1. (60 points) Consider the system shown in Fig. 1.

   a) (10 points) Derive the equations of motion (EoM) by Euler-Lagrange equations in Approach 1 (see pp.37-40 of `SDM283-Spring2021-Lecture02-3D-Mechanics.pdf`).
   b) (10 points) Derive the EoM by Euler-Lagrange equations in Approach 2 (see pp.41-44 of `SDM283-Spring2021-Lecture02-3D-Mechanics.pdf`).
   c) (10 points) Derive the EoM by Newton-Euler equations.
   d) (15 points) Show the equivalence of a) & b).
   e) (15 points) Show the equivalence of a) (b)) & c).

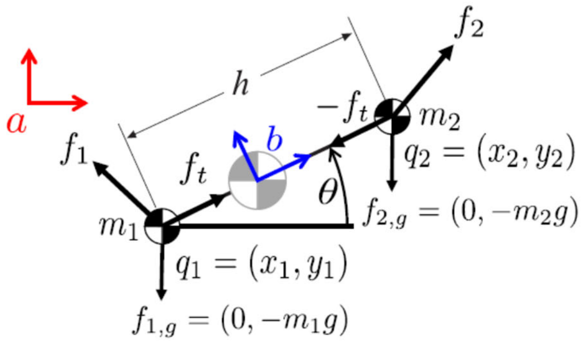

图 1: Figure for Exercise 1.

笛卡尔坐标表示的汉密尔顿原理:

$$\int_{t0}^{tf} (\delta L + f1 * \delta q1 + f2 * \delta q2) dt = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q1}}\right) - \frac{\partial L}{\partial q1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q2}}\right) - \frac{\partial L}{\partial q2}$$

$$L(t, q1, \dot{q1}, q2, \dot{q2}) = T - V = \frac{1}{2} m1 \dot{q1} * \dot{q1} + \frac{1}{2} m2 \dot{q2} * \dot{q2} - m1gy1 - m2gy2$$

(a) Approach1 使用广义坐标$[q1, \theta]$替代笛卡尔坐标

笛卡尔坐标与广义坐标的变换关系: $q2 = q1 + h \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$

虚位移变换关系: $\delta q2 = \delta q1 + h \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix} \delta\theta$

线速度变换关系: $\dot{q2} = \dot{q1} + h \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix} \dot{\theta} = \begin{bmatrix} \dot{q_{1,x}} - h\, sin\theta\dot{\theta} \\ \dot{q_{1,y}} + h\, cos\theta\dot{\theta} \end{bmatrix}$

带入汉密尔顿原理表达式中得到广义力:

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$$\delta A = \int_{t0}^{tf} (\delta L + f1 * \delta q1 + f2 * \delta q2) dt$$

$$= \int_{t0}^{tf} \left[\left(\frac{\partial L}{\partial q1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q1}}\right)\right) \delta q1 + \left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\right) \delta\theta \right.$$

$$\left. + \left(f1^T \delta q1 + f2^T (\delta q1 + h \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix} \delta\theta)\right) \right] dt$$

$$= \int_{t0}^{tf} \left[\left(\frac{\partial L}{\partial q1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q1}}\right) + f1 + f2 \right) \delta q1 \right.$$
$$\left. + \left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + f2^T h \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix} \right) \delta\theta \right] dt = 0$$

由于虚位移$\delta q1$ 与$\delta\theta$线性无关，可以得到以下$\mathrm{Eular-Lagrange\ equation}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q1}}\right) - \frac{\partial L}{\partial q1} = f1 + f2$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = f2^T h \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$$

广义坐标下的拉格朗日函数:

$$L(t, q1, \dot{q1}, \theta, \dot{\theta}) = T - V = \frac{1}{2} m1 \dot{q1}^T \dot{q1} + \frac{1}{2} m2 \dot{q2}^T \dot{q2} - m1gy1 - m2gy2$$

$$= \frac{1}{2} m1 (\dot{q_{1,x}}^2 + \dot{q_{1,y}}^2) + \frac{1}{2} m2 \left((\dot{q_{1,x}} - h\, sin\theta\dot{\theta})^2 + (\dot{q_{1,y}} + h\, cos\theta\dot{\theta})^2 \right) - m1 g q_{1,y}$$
$$- m2g(q_{1,y} + hsin\theta)$$

$q_{1,x}$分量:

$$\frac{\partial L}{\partial \dot{q_{1,x}}} = m1\dot{q_{1,x}} + m2(\dot{q_{1,x}} - h\, sin\theta\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_{1,x}}}\right) = \frac{d}{dt}\left(m1\dot{q_{1,x}} + m2(\dot{q_{1,x}} - h\, sin\theta\dot{\theta})\right) = m1\ddot{q_{1,x}} + m2(\ddot{q_{1,x}} - h(cos\theta\, \dot{\theta}^2 + sin\theta\ddot{\theta}))$$

$$\frac{\partial L}{\partial q_{1,x}} = 0$$

有:

$$\textcolor{red}{Equation1:}\ \ m1\ddot{q}1 + m2\left(\ddot{q_{1,x}} - h(cos\theta\, \dot{\theta}^2 + sin\theta\ddot{\theta})\right) = f_{1,x} + f_{2,x}$$

$q_{1,y}$分量:

$$\frac{\partial L}{\partial \dot{q_{1,y}}} = m1\dot{q_{1,y}} + m2(\dot{q_{1,y}} + h\, cos\theta\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_{1,y}}}\right) = \frac{d}{dt}\left(m1\dot{q_{1,y}} + m2(\dot{q_{1,y}} + h\, cos\theta\dot{\theta})\right) = m1\ddot{q_{1,y}} + m2(\ddot{q_{1,y}} + h(-sin\theta\, \dot{\theta}^2 + cos\theta\ddot{\theta}))$$

$$\frac{\partial L}{\partial q_{1,y}} = -m1g - m2g$$

有:

$$Equation2:\ m1\ddot{q_{1,y}} + m2\left(\ddot{q_{1,y}} + h\left(-sin\theta\,\dot{\theta}^2 + cos\theta\ddot{\theta}\right)\right) + m1g + m2g = f_{1,y} + f_{2,y}$$

θ坐标:

$$\frac{\partial L}{\partial\dot{\theta}} = m2\left(\dot{q_{1,x}} - hsin\theta\dot{\theta}\right)(-hsin\theta) + m2\left(\dot{q_{1,y}} + hcos\theta\dot{\theta}\right)(hcos\theta)$$
$$= m2\left(-hsin\theta\dot{q_{1,x}} + h^2\sin^2\theta\,\dot{\theta} + hcos\theta\dot{q_{1,y}} + h^2\cos^2\theta\,\dot{\theta}\right)$$
$$= m2h(cos\theta\dot{q_{1,y}} - sin\theta\dot{q_{1,x}} + h\dot{\theta})$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = m2h(-sin\theta\dot{\theta}\dot{q_{1,y}} + cos\theta\ddot{q_{1,y}} - cos\theta\dot{\theta}\dot{q_{1,x}} - sin\theta\ddot{q_{1,x}} + h\ddot{\theta})$$
$$\frac{\partial L}{\partial\theta} = m2\left(\left(\dot{q_{1,x}} - hsin\theta\dot{\theta}\right)\left(-hcos\theta\dot{\theta}\right) + \left(\dot{q_{1,y}} + hcos\theta\dot{\theta}\right)\left(-hsin\theta\dot{\theta}\right)\right) - m2ghcos\theta$$

有:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = m2hcos\theta\ddot{q_{1,y}} - m2hsin\theta\ddot{q_{1,x}} + m2h^2\ddot{\theta} + m2ghcos\theta$$

$$f2^T h\begin{bmatrix}-sin\theta\\cos\theta\end{bmatrix} = -hf_{2,x}sin\theta + hf_{2,y}cos\theta$$

$$Equation3:\ m2cos\theta\ddot{q_{1,y}} - m2sin\theta\ddot{q_{1,x}} + m2h\ddot{\theta} = -f_{2,x}sin\theta + f_{2,y}cos\theta - m2gcos\theta$$

(b) Approach2 使用笛卡尔坐标q1, q2作为广义坐标，但是由于q1, q2存在线性相关部分，需要在增广拉格朗日函数中引入拉格朗日乘子λ乘上约束条件来消去线性相关部分。

本题中q1, q2的约束条件即为杆长：$(q1-q2)^T(q1-q2) - h^2 = 0$

增广拉格朗日函数:

$$\bar{L} = \bar{L}\left(t, q1, \dot{q1}, q2, \dot{q2}\right) = T - V$$
$$= \frac{1}{2}m1\dot{q1}^T\dot{q1} + \frac{1}{2}m2\dot{q2}^T\dot{q2} - m1gy1 - m2gy2 + \lambda((q1-q2)^T(q1-q2) - h^2)$$

Eular − Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{q1}}\right) - \frac{\partial\bar{L}}{\partial q1} = f1$$
$$\frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{q2}}\right) - \frac{\partial\bar{L}}{\partial q2} = f2$$
$$\frac{\partial\bar{L}}{\partial\dot{q1}} = m1\dot{q1}$$
$$\frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{q1}}\right) = m1\ddot{q1}$$
$$\frac{\partial\bar{L}}{\partial q1} = 2\lambda(q1-q2) - m1g\begin{bmatrix}0\\1\end{bmatrix}$$
$$\frac{\partial\bar{L}}{\partial q2} = -2\lambda(q1-q2) - m2g\begin{bmatrix}0\\1\end{bmatrix}$$

$$Equation1:\ m1\ddot{q1} - 2\lambda(q1-q2) + m1g\begin{bmatrix}0\\1\end{bmatrix} = f1$$

$$m1\ddot{q1} = f1 + 2\lambda(q1-q2) - \begin{bmatrix}0\\m1g\end{bmatrix}$$

$$Equation2:\ m2\ddot{q2} + 2\lambda(q1-q2) + m2g\begin{bmatrix}0\\1\end{bmatrix} = f2$$

$$m2\ddot{q}2 = f2 - 2\lambda(q1 - q2) - \begin{bmatrix} 0 \\ m2g \end{bmatrix}$$

约束力: $ft = 2\lambda(q1 - q2)$

由$Equation1$ 得到: $\ddot{q}1 = \frac{f1}{m1} + \frac{2\lambda}{m1}(q1 - q2) - \begin{bmatrix} 0 \\ g \end{bmatrix}$

由$Equation2$ 得到: $\ddot{q}2 = \frac{f2}{m2} - \frac{2\lambda}{m2}(q1 - q2) - \begin{bmatrix} 0 \\ g \end{bmatrix}$

$Equation1 - Equation2$ 得到$Equation3$: $\ddot{q}1 - \ddot{q}2 = \frac{f1}{m1} - \frac{f2}{m2} + (\frac{2\lambda}{m1} + \frac{2\lambda}{m2})(q1 - q2)$

对约束条件$(q1 - q2)^T(q1 - q2) - h^2 = 0$ 求两次导:

第一次: $\left(\dot{q}1 - \dot{q}2\right)^T(q1 - q2) + (q1 - q2)^T\left(\dot{q}1 - \dot{q}2\right) = 0$

第二次: $\left(\ddot{q}1 - \ddot{q}2\right)^T(q1 - q2) + \left(\dot{q}1 - \dot{q}2\right)^T\left(\dot{q}1 - \dot{q}2\right) + \left(\dot{q}1 - \dot{q}2\right)^T\left(\dot{q}1 - \dot{q}2\right) + \cdots$

$$(q1 - q2)^T\left(\ddot{q}1 - \ddot{q}2\right) = 0$$

得到$Equation4$: $(q1 - q2)^T\left(\ddot{q}1 - \ddot{q}2\right) = -\left(\dot{q}1 - \dot{q}2\right)^T\left(\dot{q}1 - \dot{q}2\right)$

对 Equation3 两边同乘$(\mathrm{q1 - q2})^{\mathrm{T}}$得到:

$$(\mathrm{q1 - q2})^{\mathrm{T}}(\ddot{\mathrm{q}}1 - \ddot{\mathrm{q}}2) = (\mathrm{q1 - q2})^{\mathrm{T}}\left(\frac{\mathrm{f1}}{\mathrm{m1}} - \frac{\mathrm{f2}}{\mathrm{m2}}\right) + (\frac{2\lambda}{\mathrm{m1}} + \frac{2\lambda}{\mathrm{m2}})\,(\mathrm{q1 - q2})^{\mathrm{T}}(\mathrm{q1 - q2})$$

带入 Equation3 和约束条件进行化简得到:

$$-\left(\dot{q}1 - \dot{q}2\right)^T\left(\dot{q}1 - \dot{q}2\right) = (\mathrm{q1 - q2})^{\mathrm{T}}\left(\frac{\mathrm{f1}}{\mathrm{m1}} - \frac{\mathrm{f2}}{\mathrm{m2}}\right) + \left(\frac{2\lambda}{\mathrm{m1}} + \frac{2\lambda}{\mathrm{m2}}\right)\mathrm{h}^2$$

继续带入 q1 与 q2 的位置、速度关系 $\mathrm{q2 = q1 + h}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\dot{\mathrm{q}}2 = \dot{\mathrm{q}}1 + \mathrm{h}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\dot{\theta}$得到:

$$-\mathrm{h}^2\dot{\theta}^2 = \begin{bmatrix} \mathrm{h}\cos\theta \\ \mathrm{h}\sin\theta \end{bmatrix}^{\mathrm{T}} \frac{\mathrm{f1m2 - f2m1}}{\mathrm{m1m2}} + \frac{2\lambda\mathrm{h}^2(\mathrm{m2 + m1})}{\mathrm{m1m2}}$$

$$-h\dot{\theta}^2 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \frac{\begin{bmatrix} \mathrm{f}_{1,\mathrm{x}} \\ \mathrm{f}_{1,\mathrm{y}} \end{bmatrix}\mathrm{m2} - \begin{bmatrix} \mathrm{f}_{2,\mathrm{x}} \\ \mathrm{f}_{2,\mathrm{y}} \end{bmatrix}\mathrm{m1}}{\mathrm{m1m2}} + \frac{2\lambda\mathrm{h}(\mathrm{m2 + m1})}{\mathrm{m1m2}}$$

$$2\lambda\mathrm{h}(\mathrm{m2 + m1}) = -hm1m2\dot{\theta}^2 - \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathrm{f}_{1,\mathrm{x}} \\ \mathrm{f}_{1,\mathrm{y}} \end{bmatrix}\mathrm{m2} + \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathrm{f}_{2,\mathrm{x}} \\ \mathrm{f}_{2,\mathrm{y}} \end{bmatrix}\mathrm{m1}$$

$$2\lambda\mathrm{h}(\mathrm{m2 + m1}) = -\mathrm{hm1m2}\dot{\theta}^2 + \cos\theta\left(\mathrm{m1f}_{2,\mathrm{x}} - \mathrm{m2f}_{1,\mathrm{x}}\right) + \sin\theta\left(\mathrm{m1f}_{2,\mathrm{y}} - \mathrm{m2f}_{1,\mathrm{y}}\right)$$

$$\lambda = -\frac{\mathrm{m1m2}}{2(m1 + m2)}\dot{\theta}^2 + \frac{\cos\theta\left(\mathrm{m1f}_{2,\mathrm{x}} - \mathrm{m2f}_{1,\mathrm{x}}\right) + \sin\theta\left(\mathrm{m1f}_{2,\mathrm{y}} - \mathrm{m2f}_{1,\mathrm{y}}\right)}{2\mathrm{h}(\mathrm{m2 + m1})}$$

将 λ 的值带回 Equation1&Equation2 中:

$$m1\ddot{q1} = f1 + [-\frac{\mathrm{m1m2}}{(m1+m2)}\dot{\theta}^2 + \frac{\cos\theta\left(\mathrm{m1f_{2,x}} - \mathrm{m2f_{1,x}}\right) + \sin\theta\left(\mathrm{m1f_{2,y}} - \mathrm{m2f_{1,y}}\right)}{\mathrm{h(m2+m1)}}](q1-q2) - \begin{bmatrix} 0 \\ m1g \end{bmatrix}$$

$$m2\ddot{q2} = f2 - [-\frac{\mathrm{m1m2}}{(m1+m2)}\dot{\theta}^2 + \frac{\cos\theta\left(\mathrm{m1f_{2,x}} - \mathrm{m2f_{1,x}}\right) + \sin\theta\left(\mathrm{m1f_{2,y}} - \mathrm{m2f_{1,y}}\right)}{\mathrm{h(m2+m1)}}](q1-q2) - \begin{bmatrix} 0 \\ m2g \end{bmatrix}$$

(c) 证明 Newton-Euler Equation

$$\widehat{V_{ab}^s} = \begin{bmatrix} \widehat{\omega_{ab}^s} & v_{ab}^s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} J\dot{\theta} & v_{ab}^s \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{q_{1a}} \\ 0 \end{bmatrix} = \begin{bmatrix} J\dot{\theta} & v_{ab}^s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_{1a} \\ 1 \end{bmatrix}$$

$$\dot{q_{1a}} = J\dot{\theta}q_{1a} + v_{ab}^s \quad \dot{q_{2a}} = J\dot{\theta}q_{2a} + v_{ab}^s$$

$$v_{ab}^s = \dot{q}_{1a} - J\dot{\theta}q_{1a}$$

STEP1: 求 T，手动整理出$\frac{1}{2}{v_{ab}^s}^T M_s v_{ab}^s$，得到$M_S$张量矩阵

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{q_{1a}}^T q_{1a} + \frac{1}{2}m_2\dot{q_{2a}}^T q_{2a} \\ &= \frac{1}{2}m_1\left(J\dot{\theta}q_{1a} + v_{ab}^s\right)^T\left(J\dot{\theta}q_{1a} + v_{ab}^s\right) + \frac{1}{2}m_2\left(J\dot{\theta}q_{2a} + v_{ab}^s\right)^T\left(J\dot{\theta}q_{2a} + v_{ab}^s\right) \\ &= \frac{1}{2}m_1 q_{1a}^T q_{1a}\dot{\theta}^2 + \frac{1}{2}m_2 q_{2a}^T q_{2a}\dot{\theta}^2 + \frac{1}{2}(m_1+m_2){v_{ab}^s}^T v_{ab}^s + m_1{v_{ab}^s}^T J q_{1a}\dot{\theta} \\ &\quad + m_2{v_{ab}^s}^T J q_{2a}\dot{\theta} \\ &= \frac{1}{2}[v_{ab}^s \quad \dot{\theta}] \begin{bmatrix} (m_1+m_2)I & m_1 J q_{1a} + m_2 J q_{2a} \\ (m_1 J q_{1a} + m_2 J q_{2a})^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix} \begin{bmatrix} v_{ab}^s \\ \dot{\theta} \end{bmatrix} \end{aligned}$$

其中 $M_s = \begin{bmatrix} (m_1+m_2)I & m_1 J q_{1a} + m_2 J q_{2a} \\ (m_1 J q_{1a} + m_2 J q_{2a})^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}$

用质心坐标化简:

$$c_a = \frac{m_1 q_{1a} + m_2 q_{2a}}{m}$$

$$M_s = \begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}$$

STEP2: 对广义动量$M_S v_{ab}^s$求导，得到牛顿欧拉方程

$$\begin{aligned} \frac{d}{dt}(M_S v_{ab}^s) &= M_s \dot{v_{ab}^s} + \dot{M}_s v_{ab}^s \\ &= \begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix} \begin{bmatrix} \dot{v_{ab}^s} \\ \ddot{\theta} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & mJ\dot{c_a} \\ (mJ\dot{c_a})^T & 2m_1 q_{1a}^T \dot{q_{1a}} + 2m_2 q_{2a}^T \dot{q_{2a}} \end{bmatrix} \begin{bmatrix} v_{ab}^s \\ \dot{\theta} \end{bmatrix} \end{aligned}$$

广义力: $F^s = \begin{bmatrix} f^s \\ \tau^s \end{bmatrix}$ 在空间中为 6×1 向量

$$F^s=\begin{bmatrix} f_1 \\ q_{1a}\times f_1\end{bmatrix}+\begin{bmatrix} f_2 \\ q_{2a}\times f_2\end{bmatrix}+\begin{bmatrix} f_{1g} \\ q_{1a}\times f_{1g}\end{bmatrix}+\begin{bmatrix} f_{2g} \\ q_{2a}\times f_{2g}\end{bmatrix}$$
$$=\begin{bmatrix} f_1 \\ f_1^TJq_{1a}\end{bmatrix}+\begin{bmatrix} f_2 \\ f_2^TJq_{2a}\end{bmatrix}+\begin{bmatrix} f_{1g} \\ f_{1g}^TJq_{1a}\end{bmatrix}+\begin{bmatrix} f_{2g} \\ f_{2g}^TJq_{2a}\end{bmatrix}$$

得到 Newton – Euler Equation：

$$\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1q_{1a}^Tq_{1a}+m_2q_{2a}^Tq_{2a}\end{bmatrix}\begin{bmatrix} \dot{v}_{ab}^s \\ \ddot{\theta}\end{bmatrix}+\begin{bmatrix} 0 & mJ\dot{c_a} \\ (mJ\dot{c_a})^T & 2m_1q_{1a}^T\dot{q_{1a}}+2m_2q_{2a}^T\dot{q_{2a}}\end{bmatrix}\begin{bmatrix} v_{ab}^s \\ \dot{\theta}\end{bmatrix}$$
$$=\begin{bmatrix} f_1 \\ f_1^TJq_{1a}\end{bmatrix}+\begin{bmatrix} f_2 \\ f_2^TJq_{2a}\end{bmatrix}+\begin{bmatrix} f_{1g} \\ f_{1g}^TJq_{1a}\end{bmatrix}+\begin{bmatrix} f_{2g} \\ f_{2g}^TJq_{2a}\end{bmatrix}$$

其中：

$$\dot{M}^sV_{ab}^s=-\begin{bmatrix} J\dot{\theta} & -Jv_{ab}^s \\ 0 & 0\end{bmatrix}M^sV_{ab}^s$$

(注：此处消去V_{ab}^s则不成立，因为V_{ab}^s是一个向量而非可逆矩阵)

(d) 要证明(a)与(b)的等价关系，尝试由(b)推导(a)，若能成功推导则说明(a)(b)等价

将(b)的方程拆分到 x, y 方向上：

得到以下四个方程：

$$m1\ddot{x1}=f1x+\lambda(x1-x2)$$
$$m2\ddot{x2}=f2x-\lambda(x1-x2)$$
$$m1\ddot{y1}=f1y+\lambda(y1-y2)-m1g$$
$$m2\ddot{y2}=f2y-\lambda(y1-y2)-m2g$$

对式子 $x2=x1+h\cos\theta$ 求两次导得到：$\ddot{x2}=\ddot{x1}-h\cos\theta\dot{\theta}^2-h\sin\theta\ddot{\theta}$

对式子 $y2=y1+h\sin\theta$ 求两次导得到：$\ddot{y2}=\ddot{y1}-h\sin\theta\dot{\theta}^2+h\cos\theta\ddot{\theta}$

由上述方程推导(a)中的 Equation1: $m1\ddot{q1}+m2\left(\ddot{q}_{1,x}-h\left(\cos\theta\,\dot{\theta}^2+\sin\theta\ddot{\theta}\right)\right)=f_{1,x}+f_{2,x}$

$$f1x+f2x=m1\ddot{x1}-\lambda(x1-x2)+m2\ddot{x2}+\lambda(x1-x2)=m1\ddot{x1}+m2\ddot{x2}$$
$$=m1\ddot{x1}+m2(\ddot{x1}-h\cos\theta\dot{\theta}^2-h\sin\theta\ddot{\theta})\text{即为(a)中的 Equation1}$$

推导(a)中的 Equation2: $m1\ddot{q}_{1,y}+m2\left(\ddot{q}_{1,y}+h\left(-\sin\theta\,\dot{\theta}^2+\cos\theta\ddot{\theta}\right)\right)+m1g+m2g=f_{1,y}+f_{2,y}$

$$f1y+f2y-(m1+m2)g=m1\ddot{y1}-\lambda(y1-y2)+m2\ddot{y2}+\lambda(y1-y2)=m1\ddot{y1}+m2\ddot{y2}$$
$$=m1\ddot{y1}+m2(\ddot{y1}-h\sin\theta\dot{\theta}^2+h\cos\theta\ddot{\theta})\text{即为(a)中的 Equation2}$$

推导(a)中的 Equation3: $m2\cos\theta\ddot{q}_{1,y}-m2\sin\theta\ddot{q}_{1,x}+m2h\ddot{\theta}=-f_{2,x}\sin\theta+f_{2,y}\cos\theta-m2g\cos\theta$

$$-f2x\sin\theta=-\sin\theta\,(m2\ddot{x2}+\lambda(x1-x2))$$
$$(f2y-m2g)\cos\theta=\cos\theta(m2\ddot{y2}+\lambda(y1-y2))$$

$$-f_{2,x}\sin\theta + f_{2,y}\cos\theta - m2g\cos\theta$$

$$= -\sin\theta\left(m2(\ddot{x}1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) + \lambda(x1 - x2)\right)$$

$$+ \cos\theta\left(m2(\ddot{y}1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) + \lambda(y1 - y2)\right)$$

$$= -sin\theta\, m2(\ddot{x}1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) + \cos\theta\, m2(\ddot{y}1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta})$$

$= -sin\theta\, m2\ddot{x}1 + \cos\theta\, m2\ddot{y}1 + h\, m2\ddot{\theta}$ 即为(a)中的 Equation3

(e) 要证明(a) (b) & (c)的等价关系，已经证明了(a) (b)的等价关系，即需证明(a) (c)的等价关系。尝试从(c)推(a)，若能推出即可证明。

（注释：$q_{1a}\times f_1 = f_1^T J q_{1a}$）

$Newton - Euler\ Equation$:

$$\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1q_{1a}^Tq_{1a} + m_2q_{2a}^Tq_{2a}\end{bmatrix}\begin{bmatrix} \dot{v}_{ab}^s \\ \ddot{\theta}\end{bmatrix} + \begin{bmatrix} 0 & mJ\dot{c}_a \\ (mJ\dot{c}_a)^T & m_1q_{1a}^T\dot{q_{1a}} + m_2q_{2a}^T\dot{q_{2a}}\end{bmatrix}\begin{bmatrix} v_{ab} \\ \dot{\theta}\end{bmatrix}$$
$$= \begin{bmatrix} f_1 \\ f_1^TJq_{1a}\end{bmatrix} + \begin{bmatrix} f_2 \\ f_2^TJq_{2a}\end{bmatrix} + \begin{bmatrix} f_{1,g} \\ f_{1,g}^TJq_{1a}\end{bmatrix} + \begin{bmatrix} f_{2,g} \\ f_{2,g}^TJq_{2a}\end{bmatrix}$$

左边等于：

MatrixL: $\begin{bmatrix} mI\dot{v}_{ab}^s + mJc_a\ddot{\theta} + mJ\dot{c}_a\dot{\theta} \\ (mJc_a)^T\dot{v}_{ab}^s + m_1q_{1a}^Tq_{1a}\ddot{\theta} + m_2q_{2a}^Tq_{2a}\ddot{\theta} + (mJ\dot{c}_a)^Tv_{ab}^s + 2m_1q_{1a}^T\dot{q_{1a}}\dot{\theta} + 2m_2q_{2a}^T\dot{q_{2a}}\dot{\theta}\end{bmatrix}$

其中v_{ab}^s与$\dot{v}_{ab}^s$为：

$$v_{ab}^s = \dot{q_{1a}} - \dot{\theta}Jq_{1a} = \begin{bmatrix}\dot{x_1} \\ \dot{y_1}\end{bmatrix} - \dot{\theta}\begin{bmatrix}-y_1 \\ x_1\end{bmatrix} = \begin{bmatrix}\dot{x_1} + y_1\dot{\theta} \\ \dot{y_1} - x_1\dot{\theta}\end{bmatrix}$$

$$\dot{v}_{ab}^s = \ddot{q_{1a}} - \ddot{\theta}Jq_{1a} - \dot{\theta}J\dot{q_{1a}} = \begin{bmatrix}\ddot{x_1} \\ \ddot{y_1}\end{bmatrix} - \ddot{\theta}\begin{bmatrix}-y_1 \\ x_1\end{bmatrix} - \dot{\theta}\begin{bmatrix}-\dot{y_1} \\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}\ddot{x_1} + y_1\ddot{\theta} + \dot{y_1}\dot{\theta} \\ \ddot{y_1} - x_1\ddot{\theta} - \dot{x_1}\dot{\theta}\end{bmatrix}$$

MatrixL第一行等于：

$$mI\dot{v}_{ab}^s + \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}m_1x_1 + m_2x_2 \\ m_1y_1 + m_2y_2\end{bmatrix}\ddot{\theta} + \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}m_1\dot{x_1} + m_2\dot{x_2} \\ m_1\dot{y_1} + m_2\dot{y_2}\end{bmatrix}\dot{\theta}$$

Matrix1: $m\begin{bmatrix}\ddot{x_1} + y_1\ddot{\theta} + \dot{y_1}\dot{\theta} \\ \ddot{y_1} - x_1\ddot{\theta} - \dot{x_1}\dot{\theta}\end{bmatrix} + \begin{bmatrix}-m_1y_1\ddot{\theta} - m_2y_2\ddot{\theta} \\ m_1x_1\ddot{\theta} + m_2x_2\ddot{\theta}\end{bmatrix} + \begin{bmatrix}-m_1\dot{y_1}\dot{\theta} - m_2\dot{y_2}\dot{\theta} \\ m_1\dot{x_1}\dot{\theta} + m_2\dot{x_2}\dot{\theta}\end{bmatrix}$

Matrix1第一行等于：

$$(m_1 + m_2)\ddot{x_1} + (m_1 + m_2)y_1\ddot{\theta} + (m_1 + m_2)\dot{y_1}\dot{\theta} - m_1y_1\ddot{\theta} - m_2y_2\ddot{\theta} + (m_1 + m_2)\dot{x_1}$$
$$+ (m_1 + m_2)y_1\dot{\theta} - m_1\dot{y_1}\dot{\theta} - m_2\dot{y_2}\dot{\theta}$$
$$= (m_1 + m_2)\ddot{x_1} + m_2y_1\ddot{\theta} + m_2\dot{y_1}\dot{\theta} - m_2y_2\ddot{\theta} + (m_1 + m_2)\dot{x_1} + (m_1 + m_2)y_1\dot{\theta}$$
$$- m_2\dot{y_2}\dot{\theta}$$

带入$y_2 = y_1 + hsin\theta \quad \dot{y_2} = \dot{y_1} + hcos\theta\dot{\theta}$得到：

$Equation1_L$: $m_1\ddot{x_1} + m_2\ddot{x_1} - m_2hsin\theta\ddot{\theta} - m_2hcos\theta\dot{\theta}^2 = m_1\ddot{x_1} + m_2(\ddot{x_1} - hsin\theta\ddot{\theta} - hcos\theta\dot{\theta}^2)$

Matrix1第二行等于：

$$(m_1 + m_2)\ddot{y_1} - (m_1 + m_2)x_1\ddot{\theta} - (m_1 + m_2)\dot{x_1}\dot{\theta} + m_1x_1\ddot{\theta} + m_2x_2\ddot{\theta} + (m_1 + m_2)\dot{y_1}$$
$$- (m_1 + m_2)x_1\dot{\theta} + m_1\dot{x_1}\dot{\theta} + m_2\dot{x_2}\dot{\theta}$$
$$= (m_1 + m_2)\ddot{x_1} + m_2y_1\ddot{\theta} + m_2\dot{y_1}\dot{\theta} - m_2y_2\ddot{\theta} + (m_1 + m_2)\dot{x_1} + (m_1 + m_2)y_1\dot{\theta}$$
$$- m_2\dot{y_2}\dot{\theta}$$

带入$x_2 = x_1 + hcos\theta \quad \dot{x_2} = \dot{x_1} - hsin\theta\dot{\theta}$得到：

$Equation2_L$: $m_1\ddot{y}_1 + m_2\ddot{y}_1 + m_2 hcos\theta\ddot{\theta} - m_2 hsin\theta\dot{\theta}^2 = m_1\ddot{y}_1 + m_2(\ddot{y}_1 + hcos\theta\ddot{\theta} - hsin\theta\dot{\theta}^2)$

MatrixL第二行等于:

$$(\mathrm{mJc_a})^T \mathrm{v}_{ab}^{\dot{s}} + \mathrm{m_1 q_{1a}^T q_{1a}}\ddot{\theta} + \mathrm{m_2 q_{2a}^T q_{2a}}\ddot{\theta} + (\mathrm{mJ\dot{c}_a})^T \mathrm{v}_{ab}^{s} + 2\mathrm{m_1 q_{1a}^T \dot{q}_{1a}}\dot{\theta} + 2\mathrm{m_2 q_{2a}^T \dot{q}_{2a}}\dot{\theta}$$

其中第一项: $(\mathrm{mJc_a})^T \mathrm{v}_{ab}^{\dot{s}} = [-m_1y_1 - m_2y_2 \quad m_1x_1 + m_2x_2]\begin{bmatrix}\ddot{x}_1 + y_1\ddot{\theta} + \dot{y}_1\dot{\theta} \\ \ddot{y}_1 - x_1\ddot{\theta} - \dot{x}_1\dot{\theta}\end{bmatrix}$

$$= -m_1y_1(\ddot{x}_1 + y_1\ddot{\theta} + \dot{y}_1\dot{\theta}) - m_2y_1(\ddot{x}_1 + y_1\ddot{\theta} + \dot{y}_1\dot{\theta})$$
$$- m_2 hsin\theta(\ddot{x}_1 + y_1\ddot{\theta} + \dot{y}_1\dot{\theta}) + m_1x_1(\ddot{y}_1 - x_1\ddot{\theta} - \dot{x}_1\dot{\theta}) + m_2x_1(\ddot{y}_1 - x_1\ddot{\theta} - \dot{x}_1\dot{\theta})$$
$$+ m_2 hcos\theta(\ddot{y}_1 - x_1\ddot{\theta} - \dot{x}_1\dot{\theta})$$

第二项: $\mathrm{m_1 q_{1a}^T q_{1a}}\ddot{\theta} = m_1(x_1^2 + y_1^2)\ddot{\theta}$

第三项: $\mathrm{m_2 q_{2a}^T q_{2a}}\ddot{\theta} = m2(x_2^2 + y_2^2)\ddot{\theta} = m_2(x_1^2 + y_1^2 + h^2 + 2hx_1cos\theta + 2hy_1sin\theta)\ddot{\theta}$

第四项: $(\mathrm{mJ\dot{c}_a})^T \mathrm{v}_{ab}^{s} = [-m_1\dot{y}_1 - m_2\dot{y}_2 \quad m_1\dot{x}_1 + m_2\dot{x}_2]\begin{bmatrix}\dot{x}_1 + y_1\dot{\theta} \\ \dot{y}_1 - x_1\dot{\theta}\end{bmatrix}$

$$= -m_1\dot{y}_1(\dot{x}_1 + y_1\dot{\theta}) - m_2\dot{y}_1(\dot{x}_1 + y_1\dot{\theta}) - m_2 hcos\theta\dot{\theta}(\dot{x}_1 + y_1\dot{\theta})$$
$$+ m_1\dot{x}_1(\dot{y}_1 - x_1\dot{\theta}) + m_2\dot{x}_1(\dot{y}_1 - x_1\dot{\theta}) - m_2 hsin\theta\dot{\theta}(\dot{y}_1 - x_1\dot{\theta})$$

第五项: $2\mathrm{m_1 q_{1a}^T \dot{q}_{1a}}\dot{\theta} = 2m_1(x_1\dot{x}_1 + y_1\dot{y}_1)\dot{\theta}$

第六项: $2\mathrm{m_2 q_{2a}^T \dot{q}_{2a}}\dot{\theta} = 2m_2(x_2\dot{x}_2 + y_2\dot{y}_2)\dot{\theta}$

$$= 2m_2x_1(\dot{x}_1\dot{\theta} - hsin\theta\ddot{\theta}) + 2m_2hcos\theta(\dot{x}_1\dot{\theta} - hsin\theta\ddot{\theta}) + 2m_2y_1(\dot{y}_1\dot{\theta} + hcos\theta\dot{\theta}^2)$$
$$+ 2m_2hsin\theta(\dot{y}_1\dot{\theta} + hcos\theta\dot{\theta}^2)$$

$Equation3_L$: $m_2(-\ddot{x}_1 hsin\theta + \ddot{y}_1 hcos\theta + h^2\ddot{\theta})$

N − E 右边等于:

MatrixR: $\begin{bmatrix} f_{1x} + f_{2x} \\ f_{1y} + f_{2y} - m_1g - m_2g \\ f_1^TJq_{1a} + f_2^TJq_{2a} + f_{1,g}^TJq_{1a} + f_{2,g}^TJq_{2a}\end{bmatrix}$

由等式: MatrixL = MatrixR

$Equation1_L$等于 MatrixR第一行得到:

$m_1\ddot{x}_1 + m_2(\ddot{x}_1 - hsin\theta\ddot{\theta} - hcos\theta\dot{\theta}^2) = f_{1x} + f_{2x}$即为(a) $Equation1$

$Equation2_L$等于 MatrixR第二行得到:

$m_1\ddot{y}_1 + m_2(\ddot{y}_1 + hcos\theta\ddot{\theta} - hsin\theta\dot{\theta}^2) = f_{1y} + f_{2y} - m_1g - m_2g$即为(a) $Equation2$

$Equation3_L$等于 MatrixR第三行得到:

$$-f_{1x}y_1 + f_{1y}x_1 - f_{2x}y_2 + f_{2y}x_2 - m_1gx_1 - m_2gx_2$$

带入: $x_2 = x_1 + hcos\theta \quad y2 = y1 + hsin\theta$

得到:

$$x_1(f_{1y} + f_{2y} - m_1g - m_2g) + y_1(-f_{1x} - f_{2x}) - f_{2x}hsin\theta + f_{2y}hcos\theta - m_2ghcos\theta$$

由于: $f_{1x} + f_{2x} = 0 \quad f_{1y} + f_{2y} - m_1g - m_2g = 0$

MatrixR 第三行即为： $-f_{2x}hsin\theta + f_{2y}hcos\theta - m_2ghcos\theta$

$m_2(-\ddot{x}_1hsin\theta + \ddot{y}_1hcos\theta + h^2\ddot{\theta}) = -f_{2x}hsin\theta + f_{2y}hcos\theta - m_2ghcos\theta$即为(a) $Equation3$ 得证。

2. (40 points) Consider the four-bar linkage shown in Fig. 2 (all centers of mass at midpoint of links).

a) (20 points) Choose suitable generalized coordinates and derive the EoM of the system by Euler-Lagrange equations.
b) (20 points) Derive the EoM for every link by Newton-Euler equations and compute both the actuation torques at joint O and constraint wrenches at O, A, B and C.

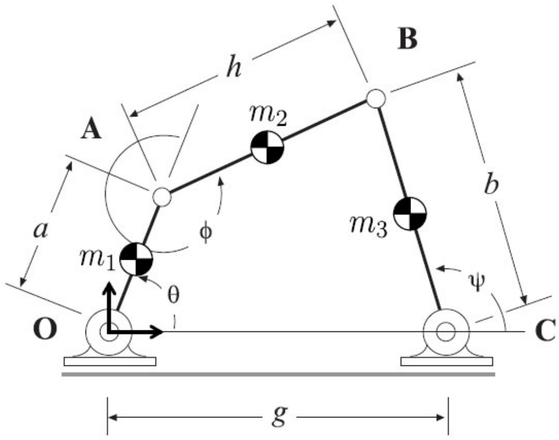

图 2: Figure for Exercise 2.

a)

Annotation: M′为 M 对 θ 的偏导数

$$Potential\ Energy:$$

$$V[\theta\_,\phi\_,\psi\_] = \mathrm{m1} * G * \frac{a}{2} * \mathrm{Sin}[\theta] + \mathrm{m2} * G * \left(a * \mathrm{Sin}[\theta] + \frac{h}{2} * \mathrm{Sin}[\theta+\phi]\right) + \mathrm{m3} * G * \frac{b}{2 * \mathrm{Sin}[\psi]};$$

$$Kinetic\ Energy:$$

$$T[\theta\_,\phi\_,\psi\_] = \frac{1}{2}\mathrm{m1} * \left(\frac{a^2}{4} * \theta\mathrm{d}^2\right) + \frac{1}{2}\mathrm{m2}\left(a^2 * \theta\mathrm{d}^2 + \frac{h^2}{4} * (\theta\mathrm{d} + \phi\mathrm{d})^2 + a * h * \mathrm{Cos}[\phi] * \theta\mathrm{d} * (\theta\mathrm{d} + \phi\mathrm{d})\right) + \frac{1}{2}\mathrm{m3} * \frac{b^2}{4} * \dot{\psi}^2;$$

$$M[\theta\_,\phi\_,\psi\_] = \frac{T[\theta,\phi,\psi]}{\theta\mathrm{d}^2};$$

$$L = T - V = \frac{1}{2}M\dot{\theta}^2 - V$$

$$Euler - Lagrange\ Equation:$$

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = M\ddot{\theta} + \frac{1}{2}M'\dot{\theta}^2 + V'$$

使用 Mathemetica 进行求解:

(*Angle*)

$$\mathrm{D1} = \frac{a * \mathrm{Sin}[\theta]}{b * \mathrm{Sin}[\theta + \phi - \psi]};$$

$$\psi\mathrm{d} = \mathrm{D1} * \theta\mathrm{d};$$

$$\mathrm{D2} = \frac{a * \mathrm{Sin}[\psi - \theta] - h * \mathrm{Sin}[\theta + \phi - \psi]}{h * \mathrm{Sin}[\theta + \phi - \psi]};$$

$$\phi\mathrm{d} = \mathrm{D2} * \theta\mathrm{d};$$

$$\mathrm{A1} = 2a * b * \mathrm{Cos}[\theta] - 2g * b;$$

$$\mathrm{B1} = 2a * b * \mathrm{Sin}[\theta];$$

$$\mathrm{C1} = b^2 + a^2 + g^2 - h^2 - 2a * g * \mathrm{Cos}[\theta];$$

$$\mathrm{A2} = 2a * h - 2g * h * \mathrm{Cos}[\theta];$$

$$\mathrm{B2} = 2g * h * \mathrm{Sin}[\theta];$$

$$\mathrm{C2} = b^2 - a^2 - g^2 - h^2 + 2a * g * \mathrm{Cos}[\theta];$$

$$\mathrm{A1p} = D[\mathrm{A1}, \theta]; \quad \mathrm{B1p} = D[\mathrm{B1}, \theta]; \quad \mathrm{C1p} = D[\mathrm{C1}, \theta];$$

$$\mathrm{A2p} = D[\mathrm{A2}, \theta]; \quad \mathrm{B2p} = D[\mathrm{B2}, \theta]; \quad \mathrm{C2p} = D[\mathrm{C2}, \theta];$$

$$\psi\mathrm{p} = \frac{\mathrm{C1p} - \mathrm{A1p} * \mathrm{Cos}[\psi] - \mathrm{B1p} * \mathrm{Sin}[\psi]}{-\mathrm{A1} * \mathrm{Sin}[\psi] + \mathrm{B1} * \mathrm{Cos}[\psi]};$$

$$\phi\mathrm{p} = \frac{\mathrm{C2p} - \mathrm{A2p} * \mathrm{Cos}[\phi] - \mathrm{B2p} * \mathrm{Sin}[\phi]}{-\mathrm{A2} * \mathrm{Sin}[\phi] + \mathrm{B2} * \mathrm{Cos}[\phi]};$$

(*PotentialEnergy*)□

$$V[\theta\_, \phi\_, \psi\_] = \mathrm{m1} * G * \frac{a}{2} * \mathrm{Sin}[\theta] + \mathrm{m2} * G * (a * \mathrm{Sin}[\theta] + \frac{h}{2} * \mathrm{Sin}[\theta + \phi]) + \mathrm{m3} * G * \frac{b}{2 * \mathrm{Sin}[\psi]};$$

$$\mathrm{Vp} = D[V[\theta, \phi, \psi], \theta] + D[V[\theta, \phi, \psi], \phi] * \phi\mathrm{p} + D[V[\theta, \phi, \psi], \psi] * \psi\mathrm{p};$$

(*KineticEnergy*)□

$$T[\theta\_, \phi\_, \psi\_] = \frac{1}{2}\mathrm{m1} * (\frac{a^2}{4} * \theta\mathrm{d}^2) + \frac{1}{2}\mathrm{m2}(a^2 * \theta\mathrm{d}^2 + \frac{h^2}{4} * (\theta\mathrm{d} + \phi\mathrm{d})^2 + a * h * \mathrm{Cos}[\phi] * \theta\mathrm{d} * (\theta\mathrm{d} + \phi\mathrm{d})) + \frac{1}{2}\mathrm{m3} * \frac{b^2}{4} * \dot{\psi}^2;$$

$$M[\theta\_, \phi\_, \psi\_] = \frac{T[\theta, \phi, \psi]}{\theta\mathrm{d}^2};$$

$$\mathrm{Mp} = D[M[\theta, \phi, \psi], \theta] + D[M[\theta, \phi, \psi], \phi] * \phi\mathrm{p} + D[M[\theta, \phi, \psi], \psi] * \psi\mathrm{p};$$

(*Euler − LagrangeEquation*)

$$\tau = M[\theta, \phi, \psi] * \theta\mathrm{dd} + \frac{1}{2} * \mathrm{Mp} * \theta\mathrm{d}^2 + \mathrm{Vp};$$

$$\mathrm{FullSimplify}[\tau]$$

得到结果如下:

$$\frac{1}{16}\left(8\,G\,h\,m2\,\text{Cos}[\theta+\phi] + 2\,a\,\text{Cos}[\theta]\left(4\,G\,(m1+2\,m2) + a\,m3\,\theta d^2\,\text{Csc}[\theta+\phi-\psi]^2\,\text{Sin}[\theta]\right) + 2\,a^2\left((m1+4\,m2)\,\theta dd + \text{Csc}[\theta+\phi-\psi]^2\left(-2\,m2\,\theta dd\,\text{Cos}[\phi]^2 + m3\left(\theta dd - \theta d^2\,\text{Cot}[\theta+\phi-\psi]\right)\text{Sin}[\theta]^2 + 2\,m2\,\text{Cos}[\phi]\left(\theta dd\,\text{Cos}[2\,\theta+\phi-2\,\psi] - \theta d^2\,\text{Sin}[\phi]\right)\right)\right) + a^2\,m2\,\text{Csc}[\theta+\phi-\psi]^3\left(\theta dd\,\text{Cos}[\phi] - \theta dd\,\text{Cos}[2\,\theta+\phi-2\,\psi] + 2\,\theta d^2\,\text{Sin}[\phi]\right)\text{Sin}[\theta-\psi]\right)$$

最后带入ϕ,ψ 与 θ 的关系：

$$\psi = \text{ArcTan}\left[\frac{\text{B1}}{\text{A1}}\right] - \text{ArcCos}\left[\frac{\text{C1}}{\sqrt{\text{A1}^2+\text{B1}^2}}\right] + \pi;$$

$$\phi = \text{ArcTan}\left[\frac{\text{B2}}{\text{A2}}\right] - \text{ArcCos}\left[\frac{\text{C2}}{\sqrt{\text{A2}^2+\text{B2}^2}}\right] + \pi;$$

得到一个不能想象的表达式。

b)

$$\textit{center of mass of each links:}$$

$$q1 = \frac{a}{2}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad \dot{q}_1 = \frac{a}{2}\begin{pmatrix}-\sin\theta\dot{\theta}\\ \cos\theta\dot{\theta}\end{pmatrix}$$

$$q2 = a\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \frac{h}{2}\begin{pmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\end{pmatrix} \quad \dot{q}_2 = a\begin{pmatrix}-\sin\theta\dot{\theta}\\ \cos\theta\dot{\theta}\end{pmatrix} + \frac{h}{2}\begin{pmatrix}-\sin(\theta+\phi)(\dot{\theta+\phi})\\ \cos(\theta+\phi)(\dot{\theta+\phi})\end{pmatrix}$$

$$q3 = \begin{pmatrix}g\\ 0\end{pmatrix} + \frac{b}{2}\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix} \quad \dot{q}_3 = \frac{b}{2}\begin{pmatrix}-\sin\psi\dot{\psi}\\ \cos\psi\dot{\psi}\end{pmatrix}$$

$$T_1 = \frac{1}{2}m_1\dot{q}_1{}^T\dot{q}_1 = \frac{1}{2}m_1\frac{a^2}{4}\dot{\theta}^2$$

$$T_2 = \frac{1}{2}m_2\dot{q}_2{}^T\dot{q}_2 = \frac{1}{2}m_2\left(a^2\dot{\theta}^2 + \frac{h^2}{4}\left(\dot{\theta}+\dot{\phi}\right)^2 + ahcos\phi\dot{\theta}\left(\dot{\theta}+\dot{\phi}\right)\right)$$

$$T_3 = \frac{1}{2}m_3\dot{q}_3{}^T\dot{q}_3 = \frac{1}{2}m_3\frac{b^2}{4}\dot{\psi}^2$$

$$V_{a1}^s = \begin{bmatrix}0\\ \dot{\theta}\end{bmatrix} \quad V_{a2}^s = \begin{bmatrix}v_{a2}^s\\ \dot{\theta+\phi}\end{bmatrix} \quad V_{a3}^s = \begin{bmatrix}0\\ \dot{\psi}\end{bmatrix}$$

$$\text{其中：}\begin{bmatrix}\dot{q}_2\\ 0\end{bmatrix} = \begin{bmatrix}J(\dot{\theta+\phi}) & v_{a2}^s\\ 0 & 0\end{bmatrix}\begin{bmatrix}q_2\\ 1\end{bmatrix}$$

$$v_{a2}^{s}=\dot{q_2}-J(\theta\dot{+}\phi)q_2$$

$$=a\begin{pmatrix}-\sin\theta\dot{\theta}\\ \cos\theta\dot{\theta}\end{pmatrix}+\frac{h}{2}\begin{pmatrix}-\sin(\theta+\phi)\,(\theta\dot{+}\phi)\\ \cos(\theta+\phi)\,(\theta\dot{+}\phi)\end{pmatrix}$$

$$-\begin{bmatrix}0&-1\\1&0\end{bmatrix}\left(a(\theta\dot{+}\phi)\begin{pmatrix}\mathrm{Cos}[\theta]\\ \mathrm{Sin}[\theta]\end{pmatrix}+\frac{h}{2}(\theta\dot{+}\phi)\begin{pmatrix}\mathrm{Cos}[\theta+\phi]\\ \mathrm{Sin}[\theta+\phi]\end{pmatrix}\right)$$

$$=\begin{pmatrix}-a\sin\theta\dot{\theta}-\frac{\mathrm{h}}{2}\sin(\theta+\phi)\,(\theta\dot{+}\phi)\\ a\cos\theta\dot{\theta}+\frac{h}{2}\cos(\theta+\phi)\,(\theta\dot{+}\phi)\end{pmatrix}-\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{pmatrix}a(\theta\dot{+}\phi)\cos\theta+\frac{h}{2}(\theta\dot{+}\phi)\cos(\theta+\phi)\\ a(\theta\dot{+}\phi)\sin\theta+\frac{h}{2}(\theta\dot{+}\phi)\sin(\theta+\phi)\end{pmatrix}$$

$$=\begin{pmatrix}-a\sin\theta\dot{\theta}-\frac{\mathrm{h}}{2}\sin(\theta+\phi)\,(\theta\dot{+}\phi)\\ a\cos\theta\dot{\theta}+\frac{h}{2}\cos(\theta+\phi)\,(\theta\dot{+}\phi)\end{pmatrix}-\begin{pmatrix}-a(\theta\dot{+}\phi)\sin\theta-\frac{h}{2}(\theta\dot{+}\phi)\sin(\theta+\phi)\\ a(\theta\dot{+}\phi)\cos\theta+\frac{h}{2}(\theta\dot{+}\phi)\cos(\theta+\phi)\end{pmatrix}$$

$$=\begin{pmatrix}-a\sin\theta\dot{\theta}-\frac{\mathrm{h}}{2}\sin(\theta+\phi)\,(\theta\dot{+}\phi)+a(\theta\dot{+}\phi)\sin\theta+\frac{h}{2}(\theta\dot{+}\phi)\sin(\theta+\phi)\\ a\cos\theta\dot{\theta}+\frac{h}{2}\cos(\theta+\phi)\,(\theta\dot{+}\phi)-a(\theta\dot{+}\phi)\cos\theta-\frac{h}{2}(\theta\dot{+}\phi)\cos(\theta+\phi)\end{pmatrix}=\begin{pmatrix}a\dot{\phi}\sin\theta\\ -a\dot{\phi}\cos\theta\end{pmatrix}$$

一个杆件的张量矩阵M^s的一般情况：

$$M^s=\begin{bmatrix}mI&mJc_a\\(mJc_a)^T&I^s\end{bmatrix}$$

I^s为惯量,质量集中于质心时为$mc_a^Tc_a$,质量分布于两端点时为$\mathrm{m}_1q_{1a}^Tq_{1a}+m_2q_{2a}^Tq_{2a}$

$$T_1=\frac{1}{2}{V_{a1}^{s}}^{T}M_1^sV_{a1}^s=\frac{1}{2}\begin{bmatrix}0&\dot{\theta}\end{bmatrix}\begin{bmatrix}m_1I_{2\times2}&m_1Jq_1\\(m_1Jq_1)^T&m_1q_1^Tq_1\end{bmatrix}\begin{bmatrix}0\\ \dot{\theta}\end{bmatrix}$$

$$T_2=\frac{1}{2}m_2\left(a^2\dot{\theta}^2+\frac{h^2}{4}(\dot{\theta}+\dot{\phi})^2+ahcos\phi\dot{\theta}(\dot{\theta}+\dot{\phi})\right)$$

$$=\frac{1}{2}{V_{a2}^{s}}^{T}M_2^sV_{a2}^s=\frac{1}{2}\begin{bmatrix}v_{a2}^s&\theta\dot{+}\phi\end{bmatrix}\begin{bmatrix}m_2I_{2\times2}&m_2Jq_2\\(m_2Jq_2)^T&m_2q_2^Tq_2\end{bmatrix}\begin{bmatrix}v_{a2}^s\\ \theta\dot{+}\phi\end{bmatrix}$$

(Annotation):

验证${V_{a2}^{s}}^{T}M_2^sV_{a2}^s$求出的结果是否等于$m_2\left(a^2\dot{\theta}^2+\frac{h^2}{4}(\dot{\theta}+\dot{\phi})^2+ahcos\phi\dot{\theta}(\dot{\theta}+\dot{\phi})\right)$

$$\begin{bmatrix}v_{a2}^s&\theta\dot{+}\phi\end{bmatrix}\begin{bmatrix}m_2I_{2\times2}&m_2Jq_2\\(m_2Jq_2)^T&m_2q_2^Tq_2\end{bmatrix}\begin{bmatrix}v_{a2}^s\\ \theta\dot{+}\phi\end{bmatrix}$$

$$=\begin{bmatrix}m_2I{v_{a2}^{s}}^{T}+(mJq_2)^T(\dot{\theta}+\dot{\phi})&{v_{ab}^{s}}^{T}m_2Jq_2+m_2q_2^Tq_2(\dot{\theta}+\dot{\phi})\end{bmatrix}\begin{bmatrix}v_{ab}^s\\ \theta\dot{+}\phi\end{bmatrix}$$

$$=mI{v_{a2}^{s}}^{T}v_{a2}^s+(mJq_2)^T(\dot{\theta}+\dot{\phi})v_{a2}^s+{v_{ab}^{s}}^{T}m_2Jq_2(\dot{\theta}+\dot{\phi})+m_2q_2^Tq_2(\dot{\theta}+\dot{\phi})^2$$

$$=m_2{v_{a2}^{s}}^{T}v_{a2}^s+m_2(\dot{\theta}+\dot{\phi})(Jq_2)^Tv_{a2}^s+m_2(\dot{\theta}+\dot{\phi}){v_{a2}^{s}}^{T}(Jq_2)+m_2q_2^Tq_2(\dot{\theta}+\dot{\phi})^2$$

$$= m_2\left(J(\dot\theta+\dot\phi)q_2+v_{a2}^s\right)^T\left(J(\dot\theta+\dot\phi)q_2+v_{a2}^s\right)$$

$$= m_2\left((\dot\theta+\dot\phi)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a\cos\theta+\frac{h}{2}\cos(\theta+\phi)\\ a\sin\theta+\frac{h}{2}\sin(\theta+\phi)\end{bmatrix}+\begin{bmatrix}a\dot\phi\sin\theta\\ -a\dot\phi\cos\theta\end{bmatrix}\right)^T(\ldots)$$

$$= m_2\left[\left(-a\sin\theta(\dot\theta+\dot\phi)-\frac{h}{2}\sin(\theta+\phi)(\dot\theta+\dot\phi)+a\dot\phi\sin\theta\right)^2 + \left(a\cos\theta(\dot\theta+\dot\phi)+\frac{h}{2}\cos(\theta+\phi)(\dot\theta+\dot\phi)-a\dot\phi\cos\theta\right)^2\right]$$

$$= m_2\left[a^2(\dot\theta+\dot\phi)^2+\frac{h^2}{4}(\dot\theta+\dot\phi)^2+a^2\dot\phi^2-2a^2(\dot\theta+\dot\phi)\dot\phi + ah(\dot\theta+\dot\phi)^2\left(\sin\theta\sin(\theta+\phi)+\cos\theta\cos(\theta+\phi)\right) - ah(\dot\theta+\dot\phi)\dot\phi\left(\sin\theta\sin(\theta+\phi)+\cos\theta\cos(\theta+\phi)\right)\right]$$

$$= m_2\left[a^2(\dot\theta+\dot\phi)^2+\frac{h^2}{4}(\dot\theta+\dot\phi)^2+a^2\dot\phi^2-2a^2(\dot\theta+\dot\phi)\dot\phi + ah(\dot\theta+\dot\phi)\cos(\theta+\phi-\theta)(\dot\theta+\dot\phi-\dot\phi)\right]$$

$$= m_2\left[a^2\dot\theta^2+\cancel{a^2\dot\phi^2}+\cancel{2a^2\dot\theta\dot\phi}+\frac{h^2}{4}(\dot\theta+\dot\phi)^2+\cancel{a^2\dot\phi^2}-\cancel{2a^2\dot\theta\dot\phi}-\cancel{2a^2\dot\phi^2}+ah(\dot\theta+\dot\phi)\dot\theta\cos\phi\right]$$

$$= m_2(a^2\dot\theta^2+\frac{h^2}{4}(\dot\theta+\dot\phi)^2+ah(\dot\theta+\dot\phi)\dot\theta\cos\phi \quad \text{得证}$$

$$T_3=\frac{1}{2}{V_{a3}^s}^T M_3^s V_{a3}^s=\frac{1}{2}\begin{bmatrix}0 & \dot\psi\end{bmatrix}\begin{bmatrix}m_3 I_{2\times2} & m_3 J q_3\\ (m_3 J q_3)^T & m_3 q_3^T q_3\end{bmatrix}\begin{bmatrix}0\\ \dot\psi\end{bmatrix}$$

$$M_1^s=\begin{bmatrix}m_1 I_{2\times2} & m_1 J q_1\\ (m_1 J q_1)^T & m_1\frac{a^2}{4}\end{bmatrix}\quad \dot M_1^s=\begin{bmatrix}0 & m_1 J\dot q_1\\ (m_1 J\dot q_1)^T & 0\end{bmatrix}$$

$$M_2^s=\begin{bmatrix}m_2 I_{2\times2} & m_2 J q_2\\ (m_2 J q_2)^T & m_2 q_2^T q_2\end{bmatrix}\quad \dot M_2^s=\begin{bmatrix}0 & m_2 J\dot q_2\\ (m_2 J\dot q_2)^T & 0\end{bmatrix}$$

$$M_3^s=\begin{bmatrix}m_3 I_{2\times2} & m_3 J q_3\\ (m_3 J q_3)^T & m_3 q_3^T q_3\end{bmatrix}\quad \dot M_3^s=\begin{bmatrix}0 & m_3 J\dot q_3\\ (m_3 J\dot q_3)^T & 0\end{bmatrix}$$

$$V_{a1}^s=\begin{bmatrix}0\\ \dot\theta\end{bmatrix}\quad V_{a2}^s=\begin{bmatrix}v_{a2}^s\\ \theta\dot{+}\phi\end{bmatrix}\quad V_{a3}^s=\begin{bmatrix}0\\ \dot\psi\end{bmatrix}$$

$$\dot V_{a1}^s=\begin{bmatrix}0\\ \ddot\theta\end{bmatrix}\quad \dot V_{a2}^s=\begin{bmatrix}\dot v_{a2}^s\\ \ddot\theta+\ddot\phi\end{bmatrix}\quad \dot V_{a3}^s=\begin{bmatrix}0\\ \ddot\psi\end{bmatrix}\quad \dot v_{a2}^s=\begin{pmatrix}a\dot\phi\cos\theta\dot\theta\\ a\dot\phi\sin\theta\dot\theta\end{pmatrix}$$

$$\mathrm{q1}=\frac{a}{2}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\quad \mathrm{q2}=a\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}+\frac{h}{2}\begin{pmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\end{pmatrix}\quad \mathrm{q3}=\begin{pmatrix}g\\ 0\end{pmatrix}+\frac{b}{2}\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}$$

$$\vec O=\begin{pmatrix}0\\ 0\end{pmatrix}\quad \vec A=\begin{pmatrix}a\cos\theta\\ a\sin\theta\end{pmatrix}\quad \vec B=\begin{pmatrix}g+b\cos\psi\\ b\sin\psi\end{pmatrix}\quad \vec C=\begin{pmatrix}g\\ 0\end{pmatrix}$$

$$Newton-Euler\ Equation:$$

$$\color{red}{Equation1:}\ M_1^s\dot{V}_{a1}^s+\dot{M}_1^sV_{a1}^s=\begin{bmatrix} f_1+f_2+f_{1g} \\ \tau+f_1^TJ\vec{O}+f_2^TJ\vec{A}+f_{1g}^TJ\overrightarrow{q_1}\end{bmatrix}$$

$$\begin{bmatrix} m_1I_{2\times2} & m_1Jq_1 \\ (m_1Jq_1)^T & m_1\frac{a^2}{4}\end{bmatrix}\begin{bmatrix}0 \\ \ddot{\theta}\end{bmatrix}+\begin{bmatrix}0 & m_1J\dot{q}_1 \\ (m_1J\dot{q}_1)^T & 0\end{bmatrix}\begin{bmatrix}0 \\ \dot{\theta}\end{bmatrix}=\begin{bmatrix} f_1+f_2+f_{1g} \\ \tau+f_1^TJ\vec{O}+f_2^TJ\vec{A}+f_{1g}^TJ\overrightarrow{q_1}\end{bmatrix}$$

$$\begin{bmatrix}-m_1y_1\ddot{\theta} \\ m_1x_1\ddot{\theta} \\ \frac{1}{4}a^2m_1\ddot{\theta}\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}m_1acos\theta\dot{\theta}^2 \\ -\frac{1}{2}m_1asin\theta\dot{\theta}^2 \\ 0\end{bmatrix}=\begin{bmatrix} f_{1x}+f_{2x} \\ f_{1y}+f_{2y}-m_1G \\ \tau+\left(-asin\theta f_{2x}+acos\theta f_{2y}\right)-\frac{1}{2}m_1Gacos\theta\end{bmatrix}$$

$$\begin{bmatrix}-m_1y_1\ddot{\theta}-\frac{1}{2}m_1acos\theta\dot{\theta}^2 \\ m_1x_1\ddot{\theta}-\frac{1}{2}m_1asin\theta\dot{\theta}^2 \\ \frac{1}{4}a^2m_1\ddot{\theta}\end{bmatrix}=\begin{bmatrix} f_{1x}+f_{2x} \\ f_{1y}+f_{2y}-m_1G \\ \tau+\left(-asin\theta f_{2x}+acos\theta f_{2y}\right)-\frac{1}{2}m_1Gacos\theta\end{bmatrix}$$

$$\color{red}{Equation2:}\ M_2^s\dot{V}_{a2}^s+\dot{M}_2^sV_{a2}^s=\begin{bmatrix} f_2+f_3+f_{2g} \\ f_2^TJ\vec{A}+f_3^TJ\vec{B}+f_{2g}^TJ\overrightarrow{q_2}\end{bmatrix}$$

$$\begin{bmatrix} m_2I_{2\times2} & m_2Jq_2 \\ (m_2Jq_2)^T & m_2q_2^Tq_2\end{bmatrix}\begin{bmatrix}\dot{v}_{a2}^s \\ \ddot{\theta}+\ddot{\phi}\end{bmatrix}+\begin{bmatrix}0 & m_2J\dot{q}_2 \\ (m_2J\dot{q}_2)^T & 0\end{bmatrix}\begin{bmatrix}v_{a2}^s \\ \dot{\theta}+\dot{\phi}\end{bmatrix}=\begin{bmatrix} f_2+f_3+f_{2g} \\ f_2^TJ\vec{A}+f_3^TJ\vec{B}+f_{2g}^TJ\overrightarrow{q_2}\end{bmatrix}$$

$$\begin{bmatrix} m_2acos\theta\dot{\phi}\dot{\theta}-m_2asin\theta\left(\ddot{\theta}+\ddot{\phi}\right)-\frac{m_2h}{2}\sin\left(\theta+\phi\right)\left(\ddot{\theta}+\ddot{\phi}\right) \\ m_2asin\theta\dot{\phi}\dot{\theta}+m_2acos\theta\left(\ddot{\theta}+\ddot{\phi}\right)+\frac{m_2h}{2}\cos\left(\theta+\phi\right)\left(\ddot{\theta}+\ddot{\phi}\right) \\ -\frac{m_2ah}{2}\dot{\theta}\dot{\phi}sin\phi+m_2(\ddot{\theta}+\ddot{\phi})(a^2+\frac{h^2}{4}+ahcos\phi)\end{bmatrix}$$

$$+\begin{bmatrix}-m_2acos\theta\dot{\theta}(\dot{\theta}+\dot{\phi})-\frac{m_2h}{2}\cos(\theta+\phi)\left(\dot{\theta}+\dot{\phi}\right)^2 \\ -m_2asin\theta\dot{\theta}(\dot{\theta}+\dot{\phi})-\frac{m_2h}{2}\sin(\theta+\phi)\left(\dot{\theta}+\dot{\phi}\right)^2 \\ \frac{m_2ah}{2}sin\phi\dot{\phi}(\dot{\theta}+\dot{\phi})\end{bmatrix}$$

$$=\begin{bmatrix} f_{2x}+f_{3x} \\ f_{2y}+f_{3y}-m_2G \\ -f_{2x}asin\theta+f_{2y}acos\theta-f_{3x}bsin\psi+f_{3y}(g+bcos\psi)-m_2G(acos\theta+\frac{h}{2}\cos\left(\theta+\phi\right))\end{bmatrix}$$

$$\begin{bmatrix} m_2acos\theta\dot{\phi}\dot{\theta}-m_2asin\theta\left(\ddot{\theta}+\ddot{\phi}\right)-\frac{m_2h}{2}\sin\left(\theta+\phi\right)\left(\ddot{\theta}+\ddot{\phi}\right)-m_2acos\theta\dot{\theta}(\dot{\theta}+\dot{\phi})-\frac{m_2h}{2}\cos(\theta+\phi)\left(\dot{\theta}+\dot{\phi}\right)^2 \\ m_2asin\theta\dot{\phi}\dot{\theta}+m_2acos\theta\left(\ddot{\theta}+\ddot{\phi}\right)+\frac{m_2h}{2}\cos\left(\theta+\phi\right)\left(\ddot{\theta}+\ddot{\phi}\right)-m_2asin\theta\dot{\theta}(\dot{\theta}+\dot{\phi})-\frac{m_2h}{2}\sin(\theta+\phi)\left(\dot{\theta}+\dot{\phi}\right)^2 \\ -\frac{m_2ah}{2}\dot{\theta}\dot{\phi}sin\phi+m_2\left(\ddot{\theta}+\ddot{\phi}\right)\left(a^2+\frac{h^2}{4}+ahcos\phi\right)+\frac{m_2ah}{2}sin\phi\dot{\phi}(\dot{\theta}+\dot{\phi})\end{bmatrix}$$

$$=\begin{bmatrix} f_{2x}+f_{3x} \\ f_{2y}+f_{3y}-m_2G \\ -f_{2x}asin\theta+f_{2y}acos\theta-f_{3x}bsin\psi+f_{3y}(g+bcos\psi)-m_2G(acos\theta+\frac{h}{2}\cos\left(\theta+\phi\right))\end{bmatrix}$$

$$Equation3\text{:}\ M_3^s\dot{V}_{a3}^s + \dot{M}_3^s V_{a3}^s = \begin{bmatrix} f_3 + f_4 + f_{3g} \\ f_3^T J\vec{B} + f_4^T J\vec{C} + f_{3g}^T J\overrightarrow{q_3} \end{bmatrix}$$

$$\begin{bmatrix} m_3 I_{2\times2} & m_3 J q_3 \\ (m_3 J q_3)^T & m_3 q_3^T q_3 \end{bmatrix}\begin{bmatrix} 0 \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & m_3 J \dot{q}_3 \\ (m_3 J \dot{q}_3)^T & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} f_3 + f_4 + f_{3g} \\ f_3^T J\vec{B} + f_4^T J\vec{C} + f_{3g}^T J\overrightarrow{q_3} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{m_3 b}{2} sin\psi\ddot{\psi} \\ m_3\left(g + \frac{b}{2}cos\psi\right)\ddot{\psi} \\ m_3\left(g^2 + \frac{b^2}{4} + gbcos\psi\right)\ddot{\psi} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}m_3 bcos\psi\dot{\psi}^2 \\ -\frac{1}{2}m_3 bsin\psi\dot{\psi}^2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_{3x} + f_{4x} \\ f_{3y} + f_{4y} - m_3 G \\ -f_{3x} bsin\psi + f_{3y}(g + bcos\psi) + f_{4x} g - m_3 G(g + \frac{b}{2}cos\psi) \end{bmatrix}$$

$$\begin{bmatrix} -\frac{m_3 b}{2} sin\psi\ddot{\psi} - \frac{1}{2}m_3 bcos\psi\dot{\psi}^2 \\ m_3\left(g + \frac{b}{2}cos\psi\right)\ddot{\psi} - \frac{1}{2}m_3 bsin\psi\dot{\psi}^2 \\ m_3\left(g^2 + \frac{b^2}{4} + gbcos\psi\right)\ddot{\psi} \end{bmatrix}$$

$$= \begin{bmatrix} f_{3x} + f_{4x} \\ f_{3y} + f_{4y} - m_3 G \\ -f_{3x} bsin\psi + f_{3y}(g + bcos\psi) + f_{4x} g - m_3 G(g + \frac{b}{2}cos\psi) \end{bmatrix}$$

整理三个 Newton – Euler Equations 得到:

$$
\begin{bmatrix}
-m_1 y_1 \ddot{\theta} - \frac{1}{2} m_1 a cos\theta \dot{\theta}^2 \\
m_1 x_1 \ddot{\theta} - \frac{1}{2} m_1 a sin\theta \dot{\theta}^2 + m_1 G \\
\frac{1}{4} a^2 m_1 \ddot{\theta} + \frac{1}{2} m_1 G a cos\theta \\
m_2 a cos\theta \dot{\phi}\dot{\theta} - m_2 a sin\theta \left(\ddot{\theta} + \ddot{\phi}\right) - \frac{m_2 h}{2} \sin(\theta + \phi)\left(\ddot{\theta} + \ddot{\phi}\right) - m_2 a cos\theta \dot{\theta}(\dot{\theta} + \dot{\phi}) - \frac{m_2 h}{2} \cos(\theta + \phi)\left(\dot{\theta} + \dot{\phi}\right)^2 \\
m_2 a sin\theta \dot{\phi}\dot{\theta} + m_2 a cos\theta \left(\ddot{\theta} + \ddot{\phi}\right) + \frac{m_2 h}{2} \cos(\theta + \phi)\left(\ddot{\theta} + \ddot{\phi}\right) - m_2 a sin\theta \dot{\theta}\left(\dot{\theta} + \dot{\phi}\right) - \frac{m_2 h}{2} \sin(\theta + \phi)\left(\dot{\theta} + \dot{\phi}\right)^2 + m_2 G \\
-\frac{m_2 a h}{2} \dot{\theta}\dot{\phi} sin\phi + m_2 \left(\ddot{\theta} + \ddot{\phi}\right)\left(a^2 + \frac{h^2}{4} + a h cos\phi\right) + \frac{m_2 a h}{2} sin\phi \dot{\phi}\left(\dot{\theta} + \dot{\phi}\right) + m_2 G\left(a cos\theta + \frac{h}{2} \cos(\theta + \phi)\right) \\
-\frac{m_3 b}{2} sin\psi \ddot{\psi} - \frac{1}{2} m_3 b cos\psi \dot{\psi}^2 \\
m_3 \left(g + \frac{b}{2} cos\psi\right) \ddot{\psi} - \frac{1}{2} m_3 b sin\psi \dot{\psi}^2 - m_3 G \\
m_3 \left(g^2 + \frac{b^2}{4} + g b cos\psi\right) \ddot{\psi} + m_3 G\left(g + \frac{b}{2} cos\psi\right)
\end{bmatrix}
$$

$$
= \begin{bmatrix}
f_{1x} + f_{2x} \\
f_{1y} + f_{2y} \\
\tau - a sin\theta f_{2x} + a cos\theta f_{2y} \\
f_{2x} + f_{3x} \\
f_{2y} + f_{3y} \\
-f_{2x} a sin\theta + f_{2y} a cos\theta - f_{3x} b sin\psi + f_{3y}(g + b cos\psi) \\
f_{3x} + f_{4x} \\
f_{3y} + f_{4y} \\
-f_{3x} b sin\psi + f_{3y}(g + b cos\psi) + f_{4x} g
\end{bmatrix}
$$

化为 $Ax = B$ 形式的矩阵方程:

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -asin\theta & acos\theta & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -asin\theta & acos\theta & -bsin\psi & g+bcos\psi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -bsin\psi & g+bcos\psi & g & 0 & 0
\end{bmatrix}
\begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \\ \tau \end{bmatrix}
$$

$$
= \begin{bmatrix}
-m_1 y_1 \ddot{\theta} - \frac{1}{2} m_1 acos\theta \dot{\theta}^2 \\
m_1 x_1 \ddot{\theta} - \frac{1}{2} m_1 asin\theta \dot{\theta}^2 + m_1 G \\
\frac{1}{4} a^2 m_1 \ddot{\theta} + \frac{1}{2} m_1 G acos\theta \\
m_2 acos\theta \dot{\phi} \dot{\theta} - m_2 asin\theta(\ddot{\theta} + \ddot{\phi}) - \frac{m_2 h}{2} \sin(\theta+\phi)(\ddot{\theta} + \ddot{\phi}) - m_2 acos\theta \dot{\theta}(\dot{\theta} + \dot{\phi}) - \frac{m_2 h}{2} \cos(\theta+\phi) (\dot{\theta} + \dot{\phi})^2 \\
m_2 asin\theta \dot{\phi} \dot{\theta} + m_2 acos\theta(\ddot{\theta} + \ddot{\phi}) + \frac{m_2 h}{2} \cos(\theta+\phi)(\ddot{\theta} + \ddot{\phi}) - m_2 asin\theta \dot{\theta}(\dot{\theta} + \dot{\phi}) - \frac{m_2 h}{2} \sin(\theta+\phi) (\dot{\theta} + \dot{\phi})^2 + m_2 G \\
-\frac{m_2 a h}{2} \dot{\theta} \dot{\phi} sin\phi + m_2(\ddot{\theta} + \ddot{\phi}) \left(a^2 + \frac{h^2}{4} + ahcos\phi \right) + \frac{m_2 a h}{2} sin\phi \dot{\phi}(\dot{\theta} + \dot{\phi}) + m_2 G(acos\theta + \frac{h}{2} \cos(\theta+\phi)) \\
-\frac{m_3 b}{2} sin\psi \ddot{\psi} - \frac{1}{2} m_3 bcos\psi \dot{\psi}^2 \\
m_3 \left(g + \frac{b}{2} cos\psi \right) \ddot{\psi} - \frac{1}{2} m_3 bsin\psi \dot{\psi}^2 - m_3 G \\
m_3 \left(g^2 + \frac{b^2}{4} + gbcos\psi \right) \ddot{\psi} + m_3 G(g + \frac{b}{2} cos\psi)
\end{bmatrix}
$$