The Problem Setup

The subject of this study is the **steady-state problem** of **2D incompressible fluid flow**. The design variable is γ (where $\gamma = 1$ represents fluid and $\gamma = 0$ represents solid). The objective is to **minimize the dissipated energy** of the structure, while the state variables $\boldsymbol{u}, p$ must satisfy the *Brinkman equation*. The sensitivity analysis in this manuscript is derived using the continuous adjoint method.

Objective (The dissipated energy):

$$\Psi = \frac{1}{2}\int_{\Omega}\left[\left(\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) + \alpha(\gamma)u_i u_i\right] d\Omega$$

In this equation, $\alpha(\gamma)$ represents the Brinkman penalty factor and is also a spatially-varying parameter. Its expression is shown below:

$$\alpha(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \Omega_f \\ \infty & \text{if } \boldsymbol{x} \in \Omega_s \end{cases} \qquad\qquad \alpha(\gamma(\boldsymbol{x})) = \alpha_{max}\frac{q_\alpha(1-\gamma(\boldsymbol{x}))}{q_\alpha + \gamma(\boldsymbol{x})}$$

Governing Equations (Brinkman equation):

$$-\nabla \cdot \boldsymbol{u} = 0$$

$$\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\nabla\cdot\nabla\boldsymbol{u} + \nabla p + \alpha\boldsymbol{u} = 0$$

Where $\alpha\boldsymbol{u}$ represents a resistance term which combines solid and fluid into NS equation.

Boundary Conditions:

$$\boldsymbol{u} = \boldsymbol{u}_{in} \qquad on\ \Gamma_{in}$$

$$\boldsymbol{u} = 0 \qquad on\ \Gamma_{wall}$$

$$\left(-p\boldsymbol{I} + \frac{\eta(\nabla\boldsymbol{u})}{\rho}\right)\cdot\boldsymbol{n} = p_0\cdot\boldsymbol{n} \qquad on\ \Gamma_{out}$$

The interpretation of Sensitivity Analysis

To ensure that the structure satisfies the constraint equations and boundary conditions while updating the design variables in the direction of decreasing the objective function, we need an augmented Lagrangian function.

The Lagrangian Function $\boldsymbol{\mathcal{L}(u, p, \gamma)}$:

$$\mathcal{L} = \Psi + \int_{\Omega} (\nabla \cdot \boldsymbol{u})\lambda_1 d\Omega + \int_{\Omega} [\rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \mu\nabla \cdot \nabla\boldsymbol{u} + \nabla p + \alpha\boldsymbol{u}] \cdot \boldsymbol{\lambda_2} d\Omega + \int_{\Gamma_{in}} (\boldsymbol{u} - \boldsymbol{u_{in}}) \cdot \boldsymbol{\lambda_{in}} d\Gamma$$

$$+ \int_{\Gamma_{\text{wall}}} \boldsymbol{u} \cdot \boldsymbol{\lambda_{wall}} d\Gamma + \int_{\Gamma_{\text{out}}} \left(-p\boldsymbol{I} + \frac{\eta(\nabla\boldsymbol{u})}{\rho}\right) \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_{out}} d\Gamma$$

The adjoint variables $\boldsymbol{\lambda}$ are: $\lambda_1, \lambda_{2,1}, \lambda_{2,2}, \lambda_{in,1}, \lambda_{in,2}, \lambda_{wall,1}, \lambda_{wall,2}, \lambda_{out,1}, \lambda_{out,2}$.

Here, we simplify it to (u represents the state variables, and $\boldsymbol{R}$ represents all the governing equations and boundary conditions):

$$\mathcal{L} = \Psi(\mathrm{u}, \gamma) + \boldsymbol{\lambda^T} \boldsymbol{R}(u, \gamma)$$

Variation of the Lagrangian Function

$$\delta\mathcal{L} = \frac{\partial\Psi}{\partial u}\frac{\partial u}{\partial\gamma}\delta\gamma + \frac{\partial\Psi}{\partial\gamma}\delta\gamma + \boldsymbol{\lambda^T}\left(\frac{\partial\mathbf{R}}{\partial u}\frac{\partial u}{\partial\gamma}\delta\gamma + \frac{\partial\mathbf{R}}{\partial\gamma}\delta\gamma\right)$$

$$= \underbrace{\left(\frac{\partial\Psi}{\partial u} + \boldsymbol{\lambda^T}\frac{\partial\mathbf{R}}{\partial u}\right)}_{\substack{\textbf{Adjoint equations} \\ \textbf{and Boundary conditions}}}\frac{\partial u}{\partial\gamma}\delta\gamma + \underbrace{\left(\frac{\partial\Psi}{\partial\gamma} + \boldsymbol{\lambda^T}\frac{\partial\mathbf{R}}{\partial\gamma}\right)}_{\textbf{Sensitivity}}\delta\gamma$$

At this point, if the adjoint variables $\boldsymbol{\lambda}$ satisfy:

$$\frac{\partial\Psi}{\partial u} + \boldsymbol{\lambda^T}\frac{\partial\mathbf{R}}{\partial u} = 0$$

The above is thus the **adjoint equation**.

Then the derivative of the Lagrangian function with respect to the design variables, which is the **sensitivity** of this optimization problem, is:

$$\frac{\partial\mathcal{L}}{\partial\gamma} = \frac{\partial\Psi}{\partial\gamma} + \boldsymbol{\lambda^T}\frac{\partial\mathbf{R}}{\partial\gamma}$$

Therefore, the subsequent work is divided into two parts:

1. Compute **the Gateaux derivatives (GD)** of the objective function, governing equations, and boundary conditions with respect to the state variables ($\frac{\partial\Psi}{\partial u}, \frac{\partial\mathbf{R}}{\partial u}$) to obtain **the adjoint equations and adjoint boundary conditions**, ultimately calculating **the adjoint variables**.
2. Compute the derivative of the objective function, governing equations, and boundary conditions with respect to the design variables ($\frac{\partial\Psi}{\partial\gamma}, \frac{\partial\mathbf{R}}{\partial\gamma}$) to obtain **the sensitivity of the optimization problem**, which will be used to update the design variables.

The GD of $\mathcal{L}$ with respect to $\boldsymbol{u}, \boldsymbol{p}$

1. The GD of the mass conservation equation with respect to $\boldsymbol{u}$

The mass conservation equation and its tensor form:

$$\textit{Symbolic form} \qquad \nabla \cdot \boldsymbol{u} = 0$$

$$\textit{Tensor form} \qquad \frac{\partial u_i}{\partial x_i} = 0 \quad i = 1{,}2$$

Multiplying by the adjoint variables, i.e., the Lagrange multipliers, we get:

$$R_1(\lambda_1, \boldsymbol{u}) = \int_\Omega \lambda_1 \cdot \left(\frac{\partial u_i}{\partial x_i}\right) d\Omega = -\int_\Omega \frac{\partial \lambda_1}{\partial x_i} \cdot u_i d\Omega + \int_\Gamma \lambda_1 u_i n_i d\Gamma$$

The GD of the velocity $\boldsymbol{u}$ is:

$$\delta_u R_1(\lambda_1, \boldsymbol{u}) = -\int_\Omega \frac{\partial \lambda_1}{\partial x_i} \cdot \delta u_i d\Omega + \int_\Gamma \lambda_1 \delta u_i n_i d\Gamma$$

δu_i is the perturbation of u_i

The symbolic expression:

$$\delta_u R_1(\lambda_1, \boldsymbol{u}) = -\int_\Omega \nabla \lambda_1 \cdot \delta \boldsymbol{u} d\Omega + \int_\Gamma \lambda_1 \boldsymbol{n} \cdot \delta \boldsymbol{u} d\Gamma$$

2. The GD of the Brinkman equation with respect to the state field u&p

The Brinkman equation:

$$R_{2,i} = \left(\rho u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i}\right) + \alpha(\gamma) u_i = 0$$

therefore:

$$\int_\Omega \lambda_{2,i} R_{2,i} d\Omega = \int_\Omega \left[\left(\rho u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i}\right) + \alpha(\gamma) u_i\right] \lambda_{2,i} d\Omega$$

$$= \underbrace{\int_\Omega \rho \lambda_{2,i} u_j \frac{\partial u_i}{\partial x_j} d\Omega}_{\text{Convective term}} + \underbrace{\int_\Omega \left[-\lambda_{2,i} \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right)\right] d\Omega}_{\text{Viscous term}} + \underbrace{\int_\Omega \lambda_{2,i} \frac{\partial p}{\partial x_i} d\Omega}_{\text{Pressure gradient term}} + \underbrace{\int_\Omega \alpha(\gamma) \lambda_{2,i} u_i d\Omega}_{\text{Darcy term}}$$

2.1 The GD of convective term:

The convective term, after applying Gauss's theorem, becomes:

$$\delta_u \int_\Omega \rho \lambda_{2,i} u_j \frac{\partial u_i}{\partial x_j} d\Omega = \int_\Omega \rho \lambda_{2,i} \delta u_j \frac{\partial u_i}{\partial x_j} d\Omega + \int_\Omega \rho \lambda_{2,i} u_j \frac{\partial (\delta u_i)}{\partial x_j} d\Omega$$

$$= \int_\Omega \rho \lambda_{2,i} \delta u_j \frac{\partial u_i}{\partial x_j} d\Omega + \int_\Gamma \rho \lambda_{2,i} u_j n_j \delta u_i d\Gamma - \int_\Omega \rho \frac{\partial}{\partial x_j}\left(\lambda_{2,i} u_j\right) \delta u_i d\Omega$$

$$= \int_\Omega \rho \lambda_{2,i} \delta u_j \frac{\partial u_i}{\partial x_j} d\Omega + \int_\Gamma \rho \lambda_{2,i} u_j n_j \delta u_i d\Gamma - \int_\Omega \rho \frac{\partial \lambda_{2,i}}{\partial x_j} u_j \delta u_i d\Omega - \int_\Omega \rho \frac{\partial u_j}{\partial x_j} \lambda_{2,i} \delta u_i d\Omega$$

$$\because \nabla\cdot\boldsymbol{u}=0, \therefore\ \frac{\partial u_j}{\partial x_j}=0$$

$$\therefore\ \delta_u\int_\Omega \rho\lambda_{2,i}u_j\frac{\partial u_i}{\partial x_j}d\Omega=\int_\Omega \rho\lambda_{2,i}\delta u_j\frac{\partial u_i}{\partial x_j}d\Omega+\int_\Gamma \rho\lambda_{2,i}u_jn_j\delta u_i d\Gamma-\int_\Omega \rho\frac{\partial \lambda_{2,i}}{\partial x_j}u_j\delta u_i d\Omega$$

$$=\int_\Omega \rho\left(\lambda_{2,i}\delta u_j\frac{\partial u_i}{\partial x_j}-\frac{\partial \lambda_{2,i}}{\partial x_j}u_j\delta u_i\right)d\Omega+\int_\Gamma \rho\lambda_{2,i}u_jn_j\delta u_i d\Gamma$$

$$=\int_\Omega (\rho\nabla\boldsymbol{u}^{\boldsymbol{T}}\cdot\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u}-\rho\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{u}\cdot\delta\boldsymbol{u})d\Omega+\int_\Gamma \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u}d\Gamma$$

$$=\int_\Omega (\rho\nabla\boldsymbol{u}^{\boldsymbol{T}}\cdot\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u}-\rho(\mathbf{u}\cdot\nabla)\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u})d\Omega+\int_\Gamma \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u}d\Gamma$$

2.2 The GD of viscous term:

The viscous term, after applying Gauss's theorem, becomes:

$$\int_\Omega \lambda_{2,i}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)d\Omega=\int_\Gamma \mu\frac{\partial u_i}{\partial x_j}\lambda_{2,i}n_j d\Gamma-\int_\Omega \mu\frac{\partial u_i}{\partial x_j}\frac{\partial \lambda_{2,i}}{\partial x_j}d\Omega$$

$$=\int_\Gamma \mu\frac{\partial u_i}{\partial x_j}\lambda_{2,i}n_j d\Gamma-\int_\Gamma \mu\frac{\partial \lambda_{2,i}}{\partial x_j}u_in_j d\Gamma+\int_\Omega u_i\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \lambda_{2,i}}{\partial x_j}\right)d\Omega$$

Therefore, the GD of the viscous term is:

$$\delta_u\left(\int_\Omega \lambda_{2,i}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)d\Omega\right)=\int_\Gamma \mu\frac{\partial(\delta u_i)}{\partial x_j}\lambda_{2,i}n_j d\Gamma-\int_\Gamma \mu\frac{\partial \lambda_{2,i}}{\partial x_j}\delta u_in_j d\Gamma+\int_\Omega \delta u_i\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \lambda_{2,i}}{\partial x_j}\right)d\Omega$$

$$=\int_\Gamma \mu(\nabla(\delta\boldsymbol{u})\cdot\boldsymbol{n})\cdot\boldsymbol{\lambda_2}d\Gamma-\int_\Gamma \mu\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{n}\cdot\delta\boldsymbol{u}d\Gamma+\int_\Omega \mu(\nabla\cdot\nabla)\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u}d\Omega$$

2.3 The GD of the pressure gradient term:

$$\int_\Omega \frac{\partial p}{\partial x_i}\lambda_{2,i}d\Omega=\int_\Gamma p\lambda_{2,i}n_i d\Gamma-\int_\Omega \frac{\partial \lambda_{2,i}}{\partial x_i}p d\Omega$$

$$\delta_p\left(\int_\Omega \frac{\partial p}{\partial x_i}\lambda_{2,i}d\Omega\right)=\int_\Gamma \lambda_{2,i}n_i\delta p d\Gamma-\int_\Omega \frac{\partial \lambda_{2,i}}{\partial x_i}\delta p d\Omega=\int_\Gamma \boldsymbol{\lambda_2}\cdot\boldsymbol{n_i}\delta p d\Gamma-\int_\Omega \nabla\cdot\boldsymbol{\lambda_2}\delta p d\Gamma$$

2.4 The GD of the darcy term:

$$\delta_u\int_\Omega \boldsymbol{\lambda_2}\cdot\alpha(\gamma)\cdot\boldsymbol{u}d\Omega=\int_\Omega \alpha(\gamma)\lambda_{2,i}\delta u_i d\Omega=\int_\Omega \alpha(\gamma)\boldsymbol{\lambda_2}\cdot\delta\boldsymbol{u}d\Omega$$

3. The GD of the Inlet BC:

$$\int_{\Gamma} u_i - u_{i_{in}} d\Gamma = 0$$

After multiplying by the Lagrange multiplier $\boldsymbol{\lambda_{in}}$:

$$\int_{\Gamma} \left(u_i - u_{i_{in}}\right) \cdot \lambda_{i_{in}} d\Gamma = 0$$

The GD:

$$\int_{\Gamma_{\mathrm{in}}} \lambda_{i_{in}} \delta u_i d\Gamma = \int_{\Gamma_{in}} \boldsymbol{\lambda}_{in} \cdot \delta \boldsymbol{u} d\Gamma \quad i = 1,2$$

4. The GD of the Wall BC:

Tensor form

$$\delta \int_{\Gamma_{\mathrm{wall}}} \lambda_{i_{wall}} u_i d\Gamma = \int_{\Gamma} \lambda_{i_{wall}} \delta u_i d\Gamma \quad i = 1,2$$

Symbolic form

$$\int_{\Gamma_{wall}} \boldsymbol{\lambda}_{wall} \cdot \delta \boldsymbol{u} d\Gamma$$

5. The GD of the Outlet BC:

$$\left[-p\boldsymbol{I} + \frac{\eta(\nabla \boldsymbol{u})}{\rho}\right] \cdot \boldsymbol{n} = p_0 \cdot \boldsymbol{n} \quad p_0 = 0$$

$$\int_{\Gamma_{\mathrm{out}}} \left[-p\boldsymbol{I} + \frac{\eta(\nabla \boldsymbol{u})}{\rho}\right] \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_{out}} d\Gamma = \int_{\Gamma_{out}} \left(-p + \frac{\eta}{\rho}\frac{\partial u_j}{\partial x_i}\right) n_i \lambda_{j_{out}} d\Gamma$$

The GD of velocity:

$$\delta_u \int_{\boldsymbol{\Gamma_{out}}} \left(-p + \frac{\eta}{\rho}\frac{\partial u_j}{\partial x_i}\right) n_i \lambda_{out} d\Gamma = \int_{\boldsymbol{\Gamma_{out}}} \frac{\eta}{\rho}\frac{\partial(\delta u_j)}{\partial x_i} n_i \lambda_{j_{out}} d\Gamma = \int_{\Gamma_{out}} \mu \nabla \delta \boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda} d\Gamma$$

The GD of pressure:

$$\delta_{\mathrm{p}} \int_{\Gamma_{out}} \left(-p\boldsymbol{I} + \frac{\eta}{\rho}\frac{\partial u_j}{\partial x_i}\right) n_i \lambda_{i_{out}} d\Gamma = \int_{\Gamma_{out}} -\lambda_{i_{out}} n_i \delta p d\Gamma = \int_{\Gamma_{\mathrm{out}}} -\boldsymbol{\lambda_{out}} \cdot \boldsymbol{n} \delta p d\Gamma$$

6. The GD of the Dissipated Energy

$$\Psi = \frac{1}{2}\int_{\Omega}\left(\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) + \alpha(\boldsymbol{x})u_i u_i d\Omega$$

6.1 The GD of 1st term of DE of velocity:

$$\delta\left(\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right) = \mu\frac{\partial(\delta u_i)}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_i}{\partial x_j}\frac{\partial(\delta u_i)}{\partial x_j} = 2\mu\frac{\partial u_i}{\partial x_j}\frac{\partial(\delta u_i)}{\partial x_j}$$

Therefore:

$$\delta\left(\int_{\Omega}\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}d\Omega\right) = \int_{\Omega}2\mu\frac{\partial u_i}{\partial x_j}\frac{\partial(\delta u_i)}{\partial x_j}d\Omega = -\int_{\Omega}2\mu\frac{\partial^2 u_i}{\partial x_j^2}\delta u_i d\Omega + \int_{\Gamma}2\mu\frac{\partial u_i}{\partial x_j}n_j\delta u_i d\Gamma$$

$$= -\int_{\Omega}2\mu(\nabla\cdot\nabla)\boldsymbol{u}\cdot\delta\boldsymbol{u}d\Omega + \int_{\Gamma}2\mu\nabla\boldsymbol{u}\cdot\boldsymbol{n}\cdot\delta\boldsymbol{u}d\Gamma$$

6.2 The GD of 2nd term of DE of velocity

$$\delta\left(\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) = \mu(\frac{\partial(\delta u_i)}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial(\delta u_j)}{\partial x_i})$$

$$\delta_u\left(\int_{\Omega}\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}d\Omega\right) = \int_{\Omega}\mu\left(\frac{\partial(\delta u_i)}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial(\delta u_j)}{\partial x_i}\right)d\Omega$$

$$= -\int_{\Omega}\mu\left[\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}\right)\delta u_j + \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)\delta u_i\right]d\Omega + \int_{\Gamma}\mu\left(\frac{\partial u_i}{\partial x_j}n_i\delta u_j + \frac{\partial u_j}{\partial x_i}n_j\delta u_i\right)d\Gamma$$

Where:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}\right)\delta u_j = \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right)\delta u_1 + \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2}\right)\delta u_2 + \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}\right)\delta u_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_2}\right)\delta u_2$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\delta u_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\delta u_2 = 0 \quad \left(\because \nabla\cdot\boldsymbol{u} = \boldsymbol{0} \therefore \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0\right)$$

Therefore:

$$\delta\left(\int_{\Omega}\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}d\Omega\right) = -\int_{\Omega}\mu\left[\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}\right)\delta u_j + \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)\delta u_i\right]d\Omega + \int_{\Gamma}\mu\left(\frac{\partial u_i}{\partial x_j}n_i\delta u_j + \frac{\partial u_j}{\partial x_i}n_j\delta u_i\right)d\Gamma$$

$$= -\int_{\Omega}2\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}\right)\delta u_j d\Omega + \int_{\Gamma}2\mu\left(\frac{\partial u_i}{\partial x_j}n_i\delta u_j\right)d\Gamma = \int_{\Gamma}2\mu\left(\frac{\partial u_i}{\partial x_j}n_i\delta u_j\right)d\Gamma$$

$$= \int_{\Gamma}2\mu\nabla\boldsymbol{u}^{\boldsymbol{T}}\cdot\boldsymbol{n}\cdot\delta\boldsymbol{u}d\Gamma$$

6.3 The GD of 3rd term of DE of velocity:

$$\delta(\alpha(\gamma)u_i u_i) = 2\alpha(\gamma)u_i\delta u_i$$

Integral form:

$$\int_\Omega 2\alpha(\gamma)u_i\delta u_i d\Omega$$

The final form:

$$\Psi_u = \delta\left(\frac{1}{2}\int_\Omega\left(\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) + \alpha(\gamma)u_i u_i d\Omega\right)$$

$$= -\int_\Omega \mu\frac{\partial^2 u_i}{\partial x_j^2}\delta u_i d\Omega + \int_\Gamma \mu\frac{\partial u_i}{\partial x_j}n_j\delta u_i d\Gamma + \int_\Gamma \mu\frac{\partial u_i}{\partial x_j}n_i\delta u_j d\Gamma + \int_\Omega \alpha(\gamma)u_i\delta u_i d\Omega$$

$$= \int_\Omega\left(\alpha(\gamma)u_i - \mu\frac{\partial^2 u_i}{\partial x_j^2}\right)\delta u_i d\Omega + \int_\Gamma \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)n_j\delta u_i d\Gamma$$

$$= \int_\Omega [\alpha(\gamma)\boldsymbol{u} - \mu(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u}]\cdot\boldsymbol{\delta u}d\Omega + \int_\Gamma \mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T})\cdot\boldsymbol{n}\cdot\delta\boldsymbol{u}d\Gamma$$

Where:

$$\int_\Gamma 2\mu\frac{\partial u_i}{\partial x_j}n_j\delta u_i d\Gamma = \int_\Gamma 2\mu\left(\frac{\partial u_1}{\partial x_1}n_1\delta u_1 + \frac{\partial u_1}{\partial x_2}n_2\delta u_1 + \frac{\partial u_2}{\partial x_1}n_1\delta u_2 + \frac{\partial u_2}{\partial x_2}n_2\delta u_2\right)d\Gamma = \int_\Gamma 2\mu\nabla\boldsymbol{u}\cdot\boldsymbol{n}\cdot\delta\boldsymbol{u}d\Gamma$$

$$\int_\Gamma 2\mu\left(\frac{\partial u_i}{\partial x_j}n_i\delta u_j\right)d\Gamma = \int_\Gamma 2\mu\left(\frac{\partial u_j}{\partial x_i}n_j\delta u_i\right)d\Gamma$$

$$= \int_\Gamma 2\mu\left(\frac{\partial u_1}{\partial x_1}n_1\delta u_1 + \frac{\partial u_1}{\partial x_2}n_1\delta u_2 + \frac{\partial u_2}{\partial x_1}n_2\delta u_1 + \frac{\partial u_2}{\partial x_2}n_2\delta u_2\right)d\Gamma = \int_\Gamma 2\mu\nabla\boldsymbol{u^T}\cdot\boldsymbol{n}\cdot\delta\boldsymbol{u}d\Gamma$$

The GD of $\boldsymbol{\mathcal{L}}$ with respect to $\boldsymbol{u}$:

$$L_u = \Psi_u + \int_\Omega -\frac{\partial\lambda_1}{\partial x_i}\cdot\delta u_i d\Omega + \int_\Gamma \lambda_1\delta u_i n_i d\Gamma + \int_\Omega \rho\left(-\frac{\partial\lambda_{2,i}}{\partial x_j}u_j\delta u_i + \lambda_{2,i}\delta u_j\frac{\partial u_i}{\partial x_j}\right)d\Omega + \int_\Gamma \rho\lambda_{2,i}u_j n_j\delta u_i d\Gamma$$

$$-\int_\Gamma \mu\frac{\partial(\delta u_i)}{\partial x_j}\lambda_{2,i}n_j d\Gamma + \int_\Gamma \mu\frac{\partial\lambda_{2,i}}{\partial x_j}\delta u_i n_j d\Gamma - \int_\Omega \delta u_i\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\lambda_{2,i}}{\partial x_j}\right)d\Omega + \int_\Omega \alpha(\gamma)\lambda_{2,i}\delta u_i d\Omega$$

$$+\int_\Gamma \lambda_{i_{in}}\cdot\delta u_i d\Gamma + \int_\Gamma \lambda_{i_{wall}}\delta u_i d\Gamma + \int_{\boldsymbol{\Gamma_{out}}} \frac{\eta}{\rho}\frac{\partial(\delta u_j)}{\partial x_i}n_i\lambda_{j_{out}}d\Gamma$$

$$= \Psi_u + \int_\Omega -\nabla\lambda_1 \cdot \delta\boldsymbol{u} d\Omega + \int_\Gamma \lambda_1 \boldsymbol{n} \cdot \delta\boldsymbol{u} d\Gamma + \int_\Omega (\rho\nabla\boldsymbol{u}^{\boldsymbol{T}} \cdot \boldsymbol{\lambda_2} \cdot \delta\boldsymbol{u} - \rho(\mathbf{u}\cdot\nabla)\boldsymbol{\lambda_2} \cdot \delta\boldsymbol{u}) d\Omega$$
$$+ \int_\Gamma \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} \cdot \delta\boldsymbol{u} d\Gamma - \int_\Gamma \mu(\nabla\delta\boldsymbol{u}\cdot\boldsymbol{n}) \cdot \boldsymbol{\lambda_2} d\Gamma + \int_\Gamma \mu\nabla\boldsymbol{\lambda_2} \cdot \boldsymbol{n} \cdot \delta\boldsymbol{u} d\Gamma - \int_\Omega \mu(\nabla\cdot\nabla)\boldsymbol{\lambda_2} \cdot \delta\boldsymbol{u} d\Omega$$
$$+ \int_\Omega \alpha(\gamma)\boldsymbol{\lambda_2} \cdot \delta\boldsymbol{u} d\Omega + \int_{\Gamma_{in}} \boldsymbol{\lambda}_{in} \cdot \delta\boldsymbol{u} d\Gamma + \int_{\Gamma_{wall}} \boldsymbol{\lambda}_{wall} \cdot \delta\boldsymbol{u} d\Gamma + \int_{\Gamma_{out}} \mu\nabla\delta\boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda} d\Gamma$$
$$= \Psi_u + \int_\Omega [-\nabla\lambda_1 + \rho(\nabla\boldsymbol{u}^{\boldsymbol{T}} \cdot \boldsymbol{\lambda_2} - \rho(\mathbf{u}\cdot\nabla)\boldsymbol{\lambda_2}) - \mu(\nabla\cdot\nabla)\boldsymbol{\lambda_2} + \alpha(\gamma)\boldsymbol{\lambda_2}] \cdot \delta\boldsymbol{u} d\Omega$$
$$+ \int_\Gamma [\lambda_1\boldsymbol{n} + \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} + \mu\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{n}] \cdot \delta\boldsymbol{u} d\Gamma - \int_\Gamma \mu(\nabla\delta\boldsymbol{u}\cdot\boldsymbol{n}) \cdot \boldsymbol{\lambda_2} d\Gamma + \int_{\Gamma_{in}} \boldsymbol{\lambda}_{in} \cdot \delta\boldsymbol{u} d\Gamma$$
$$+ \int_{\Gamma_{wall}} \boldsymbol{\lambda}_{wall} \cdot \delta\boldsymbol{u} d\Gamma + \int_{\Gamma_{out}} \mu\nabla\delta\boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_{out}} d\Gamma$$

Substituting the GD of the dissipated energy with respect to velocity, we get:

$$L_u = \underbrace{\int_\Omega \left(\alpha(\gamma)u_i - 2\mu\frac{\partial^2 u_i}{\partial x_j^2}\right)\delta u_i d\Omega + \int_\Gamma \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j \delta u_i d\Gamma}_{\Psi_u} + \int_\Omega -\frac{\partial\lambda_1}{\partial x_i} \cdot \delta u_i d\Omega + \int_\Gamma \lambda_1 \delta u_i n_i d\Gamma$$
$$+ \int_\Omega \rho\left(-\frac{\partial\lambda_{2,i}}{\partial x_j} u_j \delta u_i + \lambda_{2,i}\delta u_j \frac{\partial u_i}{\partial x_j}\right) d\Omega + \int_\Gamma \rho\lambda_{2,i} u_j n_j \delta u_i d\Gamma - \int_\Gamma \mu\frac{\partial(\delta u_i)}{\partial x_j}\lambda_{2,i} n_j d\Gamma$$
$$+ \int_\Gamma \mu\frac{\partial\lambda_{2,i}}{\partial x_j}\delta u_i n_j d\Gamma - \int_\Omega \delta u_i \frac{\partial}{\partial x_j}\left(\mu\frac{\partial\lambda_{2,i}}{\partial x_j}\right) d\Omega + \int_\Omega \alpha(\gamma)\lambda_{2,i}\delta u_i d\Omega + \int_\Gamma \lambda_{i_{in}} \cdot \delta u_i d\Gamma$$
$$+ \int_\Gamma \lambda_{i_{wall}} \delta u_i d\Gamma + \int_{\boldsymbol{\Gamma_{out}}} \frac{\eta}{\rho}\frac{\partial(\delta u_j)}{\partial x_i} n_i \lambda_{j_{out}} d\Gamma$$
$$= \int_\Omega [\alpha(\gamma)\boldsymbol{u} - \mu(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u}] \cdot \delta\boldsymbol{u} d\Omega + \int_\Gamma \mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T}) \cdot \boldsymbol{n} \cdot \delta\boldsymbol{u} d\Gamma$$
$$+ \int_\Omega [-\nabla\lambda_1 + \rho(\nabla\boldsymbol{u}^{\boldsymbol{T}} \cdot \boldsymbol{\lambda_2} - (\mathbf{u}\cdot\nabla)\boldsymbol{\lambda_2}) - \mu(\nabla\cdot\nabla)\boldsymbol{\lambda_2} + \alpha(\gamma)\boldsymbol{\lambda_2}] \cdot \delta\boldsymbol{u} d\Omega$$
$$+ \int_\Gamma [\lambda_1\boldsymbol{n} + \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} + \mu\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{n}] \cdot \delta\boldsymbol{u} d\Gamma - \int_\Gamma \mu(\nabla\delta\boldsymbol{u}\cdot\boldsymbol{n}) \cdot \boldsymbol{\lambda_2} d\Gamma + \int_{\Gamma_{in}} \boldsymbol{\lambda}_{in} \cdot \delta\boldsymbol{u} d\Gamma$$
$$+ \int_{\Gamma_{wall}} \boldsymbol{\lambda}_{wall} \cdot \delta\boldsymbol{u} d\Gamma + \int_{\Gamma_{out}} \mu\nabla\delta\boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_{out}} d\Gamma$$
$$= \int_\Omega [\alpha(\gamma)\boldsymbol{u} - \mu(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \nabla\lambda_1 + \rho(\nabla\boldsymbol{u}^{\boldsymbol{T}} \cdot \boldsymbol{\lambda_2} - (\mathbf{u}\cdot\nabla)\boldsymbol{\lambda_2}) - \mu(\nabla\cdot\nabla)\boldsymbol{\lambda_2} + \alpha(\gamma)\boldsymbol{\lambda_2}] \cdot \delta\boldsymbol{u} d\Omega$$
$$+ \int_\Gamma [\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T}) + \lambda_1\boldsymbol{I} + \mu\nabla\boldsymbol{\lambda_2}] \cdot \boldsymbol{n} \cdot \delta\boldsymbol{u} d\Gamma + \int_\Gamma \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} \cdot \delta\boldsymbol{u} d\Gamma + \int_\Gamma -\mu\nabla\delta\boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_2} d\Gamma$$
$$+ \int_{\Gamma_{in}} \boldsymbol{\lambda}_{in} \cdot \delta\boldsymbol{u} d\Gamma + \int_{\Gamma_{wall}} \boldsymbol{\lambda}_{wall} \cdot \delta\boldsymbol{u} d\Gamma + \int_{\Gamma_{out}} \mu\nabla\delta\boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_{out}} d\Gamma$$

The GD of $\mathcal{L}$ with respect to p:

$$L_p = \Psi_p + \overbrace{\int_{\Gamma} \lambda_{2,i} n_i \delta p d\Gamma - \int_{\Omega} \frac{\partial \lambda_{2,i}}{\partial x_i} \delta p d\Omega}^{R_2,p} + \overbrace{\int_{\Gamma_{out}} -\lambda_{i_{out}} n_i \delta p d\Gamma}^{\Gamma_{out,p}}$$
$$= \int_{\Gamma} \boldsymbol{\lambda_2} \cdot \boldsymbol{n} \delta p d\Gamma - \int_{\Omega} \nabla \cdot \boldsymbol{\lambda_2} \delta p d\Gamma + \int_{\Gamma_{\text{out}}} -\boldsymbol{\lambda_{out}} \cdot \boldsymbol{n} \delta p d\Gamma$$
$$= -\int_{\Omega} \nabla \cdot \boldsymbol{\lambda_2} \delta p d\Gamma + \int_{\Gamma_{\text{in\&wall}}} \boldsymbol{\lambda_2} \cdot \boldsymbol{n} \delta p d\Gamma + \int_{\Gamma_{\text{out}}} (\boldsymbol{\lambda_2} - \boldsymbol{\lambda_{out}}) \cdot \boldsymbol{n} \delta p d\Gamma$$

Adjoint Equation:

The adjoint equation derived from the pressure：

$$L_p = -\int_{\Omega} \nabla \cdot \boldsymbol{\lambda_2} \delta p d\Gamma + \int_{\Gamma_{\text{in\&wall}}} \boldsymbol{\lambda_2} \cdot \boldsymbol{n} \delta p d\Gamma + \int_{\Gamma_{\text{out}}} (\boldsymbol{\lambda_2} - \boldsymbol{\lambda_{out}}) \cdot \boldsymbol{n} \delta p d\Gamma$$

Symbolic form	Tensor form	Zone
$\nabla \cdot \boldsymbol{\lambda_2} = 0$	$\frac{\partial \lambda_{2,1}}{\partial x_1} + \frac{\partial \lambda_{2,2}}{\partial x_2} = 0$	Ω
$\boldsymbol{\lambda_2} \cdot \boldsymbol{n} = 0$	$\lambda_{2,1} n_1 + \lambda_{2,2} n_2 = 0$	$\Gamma_{in\&wall}$
$\boldsymbol{\lambda_2} - \boldsymbol{\lambda_{out}} = 0$	$\lambda_{2,1} = \lambda_{out,1}$ $\lambda_{2,2} = \lambda_{out,2}$	Γ_{out}

The adjoint equation derived from the velocity:

$$L_u = \int_{\Omega} [\alpha(\gamma)\boldsymbol{u} - \mu(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \nabla\lambda_1 + \rho(\nabla \boldsymbol{u^T} \cdot \boldsymbol{\lambda_2} - (\mathbf{u} \cdot \nabla)\boldsymbol{\lambda_2}) - \mu(\nabla \cdot \nabla)\boldsymbol{\lambda_2} + \alpha(\gamma)\boldsymbol{\lambda_2}] \cdot \delta \boldsymbol{u} d\Omega$$
$$+ \int_{\Gamma} [2\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T}) + \lambda_1 \boldsymbol{I} + \mu \nabla \boldsymbol{\lambda_2}] \cdot \boldsymbol{n} \cdot \delta \boldsymbol{u} d\Gamma + \int_{\Gamma} \rho(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{\lambda_2} \cdot \delta \boldsymbol{u} d\Gamma$$
$$+ \int_{\Gamma} -\mu \nabla \delta \boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_2} d\Gamma + \int_{\Gamma_{in}} \boldsymbol{\lambda}_{in} \cdot \delta \boldsymbol{u} d\Gamma + \int_{\Gamma_{wall}} \boldsymbol{\lambda}_{wall} \cdot \delta \boldsymbol{u} d\Gamma + \int_{\Gamma_{out}} \mu \nabla \delta \boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda_{out}} d\Gamma$$
$$= \int_{\Omega} [\alpha(\gamma)\boldsymbol{u} - \mu(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \nabla\lambda_1 + \rho(\nabla \boldsymbol{u^T} \cdot \boldsymbol{\lambda_2} - (\mathbf{u} \cdot \nabla)\boldsymbol{\lambda_2}) - \mu(\nabla \cdot \nabla)\boldsymbol{\lambda_2} + \alpha(\gamma)\boldsymbol{\lambda_2}] \cdot \delta \boldsymbol{u} d\Omega$$
$$+ \int_{\Gamma_{in}} [\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T}) \cdot \boldsymbol{n} + \lambda_1 \boldsymbol{n} + \mu \nabla \boldsymbol{\lambda_2} \cdot \boldsymbol{n} + \rho(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{\lambda_2} + \boldsymbol{\lambda}_{in}] \cdot \delta \boldsymbol{u} d\Gamma$$
$$+ \int_{\Gamma_{wall}} [\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T}) \cdot \boldsymbol{n} + \lambda_1 \boldsymbol{n} + \mu \nabla \boldsymbol{\lambda_2} \cdot \boldsymbol{n} + \rho(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{\lambda_2} + \boldsymbol{\lambda}_{wall}] \cdot \delta \boldsymbol{u} d\Gamma$$
$$+ \int_{\Gamma_{\text{out}}} [\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T}) \cdot \boldsymbol{n} + \lambda_1 \boldsymbol{n} + \mu \nabla \boldsymbol{\lambda_2} \cdot \boldsymbol{n} + \rho(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{\lambda_2}] \cdot \delta \boldsymbol{u} d\Gamma$$
$$+ \int_{\Gamma_{\text{out}}} \mu \nabla \delta \boldsymbol{u} \cdot \boldsymbol{n} \cdot (\boldsymbol{\lambda_{out}} - \boldsymbol{\lambda}_2) d\Gamma + \int_{\Gamma_{\text{in\&wall}}} -\mu \nabla \delta \boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\lambda}_2 d\Gamma$$

The adjoint equation and boundary equation:

$$\alpha(\gamma)\boldsymbol{u} - \mu(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \nabla\lambda_1 + \rho(\nabla\boldsymbol{u}^{\boldsymbol{T}}\cdot\boldsymbol{\lambda_2} - (\mathbf{u}\cdot\nabla)\boldsymbol{\lambda_2}) - \mu(\nabla\cdot\nabla)\boldsymbol{\lambda_2} + \alpha(\gamma)\boldsymbol{\lambda_2} = 0 \qquad \Omega$$

$$\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T})\cdot\boldsymbol{n} + \lambda_1\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{n} + \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} + \boldsymbol{\lambda}_{in} = 0 \qquad \Gamma_{in}$$

$$\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T})\cdot\boldsymbol{n} + \lambda_1\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{n} + \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} + \boldsymbol{\lambda}_{wall} = 0 \qquad \Gamma_{wall}$$

$$\mu(\boldsymbol{\nabla u} + \boldsymbol{\nabla u^T})\cdot\boldsymbol{n} + \lambda_1\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda_2}\cdot\boldsymbol{n} + \rho(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\lambda_2} = 0 \qquad \Gamma_{\mathrm{out}}$$

Tensor form:

$$\left.\begin{aligned}
&\alpha(\gamma)u_1 - \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u_1 - \frac{\partial\lambda_1}{\partial x_1} + \rho\left(\lambda_{2,1}\frac{\partial u_1}{\partial x_1} + \lambda_{2,2}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial\lambda_{2,1}}{\partial x_1} - u_2\frac{\partial\lambda_{2,1}}{\partial x_2}\right)\\
&\qquad - \mu\frac{\partial^2\lambda_{2,1}}{\partial x_1^2} - \mu\frac{\partial^2\lambda_{2,1}}{\partial x_2^2} + \alpha(\gamma)\lambda_{2,1} = 0\\
&\alpha(\gamma)u_2 - \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u_2 - \frac{\partial\lambda_1}{\partial x_2} + \rho\left(\lambda_{2,1}\frac{\partial u_1}{\partial x_2} + \lambda_{2,2}\frac{\partial u_2}{\partial x_2} - u_1\frac{\partial\lambda_{2,2}}{\partial x_1} - u_2\frac{\partial\lambda_{2,2}}{\partial x_2}\right)\\
&\qquad - \mu\frac{\partial^2\lambda_{2,2}}{\partial x_1^2} - \mu\frac{\partial^2\lambda_{2,2}}{\partial x_2^2} + \alpha(\gamma)\lambda_{2,2} = 0
\end{aligned}\right\} \quad \Omega$$

$$\left.\begin{aligned}
&\mu\left(\frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_1}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_1}n_2\right) + \lambda_1 n_1 + \mu\left(\frac{\partial\lambda_{2,1}}{\partial x_1}n_1 + \frac{\partial\lambda_{2,1}}{\partial x_2}n_2\right)\\
&\qquad + \rho(u_1n_1 + u_2n_2)\lambda_{2,1} + \lambda_{in,1} = 0\\
&\mu\left(\frac{\partial u_2}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_2}n_1 + \frac{\partial u_2}{\partial x_2}n_2\right) + \lambda_1 n_2 + \mu\left(\frac{\partial\lambda_{2,2}}{\partial x_1}n_1 + \frac{\partial\lambda_{2,2}}{\partial x_2}n_2\right)\\
&\qquad + \rho(u_1n_1 + u_2n_2)\lambda_{2,2} + \lambda_{in,2} = 0
\end{aligned}\right\} \quad \Gamma_{in}$$

$$\left.\begin{aligned}
&\mu\left(\frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_1}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_1}n_2\right) + \lambda_1 n_1 + \mu\left(\frac{\partial\lambda_{2,1}}{\partial x_1}n_1 + \frac{\partial\lambda_{2,1}}{\partial x_2}n_2\right)\\
&\qquad + \rho(u_1n_1 + u_2n_2)\lambda_{2,1} + \lambda_{wall,1} = 0 \quad in\ \Gamma_{wall}\\
&\mu\left(\frac{\partial u_2}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_2}n_1 + \frac{\partial u_2}{\partial x_2}n_2\right) + \lambda_1 n_2 + \mu\left(\frac{\partial\lambda_{2,2}}{\partial x_1}n_1 + \frac{\partial\lambda_{2,2}}{\partial x_2}n_2\right)\\
&\qquad + \rho(u_1n_1 + u_2n_2)\lambda_{2,2} + \lambda_{wall,2} = 0
\end{aligned}\right\} \quad \Gamma_{wall}$$

$$\left.\begin{aligned}
&\mu\left(\frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_1}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_1}n_2\right) + \lambda_1 n_1 + \mu\left(\frac{\partial\lambda_{2,1}}{\partial x_1}n_1 + \frac{\partial\lambda_{2,1}}{\partial x_2}n_2\right)\\
&\qquad + \rho(u_1n_1 + u_2n_2)\lambda_{2,1} = 0\\
&\mu\left(\frac{\partial u_2}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_2}n_1 + \frac{\partial u_2}{\partial x_2}n_2\right) + \lambda_1 n_2 + \mu\left(\frac{\partial\lambda_{2,2}}{\partial x_1}n_1 + \frac{\partial\lambda_{2,2}}{\partial x_2}n_2\right)\\
&\qquad + \rho(u_1n_1 + u_2n_2)\lambda_{2,2} = 0
\end{aligned}\right\} \quad \Gamma_{\mathrm{out}}$$

The Final Form of the Adjoint Equation

$$\frac{\partial \lambda_{2,1}}{\partial x_1} + \frac{\partial \lambda_{2,2}}{\partial x_2} = 0$$

$$\alpha(\gamma)u_1 - \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u_1 - \frac{\partial \lambda_1}{\partial x_1} + \rho\left(\lambda_{2,1}\frac{\partial u_1}{\partial x_1} + \lambda_{2,2}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial \lambda_{2,1}}{\partial x_1} - u_2\frac{\partial \lambda_{2,1}}{\partial x_2}\right) - \mu\frac{\partial^2 \lambda_{2,1}}{\partial x_1^2} - \mu\frac{\partial^2 \lambda_{2,1}}{\partial x_2^2} + \alpha(\gamma)\lambda_{2,1} = 0$$

$$\alpha(\gamma)u_2 - \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u_2 - \frac{\partial \lambda_1}{\partial x_2} + \rho\left(\lambda_{2,1}\frac{\partial u_1}{\partial x_2} + \lambda_{2,2}\frac{\partial u_2}{\partial x_2} - u_1\frac{\partial \lambda_{2,2}}{\partial x_1} - u_2\frac{\partial \lambda_{2,2}}{\partial x_2}\right) - \mu\frac{\partial^2 \lambda_{2,2}}{\partial x_1^2} - \mu\frac{\partial^2 \lambda_{2,2}}{\partial x_2^2} + \alpha(\gamma)\lambda_{2,2} = 0$$

Adjoint BC:

$$\lambda_{2,1}n_1 + \lambda_{2,2}n_2 = 0 \qquad \Gamma_{in\&wall}$$

$$\mu\left(\frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_1}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_1}n_2\right) + \lambda_1 n_1 + \mu\left(\frac{\partial \lambda_{2,1}}{\partial x_1}n_1 + \frac{\partial \lambda_{2,1}}{\partial x_2}n_2\right) + \rho(u_1n_1 + u_2n_2)\lambda_{2,1} = 0$$

$$\mu\left(\frac{\partial u_2}{\partial x_1}n_1 + \frac{\partial u_2}{\partial x_2}n_2 + \frac{\partial u_1}{\partial x_2}n_1 + \frac{\partial u_2}{\partial x_2}n_2\right) + \lambda_1 n_2 + \mu\left(\frac{\partial \lambda_{2,2}}{\partial x_1}n_1 + \frac{\partial \lambda_{2,2}}{\partial x_2}n_2\right) + \rho(u_1n_1 + u_2n_2)\lambda_{2,2} = 0 \qquad \Gamma_{out}$$

The variation of the Lagrangian function with respect to γ:

The variation of the dissipated energy with respect to γ:

$$\Psi = \frac{1}{2}\int_{\Omega}\left[\mu\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\alpha(\gamma)u_iu_i\right]d\Omega$$

Here, the flow resistance of the artificial porous medium $\alpha(\gamma)$is interpolated using the **RAMP** (Rational Approximation of Material Properties) scheme:

$$\alpha(\gamma) = \alpha_{max}\frac{q_\alpha(1-\gamma)}{q_\alpha+\gamma}$$

Therefore, the variation of the dissipated energy with respect to γ:

$$\Psi_\gamma = \frac{1}{2}\int_{\Omega} -\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}(u_1^2+u_2^2)d\Omega$$

The variation of the governing equation with respected to γ:

$$R_2 = [\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\mu\nabla\cdot(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\boldsymbol{T}})+\nabla p]+\alpha_{max}\frac{q_\alpha(1-\gamma)}{q_\alpha+\gamma}\boldsymbol{u}$$

$$= \left[\rho u_j\frac{\partial u_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)+\frac{\partial p}{\partial x_i}\right]+\alpha_{max}\frac{q_\alpha(1-\gamma)}{q_\alpha+\gamma}u_i$$

$$R_{2_\gamma} = \left[\rho u_j\frac{\partial u_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)+\frac{\partial p}{\partial x_i}\right]-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_i$$

$$\boldsymbol{\lambda}_2^T\cdot\boldsymbol{R}_{\boldsymbol{2}_\gamma} = [\lambda_{2_1}\quad\lambda_{2_2}]\cdot\begin{bmatrix}\left(\rho u_1\frac{\partial u_1}{\partial x_1}+\rho u_2\frac{\partial u_1}{\partial x_2}-\mu\frac{\partial^2u_1}{\partial x_1^2}-\mu\frac{\partial^2u_1}{\partial x_2^2}+\frac{\partial p}{\partial x_1}\right)-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_1\\ \left(\rho u_1\frac{\partial u_2}{\partial x_1}+\rho u_2\frac{\partial u_2}{\partial x_2}-\mu\frac{\partial^2u_2}{\partial x_1^2}-\mu\frac{\partial^2u_2}{\partial x_2^2}+\frac{\partial p}{\partial x_2}\right)-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_2\end{bmatrix}$$

$$= \lambda_{2_1}\left[\left(\rho u_1\frac{\partial u_1}{\partial x_1}+\rho u_2\frac{\partial u_1}{\partial x_2}-\mu\frac{\partial^2u_1}{\partial x_1^2}-\mu\frac{\partial^2u_1}{\partial x_2^2}+\frac{\partial p}{\partial x_1}\right)-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_1\right]$$
$$+\lambda_{2_2}\left[\left(\rho u_1\frac{\partial u_2}{\partial x_1}+\rho u_2\frac{\partial u_2}{\partial x_2}-\mu\frac{\partial^2u_2}{\partial x_1^2}-\mu\frac{\partial^2u_2}{\partial x_2^2}+\frac{\partial p}{\partial x_2}\right)-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_2\right]$$

The sensitivity of the Lagrangian function with respect to the design variables γ is

$$
\begin{aligned}
L_\gamma = \Psi_\gamma + \boldsymbol{\lambda}_2^T \cdot \boldsymbol{R}_{\boldsymbol{2}_\gamma} \\
&= -\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{2(q_\alpha+\gamma)^2}(u_1^2+u_2^2) \\
&+ \lambda_{2_1}\left[\left(\rho u_1\frac{\partial u_1}{\partial x_1}+\rho u_2\frac{\partial u_1}{\partial x_2}-\mu\frac{\partial^2 u_1}{\partial x_1^2}-\mu\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial p}{\partial x_1}\right)-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_1\right] \\
&+ \lambda_{2_2}\left[\left(\rho u_1\frac{\partial u_2}{\partial x_1}+\rho u_2\frac{\partial u_2}{\partial x_2}-\mu\frac{\partial^2 u_2}{\partial x_1^2}-\mu\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial p}{\partial x_2}\right)-\alpha_{max}\frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2}u_2\right]
\end{aligned}
$$

Where $\lambda_{2_1}, \lambda_{2_2}$ need to be obtained through the adjoint equation.

The sensitivity and adjoint equation of the Pipe Bend Problem

	n_1	n_2
$\Gamma_{wall,left}$	-1	0
$\Gamma_{wall,Bottom}$	0	-1
$\Gamma_{wall,right}$	1	0
$\Gamma_{wall,up}$	0	1
Γ_{inlet}	-1	0
Γ_{outlet}	0	-1

Substituting into the adjoint equation, we get:

$$\begin{cases} \dfrac{\partial \lambda_{2,1}}{\partial x_1} + \dfrac{\partial \lambda_{2,2}}{\partial x_2} = 0 \\ \alpha_{max} \dfrac{q_\alpha(1-\gamma)}{q_\alpha+\gamma} u_1 - \mu\left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right) u_1 - \dfrac{\partial \lambda_1}{\partial x_1} + \rho\left(\lambda_{2,1}\dfrac{\partial u_1}{\partial x_1} + \lambda_{2,2}\dfrac{\partial u_2}{\partial x_1} - u_1 \dfrac{\partial \lambda_{2,1}}{\partial x_1} - u_2 \dfrac{\partial \lambda_{2,1}}{\partial x_2}\right) \\ \qquad -\mu \dfrac{\partial^2 \lambda_{2,1}}{\partial x_1^2} - \mu \dfrac{\partial^2 \lambda_{2,1}}{\partial x_2^2} + \alpha_{max} \dfrac{q_\alpha(1-\gamma)}{q_\alpha+\gamma} \lambda_{2,1} = 0 \\ \alpha_{max} \dfrac{q_\alpha(1-\gamma)}{q_\alpha+\gamma} u_2 - \mu\left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right) u_2 - \dfrac{\partial \lambda_1}{\partial x_2} + \rho\left(\lambda_{2,1}\dfrac{\partial u_1}{\partial x_2} + \lambda_{2,2}\dfrac{\partial u_2}{\partial x_2} - u_1 \dfrac{\partial \lambda_{2,2}}{\partial x_1} - u_2 \dfrac{\partial \lambda_{2,2}}{\partial x_2}\right) \\ \qquad -\mu \dfrac{\partial^2 \lambda_{2,2}}{\partial x_1^2} - \mu \dfrac{\partial^2 \lambda_{2,1}}{\partial x_2^2} + \alpha_{max} \dfrac{q_\alpha(1-\gamma)}{q_\alpha+\gamma} \lambda_{2,2} = 0 \end{cases}$$

Adjoint boundary conditions:

$$\lambda_{2,1} n_1 + \lambda_{2,2} n_2 = 0 \qquad \Gamma_{in\&wall}$$

$$\mu\left(-\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) + \mu\left(-\frac{\partial \lambda_{2,1}}{\partial x_2}\right) + \rho(-u_2)\lambda_{2,1} \qquad \Gamma_{out}$$

$$\mu\left(-\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right) - \lambda_1 + \mu\left(-\frac{\partial \lambda_{2,2}}{\partial x_2}\right) + \rho(-u_2)\lambda_{2,2} \qquad \Gamma_{out}$$

Through the above adjoint equations and the adjoint boundary conditions, the adjoint variables are solved and brought into the following equations, the sensitivity can be obtained. The sensitivity of the Lagrangian function with respect to the design variables γ is:

$$\begin{aligned} L_\gamma &= \Psi_\gamma + \boldsymbol{\lambda}_2^T \cdot \boldsymbol{R_2}_\gamma \\ &= -\alpha_{max} \frac{q_\alpha(q_\alpha+1)}{2(q_\alpha+\gamma)^2}(u_1^2+u_2^2) \\ &+ \lambda_{2_1}\left[\left(\rho u_1 \frac{\partial u_1}{\partial x_1} + \rho u_2 \frac{\partial u_1}{\partial x_2} - \mu \frac{\partial^2 u_1}{\partial x_1^2} - \mu \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial p}{\partial x_1}\right) - \alpha_{max} \frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2} u_1\right] \\ &+ \lambda_{2_2}\left[\left(\rho u_1 \frac{\partial u_2}{\partial x_1} + \rho u_2 \frac{\partial u_2}{\partial x_2} - \mu \frac{\partial^2 u_2}{\partial x_1^2} - \mu \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial p}{\partial x_2}\right) - \alpha_{max} \frac{q_\alpha(q_\alpha+1)}{(q_\alpha+\gamma)^2} u_2\right] \end{aligned}$$

Appendix

1. Definition of Gateaux Derivative:

The Gateaux derivative is a generalization of the directional derivative for functional analysis. It represents the rate of change of a functional in the direction of a perturbation.

Given a functional $F(u)$, where u is a function (or vector field, or state variable), the Gateaux derivative of $F(u)$ at a point u in the direction of a perturbation δu is defined as:

$$\delta F(u; \delta u) = \lim_{\epsilon \to 0} \frac{F(u + \epsilon \delta u) - F(u)}{\epsilon}$$

If F is a functional and u is the variable of interest, the Gateaux derivative in the direction of δu is typically written as:

$$\delta_u F = \frac{d}{d\epsilon} F(u + \epsilon \delta u)\bigg|_{\epsilon=0}$$

2. Converting tensor expressions into symbolic expressions:

$$\lambda_{2,i} \delta u_j \frac{\partial u_i}{\partial x_j} = \lambda_{2,1} \delta u_1 \frac{\partial u_1}{\partial x_1} + \lambda_{2,1} \delta u_2 \frac{\partial u_1}{\partial x_2} + \lambda_{2,2} \delta u_1 \frac{\partial u_2}{\partial x_1} + \lambda_{2,2} \delta u_2 \frac{\partial u_2}{\partial x_2} = \begin{bmatrix} \lambda_{2,1} \frac{\partial u_1}{\partial x_1} + \lambda_{2,2} \frac{\partial u_2}{\partial x_1} \\ \lambda_{2,1} \frac{\partial u_1}{\partial x_2} + \lambda_{2,2} \frac{\partial u_2}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} \lambda_{2,1} \\ \lambda_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} = \nabla \boldsymbol{u}^{\boldsymbol{T}} \cdot \boldsymbol{\lambda_2} \cdot \delta \boldsymbol{u}$$

$$\nabla \boldsymbol{u} \cdot \boldsymbol{\lambda_2} \cdot \delta \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} \lambda_{2,1} \\ \lambda_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} = \lambda_{2,1} \delta u_1 \frac{\partial u_1}{\partial x_1} + \lambda_{2,1} \delta u_2 \frac{\partial u_2}{\partial x_1} + \lambda_{2,2} \delta u_1 \frac{\partial u_1}{\partial x_2} + \lambda_{2,2} \delta u_2 \frac{\partial u_2}{\partial x_2}$$