ECE 205A Fall 2023

Matrix Analysis

Final

Name: ____

UID: ____

1. (2 points) **Solution:**

2. (10 pts) Find the least norm solution for the following system of equations

$$
\begin{aligned}
3x_1 + x_3 &= 1 \\
x_1 + x_2 + x_3 &= 0
\end{aligned}
$$

Solution:

Let A, b be the matrix and the vector in the constraint $Ax = b$.

$$
A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

The least norm solution can be calculated as:

$$
\begin{aligned}
x &= A^T(AA^T)^{-1}b \\
&= \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{14} \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{14} \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}
\end{aligned}
$$

3. (15 points) Perform QR decomposition of the following matrix A:

$$A = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix}$$

Solution: The Q matrix can be obtained from the orthogonalization process:

$$e_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 21/25 \\ -28/25 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

Consequently,

$$A = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}\begin{bmatrix} 5 & 1/5 \\ 0 & 7/5 \end{bmatrix}$$

4. (10 points) Prove that matrix A and its transpose A^T have the same eigenvalues.

   Solution:

   Let λ be an eigenvalue of A. Then the matrix $A - \lambda I$ is singular. Notice that:

$$(A - \lambda I)^T = A^T - \lambda I$$

   Therefore, the matrix $A^T - \lambda I$ has the same rank as that of $A - \lambda I$. Consequently, this indicate that $A^T - \lambda I$ is also a singular matrix. Therefore, every eigenvalue of A must also be an eigenvalue of A^T.

   The above proof also applies to the reverse statement, i.e., every eigenvalue of A^T is also an eigenvalue of A. Therefore, the matrix A and A^T has the same eigenvalues.

5. (15 points)

(a) Prove that if A skew-symmetric (i.e., $-A = A^T$), then e^{tA} is orthogonal.

(b) Suppose A is an $n \times n$ matrix such that

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

Let $e^{tA}(i,j)$ be the (i,j) entry in e^tA. Compute $e^{tA}(i,j)$ for all (i,j) pairs.

Solution:

(a) Consider the following:

$$\begin{aligned} \left(e^{tA}\right)^T e^{tA} &= e^{tA^T} e^{tA} \\ &= e^{-tA} e^{tA} \\ &= e^{(-t+t)A} \\ &= e^{0A} \\ &= I \end{aligned}$$

(b) For this part, the important trick is to notice that: $A^n = 0$. Moreover, for $i = m$, it's critical to figure out the following:

$$\begin{aligned} \text{If} \quad i + m = j, &\quad A^m(i,j) = 1 \\ \text{else,} &\quad A^m(i,j) = 0 \end{aligned}$$

The expansion of e^{tA} can be expressed as:

$$e^{tA} = \sum_{m=0}^{\infty} \frac{t^m}{m!} A^m$$

Therefore, all elements along a diagonal line of e^{tA} are the same.

$$\begin{aligned} \forall i > j, &\quad e^{tA}(i,j) = 0 \\ \forall i = j, &\quad e^{tA}(i,j) = 1 \\ \forall i < j, &\quad e^{tA}(i,j) = \frac{t^{j-i}}{(j-i)!} \end{aligned}$$

6. (20pts)

   (a) Consider SVD decomposition of an arbitrary matrix A, using the notation from the class. In an SVD decomposition of A, prove that the columns of the matrix U_1 correspond to an orthogonal basis of $\mathcal{C}(A)$, the column space of A. Prove both directions.

   (b) Suppose A is a diagonalizable, symmetric $n \times n$ matrix with positive eigenvalues $\lambda_1 > \lambda_2 > ... > \lambda_n > 0$.

      i. Express A as a summation of n terms, each of which involving exactly one λ_i.

      ii. What does A^k scale as?

Solutions:

(a) Let $y \in \mathcal{C}(A)$ which can be expressed as $y = Ax$. Then, Since $A = U_1\Sigma_1 V_1^T$, it's clear that $y = U_1 x'$ where $x' = \Sigma_1 V_1^T x$. Therefore, any vector from $\mathcal{C}(A)$ can be expressed as linear combination of columns of U_1. On the other hand, let u_i, v_i be the i-th column of U_1, V_1 respectively. Let σ_i be the i-th diagonal entry in Σ_1. Then there is:

$$u_i = U_1 e_i = U_1 \Sigma_1 V_1^T v_i \frac{1}{\sigma_i} = A(\frac{v_i}{\sigma_i})$$

therefore $u_i \in \mathcal{C}(A)$. This concludes the proof.

(b) Let x_i be the corresponding eigenvector associated with λ_i.

   i. Since A is symmetric, different eigenvectors are orthogonal to each other. Consequently,

$$A = \sum_{i=1}^{n} \lambda_i x_i x_i^T$$

   ii. Notice that:

$$A^k = \left(\sum_{i=1}^{n} \lambda_i x_i x_i^T\right)^k = \sum_{i=1}^{n} \lambda_i^k x_i x_i^T = \lambda_1^k \sum_{i=1}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^k x_i x_i^T$$

   $\forall i > 1$, as $k \to \infty$, $\frac{\lambda_i}{\lambda_1} \to 0$. Therefore, A^k scales as $A \sim \lambda_1^k x_1 x_1^T$.

7. (10 pts) Suppose a symmetric and diagonalizable matrix $A \in \mathbb{R}^{n\times n}$ is idempotent, that is, $A^2 = A$. Prove or disprove: Each eigenvalue of A is either 0 or 1.

Solutions:

The statement is TRUE. Let λ_i be the eigenvalue of A. Since A is symmetric and diagonalizable, let $x_i \neq 0$ be the eigenvector of A such that $Ax_i = \lambda_i x_i$. Moreover, there is:

$$\lambda_i x_i = Ax_i = A^2 x_i = \lambda_i A x_i = \lambda_i^2 x_i$$

This means that $(\lambda_i^2 - \lambda_i)\, x_i = 0$. As $x_i \neq 0$, there must be $\lambda_i - \lambda^2 = 0$, which indicates that $\lambda_i = 0$ or $\lambda_i = 1$.

8. (18 pts) True or False

   (a) Determinant of a 2×2 rotation matrix is 1. (TRUE)
   (b) For a given matrix A, eigenvalues of $A^T A$ and AA^T are inverses of each other. (FALSE)
   (c) Each eigenvector is associated with a distinct eigenvalue. (FALSE)
   (d) Matrix norm of an $n \times n$ identity matrix is n. (FALSE)
   (e) Every orthogonal matrix is non-singular. (TRUE)
   (f) A matrix that has all real-valued entries must have all real-valued eigenvalues. (FALSE)

9. (10pts) Consider a matrix $A \in \mathbb{R}^{m \times n}$. Suppose its largest singular value is denoted by σ_1 and its eigenvalue with the maximum absolute value by $|\lambda_{\max}|$. Show that $|\lambda_{\max}| \le \sigma_1$. Hint: consider SVD of A in the standard notation, given by $U\Sigma V^T$.

   Solution:

   Notice that for any vector x. There is:

$$
\begin{aligned}
\frac{\|Ax\|_2^2}{\|x\|_2^2} &= \frac{x^T A^T A x}{x^T x} \\
&= \frac{x^T U^T \Sigma^2 U x}{x^T x}
\end{aligned}
$$

   Since σ_1 is the largest singular value, it's straight forward to see that:

$$
\begin{aligned}
\frac{\|Ax\|_2^2}{\|x\|_2^2} &\le \frac{\sigma_1^2 x^T U^T U x}{x^T x} \\
&= \sigma^2 \frac{x^T x}{x^T x} \\
&= \sigma^2
\end{aligned}
$$

   Therefore, let y be the eigenvector for $\lambda_{\max}$, i.e., $Ay = \lambda_{\max} y$. Consequently,

$$
\frac{\|Ay\|_2^2}{\|y\|_2^2} = |\lambda_{\max}|^2 \frac{\|y\|_2^2}{\|y\|_2^2} = |\lambda_{\max}|^2 \le \sigma_1^2
$$

   which indicates that $|\lambda_{\max}| \le \sigma_1$.