda for package management,

```
conda install -c anaconda nbconvert
```