FINITE ELEMENT METHOD

Coursework

Zijun Fang
CID: 01811420

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Q1:
Analytically construct the Galerkin weak form of equation (1) in the region $\boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}$ describing how to impose the Neumann and Dirichlet boundary conditions where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are known constants.

$$\boldsymbol{u}(\boldsymbol{a}) = \boldsymbol{\alpha}, \quad \frac{\boldsymbol{du}}{\boldsymbol{dx}}(\boldsymbol{b}) = \boldsymbol{\beta} \qquad \textbf{[1]}$$

* The Helmholtz problem (with $\sigma = constant$):

$$\sigma \frac{\partial^2 u}{\partial x^2} - \lambda u = f(x) \tag{1.1}$$

With imposing the **Neumann and Dirichlet boundary conditions** to the one-dimensional Helmholtz problem analysis, rearrange: λ is a real positive constant;

$$\mathbb{L}(u) = \sigma \frac{\partial^2 u}{\partial x^2} - \lambda u - f(x) = 0 \tag{1.2}$$

The equation is presumed to be supplemented with appropriate boundary conditions:

$$u(a) = g_{\mathcal{D}}, \quad \frac{du}{dx}(b) = g_{\mathcal{N}}$$

Multiplying equation (1.2) by an arbitrary test function $v(x)$, and integrating over the domain Ω:

$$\int_a^b \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \int_a^b \lambda v u dx = -\int_a^b v f dx + [v \frac{\partial u}{\partial x}]_a^b$$

$$\int_a^b \frac{\partial v}{\partial x}\frac{\partial (u^{\mathcal{D}} + u^{\mathcal{H}})}{\partial x} + \int_a^b \lambda v (u^{\mathcal{D}} + u^{\mathcal{H}}) dx = -\int_a^b v f dx + v(a)\frac{\partial u}{\partial x}(a) - v(b)\beta \tag{1.3}$$

For the equation (1.3) against a numerical test function v^{δ} and integrating by parts:

$$\left[\int_a^b \frac{\partial v^{\delta}}{\partial x}\frac{\partial (u^{\mathcal{H}})}{\partial x} + \lambda v u^{\mathcal{H}} dx\right] = -\left[\int_a^b v^{\delta} f + \frac{\partial v^{\delta}}{\partial x}\frac{\partial (u^{\mathcal{D}})}{\partial x} + \lambda v^{\delta} u^{\mathcal{D}} dx\right] + v^{\delta}(a)\frac{\partial u^{\mathcal{D}}}{\partial x}(a) - v^{\delta}(b)\beta \tag{1.4}$$

If for the notation of the equation (1.4), which could be written as:

$$a\left(v^{\delta}, u^{\mathcal{H}}\right) = \left[\int_a^b \frac{\partial v^{\delta}}{\partial x}\frac{\partial (u^{\mathcal{H}})}{\partial x} + \lambda v u^{\mathcal{H}} dx\right]$$

$$f\left(v^{\delta}, u^{\mathcal{D}}\right) = -\left[\int_a^b v^{\delta} f + \frac{\partial v^{\delta}}{\partial x}\frac{\partial (u^{\mathcal{D}})}{\partial x} + \lambda v^{\delta} u^{\mathcal{D}} dx\right] + v^{\delta}(a)\frac{\partial u^{\mathcal{D}}}{\partial x}(a) - v^{\delta}(b)\beta$$

$$a\left(v^{\delta}, u^{\mathcal{H}}\right) = f\left(v^{\delta}, u^{\mathcal{D}}\right) \tag{1.5}$$

Q2:
Discuss how to enforce the alternative mixed boundary condition

$$\boldsymbol{\sigma}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}(\boldsymbol{a}) + \boldsymbol{u}(\boldsymbol{a}) = \boldsymbol{\gamma} \qquad \textbf{[2]}$$

where γ is a constant.

As known in the Dirichlet boundary condition the $\boldsymbol{u}(\boldsymbol{a}) = \boldsymbol{\alpha}$**;**
If for the σ is constant that the first derivative of function $\boldsymbol{u}(\boldsymbol{x})$ could be separated to analysis in the equation (1.1), to integrate the function:

$$\int \sigma \frac{\partial^2 u}{\partial x^2} dx - \int \lambda u dx = \int f(x) dx$$

$$\left[\sigma \frac{\partial u}{\partial x}\right]_a^b = \sigma \frac{\partial u}{\partial x}(b) - \sigma \frac{\partial u}{\partial x}(a) = \int [\lambda u + f(x)] dx \qquad (2.1)$$

For the Neumann boundary condition constant value: $\frac{\boldsymbol{du}}{\boldsymbol{dx}}(\boldsymbol{b}) = \boldsymbol{\beta}$;
For the boundary condition is mixing the Neumann and Dirichlet, at $x = a$, where the initial stage to integrate the Helmholtz problem. To enforce the relationship from equation (2.1) shows that the mixture boundary conditions, which this case is belonging to the Robin boundary conditions. These are a weighted combination of Dirichlet and Neumann boundary conditions, which are of different types specified on different subsets of the boundary.

To enforce the alternative boundary condition that, concerning the boundary conditions which observing that they are naturally embedded in the setup of the problem bilinear form. Some alternatives for the treatment of natural boundary conditions have already been recently addressed in. With the initial condition has been given that all the mixed results are constant, so the first derivative with the unknown value product is also a constant, for providing the fixed constant output; which should be confirmed by the function (2.1), showing that the above enforced function is possible to be concerned in the simulation system. At last, the Robin mixed boundary condition is useful for heat transferring issues analysing, therefore, this model could be placed as a transforming model to be analysed.

Q3:

For the region $\mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}$, show that the solution $\boldsymbol{u}^{\boldsymbol{ex}}(\boldsymbol{x}) = \mathbf{cos}\ (\boldsymbol{2\pi x})$ satisfies equation (1.1) when $\boldsymbol{f}(\boldsymbol{x}) = -(\boldsymbol{4\sigma\pi^2} + \boldsymbol{\lambda})\mathbf{cos}\ (\boldsymbol{2\pi x})$ and determine the value of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in the boundary conditions in the equations [1] and [2].

For considering the value of σ is not a constant, which the function $\boldsymbol{u}^{\boldsymbol{ex}}(\boldsymbol{x})$ through the equation (1.1) could obtain:

$$\frac{\partial}{\partial x}(\sigma(-2\pi \sin(2\pi x)) - \lambda(\cos(2\pi x)) = f(x)$$

For the method derivative by parts:

$$\frac{\partial \sigma}{\partial x}(-2\pi \sin(2\pi x)) + \sigma \frac{\partial(-2\pi \sin(2\pi x))}{\partial x} - \lambda(\cos(2\pi x)) = f(x) \tag{3.1}$$

For applying with the general solution and the boundary conditions that could conform the details of the constants for the original function:

$$\frac{\partial \sigma}{\partial x} = 0\ ; a = 0\ ; b = 1 \tag{3.2}$$

For the constant $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} = \cos(2\pi * 0) = 1$$

For the constant $\boldsymbol{\beta}$:

$$\boldsymbol{\beta} = \frac{\partial \cos(2\pi x)}{\partial x}(1) = -2\pi \sin(2\pi * 1) = 0$$

For the constant $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma} = \sigma \frac{\partial u}{\partial x}(a) + u(a) = \sigma(-2\pi \sin(2\pi * 0)) + \cos(2\pi * 0) = 0 + 1 = 1$$

For the final solutions that:

$$\alpha = 1\ ;\ \beta = 0\ ;\ \gamma = 1 \tag{3.3}$$

Q4:

(a) Discuss how integration, differentiation and global matrix assembly are performed in your implementation explaining alternative methods of appropriate.

For the numerical integration, the Gaussian Quadrature is a particularly accurate method for treating integrals where the integrand, $u(\xi)$, is smooth. In this technique the integrand is represented as Lagrange polynomial using the Q points ξ_i, which are to be specified; for the Lagrange polynomial had a value of 1 at the coordinate ξ_i, and is zero at all others. Where is local coordinate ξ_i is different from the global coordinate x_i, which the distinct points in the interval $-1 \leq \xi_i \leq 1$. As the function list below (4.1) is presenting the Lagrange integration with adding the weight function:

$$\int_{-1}^{1} u(\xi)\mathrm{d}\xi = \sum_{i=0}^{Q-1} w_i u(\xi_i) + R(u) \tag{4.1}$$

The above function should be considered with the weight function w_i and the highest polynomial order number Q by using the Gauss-Legendre justification method. The Gauss-Legendre details list below:

$$\xi_i = \xi_{i,Q}^{0,0}\ , w_i^{0,0} = \frac{1}{[1-(\xi_i)^2]}\left[\frac{d}{d\xi}\left(L_Q(\xi)\right)|_{\xi=\xi_i}\right]^{-2}$$
$$R(u) = 0 \tag{4.2}$$

To evaluate the Poisson equation, differentiating the basis expansion function which recall the normally polynomial, for approximation of the form; as the linear polynomial when $P = 1$; as the quadratic order level that $Q = 3$. To derivative the polynomial expansion under linear and quadratic situations, which directly relate to the basis expansion equations shown below:

$$\frac{du^{\delta}(\xi)}{d\xi} = \sum_{p=0}^{P} \widehat{u_p}\frac{d\phi_p(\xi)}{d\xi}$$
$$\frac{du(\xi)}{d\xi} = \sum_{i=0}^{Q-1} u(\xi_i)\frac{dh_i(\xi)}{d\xi} \tag{4.3}$$

After forming the integration and differentiation equations, discussing the FEM analysis from the local coordinate (ξ_i) to the global coordinate (x_i) with generating the global matrices from the elemental components. At the LHS equation, demonstrating the mass matrices ($\boldsymbol{M}$) and Laplacian matrices ($\boldsymbol{L}$), where

$$\boldsymbol{M}[i,j] = \int_{\Omega}\phi_i\phi_j dx \qquad \boldsymbol{M}^e[i,j] = \int_{\Omega\mathrm{st}}\phi_i\phi_j J^e d\xi \qquad J^e = \frac{dx^e}{d\xi} \tag{4.4}$$

$$\boldsymbol{L}[i,j] = \int_{\Omega}\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx \qquad \boldsymbol{L}^e[i,j] = \int_{\Omega\mathrm{st}}\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}\left(\frac{\partial\xi}{\partial x}\right)^2 J^e d\xi \tag{4.5}$$

The RHS evaluation with function ($\boldsymbol{f}$):

$$\sum_i \widehat{v_i}\int_{\Omega}\phi_i f(x)dx \Rightarrow \widehat{\boldsymbol{f_g}} \tag{4.6}$$

The general global matrix forming system function is shown as:

$$[\boldsymbol{L}^{\mathcal{H}\mathcal{H}} + \lambda\boldsymbol{M}^{\mathcal{H}\mathcal{H}}]\widehat{\boldsymbol{u}}_{\boldsymbol{g}}^{\mathcal{H}} = \widehat{\boldsymbol{f_g}} \tag{4.7}$$

Q4:

(b) Using a mesh of N_{el} =5, 10, 20, 50, 100 equispaced elements determine the L_2 norm of the error $\epsilon(x) = u^{ex}(x) - u^{\delta}(x)$, between the exact and the numerical solution defined as:

$$L_2 = \sqrt{\sum_{i=0}^{N} \frac{\epsilon(x)}{(N+1)}}$$

$N = N_{el}$ for the linear finite element expansion;
$N = 2N_{el}$ for the quadratic finite element expansion.

* The norm of the error number results:

N_{el}	5	10	20	50	100
L_2 Linear	0.4928	0.2469	0.1210	0.0462	0.0222
L_2 Quadratic	0.6543	0.3133	0.1272	0.0447	0.0229

Table 1: FEM Helmholtz problem analysis linear and quadratic expansion error norm number

* L_2 norm number against the element size plotting:

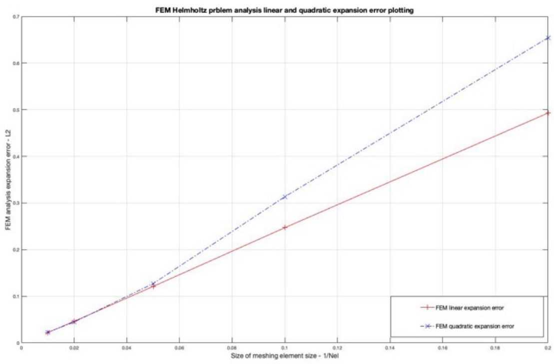

Figure 1: FEM Helmholtz problem analysis norm number against the element size plotting

* L_2 norm number against the element size log-log plotting:

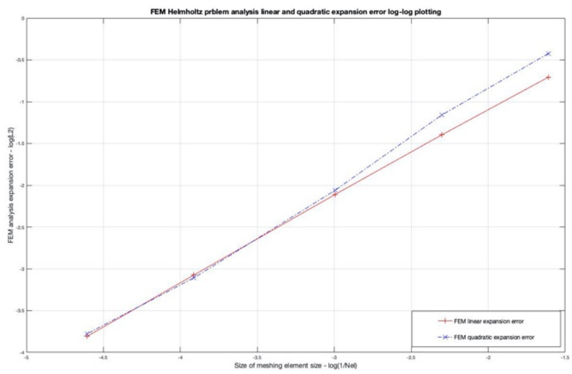

Figure 2: FEM Helmholtz problem analysis norm number against the element size log-log plotting

Q4:

(c) Comment on the slopes of error curves on the plot and whether there is an alternative way to express the solution error.

Generally discussing the figure of plotting shown upon, which present that the error is decreasing while the size of meshing element is decreasing; also, the difference of errors between the linear and quadratic expansion approaching methods is reduced. By considering the slope of log-log plotting, which the slope magnitude of meshing element nodes number from 5 to 20 has larger rate comparing the quadratic to the linear one, furthermore, the linear expansion curve is reducing smoother than the quadratic. Also, with the quadratic expansion has a less average slope from 5 to 20 than 50 to 100, which means, while the meshing size reduced, the accuracy is increasing.

To analysis as more specific aspects, firstly, the 5 nodes element analysis, the norm number is largest against to the exact solution function output, however, for the less nodes output, the exact solution also present in a low accurate situation; from 10 to 50 nodes, not much more different from quadratic and linear expansion, the accuracy is increasing by adding the element number along the range; then, at the 100-node model shows that the error norm number is further closer to the exact solution. However, analysis from 5-node to 50-node model that the norm number of the error output against quadratic and linear, which is larger in common. Linear expansion could keep the low norm number of the error output in the low-node number (5 to 20) than the quadratic one.

Appendix

* Linear and Quadratic Expansion Finite Element approximation outputs:

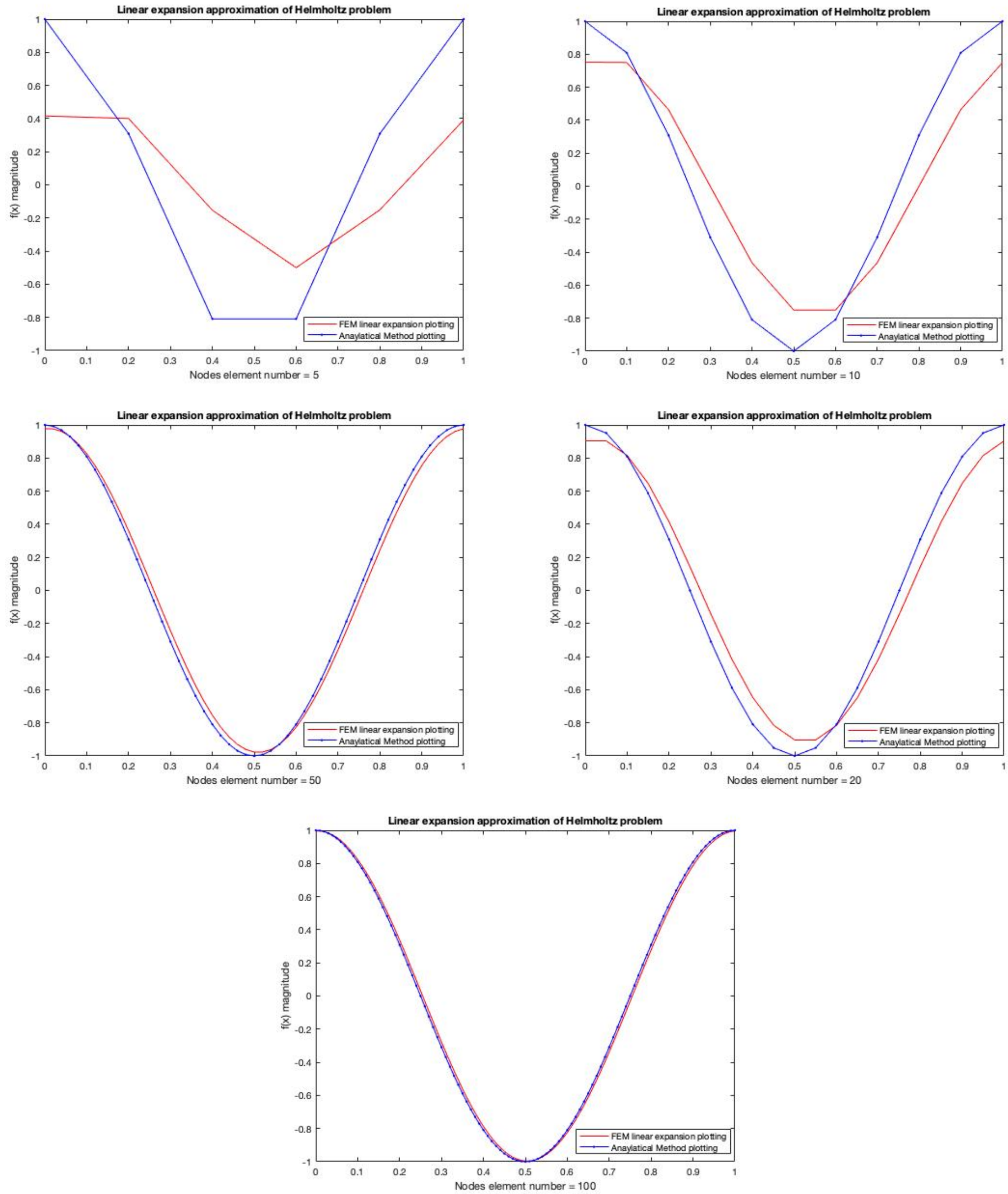

Figure 3: Linear expansion approximation output results

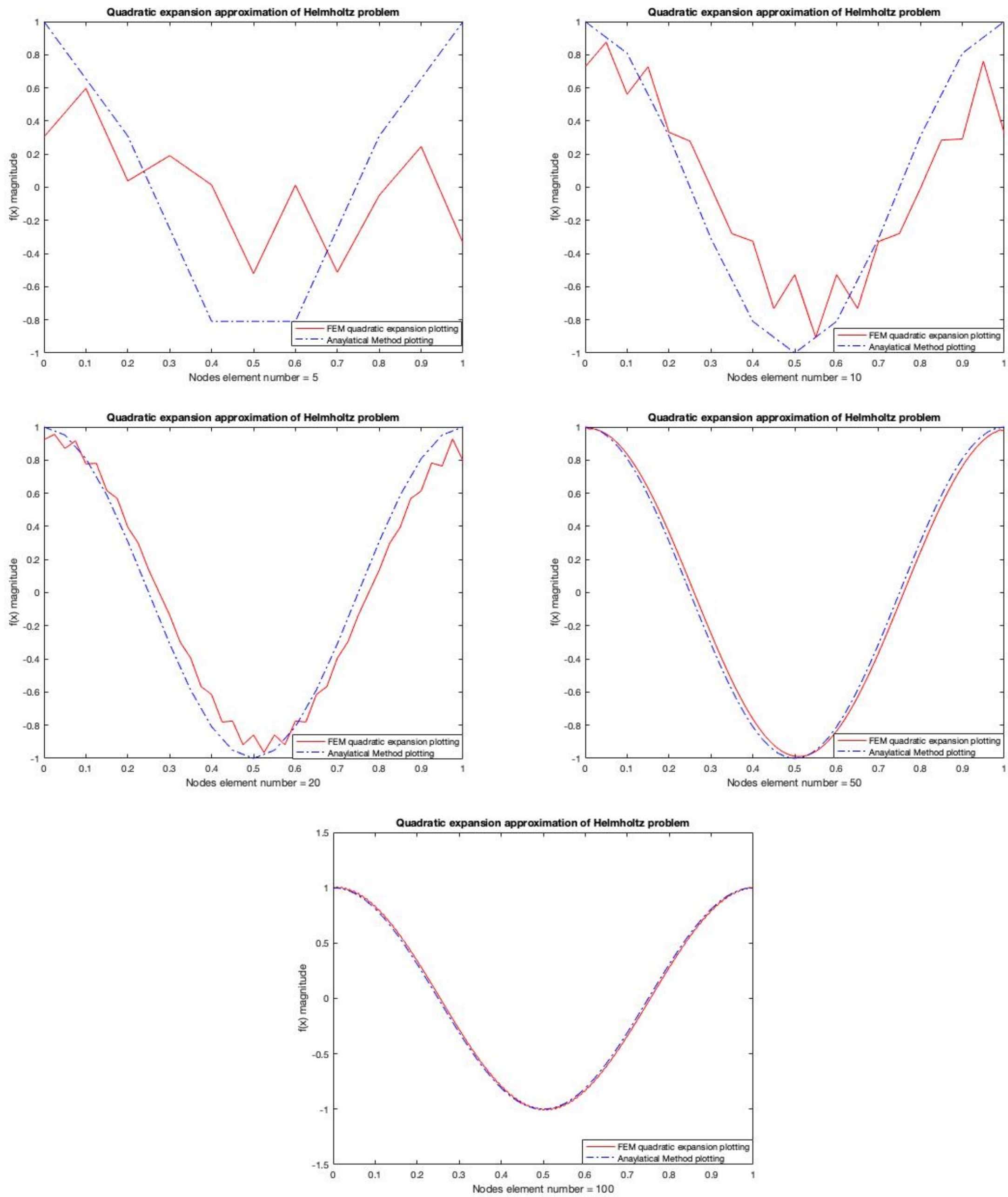

Figure 4: Quadratic expansion approximation output results