1. (10 points) Show that for any vector $\omega \in \mathbb{R}^3$ and a rotation matrix R, we have:

$$R\hat{\omega}R^T = (R\omega)^\wedge \tag{1}$$

where the operator $\wedge$ takes a vector $v \in \mathbb{R}^3$ to the corresponding skew-symmetric matrix $\hat{v}$:

$$\wedge : v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \hat{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{2}$$

in the following steps:

a) (5 points) Given two vectors $\omega, v \in \mathbb{R}^3$ and a rotation matrix R, show that:

$$(R\omega) \times (Rv) = R(\omega \times v) \tag{3}$$

(hint: the cross-product $\omega \times v$ of two vectors $\omega, v \in \mathbb{R}^3$ as seen from two frames a and b have the coordinates of the vectors related by the rotation matrix R_{ab})

Rω 表示对 ω 进行旋转 Rv 表示对 v 进行旋转

R(ω × v)表示对 ω 与 v 叉乘的结果进行旋转

由于线性变换 R 不会影响叉乘运算，不会改变两个向量之间的相对大小、方向，故可知：

$$(\mathrm{R}\omega) \times (\mathrm{Rv}) = \mathrm{R}(\omega \times \mathrm{v})$$

b) (5 points) Re-write $(R\omega) \times (Rv) = R(\omega \times v)$ as:

$$(R\omega)^\wedge Rv = R\hat{\omega}v \tag{4}$$

and conclude that $(R\omega)^\wedge R = R\hat{\omega}$.

$$(R\omega) \times (Rv) = (R\omega)^\wedge (Rv)$$
$$R(\omega \times v) = R\hat{\omega}v$$

由 a)中结论：$(\mathrm{R}\omega) \times (\mathrm{Rv}) = \mathrm{R}(\omega \times \mathrm{v})$

故：$(\mathrm{R}\omega)^\wedge(\mathrm{Rv}) = \mathrm{R}\hat{\omega}\mathrm{v}$

两边同乘 v^{-1}得: $(\mathrm{R}\omega)^\wedge \mathrm{R} = \mathrm{R}\hat{\omega}$

两边同乘R^{-1}得: $(\mathrm{R}\omega)^\wedge = \mathrm{R}\hat{\omega}\mathrm{R}^{-1}$

由于$\mathrm{R}^{-1} = \mathrm{R}^\mathrm{T}$, 得: $(\mathrm{R}\omega)^\wedge = \mathrm{R}\hat{\omega}\mathrm{R}^\mathrm{T}$

2. (5 points) Compute the matrix $R_{ab} = R_z(\alpha)R_y(\beta)R_x(\gamma)$ in ZYX Euler angle parametrization; (5 points) then compute $R_{ab} = R_x(\gamma)R_y(\beta)R_z(\alpha)$ and compare the two results. □

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\,(\alpha) & -sin\,(\alpha) \\ 0 & sin\,(\alpha) & cos\,(\alpha) \end{bmatrix}$$

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$$R_y(\beta) = \begin{bmatrix} cos\,(\beta) & 0 & sin\,(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos\,(\beta) \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} cos\,(\gamma) & -sin\,(\gamma) & 0 \\ sin(\gamma) & cos\,(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{ab} = R_z(\alpha) * R_y(\beta) * R_x(\gamma)$$

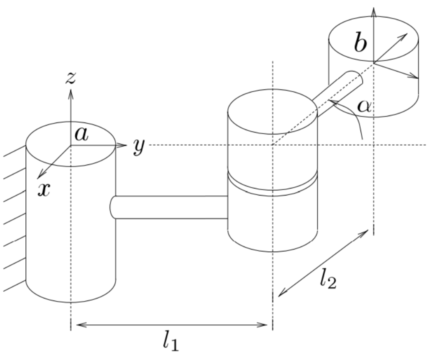
$$= \begin{bmatrix} cos(\alpha) * cos(\beta) & cos(\alpha) * sin(\beta) * sin(\gamma) - cos(\gamma) * sin(\alpha) & sin(\gamma) * sin(\alpha) + cos(\alpha) * cos(\gamma) * sin(\beta) \\ cos(\beta) * sin\,(\alpha) & cos(\gamma) * cos(\alpha) + sin(\gamma) * sin(\beta) * sin(\alpha) & cos(\gamma) * sin(\alpha) * sin(\beta) - cos(\alpha) * sin(\gamma) \\ -sin\,(\beta) & cos(\beta) * sin(\gamma) & cos(\beta) * cos\,(\gamma) \end{bmatrix}$$

$$R'_{ab} = R_x(\gamma) * R_y(\beta) * R_z(\alpha)$$

$$= \begin{bmatrix} cos(\beta) * cos\,(\gamma) & -cos(\beta) * sin(\gamma) & sin\,(\beta) \\ cos(\gamma) * sin(\alpha) + cos(\alpha) * sin(\gamma) * sin\,(\beta) & cos(\gamma) * cos(\alpha) - sin(\gamma) * sin(\beta) * sin(\alpha) & -cos(\beta) * sin\,(\gamma) \\ sin(\gamma) * sin(\alpha) - cos(\alpha) * cos(\beta) * sin(\alpha) & cos(\alpha) * sin(\gamma) + cos(\gamma) * sin(\beta) * sin(\alpha) & cos(\gamma) * cos\,(\beta) \end{bmatrix}$$

3. (5 points) Compute g_{ab} shown on Page 18 of **SDM283-Spring2021-Lecture01-3D-Kinematics.pdf**; (5 points) then carry out the computation again according to the figure on Page 22. □

- Compute g_{ab}



$$R_{ab} = \begin{bmatrix} cos\,(\alpha) & -sin\,(\alpha) & 0 \\ sin\,(\alpha) & cos\,(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_{ab} = \begin{bmatrix} -l_2 * sin\,(\alpha) \\ l_1 + l_2 * cos\,(\alpha) \\ 0 \end{bmatrix}$$

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\,(\alpha) & -sin\,(\alpha) & 0 & -l_2 * sin\,(\alpha) \\ sin\,(\alpha) & cos\,(\alpha) & 0 & l_1 + l_2 * cos\,(\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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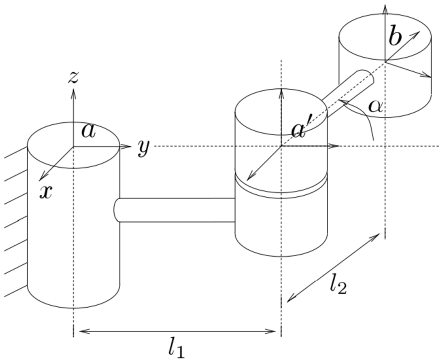
- Compute g_{ab}



$$R_{aa'} = \begin{bmatrix} cos\,(0) & -sin\,(0) & 0 \\ sin\,(0) & cos\,(0) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_{aa'} = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$$

$$g_{aa'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{a'b} = \begin{bmatrix} cos\,(\alpha) & -sin\,(\alpha) & 0 \\ sin\,(\alpha) & cos\,(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_{a'b} = \begin{bmatrix} -l_2 * sin\,(\alpha) \\ l_2 * cos\,(\alpha) \\ 0 \end{bmatrix}$$

$$g_{a'b} = \begin{bmatrix} cos\,(\alpha) & -sin\,(\alpha) & 0 & -l_2 * sin\,(\alpha) \\ sin\,(\alpha) & cos\,(\alpha) & 0 & l_2 * cos\,(\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{ab} = g_{aa'} * g_{a'b} = \begin{bmatrix} cos\,(\alpha) & -sin\,(\alpha) & 0 & -l_2 * sin\,(\alpha) \\ sin\,(\alpha) & cos\,(\alpha) & 0 & l_1 + l_2 * cos\,(\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. (20 points) Derive the Rodrigues formula in the following steps:

a) (5 points) Prove that for any vectors $u, v, w \in \mathbb{R}^3$:

$$u \times (v \times w) = (u^T w)v - (u^T v)w \tag{5}$$

Double cross 法则证明:

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$$u \times (v \times w) = \hat{u}(v \times w) = \hat{u}(\hat{v}w) = \begin{bmatrix} 0 & -u3 & u2 \\ u3 & 0 & -u1 \\ -u2 & u1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -v3 & v2 \\ v3 & 0 & -v1 \\ -v2 & v1 & 0 \end{bmatrix} \begin{bmatrix} w1 \\ w2 \\ w3 \end{bmatrix}$$

$$= \begin{bmatrix} u2*v1*w2 - w1*(u2*v2 + u3*v3) + u3*v1*w3 \\ u1*v2*w1 - w2*(u1*v1 + u3*v3) + u3*v2*w3 \\ u1*v3*w1 - w3*(u1*v1 + u2*v2) + u2*v3*w2 \end{bmatrix}$$

$$(u^T w)v - (u^T v)w = \begin{bmatrix} u1 & u2 & u3 \end{bmatrix} \begin{bmatrix} w1 \\ w2 \\ w3 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} - \begin{bmatrix} u1 & u2 & u3 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} \begin{bmatrix} w1 \\ w2 \\ w3 \end{bmatrix}$$

$$= \begin{bmatrix} u2*v1*w2 - w1*(u2*v2 + u3*v3) + u3*v1*w3 \\ u1*v2*w1 - w2*(u1*v1 + u3*v3) + u3*v2*w3 \\ u1*v3*w1 - w3*(u1*v1 + u2*v2) + u2*v3*w2 \end{bmatrix}$$

得证

b) (5 points) Use the previous result to prove that (recall that $\hat{u}v = u \times v, u, v \in \mathbb{R}^3$ by definition):

$$\hat{\omega}^2 = \omega\omega^T - \|\omega\|^2 I \tag{6}$$

对等式左侧乘 v 得： $\widehat{\omega}\,\widehat{\omega}\,v = \omega \times (\omega \times v) = (\omega * v)\omega - (\omega * \omega)v = \omega^T v\omega - \omega^T \omega v$

$$= \omega\omega^T v - \|\omega\|^2 v$$

等式左右同乘v^{-1}得： $\widehat{\omega}\,\widehat{\omega} = \omega\omega^T - \|\omega\|^2 I$

故得证： $\widehat{\omega}^2 = \omega\omega^T - \|\omega\|^2 I$

c) (5 points) Then prove that:

$$\hat{\omega}^{2k+1} = (-1)^k \|\omega\|^{2k} \hat{\omega}, \qquad k = 0, 1, 2, \ldots \tag{7}$$

and

$$\hat{\omega}^{2k} = (-1)^{k-1} \|\omega\|^{2(k-1)} \hat{\omega}^2, \qquad k = 1, 2, 3, \ldots \tag{8}$$

c)

使用数学归纳法进行证明：

当 k = 0 时： $\widehat{\omega}^1 = \widehat{\omega}$ 公式成立

当 k = 1 时： $\widehat{\omega}^3 = \widehat{\omega} * \widehat{\omega}^2 = \widehat{\omega} * (\omega\omega^T - \|\omega\|^2 I) = \omega \times \omega\omega^T - \widehat{\omega}\|\omega\|^2 I = 0 - \|\omega\|^2 \widehat{\omega}$ 公式成立

假设当 k = m 时公式成立，则有： $\widehat{\omega}^{2m+1} = (-1)^m \|\omega\|^{2m} \widehat{\omega}$

那么当 k = m+1 时有： $\widehat{\omega}^{2m+3} = \widehat{\omega}^{2m+1} * \widehat{\omega}^2 = (-1)^m \|\omega\|^{2m} \widehat{\omega} * (\omega\omega^T - \|\omega\|^2 I) = (-1)^m \|\omega\|^{2m} \widehat{\omega} * \omega\omega^T - (-1)^m \|\omega\|^{2m} \widehat{\omega} * \|\omega\|^2 I = 0 - (-1)^m \|\omega\|^{2m+2} \widehat{\omega}$ 满足公式

综上所述，公式成立。

求证： $\widehat{\omega}^{2k} = (-1)^{k-1} \|\omega\|^{2(k-1)} \widehat{\omega}^2$

即证： $\widehat{\omega}^{2k} * \widehat{\omega} = (-1)^{k-1} \|\omega\|^{2(k-1)} * \widehat{\omega} * \widehat{\omega}^2 = (-1)^{k-1} \|\omega\|^{2(k-1)} * \widehat{\omega} * (\omega\omega^T - \|\omega\|^2 I) = (-1)^{k-1} \|\omega\|^{2(k-1)} * \widehat{\omega} * \omega\omega^T - (-1)^{k-1} \|\omega\|^{2(k-1)} * \widehat{\omega} * \|\omega\|^2 I = 0 - (-1)^{k-1} \|\omega\|^{2k} \widehat{\omega}$

由之前所证公式，得证。

d) (5 points) Finally show that:

$$e^{\hat{\omega}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\hat{\omega}^k = I + \frac{\hat{\omega}}{\|\omega\|}\left(\sum_{k=0}^{\infty}(-1)^k \frac{(\|\omega\|t)^{2k+1}}{(2k+1)!}\right) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(-\sum_{k=0}^{\infty}(-1)^k \frac{(\|\omega\|t)^{2k}}{(2k)!}\right) \tag{9}$$

and by recalling that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{10}$$

show that:

$$e^{\hat{\omega}t} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|t + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|t) \tag{11}$$

□

$$sin\|\omega\|t = \|\omega\|t - \frac{(\|\omega\|t)^3}{3!} + \frac{(\|\omega\|t)^5}{5!} - \frac{(\|\omega\|t)^7}{7!} + \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{(\|\omega\|t)^{2k+1}}{(2k+1)!}$$

$$cos\|\omega\|t = 1 - \frac{(\|\omega\|t)^2}{2!} + \frac{(\|\omega\|t)^4}{4!} - \frac{(\|\omega\|t)^6}{6!} + \cdots = 1 + \sum_{k=0}^{\infty}(-1)^k \frac{(\|\omega\|t)^{2k}}{(2k)!}$$

$$e^{\hat{\omega}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\hat{\omega}^k = I + \frac{\hat{\omega}}{\|\omega\|}\left(\sum_{k=0}^{\infty}(-1)^k \frac{(\|\omega\|t)^{2k+1}}{(2k+1)!}\right) + \frac{\hat{\omega}^2}{\|\omega\|^2}(-\sum_{k=0}^{\infty}(-1)^k \frac{(\|\omega\|t)^{2k}}{(2k)!})$$

代入之前计算结果：$e^{\hat{\omega}t} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|t + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|t)$ 得证

5. (20 points) Given a constant spatial velocity:

$$\hat{V} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \tag{12}$$

Assuming that $\|\omega\| \neq 0$, compute the corresponding rigid motion $e^{\hat{V}t}$ in the following steps:

a) (5 points) Show that:

$$\hat{V}^k = \begin{bmatrix} \hat{\omega}^k & \hat{\omega}^{k-1}v \\ 0 & 0 \end{bmatrix}, \qquad k = 1, 2, 3, \ldots \tag{13}$$

b) (5 points) Show that:

$$\hat{\omega}^2 v = (\omega^T v)\omega - \|\omega\|^2 v \tag{14}$$

and therefore:

$$v = \frac{\omega^T v}{\|\omega\|^2}\omega - \frac{\hat{\omega}^2}{\|\omega\|^2}v \tag{15}$$

c) (10 points) Show that:

$$\hat{\omega}^k v = \hat{\omega}^k\left(\frac{\omega^T v}{\|\omega\|^2}\omega - \frac{\hat{\omega}^2}{\|\omega\|^2}v\right) = -\hat{\omega}^{k+1}\frac{\hat{\omega}v}{\|\omega\|^2}, \qquad k = 1, 2, 3, \ldots \tag{16}$$

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and hence that:

$$e^{\hat{V}t} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}\hat{\omega}^k & \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!}\hat{\omega}^k v \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}\hat{\omega}^k & -\sum_{k=1}^{\infty} \frac{t^k}{k!}\hat{\omega}^k \frac{\hat{\omega}v}{\|\omega\|^2} + \frac{\omega^T v}{\|\omega\|^2}\omega t \\ 0 & 1 \end{bmatrix} \tag{17}$$
$$= \begin{bmatrix} e^{\hat{\omega}t} & (I - e^{\hat{\omega}t})\frac{\hat{\omega}v}{\|\omega\|^2} + \frac{\omega^T v}{\|\omega\|^2}\omega t \\ 0 & 1 \end{bmatrix}$$

□

a) 使用数学归纳法进行证明:

当 k = 1 时: $\hat{V}^1 = \hat{V} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$ 等式成立

假设当 k = m 时等式成立，则有: $\hat{V}^m = \begin{bmatrix} \hat{\omega}^m & \hat{\omega}^{m-1}v \\ 0 & 0 \end{bmatrix}$

那么当 k = m+1 时有: $\hat{V}^{m+1} = \hat{V}^m * \hat{V} = \begin{bmatrix} \hat{\omega}^m & \hat{\omega}^{m-1}v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}^{m+1} & \hat{\omega}^m v \\ 0 & 0 \end{bmatrix}$ 满足等式

综上所述，等式成立。

b)

由向量叉乘的 Double cross 法则:

$$\hat{\omega}^2 v = \omega \times (\omega \times v) = (\omega^T v)\omega - \omega^T \omega v = (\omega^T v)\omega - \|\omega\|^2 v$$

两边同除$\|\omega\|^2$得:

$$\frac{\hat{\omega}^2 v}{\|\omega\|^2} = \frac{(\omega^T v)\omega}{\|\omega\|^2} - v$$

移项得:

$$v = \frac{\omega^T v}{\|\omega\|^2}\omega - \frac{\hat{\omega}^2}{\|\omega\|^2}v$$

得证。

c) 使用数学归纳法进行证明:

当 k = 1 时:

等式左端代入 b)中结论得: $\hat{\omega}v = \hat{\omega}\frac{\omega^T v}{\|\omega\|^2}\omega - \frac{\hat{\omega}^3}{\|\omega\|^2}v$

$$\hat{\omega}v = \hat{\omega}\omega\frac{\omega^T v}{\|\omega\|^2} - \hat{\omega}^2 * \frac{\hat{\omega}v}{\|\omega\|^2} = \omega \times \omega\frac{\omega^T v}{\|\omega\|^2} - \hat{\omega}^2 * \frac{\hat{\omega}v}{\|\omega\|^2} = -\hat{\omega}^2 * \frac{\hat{\omega}v}{\|\omega\|^2}$$ 满足公式

假设当 k = m 时等式成立，则有: $\hat{\omega}^m v = -\hat{\omega}^{m+1}(\frac{\hat{\omega}v}{\|\omega\|^2})$

那么当 k = m+1 时有: $\hat{\omega}^{m+1}v = \hat{\omega} * \left(-\hat{\omega}^{m+1}\left(\frac{\hat{\omega}v}{\|\omega\|^2}\right)\right) = -\hat{\omega}^{m+2}\left(\frac{\hat{\omega}v}{\|\omega\|^2}\right)$ 满足公式

综上所述，公式成立。

$$e^{\hat{V}t} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\hat{V}^k = \hat{V}^0 + \sum_{k=1}^{\infty}\frac{t^k}{k!}\begin{bmatrix}\hat{\omega}^k & \hat{\omega}^{k-1}v \\ 0 & 0\end{bmatrix} = I + \begin{bmatrix}\sum_{k=1}^{\infty}\frac{t^k}{k!}\hat{\omega}^k & \sum_{k=0}^{\infty}\frac{t^{k+1}}{(k+1)!}\hat{\omega}^k v \\ 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\sum_{k=0}^{\infty}\frac{t^k}{k!}\hat{\omega}^k & \hat{\omega}^{-1} * \sum_{k=0}^{\infty}\frac{t^{k+1}}{(k+1)!}\hat{\omega}^{k+1}v \\ 0 & 1\end{bmatrix} = \begin{bmatrix}e^{\hat{\omega}t} & \hat{\omega}^{-1} * \sum_{k=1}^{\infty}\frac{t^k}{(k)!}\hat{\omega}^k v \\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}R(t) & \hat{\omega}^{-1} * \sum_{k=1}^{\infty}\frac{t^k}{(k)!} * (-\hat{\omega}^{k+1} * \frac{\hat{\omega}v}{\|\omega\|^2}) \\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}R(t) & -\hat{\omega}^{-1} * \hat{\omega} * \frac{\hat{\omega}v}{\|\omega\|^2} * \sum_{k=1}^{\infty}\frac{t^k}{(k)!} * (\hat{\omega}^k) \\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}R(t) & -\frac{\hat{\omega}v}{\|\omega\|^2} * \sum_{k=0}^{\infty}\frac{t^k}{(k)!} * (\hat{\omega}^k) + \frac{\hat{\omega}v}{\|\omega\|^2} * \hat{\omega}^0 \\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}R(t) & -\frac{\hat{\omega}v}{\|\omega\|^2} * R(t) + \frac{\hat{\omega}v}{\|\omega\|^2} * I \\ 0 & 1\end{bmatrix} = \begin{bmatrix}R(t) & \frac{\hat{\omega}v}{\|\omega\|^2} * (I - R(t)) \\ 0 & 1\end{bmatrix}$$

6. (15 points) Given a rotation $R_{ab}(t)$ in the form of XYZ Euler angles:

$$R_{ab}(t) = R_x(\alpha(t))R_y(\beta(t))R_z(\gamma(t)) \tag{18}$$

compute the angular velocity ω_{ab} as a function of $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ in the following steps:

a) (5 points) Compute $\dot{R}_x R_x^T$, $\dot{R}_y R_y^T$ and $\dot{R}_z R_z^T$.
b) (5 points) Compute $\hat{\omega}_{ab} = \dot{R}_{ab}R_{ab}^T$ using result of a) and Problem 1.
c) (5 points) Transform $\hat{\omega}_{ab}$ into vector form ω_{ab} (i.e., remove the hat operator $\wedge$); in particular, write ω_{ab} as the product of a 3×3 matrix and the vector $\begin{bmatrix}\dot{\alpha} & \dot{\beta} & \dot{\gamma}\end{bmatrix}^T$.

□

a)

$$R_x = \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t))\end{bmatrix}$$

$$\dot{R}_x = \begin{bmatrix}0 & 0 & 0 \\ 0 & -\sin(\alpha(t)) * \dot{\alpha} & -\cos(\alpha(t)) * \dot{\alpha} \\ 0 & \cos(\alpha(t)) * \dot{\alpha} & -\sin(\alpha(t)) * \dot{\alpha}\end{bmatrix} = \dot{\alpha} * \begin{bmatrix}0 & 0 & 0 \\ 0 & -\sin(\alpha(t)) & -\cos(\alpha(t)) \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t))\end{bmatrix}$$

$$R_x^T = \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t))\end{bmatrix}$$

$$\dot{R}_x R_x^T = \dot{\alpha} * \begin{bmatrix}0 & 0 & 0 \\ 0 & -\sin(\alpha(t)) & -\cos(\alpha(t)) \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t))\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t))\end{bmatrix} = \dot{\alpha} * \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}$$

$$R_y = \begin{bmatrix}\cos(\beta(t)) & 0 & \sin(\beta(t)) \\ 0 & 1 & 0 \\ -\sin(\beta(t)) & 0 & \cos(\beta(t))\end{bmatrix}$$

$$\dot{R_y} = \dot{\beta} * \begin{bmatrix} -\sin(\beta(t)) & 0 & \cos(\beta(t)) \\ 0 & 0 & 0 \\ -\cos(\beta(t)) & 0 & -\sin(\beta(t)) \end{bmatrix}$$

$$R_y^T = \begin{bmatrix} \cos(\beta(t)) & 0 & -\sin(\beta(t)) \\ 0 & 1 & 0 \\ \sin(\beta(t)) & 0 & \cos(\beta(t)) \end{bmatrix}$$

$$\dot{R_y}R_y^T = \dot{\beta} * \begin{bmatrix} -\sin(\beta(t)) & 0 & \cos(\beta(t)) \\ 0 & 0 & 0 \\ -\cos(\beta(t)) & 0 & -\sin(\beta(t)) \end{bmatrix} \begin{bmatrix} \cos(\beta(t)) & 0 & -\sin(\beta(t)) \\ 0 & 1 & 0 \\ \sin(\beta(t)) & 0 & \cos(\beta(t)) \end{bmatrix} = \dot{\beta} * \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos(\gamma(t)) & -\sin(\gamma(t)) & 0 \\ \sin(\gamma(t)) & \cos(\gamma(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{R_z} = \dot{\gamma} * \begin{bmatrix} -\sin(\gamma(t)) & -\cos(\gamma(t)) & 0 \\ \cos(\gamma(t)) & -\sin(\gamma(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_z^T = \begin{bmatrix} \cos(\gamma(t)) & \sin(\gamma(t)) & 0 \\ -\sin(\gamma(t)) & \cos(\gamma(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{R_z}R_z^T = \dot{\gamma} * \begin{bmatrix} -\sin(\gamma(t)) & -\cos(\gamma(t)) & 0 \\ \cos(\gamma(t)) & -\sin(\gamma(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\gamma(t)) & \sin(\gamma(t)) & 0 \\ -\sin(\gamma(t)) & \cos(\gamma(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\gamma} * \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b)

$$\dot{R_{ab}} = (\dot{R_x R_y})R_z + (R_x R_y)\dot{R_z} = \dot{R_x}R_yR_z + R_x\dot{R_y}R_z + R_xR_y\dot{R_z}$$

$$R_{ab}^T = [(R_xR_y)R_z]^T = R_z^T(R_xR_y)^T = R_z^TR_y^TR_x^T$$

$$\begin{aligned} \dot{R_{ab}}R_{ab}^T &= (\dot{R_x}R_yR_z + R_x\dot{R_y}R_z + R_xR_y\dot{R_z})R_z^TR_y^TR_x^T \\ &= \dot{R_x}R_yR_zR_z^TR_y^TR_x^T + R_x\dot{R_y}R_zR_z^TR_y^TR_x^T + R_xR_y\dot{R_z}R_z^TR_y^TR_x^T \end{aligned}$$

已知$RR^T = I$，化简得：$\dot{R_{ab}}R_{ab}^T = \dot{R_x}R_x^T + R_x(\dot{R_y}R_y^T)R_x^T + R_xR_y(\dot{R_z}R_z^T)R_y^TR_x^T$

其中，$\dot{R_y}R_y^T = \dot{\beta} * \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$，$\dot{R_z}R_z^T = \dot{\gamma} * \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\gamma} * \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 均为反对称矩阵，故满足第一题中结论的使用条件。

$$(\mathrm{R}\omega)^{\wedge} = \mathrm{R}\hat{\omega}\mathrm{R}^{\mathrm{T}}$$

$$R_x(\dot{R_y}R_y^T)R_x^T = \dot{\beta} * R_x \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} R_x^T = \dot{\beta} * \left(\mathrm{R_x} * \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)^{\wedge}$$

$$= \dot{\beta} * \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)^{\wedge} = \dot{\beta} * \left(\begin{bmatrix} 0 \\ \cos(\alpha(t)) \\ \sin(\alpha(t)) \end{bmatrix}\right)^{\wedge}$$

$$= \dot{\beta} * \begin{bmatrix} 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \\ \sin(\alpha(t)) & 0 & 0 \\ -\cos(\alpha(t)) & 0 & 0 \end{bmatrix}$$

$$R_x R_y(\dot{R}_z R_z^T)R_y^T R_x^T = \dot{\gamma} * R_x R_y \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_y^T R_x^T = \dot{\gamma} * R_x \left(\mathrm{R}_y * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^{\wedge} R_x^T$$

$$= \dot{\gamma} * R_x \left(\begin{bmatrix} \cos(\beta(t)) & 0 & \sin(\beta(t)) \\ 0 & 1 & 0 \\ -\sin(\beta(t)) & 0 & \cos(\beta(t)) \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^{\wedge} R_x^T = \dot{\gamma} * R_x \left(\begin{bmatrix} \sin(\beta(t)) \\ 0 \\ \cos(\beta(t)) \end{bmatrix} \right)^{\wedge} R_x^T$$

$$= \dot{\gamma} * \left(R_x * \begin{bmatrix} \sin(\beta(t)) \\ 0 \\ \cos(\beta(t)) \end{bmatrix} \right)^{\wedge} = \dot{\gamma} * \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix} * \begin{bmatrix} \sin(\beta(t)) \\ 0 \\ \cos(\beta(t)) \end{bmatrix} \right)^{\wedge}$$

$$= \dot{\gamma} * \left(\begin{bmatrix} \sin(\beta(t)) \\ -\sin(\alpha(t)) * \cos(\beta(t)) \\ \cos(\alpha(t)) * \cos(\beta(t)) \end{bmatrix} \right)^{\wedge}$$

$$= \dot{\gamma} * \begin{bmatrix} 0 & -\cos(\alpha(t)) * \cos(\beta(t)) & -\sin(\alpha(t)) * \cos(\beta(t)) \\ \cos(\alpha(t)) * \cos(\beta(t)) & 0 & -\sin(\beta(t)) \\ \sin(\alpha(t)) * \cos(\beta(t)) & \sin(\beta(t)) & 0 \end{bmatrix}$$

综上:

$$\dot{R}_{ab} R_{ab}^T = \dot{R}_x R_x^T + R_x(\dot{R}_y R_y^T)R_x^T + R_x R_y(\dot{R}_z R_z^T)R_y^T R_x^T$$

$$= \dot{\alpha} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \dot{\beta} * \begin{bmatrix} 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \\ \sin(\alpha(t)) & 0 & 0 \\ -\cos(\alpha(t)) & 0 & 0 \end{bmatrix} +$$

$$\dot{\gamma} * \begin{bmatrix} 0 & -\cos(\alpha(t)) * \cos(\beta(t)) & -\sin(\alpha(t)) * \cos(\beta(t)) \\ \cos(\alpha(t)) * \cos(\beta(t)) & 0 & -\sin(\beta(t)) \\ \sin(\alpha(t)) * \cos(\beta(t)) & \sin(\beta(t)) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dot{\beta} * \sin(\alpha(t)) - \dot{\gamma} * \cos(\alpha(t)) * \cos(\beta(t)) & \dot{\beta} * \cos(\alpha(t)) - \dot{\gamma} * \sin(\alpha(t)) * \cos(\beta(t)) \\ \dot{\beta} * \sin(\alpha(t)) + \dot{\gamma} * \cos(\alpha(t)) * \cos(\beta(t)) & 0 & -\dot{\alpha} - \dot{\gamma} * \sin(\beta(t)) \\ -\dot{\beta} * \cos(\alpha(t)) + \dot{\gamma} * \sin(\alpha(t)) * \cos(\beta(t)) & \dot{\alpha} + \dot{\gamma} * \sin(\beta(t)) & 0 \end{bmatrix}$$

c)

$$\omega_{ab} = \begin{bmatrix} \dot{\alpha} + \dot{\gamma} * \sin(\beta(t)) \\ \dot{\beta} * \cos(\alpha(t)) - \dot{\gamma} * \sin(\alpha(t)) * \cos(\beta(t)) \\ \dot{\beta} * \sin(\alpha(t)) + \dot{\gamma} * \cos(\alpha(t)) * \cos(\beta(t)) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \sin(\beta(t)) \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) * \cos(\beta(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) * \cos(\beta(t)) \end{bmatrix} * \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

7. (15 points) Given a rigid motion $g_{ac}(t)$ which is given by the composition of two motions:

$$g_{ac}(t) = \begin{bmatrix} R_{ac}(t) & p_{ac}(t) \\ 0 & 1 \end{bmatrix} = g_{ab}(t) g_{bc}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{bc}(t) & p_{bc}(t) \\ 0 & 1 \end{bmatrix} \tag{19}$$

compute the spatial velocity (twist) V_{ac} as the linear combination of V_{ab} and V_{bc} in the following steps:

a) (5 points) Compute $\hat{V}_{ac} = \dot{g}_{ac} g_{ac}^{-1}$.
b) (5 points) Re-write $\hat{V}_{ab} = \dot{g}_{ab} g_{ab}^{-1}$ and $\hat{V}_{bc} = \dot{g}_{bc} g_{bc}^{-1}$ in the result of a) with $V_{ab} = [v_{ab}^T \quad \omega_{ab}^T]^T$ and $V_{bc} = [v_{bc}^T \quad \omega_{bc}^T]^T$.
c) (5 points) Re-write $\hat{V}_{ac}$ in the vector form $V_{ac} = [v_{ac}^T \quad \omega_{ac}^T]^T$.

□

a)

$$g_{ac} = g_{ab} g_{bc}$$
$$\dot{g_{ac}} = \dot{g_{ab}} g_{bc} + g_{ab} \dot{g_{bc}}$$
$$g_{ac}^{-1} = g_{bc}^{-1} g_{ab}^{-1}$$
$$\dot{g_{ac}} g_{ac}^{-1} = (\dot{g_{ab}} g_{bc} + g_{ab} \dot{g_{bc}}) g_{bc}^{-1} g_{ab}^{-1} = \dot{g_{ab}} g_{bc} g_{bc}^{-1} g_{ab}^{-1} + g_{ab} \dot{g_{bc}} g_{bc}^{-1} g_{ab}^{-1} = \dot{g_{ab}} g_{ab}^{-1} + g_{ab} \dot{g_{bc}} g_{bc}^{-1} g_{ab}^{-1}$$
$$\widehat{V_{ac}} = \widehat{V_{ab}} + g_{ab} \widehat{V_{bc}} g_{ab}^{-1}$$

b)

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$\dot{g_{ab}} = \begin{bmatrix} \dot{R_{ab}} & \dot{p_{ab}} \\ 0 & 0 \end{bmatrix}$$

$$g_{ab}^{-1} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R_{ab}^{-1} & -R_{ab}^{-1} p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$\widehat{V_{ab}} = \dot{g_{ab}} g_{ab}^{-1} = \begin{bmatrix} \dot{R_{ab}} & \dot{p_{ab}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^{-1} & -R_{ab}^{-1} p_{ab} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R_{ab}} R_{ab}^{-1} & \dot{p_{ab}} - \dot{R_{ab}} R_{ab}^{-1} p_{ab} \\ 0 & 0 \end{bmatrix}$$
$$\widehat{\omega_{ab}} = \dot{R_{ab}} R_{ab}^{-1}$$
$$v_{ab} = \dot{p_{ab}} - \widehat{\omega_{ab}} p_{ab}$$

$$\widehat{V_{ab}} = \dot{g_{ab}} g_{ab}^{-1} = \begin{bmatrix} \widehat{\omega_{ab}} & v_{ab} \\ 0 & 0 \end{bmatrix}$$

同理:

$$\widehat{V_{bc}} = \dot{g_{bc}} g_{bc}^{-1} = \begin{bmatrix} \widehat{\omega_{bc}} & v_{bc} \\ 0 & 0 \end{bmatrix}$$

c)

$$\widehat{V_{ac}} = \widehat{V_{ab}} + g_{ab} \widehat{V_{bc}} g_{ab}^{-1} = \begin{bmatrix} \widehat{\omega_{ab}} & v_{ab} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\omega_{bc}} & v_{bc} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^{-1} & -R_{ab}^{-1} p_{ab} \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \widehat{\omega_{ab}} & v_{ab} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} R_{ab} \widehat{\omega_{bc}} R_{ab}^{-1} & R_{ab} v_{bc} - R_{ab} \widehat{\omega_{bc}} R_{ab}^{-1} p_{ab} \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \widehat{\omega_{ab}} + \widehat{R_{ab}\omega_{bc}} & v_{ab} + R_{ab} v_{bc} - \widehat{R_{ab}\omega_{bc}} p_{ab} \\ 0 & 0 \end{bmatrix}$$

$$v_{ac} = v_{ab} + R_{ab} v_{bc} - \widehat{R_{ab}\omega_{bc}} p_{ab}$$
$$\omega_{ac} = \omega_{ac} + R_{ab} \omega_{bc}$$
$$V_{ac} = \begin{bmatrix} v_{ab} + R_{ab} v_{bc} - \widehat{R_{ab}\omega_{bc}} p_{ab} \\ \omega_{ac} + R_{ab} \omega_{bc} \end{bmatrix}$$

$$V_{ab} = \begin{bmatrix} \dot{p_{ab}} - \widehat{\omega_{ab}} p_{ab} \\ \omega_{ab} \end{bmatrix}$$

$$V_{bc} = \begin{bmatrix} \dot{p_{bc}} - \widehat{\omega_{bc}} p_{bc} \\ \omega_{bc} \end{bmatrix}$$

$$Adg_{ab} = \begin{bmatrix} R_{ab} & \widehat{p_{ab}} R_{ab} \\ 0 & R_{ab} \end{bmatrix}$$

$$V_{ac}=V_{ab}+Adg_{ab}*V_{bc}=\begin{bmatrix}\dot{p_{ab}}-\widehat{\omega_{ab}}p_{ab}\\ \omega_{ab}\end{bmatrix}+\begin{bmatrix}R_{ab} & \widehat{p_{ab}}R_{ab}\\ 0 & R_{ab}\end{bmatrix}\begin{bmatrix}\dot{p_{bc}}-\widehat{\omega_{bc}}p_{bc}\\ \omega_{bc}\end{bmatrix}$$

$$=\begin{bmatrix}\dot{p_{ab}}-\widehat{\omega_{ab}}p_{ab}\\ \omega_{ab}\end{bmatrix}+\begin{bmatrix}R_{ab}\dot{p_{bc}}-R_{ab}\widehat{\omega_{bc}}p_{bc}+\widehat{p_{ab}}R_{ab}\omega_{bc}\\ R_{ab}\omega_{bc}\end{bmatrix}$$

$$=\begin{bmatrix}\dot{p_{ab}}-\widehat{\omega_{ab}}p_{ab}+R_{ab}(\dot{p_{bc}}-\widehat{\omega_{bc}}p_{bc})-\widehat{R_{ab}\omega_{bc}}p_{ab}\\ \omega_{ab}+R_{ab}\omega_{bc}\end{bmatrix}=\begin{bmatrix}v_{ab}+R_{ab}v_{ab}-\widehat{R_{ab}\omega_{bc}}p_{ab}\\ \omega_{ab}+R_{ab}\omega_{bc}\end{bmatrix}$$

得证

(注：$a\times b=-b\times a \quad \hat{a}b=-\hat{b}a$)