From D'Alembert principle to Hamilton's principle

If we restrict the virtual displacements δq_i to those such that $\delta q_i(t_0) = \delta q_i(t_f) = 0, \mathrm{i} = 1, \dots, n,$ we see that PVW leads to:

$$\int_{t_0}^{t_f} \delta L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = \delta \underbrace{\int_{t_0}^{t_f} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt}_{A} = 0$$

where $\boxed{A \triangleq \int_{t_0}^{t_f} L\, dt \text{ is called the } \textit{action integral}}$. Note that A is a function of the trajectory $q_i(t)$'s, $i = 1, \dots, n$.

Question: Given the beginning and end $q_i(t_0), q_i(t_f), i = 1, \dots, n$ of the trajectory of the system, which one is the actual trajectory that's happening?

Hamilton says, if the action integral A is stationary for a particular trajectory, i.e. $\delta A = 0$, then it is the actual trajectory of the system! This is called *the Hamilton's principle*.

William Rowan Hamilton (1805-1865)

From Hamilton's principle to Euler-Lagrange equation

Starting with a given Lagrangian $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$ w.r.t. to generalized coordinates $(q_1, \dots, q_n) \in \mathbb{R}^n$, Hamilton's principle leads to:

$$\delta A = \int_{t_0}^{t_f} \delta L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = \int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

$$= \int_{t_0}^{t_f} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i dt + \int_{t_0}^{t_f} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\delta q_i(t) = \int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right) dt + \cancelto{0}{\left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_0}^{t_f}}$$

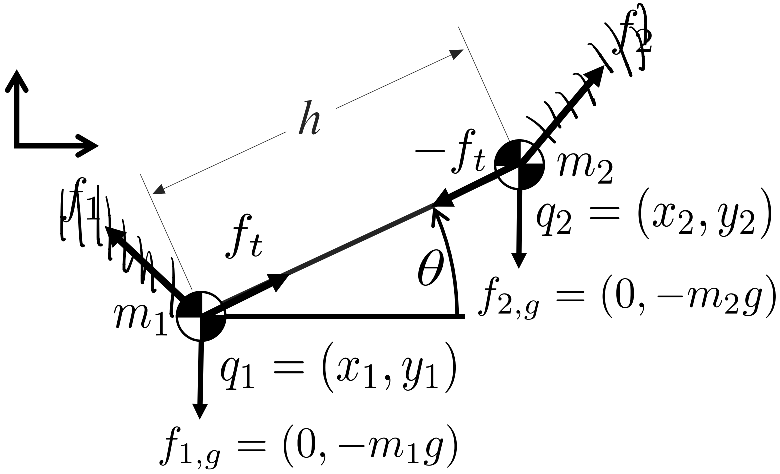
$$= \boxed{\int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i dt = 0}$$

Since δq_i's are arbitrarily chosen other than the fact that $\delta q_i(t_0) = \delta q_i(t_f) = 0$, we conclude that:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad i = 1, \dots, n$$

This is called *Euler-Lagrange equations*, a set of second order ordinary differential equations.

In Hamilton's principle and Euler-Lagrange equation, $q_1, \dots q_n$; $\dot{q}_1, \dots, \dot{q}_n$ can be chosen to be any generalized coordinates, not just Cartesian coordinates!



Example: in the two-particle system, q_1, q_2 are not independent:

$$q_2 = q_1 + h \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

and its Lagrangian becomes a function of $q_1, \dot{q}_1, \theta, \dot{\theta}$; we have:

$L(q_1, \dot{q}_1, \theta, \dot{\theta})$

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right)^T \delta q_1 + \left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\right) \delta\theta \right] dt = 0$$

Since δq_1 (it is a vector and has the component $\delta x_1, \delta y_1$) and $\delta\theta$ are linearly independent:

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

广义坐标均线性无关？

非保守广义力Q 未包含进L表达式中

Euler-Lagrange equation – generalization

系统内力成对出现，做功抵消，故不出现在表达式中

For systems with non-conservative generalized forces Q_i's (we will see how to derive Q_i's in a moment) associated with generalized coordinates q_i's:

δq_i

$$\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + Q_i\right) = 0, \quad i = 1, \ldots, n \qquad \text{or} \qquad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i,} \quad i = 1, \ldots, n$$

For systems with holonomic constraints $h_1(q_1, \ldots, q_n) = 0, \ldots, h_m(q_1, \ldots, q_n) = 0$, not all virtual displacements δq_i's are linear independent; we use *Lagrange multipliers $\lambda_1, \ldots, \lambda_m$ (they are new unknown variables now!)* to eliminate redundant virtual displacements.

$$\int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i}\right) \delta q_i \, dt = 0$$

which is the same as (because $\delta(\lambda_j h_j) = \lambda_j \delta h_j + \delta\lambda_j h_j = \lambda_j \delta h_j$ since $h_j = 0$):

$$\int_{t_0}^{t_f} \delta L + \sum_{j=1}^{m} \lambda_j \delta h_j \, dt = \delta \int_{t_0}^{t_f} \underbrace{\left(L + \sum_{j=1}^{m} \lambda_j h_j\right)}_{\bar{L}} dt = 0, \qquad h_1 = h_2 = \cdots = h_m = 0$$

Euler-Lagrange equation – generalization

We define *augmented Lagrangian* $\bar{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, \lambda_1, \dots, \lambda_m) \triangleq L + \lambda_1 h_1 + \cdots + \lambda_m h_m$, then Hamilton's principle still holds for the augmented Lagrangian $\bar{L}$ in place of L:

$$\int_{t_0}^{t_f} (\delta L + \sum_{j=1}^{m} \lambda_j \delta h_j) dt = \delta \int_{t_0}^{t_f} \underbrace{(L + \sum_{j=1}^{m} \lambda_j h_j)}_{\bar{L}}\, dt = 0, \qquad h_1 = h_2 = \cdots = h_m = 0$$

Now the Euler-Lagrange equation becomes (along with $h_1 = \cdots = h_m = 0$):

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + Q_i + \lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i} = 0, \quad i = 1, \dots, n$$

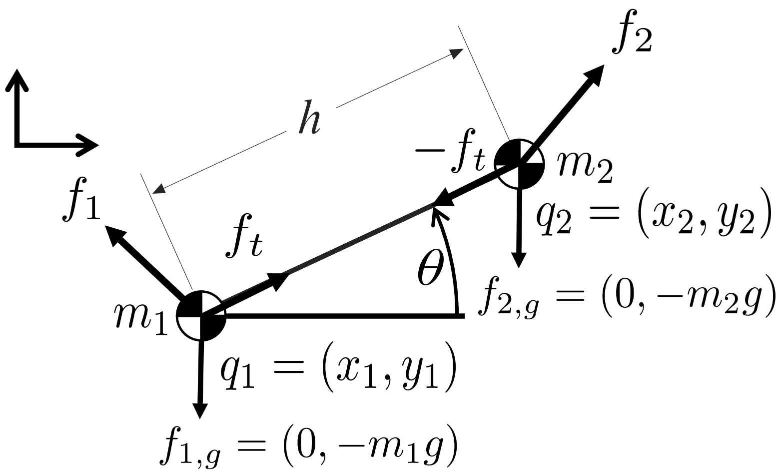
or

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}}_{\text{All inertial force and conservative forces go here}} = \underbrace{Q_i}_{\text{Whatever forces not covered by } L} + \underbrace{\lambda_1 \frac{\partial h_1}{\partial q_i} + \cdots + \lambda_m \frac{\partial h_m}{\partial q_i}}_{\text{Constraint forces}}, \quad i = 1, \dots, n$$

Back to the example with two particles – Approach 1

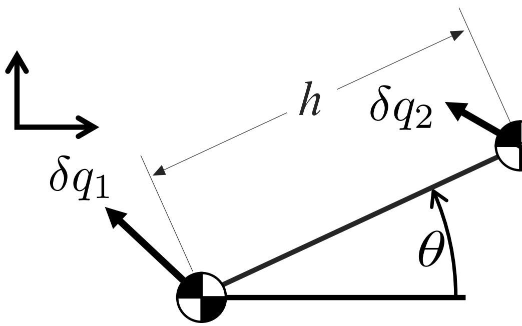
We now recover Newton's equation for the example with two particles. We have:

$$L = T - V = \frac{1}{2}m_1\dot{q}_1^T\dot{q}_1 + \frac{1}{2}m_2\dot{q}_2^T\dot{q}_2 - m_1gy_1 - m_2gy_2$$



We first express L as function of the generalized coordinates x_1, y_1, θ:

$$\dot{q}_2 = \begin{bmatrix} \dot{x}_1 - h\sin\theta\dot{\theta} \\ \dot{y}_1 + h\cos\theta\dot{\theta} \end{bmatrix}, \qquad y_2 = y_1 + h\sin\theta$$

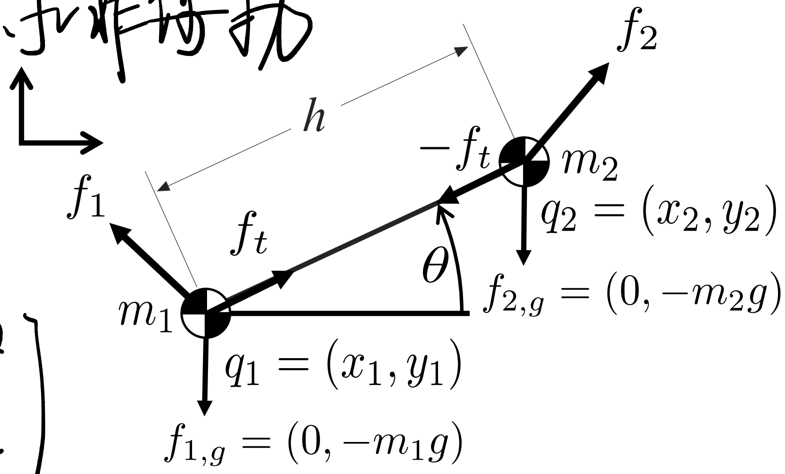


$$L = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2((\dot{x}_1 - h\sin\theta\dot{\theta})^2 + \cdots$$
$$(\dot{y}_1 + h\cos\theta\dot{\theta})^2) - m_1gy_1 - m_2g(y_1 + h\sin\theta)$$

Back to the example with two particles – Approach 1

To derive generalized force from f_1 and f_2 , we go back one step in Hamilton's principle:

f_1, f_2 非保守力



$$\int_{t_0}^{t_f} (\delta L + f_1^T \delta \cancel{q}_1 + f_2^T \delta \cancel{q}_2) dt = 0$$

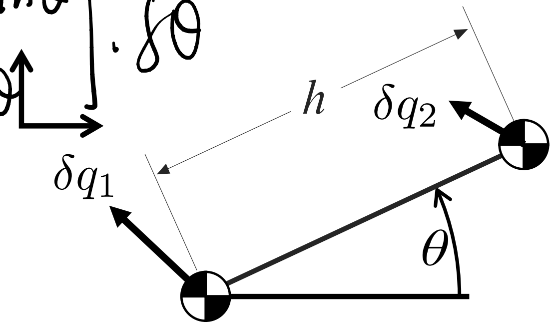
We recall that:

$$q_2 = q_1 + h \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\delta q_2 = \delta q_1 + h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \delta\theta$$

$$\delta q_2 = \delta q_1 + h \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \cdot \delta\theta$$

Then we



$$\int_{t_0}^{t_f} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1}\right) + f_1 + f_2 \right)^T \delta q_1 + \cdots$$
$$\left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) + f_2^T h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right) \delta\theta dt = 0$$

广义坐标θ对应的非保守力不确定，积分计算

$$\int_{t_0}^{t_f} (\delta L + f_1^T \delta q_1 + f_2^T \delta q_2)\, dt = 0$$

$$\int_{t_0}^{t_f} \Big[\Big(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_1}\Big)\Big) \delta q_1 + \Big(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big)\Big) \delta\theta + \cdots$$

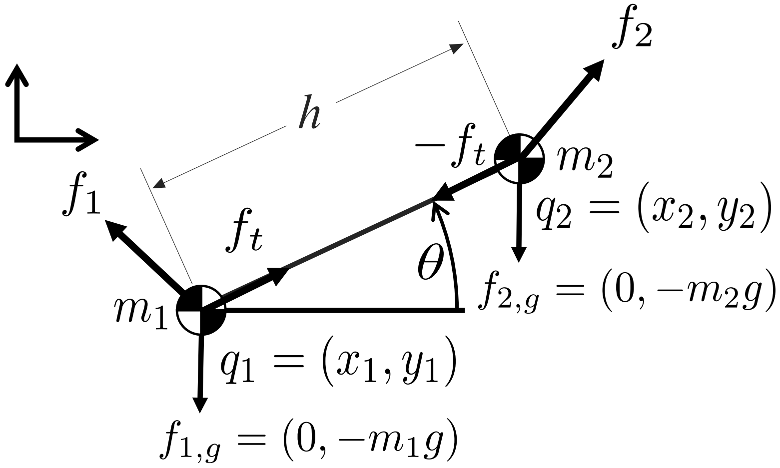
$$+ f_1^T \cdot \delta q_1 + f_2^T \cdot \Big(\delta q_1 + h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \delta\theta \Big) \Big] dt$$

$$= \int_{t_0}^{t_f} \Bigg(\Big(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_1}\Big) + f_1^T + f_2^T \Big) \cdot \delta q_1 + \cdots$$

$$\Big(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big) + f_2^T \cdot h \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \Big) \cdot \delta\theta \Bigg)\, dt$$

Back to the example with two particles – Approach 1

The corresponding Euler-Lagrange equations are (they are three equations):



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = f_1 + f_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = f_2^T h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

方程

which leads to

$$\begin{aligned}
m_1\ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) &= f_{1,x} + f_{2,x} \\
m_1\ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) &= f_{1,y} + f_{2,y} - (m_1 + m_2)g \\
m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) &= -f_{2,x} h\sin\theta + f_{2,y} h\cos\theta - m_2 g h\cos\theta
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \vec{f}_1 + \vec{f}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = f_2^T h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$L = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2((\dot{x}_1 - h\sin\theta\dot{\theta})^2 + \cdots$$
$$(\dot{y}_1 + h\cos\theta\dot{\theta})^2) - m_1 g y_1 - m_2 g(y_1 + h\sin\theta)$$

$$\dot{q}_2 = \begin{bmatrix} q_{1,x} - h\sin\theta \\ q_{1,y} + h\cos\theta \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = f_{1,x} + f_{2,x}$$

$$\frac{\partial L}{\partial \dot{y}_1} = m_1 \dot{y}_1 + m_2 \cdot (\dot{y}_1 + h\cos\theta \cdot \dot{\theta})$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 + m_2 \cdot (\dot{x}_1 - h\sin\theta\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \cdot \ddot{x}_1 + m_2 \cdot (\ddot{x}_1 - h \cdot (\cos\theta \cdot \dot{\theta} \cdot \dot{\theta} + \sin\theta \cdot \ddot{\theta}))$$

$$\frac{\partial L}{\partial x_1} = 0$$

$$L = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2((\dot{x}_1 - h\sin\theta\dot{\theta})^2 + \cdots$$
$$(\dot{y}_1 + h\cos\theta\dot{\theta})^2) - m_1 g y_1 - m_2 g(y_1 + h\sin\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2(\dot{x}_1 - h\sin\theta \cdot \dot{\theta}) \cdot (-h\sin\theta) + m_2(\dot{y}_1 + h\cos\theta\,\dot{\theta}) \cdot (h\cos\theta)$$
$$= m_2(-h\sin\theta\,\dot{x}_1 + h^2\sin^2\theta\,\dot{\theta} + h\cos\theta\,\dot{y}_1 + h^2\cos^2\theta\,\dot{\theta})$$
$$= m_2 h\left[(\cos\theta\,\dot{y}_1 - \sin\theta\,\dot{x}_1) + h\dot{\theta}\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 h\left[(-\sin\theta \cdot \dot{\theta}\,\dot{y}_1 + \cos\theta\,\ddot{y}_1) - (\cos\theta\,\dot{\theta}\,\dot{x}_1 + \sin\theta \cdot \ddot{x}_1) + h\ddot{\theta}\right]$$

$$\frac{\partial L}{\partial \theta} = m_2 \cdot \left[(\dot{x}_1 - h\sin\theta\,\dot{\theta}) \cdot (-h\cos\theta \cdot \dot{\theta}) + (\dot{y}_1 + h\cos\theta\,\dot{\theta}) \cdot (-h\sin\theta\,\dot{\theta})\right]$$
$$- m_2 g h\cos\theta$$

$$= m_2\left(-h\dot{x}_1\cos\theta\,\dot{\theta} - h\dot{y}_1\sin\theta\,\dot{\theta}\right) - m_2 g h\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m_2 h\left[\cancel{-\sin\theta\,\dot{\theta}\,\dot{y}_1} + \cos\theta\,\ddot{y}_1 \cancel{- \cos\theta\,\dot{\theta}\,\dot{x}_1} - \sin\theta\,\ddot{x}_1 + h\ddot{\theta}\right)$$

$$- m_2 h\left(\cancel{-\cos\theta\,\dot{\theta}\,\dot{x}_1} \cancel{- \sin\theta\,\dot{\theta}\,\dot{y}_1}\right) + m_2 g h\cos\theta$$

$$= m_2 h\left(\cos\theta\,\ddot{y}_1 - \sin\theta\,\ddot{x}_1 + h\ddot{\theta}\right) + m_2 g h\cos\theta$$

$$= h\left[f_{21x},\ f_{21y}\right]\cdot\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$m_2\cos\theta\,\ddot{y}_1 - m_2\sin\theta\,\ddot{x}_1 + m_2 h\ddot{\theta} = -f_{21x}\sin\theta + f_{21y}\cos\theta - m_2 g\cos\theta$$

Back to the example with two particles – Approach 1

We can see that:

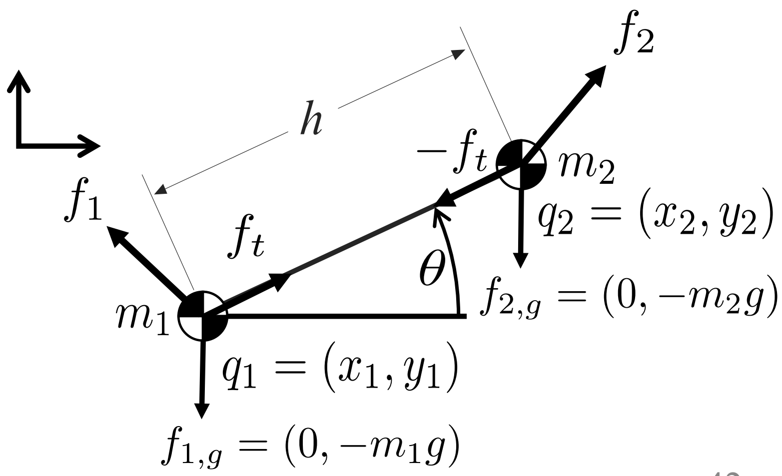
牛

$$m_1\ddot{x}_1 + m_2\underbrace{\left(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}\right)}_{\ddot{x}_2} = f_{1,x} + f_{2,x}$$

$$m_1\ddot{y}_1 + m_2\underbrace{\left(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}\right)}_{\ddot{y}_2} = f_{1,y} + f_{2,y} - (m_1 + m_2)g$$

欧 $\Leftarrow$

$$\underbrace{m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta})}_{-m_2\ddot{x}_2 h\sin\theta + m_2\ddot{y}_2 h\cos\theta} = -f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2 g h\cos\theta$$



Question: do you know what these equations mean?

Question: How to derive f_t? 解方程，得到 $\ddot{x}$, $\ddot{y}$

线性方程组 $\ddot{x}$, $\ddot{y}$, $\ddot{\theta}$

Back to the example with two particles – Approach 2

Just use Cartesian coordinates q_1, q_2, along with the constraint:

因q1 q2作为广义坐标，存在线性相关项，

$$(q_1 - q_2)^T (q_1 - q_2) - h^2 = 0$$

入消去

Now we have to use augmented Lagrangian:

q1，q2 认为不是
独立的
打破了

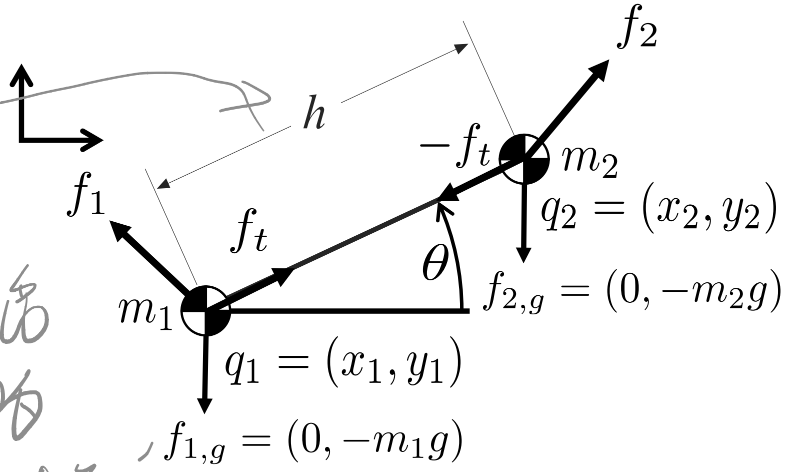
$$\bar{L} = \frac{1}{2} m_1 \dot{q}_1^T \dot{q}_1 + \frac{1}{2} m_2 \dot{q}_2^T \dot{q}_2 - m_1 g y_1 - m_2 g y_2 + \cdots$$
$$\lambda((q_1 - q_2)^T (q_1 - q_2) - h^2)$$

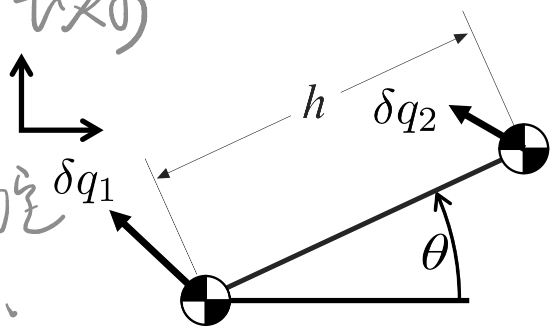
We will get four Euler-Lagrange equations:

每个广义坐标
对应施加非保守力确定
直接写方程即可、

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \bar{L}}{\partial \dot{q}_1}\right) - \dfrac{\partial \bar{L}}{\partial q_1} = f_1 \\ \dfrac{d}{dt}\left(\dfrac{\partial \bar{L}}{\partial \dot{q}_2}\right) - \dfrac{\partial \bar{L}}{\partial q_2} = f_2 \end{cases}$$

$q_1^2 - q_2^2 - h^2$





$$\bar{L} = \frac{1}{2} m_1 \dot{q}_1^T \dot{q}_1 + \frac{1}{2} m_2 \dot{q}_2^T \dot{q}_2 - m_1 g y_1 - m_2 g y_2 + \cdots$$

$$\lambda(\overbrace{(q_1 - q_2)^T (q_1 - q_2)} - h^2)$$

$$m_1 g \begin{bmatrix} 0 \\ q_{1y} \end{bmatrix} \qquad \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}_1}\right) - \frac{\partial \bar{L}}{\partial q_1} = f_1$$

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}_2}\right) - \frac{\partial \bar{L}}{\partial q_2} = f_2$$

$$\frac{\partial \bar{L}}{\partial \dot{q}_1} = \frac{\partial \left(\frac{1}{2} m_1 \dot{q}_1 \cdot \dot{q}_1\right)}{\partial \dot{q}_1} = m_1 \dot{q}_1 \qquad \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}_1}\right) = m_1 \ddot{q}_1$$

$$\left[q_{1x} - q_{2x},\ q_{1y} - q_{2y}\right]$$

$$\frac{\partial \bar{L}}{\partial q_1} = \lambda \cdot 2(q_1 - q_2) - m_1 g \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{(q_{1x} - q_{2x})^2 + (q_{1y} - q_{2y})^2}{\partial \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}}$$

$$\frac{\partial \bar{L}}{\partial \dot{q}_2} = m_2 \dot{q}_2 \qquad \frac{d}{dt}(\ ) = m_2 \ddot{q}_2 \qquad = \begin{pmatrix} 2(q_{1x} - q_{2x}) \\ 2(q_{1y} - q_{2y}) \end{pmatrix} =$$

$$\frac{\partial \bar{L}}{\partial q_2} = -m_2 g \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\lambda \cdot (q_1 - q_2).$$

$$m_1 \ddot{q}_1 = f_1 + 2\lambda (q_1 - q_2) - \begin{bmatrix} 0 \\ m_1 g \end{bmatrix}$$

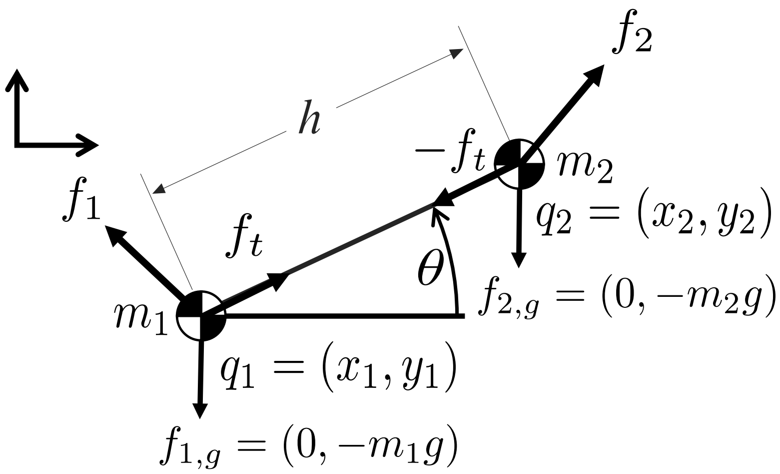
$$m_2 \ddot{q}_2 = f_2 - 2\lambda (q_1 - q_2) - \begin{bmatrix} 0 \\ m_2 g \end{bmatrix}$$

Back to the example with two particles – Approach 2

The Euler-Lagrange equations become:

$$m_1\ddot{q}_1 = f_1 + 2\lambda(q_1 - q_2) - \begin{bmatrix} 0 \\ m_1 g \end{bmatrix}$$

$$m_2\ddot{q}_2 = f_2 - 2\lambda(q_1 - q_2) - \begin{bmatrix} 0 \\ m_2 g \end{bmatrix}$$



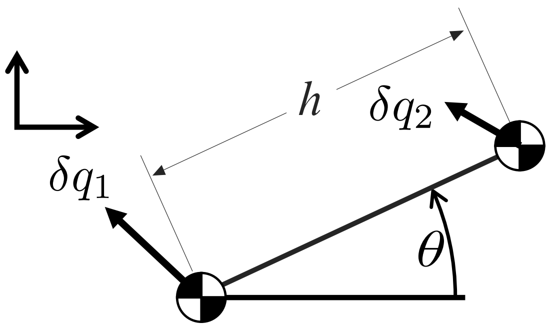
You can easily see now that $f_t = \lambda(q_1 - q_2)$!

Question: But how do we solve λ?

$$(q_1 - q_2)^T(q_1 - q_2) - h^2 = 0 \Rightarrow$$

2次求导

$$(q_1 - q_2)^T(\ddot{q}_1 - \ddot{q}_2) = -(\dot{q}_1 - \dot{q}_2)^T(\dot{q}_1 - \dot{q}_2)$$



$$(\dot{q}_1 - \dot{q}_2)^T(q_1 - q_2) + (q_1 - q_2)^T\cdot(\dot{q}_1 - \dot{q}_2) = 0$$

$$(\ddot{q}_1 - \ddot{q}_2)^T(q_1 - q_2) + (\dot{q}_1 - \dot{q}_2)^T(\dot{q}_1 - \dot{q}_2) + (\dot{q}_1 - \dot{q}_2)^T(\dot{q}_1 - \dot{q}_2) + (q_1 - q_2)^T(\ddot{q}_1 - \ddot{q}_2)$$

故 $$2(q_1-q_2)^T(\ddot{q}_1-\ddot{q}_2) = -2(\dot{q}_1-\dot{q}_2)^T(\dot{q}_1-\dot{q}_2)$$

由拉-格推出的2个方程；

$$m_1\ddot{q}_1 = f_1 + 2\lambda(q_1-q_2) - \begin{bmatrix} 0 \\ m_1 g \end{bmatrix}$$

$$m_2\ddot{q}_2 = f_2 - 2\lambda(q_1-q_2) - \begin{bmatrix} 0 \\ m_2 g \end{bmatrix}$$

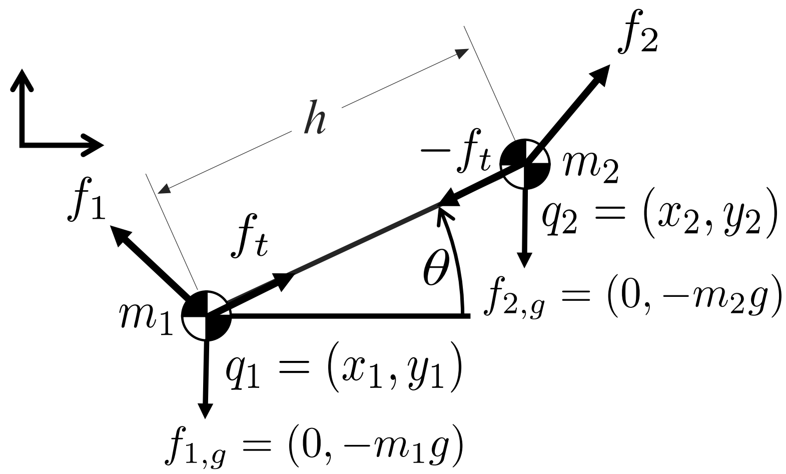
$$\ddot{q}_1 - \ddot{q}_2 = \frac{f_1}{m_1} + \frac{2\lambda}{m_1}(q_1-q_2) - \begin{bmatrix} 0 \\ g \end{bmatrix} - \frac{f_2}{m_2} + \frac{2\lambda}{m_2}(q_1-q_2) + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\ddot{q}_1 - \ddot{q}_2 = \frac{f_1}{m_1} - \frac{f_2}{m_2} + (q_1-q_2)\left(\frac{2\lambda}{m_1} + \frac{2\lambda}{m_2}\right)$$

Back to the example with two particles – Approach 2

From the E-L equations:

$$\ddot{q}_1 - \ddot{q}_2 = \frac{f_1}{m_1} - \frac{f_2}{m_2} + \lambda\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(q_1 - q_2)$$



Multiply on both sides $(q_1 - q_2)^T$:

$$-(\dot{q}_1 - \dot{q}_2)^T(\dot{q}_1 - \dot{q}_2) = (q_1 - q_2)^T \frac{m_2 f_1 - m_1 f_2}{m_1 m_2} + \lambda \frac{h^2(m_1 + m_2)}{m_1 m_2}$$

$$\dot{q}_2 - \dot{q}_1 = h \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \dot{\theta}$$

$$\lambda = -\frac{m_1 m_2}{m_1 + m_2}\dot{\theta}^2 + \frac{(m_2 f_{1,x} - m_1 f_{2,x})\cos\theta + (m_2 f_{1,y} - m_1 f_{2,y})\sin\theta}{h(m_1 + m_2)}$$

$$\begin{pmatrix} -h\sin\theta\,\dot\theta \\ h\cos\theta\,\dot\theta \end{pmatrix} - (\ \ J_1\ \ ) = -[h\sin\theta\,\dot\theta \quad -h\cos\theta\,\dot\theta] \begin{bmatrix} h\sin\theta\,\dot\theta \\ h\cos\theta\,\dot\theta \end{bmatrix}$$

$$q_1 - q_2 = -h\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad = -h^2\sin^2\theta\,\dot\theta^2 - h^2\cos^2\theta\,\dot\theta^2 = -h^2$$

$$-h\dot\theta^2 = [\cos\theta \quad \sin\theta]\frac{\begin{bmatrix} f_{1x} \\ f_{1y} \end{bmatrix} m_2 - \begin{bmatrix} f_{2y} \\ f_{2y} \end{bmatrix} m_1}{m_1 m_2} + \frac{2\lambda h(m_2+m_1)}{m_1 m_2}$$

$$2\lambda h(m_2+m_1) = -h m_1 m_2 \dot\theta^2 - \left([\cos\theta \ \ \sin\theta]\begin{bmatrix} f_{1x} \\ f_{1y} \end{bmatrix} m_2 - [\cos\theta \ \ \sin\theta]\begin{pmatrix} f_{2y} \\ f_{2y} \end{pmatrix} m_1\right)$$

$$2\lambda h(m_2+m_1) = -h m_1 m_2 \dot\theta^2 - m_2(f_{1x}\cos\theta + f_{1y}\sin\theta) + m_1(f_{2x}\cos\theta + f_{2y}\sin\theta)$$

$$2\lambda h(m_2+m_1) = -h m_1 m_2 \dot\theta^2 - \cos\theta(m_2 f_{1x} - m_1 f_{2x}) - \sin\theta(m_2 f_{1y} - m_1 f_{2y})$$

$$\lambda = \frac{-h m_1 m_2 \dot{\theta}^2}{2h(m_2+m_1)} - \cos\theta \cdot \frac{m_2 f_{1x} - m_1 f_{2x}}{2h(m_2+m_1)} - \sin\theta \cdot \frac{m_2 f_{1y} - m_1 f_{2y}}{2h(m_1+m_2)}$$

$$\lambda = \frac{-m_1 m_2 \dot{\theta}^2}{2(m_2+m_1)} + \frac{\cos\theta(m_1 f_{2x} - m_2 f_{1x}) + \sin\theta(m_1 f_{2y} - m_2 f_{1y})}{2h(m_1+m_2)}$$

Back to the example with two particles – Approach 2

leading to:

$$m_1\ddot{q}_1 = f_1 + \left(-\frac{m_1 m_2}{m_1+m_2}\dot{\theta}^2 + \frac{(m_2 f_{1,x} - m_1 f_{2,x})\cos\theta + (m_2 f_{1,y} - m_1 f_{2,y})\sin\theta}{h(m_1+m_2)}\right)(q_1 - q_2) + \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix}$$

$$m_2\ddot{q}_2 = f_2 - \left(-\frac{m_1 m_2}{m_1+m_2}\dot{\theta}^2 + \frac{(m_2 f_{1,x} - m_1 f_{2,x})\cos\theta + (m_2 f_{1,y} - m_1 f_{2,y})\sin\theta}{h(m_1+m_2)}\right)(q_1 - q_2) + \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

But is it equivalent to our previous derivations?

$$\begin{aligned}
m_1\ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) \quad &= f_{1,x} + f_{2,x} \qquad \text{EQ1} \\
m_1\ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) \quad &= f_{1,y} + f_{2,y} - (m_1+m_2)g \qquad \text{EQ2} \\
m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) \quad &= -f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2 g h\cos\theta \qquad \text{EQ3}
\end{aligned}$$

You are about to find out in the written assignment… ;-)

$$m_1 \ddot{x}_1 = f_{1x} + (\ \ ) \cdot (x_1 - x_2) \qquad m_1 \ddot{y}_1 = f_{1y} + (\ \ ) \cdot (y_1 - y_2) - m_1 g$$

$$m_2 \ddot{x}_2 = f_{2x} - (\ \ ) \cdot (x_1 - x_2) \qquad m_2 \ddot{y}_2 = f_{2y} - (\ \ ) \cdot (y_1 - y_2) - m_2 g$$

$$f_{1x} + f_{2x} = m_1 \ddot{x}_1 - (\ \ )(x_1 - x_2) + m_2 \ddot{x}_2 + (\ \ )(\cancel{x_1 - x_2}) = m_1 \ddot{x}_1 + m_2 \ddot{x}_2$$

$$x_2 = x_1 + h\cos\theta$$

$$\dot{x}_2 = \dot{x}_1 - h\sin\theta \cdot \dot{\theta} \qquad \ddot{x}_2 = \ddot{x}_1 - h(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}) \Rightarrow 式1)$$

$$f_{1y} + f_{2y} - (m_1 g + m_2 g) = m_1 \ddot{y}_1 + m_2 \ddot{y}_2$$

$$\Rightarrow 式2$$

$$y_2 = y_1 + h\sin\theta$$

$$\dot{y}_2 = \dot{y}_1 + h\cos\theta\,\dot{\theta} \qquad \ddot{y}_2 = \ddot{y}_1 + h(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta})$$

$$f_{2x} = m_2 \ddot{x}_2 + \lambda(x_1 - x_2)$$

$$f_{2y} - m_2 g = m_2 \ddot{y}_2 + \lambda(y_1 - y_2)$$

$$-f_{2x} h\sin\theta = -h\sin\theta\left(m_2 \ddot{x}_2 + \lambda(x_1 - x_2)\right)$$

$$\cdots = h\cos\theta\left(m_2 \ddot{y}_2 + \lambda(y_1 - y_2)\right)$$

$$-f_{2x} h\sin\theta + f_{2y} h \cdots = -h\sin\theta\left(m_2\left[\ddot{x}_1 - h(\cos\theta\,\dot{\theta} + \sin\theta\,\ddot{\theta})\right] + \lambda(-h\cos\theta)\right)$$

$$+ h\cos\theta\left(m_2\left[+\ddot{y}_1 + h(-\sin\theta\,\dot{\theta} + \cos\theta\,\ddot{\theta})\right] + \lambda(-h\sin\theta)\right)$$

$$= -h\sin\theta\, m_2 \left(\ddot{x}_1 - h\cos\theta\,\dot{\theta} - h\sin\theta\,\ddot{\theta}\right) +$$

$$h\cos\theta\, m_2 \left(-\ddot{y}_1 - h\sin\theta\,\dot{\theta} + h\cos\theta\,\ddot{\theta}\right)$$

$$= -h\sin\theta\, m_2 \ddot{x}_1 + h\cos\theta\, m_2 \ddot{y}_1 + h^2 m_2 \ddot{\theta}$$

刚体速度 $\hat{V}_{ab}$ 是一个变换矩阵 4x4

其乘上刚体上任一点坐标即可得到该点速度

4 Newton-Euler equation

惯量 $\int m r^2 dr$ ⇒ 平行轴/… 定理 ⇒ 惯量在世界坐标架下值会变

但在刚体坐标架下值不变

参考坐标架　　刚体坐标架

Prerequisite: Spatial velocity v.s. body velocity

Recall the definition of spatial velocity of a rigid body:

$$q_a(t) \mapsto \dot{q}_a(t) = \hat{V}_{ab}(t) q_a(t) = (\dot{g}_{ab}(t) g_{ab}^{-1}(t)) q_a = \hat{\omega}_{ab}(t) q_a + v_{ab}(t)$$

Sometimes, its easier to work with the b-frame (body frame). We define:

$q_{1a}(t) = g_{ab} \cdot q_{1b}(t)$

$$\boxed{\dot{q}_b(t) \triangleq g_{ab}^{-1} \dot{q}_a(t)} = \underbrace{g_{ab}^{-1} \dot{g}_{ab}}_{\hat{V}_{ab}^b} q_b$$

b参考系为刚体移动之前的b位置

$\dot{q}_{1a}(t) = \hat{V}_{ab} \cdot q_{1a}(t)$ $\dot{q}_a(t)$ 变换到b参考系

That is, $\dot{q}_b$ is the velocity of point q w.r.t. to a-frame (ref. frame) but observed in b-frame. Here, the *body velocity* $\hat{V}_{ab}^b = g_{ab}^{-1} \dot{g}_{ab}$ is defined similarly to the *spatial velocity* $\hat{V}_{ab} = \dot{g}_{ab} g_{ab}^{-1}$:

$\dot{q}_{1a}(t) = g_{ab}^{-1} \cdot \dot{q}_{1b}(t) =$

$$\hat{V}_{ab}^b = g_{ab}^{-1} \dot{g}_{ab} = \begin{bmatrix} \hat{\omega}_{ab}^b & v_{ab}^b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{ab}^T \dot{R}_{ab} & R_{ab}^T \dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\hat{V}_{ab}^{s(patial)}$$

$$\hat{V}_{ab}^{b(ody)}$$

Note: In MLS94, $\hat{V}_{ab}$ is written as $\hat{V}_{ab}^s$ to further distinguish from $\hat{V}_{ab}^b$. We didn't write it that way until now to avoid confusion. From now on, we shall write $\hat{V}_{ab}^s$ in place of $\hat{V}_{ab}$.

Prerequisite: Spatial velocity v.s. body velocity

The relation between $\hat{V}^s_{ab}$ and $\hat{V}^b_{ab}$ is given by:

$$\boxed{\hat{V}^s_{ab} = \dot{g}_{ab} g^{-1}_{ab} = g_{ab} g^{-1}_{ab} \dot{g}_{ab} g^{-1}_{ab} = g_{ab} \hat{V}^b_{ab} g^{-1}_{ab}}$$

$$\hat{V}^s_{ab} = \begin{bmatrix} \hat{\omega}^s_{ab} & v^s_{ab} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b_{ab} & v^b_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T_{ab} & -R^T_{ab} p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab} \hat{\omega}^b_{ab} R^T_{ab} & -R_{ab} \hat{\omega}^b_{ab} R^T_{ab} p_{ab} + R_{ab} v^b_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\omega^s_{ab} = R_{ab} \omega^b_{ab}, \qquad v^s_{ab} = \hat{p}_{ab} R_{ab} \omega^b_{ab} + R_{ab} v^b_{ab}$$

效果同 g_{ab}，对刚体速度变换

扩

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \underset{6\times 6}{\begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}} \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \underbrace{Ad_{g_{ab}}}_{\text{Adjoint transformation}} V^b_{ab}$$

$V_a = \underset{6\times 6}{Adg_{ab}} \times V_{b\ 6\times 1}$

伴随变换

Planar case:

$$\hat{V}^s_{ab} = \begin{bmatrix} J\dot{\theta} & \dot{p}_{ab} - \dot{\theta} J p_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\hat{V}^b_{ab} = \begin{bmatrix} J\dot{\theta} & R^T_{ab} \dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$v^s_{ab} = \dot{p}_{ab} - \dot{\theta} J p_{ab}, \omega^s_{ab} = \dot{\theta}$$

$$v^b_{ab} = R^T_{ab} \dot{p}_{ab}, \omega^b_{ab} = \dot{\theta}$$

$$J = R(\frac{\pi}{2})$$

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab} & -J p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^b_{ab} \\ \dot{\theta} \end{bmatrix}$$

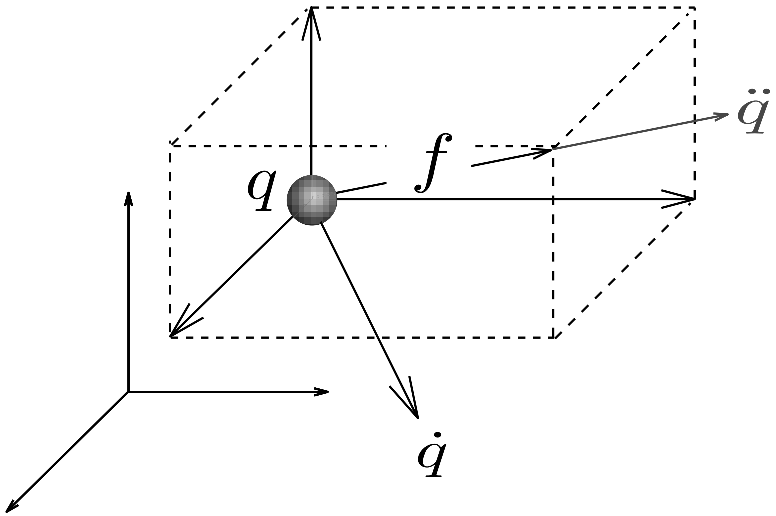
$$= Ad_{g_{ab}} V^b_{ab}$$

Newton-Euler equation – Review of Newton's equation

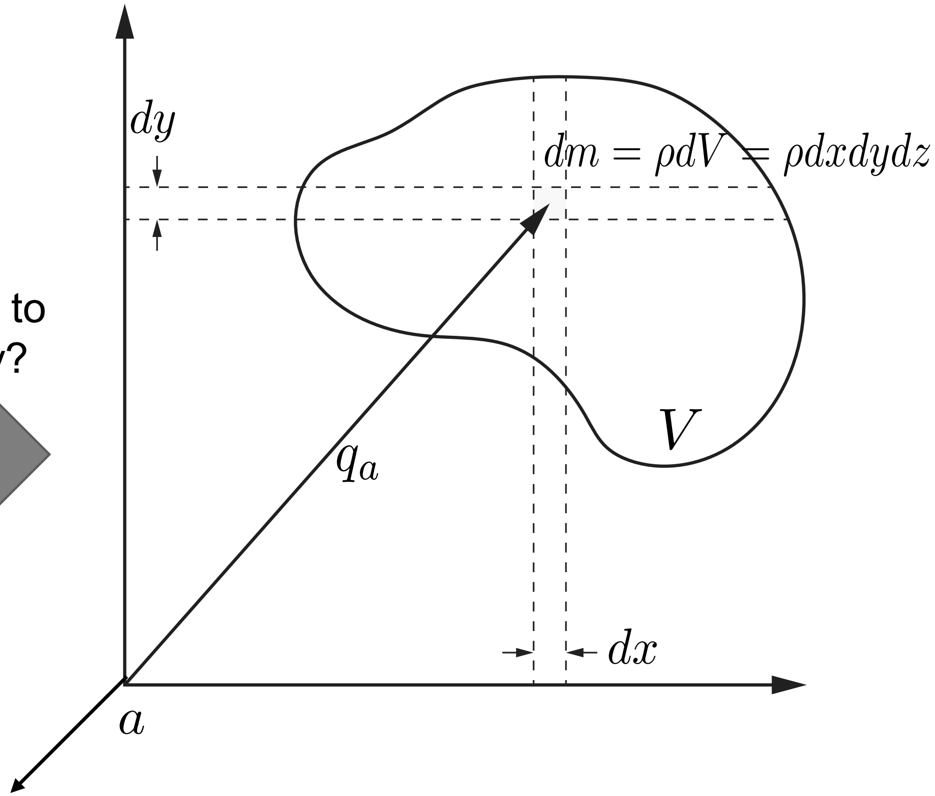
Consider Newton's equation for a point mass:

$$f = \frac{d}{dt}p = \frac{d}{dt}(m\dot{q})$$

where $p = m\dot{q}$ is called the *linear momentum*.



How to generalize to rigid body?



平动-转动的刚体方程

Newton-Euler equation – generalized inertia matrix, generalized momentum

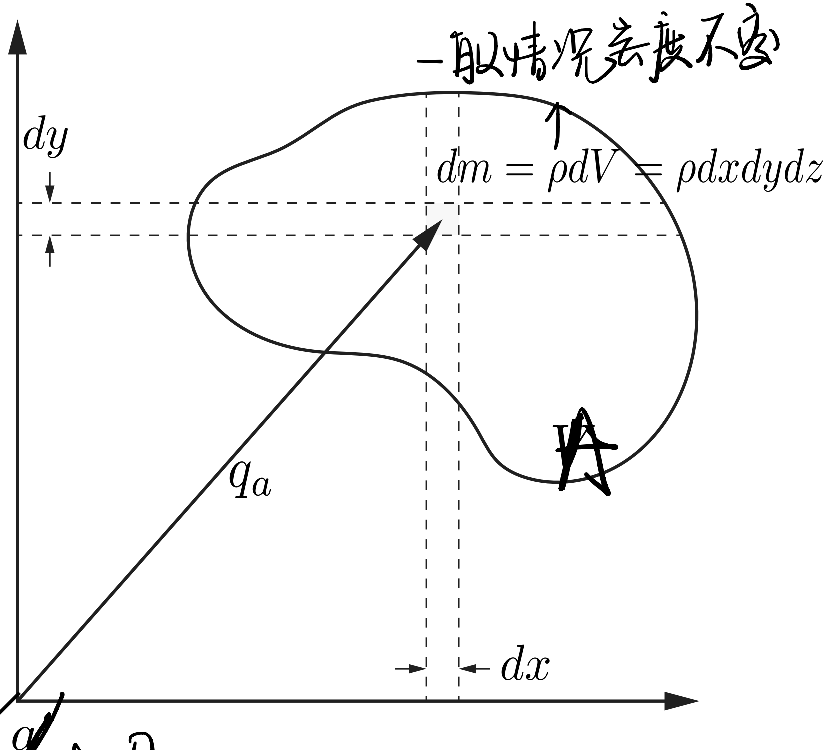
The kinetic energy of a rigid in motion: $dT = \frac{1}{2} dm \cdot (\dot{q}_a^T \dot{q}_a)$ (v^2)

$$T = \frac{1}{2}\int_V \rho \dot{q}_a^T \dot{q}_a dV \qquad T = \frac{1}{2}\int \dot{q}_a^T \dot{q}_a \, dm$$

$$= \frac{1}{2}\int_V \rho(\hat{\omega}^s_{ab} q_a + v^s_{ab})^T(\hat{\omega}^s_{ab} q_a + v^s_{ab}) dV$$

$$= \frac{1}{2}\underbrace{\begin{bmatrix} {v^s_{ab}}^T & {\omega^s_{ab}}^T \end{bmatrix}}_{1\times 6} \underbrace{\begin{bmatrix} mI_{3\times 3} & -\int_V \rho \hat{q}_a dV \\ \int_V \rho \hat{q}_a dV & -\int_V \rho \hat{q}_a^2 dV \end{bmatrix}}_{M^s \text{ 对称阵},\ 6\times 6} \underbrace{\begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}}_{6\times 1}$$

$$= \frac{1}{2}{V^s_{ab}}^T M^s V^s_{ab}, \qquad m = \int_V \rho dV$$



广义惯量矩阵/张量

M^s is called the *generalized inertia matrix* (superscript s means computed in spatial frame a).

We define *generalized momentum* by $M^s V^s_{ab}$.

广义动量 $\quad M^s \to 6\times 6 \quad V^s_{ab} \to 6\times 1$

合并线性动量与角动量

减少了坐标变换次数.(伴随变换)

The $\boxed{\text{Newton-Euler equation}}$ is given by: 力旋量 wrench

$$\frac{d}{dt}\left(M^s V^s_{ab}\right) = M^s \dot{V}^s_{ab} + \boxed{\dot{M}^s V^s_{ab}} = F^s = \begin{bmatrix} f^s \\ \tau^s \end{bmatrix}$$

线性力 3×1

转矩 3×1

6×1

M^s 关于时间变动

(F^s denotes wrench in a-frame)

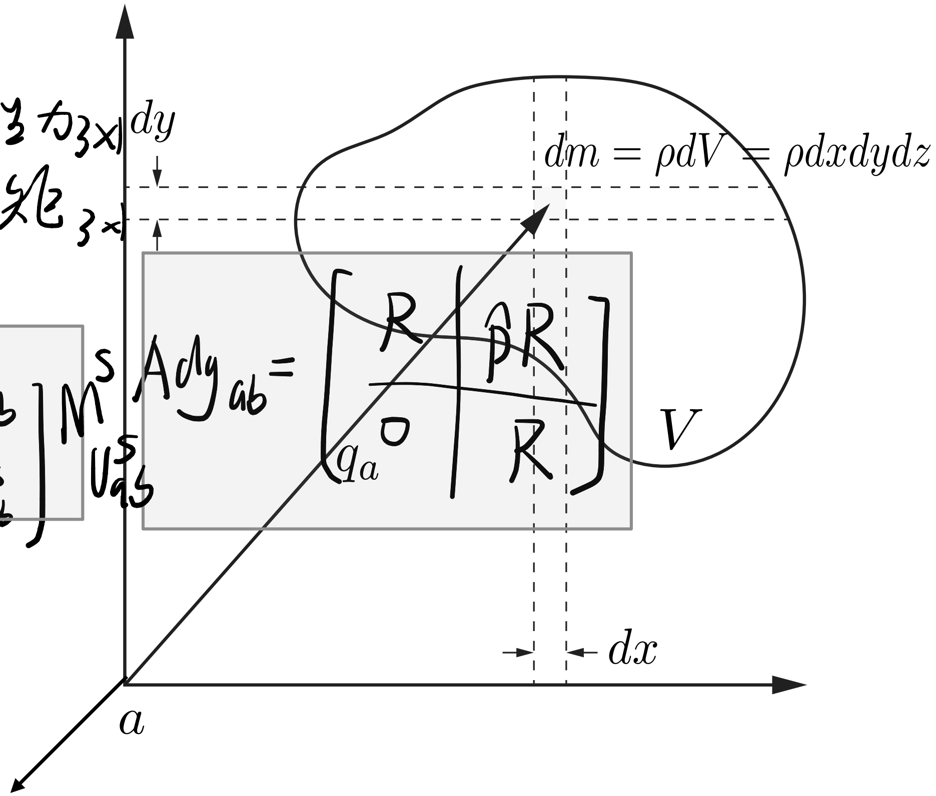
$$\dot{M}^s V^s_{ab} = -\begin{bmatrix} \hat{\omega}^s_{ab} & \hat{v}^s_{ab} \\ 0 & \hat{\omega}^s_{ab} \end{bmatrix} M^s V^s_{ab}$$

which leads to:

$$M^s \dot{V}^s_{ab} - \begin{bmatrix} \hat{\omega}^s_{ab} & \hat{v}^s_{ab} \\ 0 & \hat{\omega}^s_{ab} \end{bmatrix}^T M^s V^s_{ab} = F^s \qquad \text{(3-dimension)}$$

$$M^s \dot{V}^s_{ab} - \begin{bmatrix} J\dot{\theta} & -Jv^s_{ab} \\ 0 & 0 \end{bmatrix}^T M^s V^s_{ab} = F^s \qquad \text{(2-dimension)}$$

牛顿欧拉方程



$$Ad_{g_{ab}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

The Newton-Euler equation is useful for deriving constraint force (Lagrange's equation eats away all constraint forces).

求解约束力 ⇒ 关节是否能承受的力矩 (Adams 原理)

Newton-Euler equation – Simple example (all coordinates w.r.t to a)

$\omega : J\dot{\theta}$?

Recall the planar spatial velocity ($J = R(\pi/2)$): 绕原点旋转

b : CoM 质心 of (m_1 m_2)

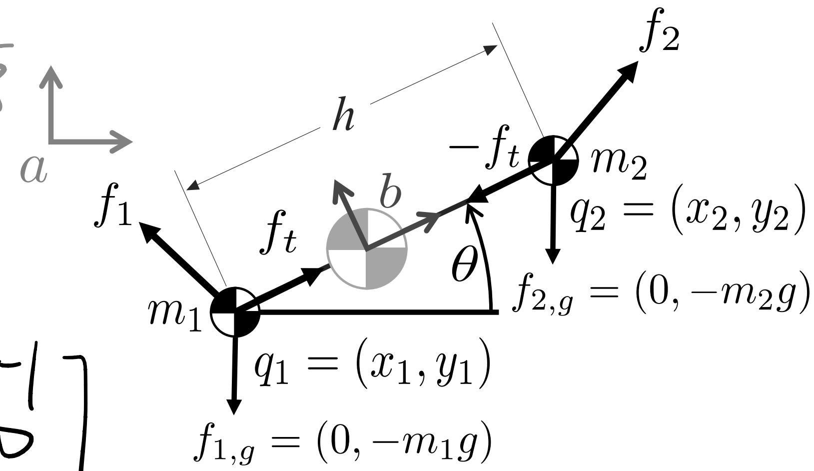
$$\hat{V}^s_{ab} = \begin{bmatrix} \boxed{J}\dot{\theta} & v^s_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_{1a} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{q}_{1a} \\ 0 \end{bmatrix} = \hat{V}^s_{ab} \cdot \begin{bmatrix} q_{1a} \\ 1 \end{bmatrix}$$

$$\dot{q}_{1a} = J\dot{\theta}q_{1a} + v^s_{ab}$$

$$\dot{q}_{2a} = J\dot{\theta}q_{2a} + v^s_{ab}$$

$$\dot{q}_{1a} - J\dot{\theta}q_{1a} = V^s_{ab} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



which leads to:

$$T = \frac{1}{2}m_1(J\dot{\theta}q_{1a} + v^s_{ab})^T(J\dot{\theta}q_{1a} + v^s_{ab}) + \frac{1}{2}m_2(J\dot{\theta}q_{2a} + v^s_{ab})^T(J\dot{\theta}q_{2a} + v^s_{ab})$$

$$= \frac{1}{2}m_1 q_{1a}^T q_{1a}\dot{\theta}^2 + \frac{1}{2}m_2 q_{2a}^T q_{2a}\dot{\theta}^2 + \frac{1}{2}(m_1 + m_2){v^s_{ab}}^T v^s_{ab} + m_1 {v^s_{ab}}^T J q_{1a}\dot{\theta} + m_2 {v^s_{ab}}^T J q_{2a}\dot{\theta}$$

$$= \frac{1}{2}\underbrace{\begin{bmatrix} v^s_{ab} & \dot{\theta} \end{bmatrix}}_{{V^s_{ab}}^T} \underbrace{\begin{bmatrix} (m_1 + m_2)I & m_1 J q_{1a} + m_2 J q_{2a} \\ (m_1 J q_{1a} + m_2 J q_{2a})^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}}_{M^s} \underbrace{\begin{bmatrix} v^s_{ab} \\ \dot{\theta} \end{bmatrix}}_{V^s_{ab}}$$

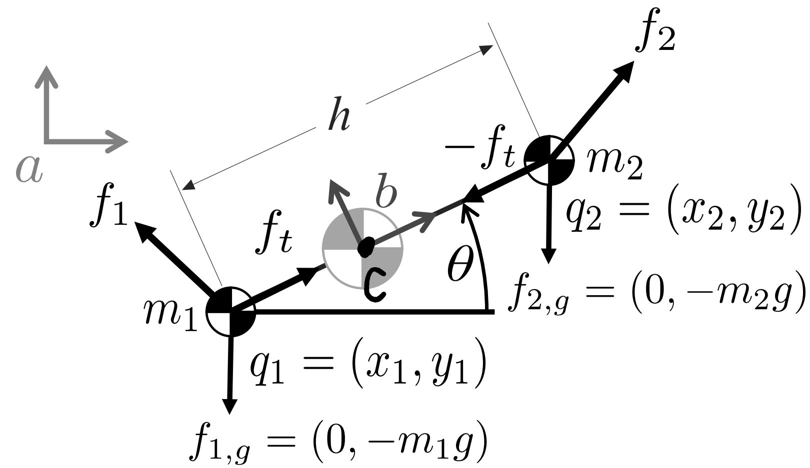
Newton-Euler equation – Simple example (all coordinates w.r.t to a)

Recall the center of mass c is given by:

连续体.

$$m\vec{c} = m_1\vec{q}_{1a} + m_2\vec{q}_{2a} \qquad \vec{c}_a = \frac{m_1\vec{q}_{1a} + m_2\vec{q}_{2a}}{m_1 + m_2} \qquad C = \frac{\int m q}{\Sigma m}$$

$$mJ\vec{c} = m_1 J\vec{q}_{1a} + m_2 J q_{2a}$$



Also denote $m_1 + m_2$ by m for simplicity; then:

$$T = \frac{1}{2}{V_{ab}^s}^T \underbrace{\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}}_{M^s} V_{ab}^s$$

M^s 手动整理

$\dot{M}^s$ 用公式

The N-E equation becomes:

$$M^s \dot{V}_{ab}^s + \dot{M}^s V_{ab}^s = F_1 + F_2 + F_{1,g} + F_{2,g}$$

三個 $\begin{bmatrix} f_1 \\ q_{1a} \times f_1 \end{bmatrix}$

除去内力

Where $F_1, F_2, F_{1,g}, F_{2,g}$ are the wrenches corresponding to $f_1, f_2, f_{1,g}, f_{2,g}$.

包括势能的力 (不同于上一推导)

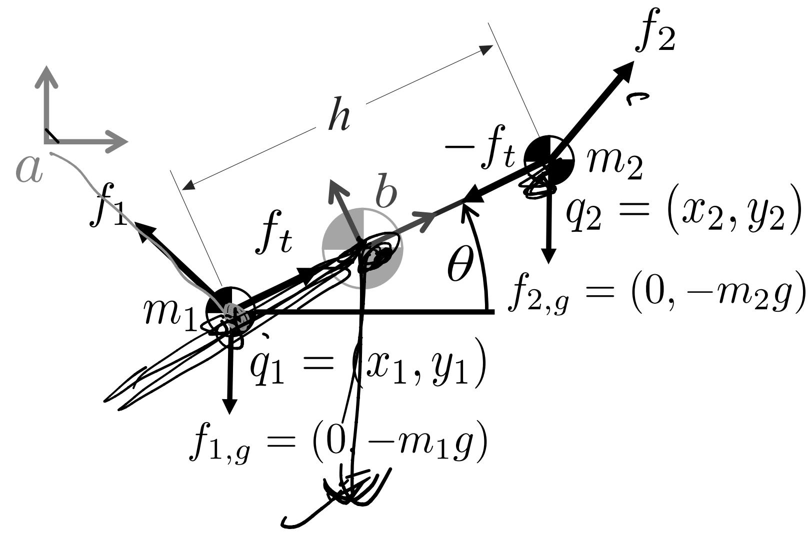
$$F_1 = \begin{bmatrix} f_1 \\ f_1^T J q_{1a} \end{bmatrix}$$

The N-E equation becomes:

$$\overbrace{\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}}^{M^s} \overbrace{\begin{bmatrix} \dot{v}_{ab}^s \\ \ddot{\theta} \end{bmatrix}}^{\dot{V}_{ab}^s} + \cdots$$

$$\overbrace{\begin{bmatrix} mI & mJ\dot{c}_a \\ (mJ\dot{c}_a)^T & 2m_1 q_{1a}^T \dot{q}_{1a} + 2m_2 q_{2a}^T \dot{q}_{2a} \end{bmatrix}}^{\dot{M}^s} \overbrace{\begin{bmatrix} v_{ab}^s \\ \dot{\theta} \end{bmatrix}}^{V_{ab}^s} = \cdots$$

$$\begin{bmatrix} f_1 \\ f_1^T J q_{1a} \end{bmatrix} + \begin{bmatrix} f_2 \\ f_2^T J q_{2a} \end{bmatrix} + \begin{bmatrix} f_{1,g} \\ f_{1,g}^T J q_{1a} \end{bmatrix} + \cancel{\begin{bmatrix} f_{2,g} \\ f_{2,g}^T J q_{2a} \end{bmatrix}}$$



Remember we say:

$$\dot{M}^s V_{ab}^s = -\begin{bmatrix} J\dot{\theta} & -Jv_{ab}^s \\ 0 & 0 \end{bmatrix}^T M^s V_{ab}^s$$

$\begin{bmatrix} f_g \\ q_c \end{bmatrix}$ $v_{ab}^s = \dot{p}_{ab} - \dot{\theta} J p_{ab}$

$= \dot{q}_{1a} - \dot{\theta} J q_{1a}$

But is this true?...You are about to find out now ☺.

Exercise: prove the above equation.

① 求 T，得到 $\frac{1}{2} V_{ab}^{s\,T} M^s V_{ab}^s$ 表达式

手动整理得到 M^s 张量矩阵

② 广义动量：$M^s \cdot V_{ab}^s$

③ $$\frac{d}{dt}(M^s \cdot V_{ab}^s) = M^s \dot{V}_{ab}^s + \dot{M}^s V_{ab}^s = F^s = \begin{bmatrix} f^s \\ \tau^s \end{bmatrix}_{6\times1}$$

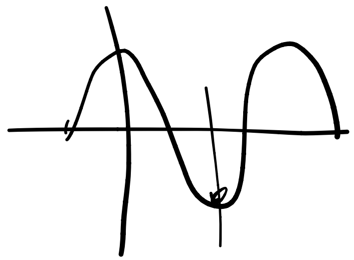
牛顿欧拉方程

④ $$\dot{M}^s V_{ab}^s = -\begin{bmatrix} \hat{\omega}_{ab} & \hat{v}_{ab} \\ 0 & \hat{\omega}_{ab} \end{bmatrix} M^s V_{ab}^s$$

$$-\begin{bmatrix} \dot{J_\theta} & 0 \\ -J_{V_{ab}^s} & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} mI & mJ_{ca} \\ (mJ_{ca})^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}}_{M^s}$$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$= -\begin{bmatrix} \dot{J_\theta}\, mI & \dot{J_\theta}\, mJ_{ca} \\ -J_{V_{ab}^s} (mJ_{ca})^T & -J_{V_{ab}^s} \cdot mJ_{ca} \end{bmatrix}$$



$$= \begin{bmatrix} -\dot{J_\theta}\, mI & -\dot{J_\theta}\, mJ_{ca} \\ J_{V_{ab}^s} (mJ_{ca})^T & J_{V_{ab}^s}\, mJ_{ca} \end{bmatrix} \begin{bmatrix} V_{ab}^s \\ \dot{\theta} \end{bmatrix}$$

$\dot{q}_{1a} = J\dot{c}_a$

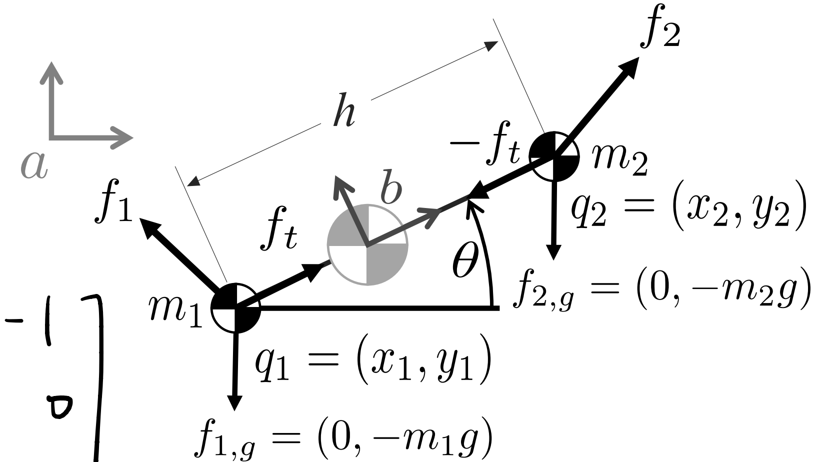
$x^T J x = 0$

二次型

$\dot{M}^s V_{ab}^s = -[\quad] M^s \cancel{V_{ab}^s}$

Newton-Euler equation – Simple example – Written assignment

Written assignment: prove for that in the derivation of equation of motion for the two-particle system, the Euler-Lagrange equation (with generalized coordinates x_1, y_1, θ) is equivalent to the Newton-Euler equation.



$$J = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Euler-Lagrange equation

$$m_1\ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) = \cdots$$
$$f_{1,x} + f_{2,x}$$
$$m_1\ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) = \cdots$$
$$f_{1,y} + f_{2,y} - (m_1 + m_2)g$$
$$m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) = \cdots$$
$$-f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2 gh\cos\theta$$

Newton-Euler equation

$$\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{2a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix} \begin{bmatrix} \dot{v}_{ab}^s \\ \ddot{\theta} \end{bmatrix} + \cdots$$
$$\begin{bmatrix} \boxed{mI}\ 0 & mJ\dot{c}_a \\ (mJ\dot{c}_a)^T & 2m_1 q_{1a}^T \dot{q}_{1a} + 2m_2 q_{2a}^T \dot{q}_{2a} \end{bmatrix} \begin{bmatrix} v_{ab}^s \\ \dot{\theta} \end{bmatrix} = \cdots$$
$$\begin{bmatrix} f_1 \\ f_1^T J q_{1a} \end{bmatrix} + \begin{bmatrix} f_2 \\ f_2^T J q_{2a} \end{bmatrix} + \begin{bmatrix} f_{1,g} \\ f_{1,g}^T J q_{1a} \end{bmatrix} + \begin{bmatrix} f_{2,g} \\ f_{2,g}^T J q_{2a} \end{bmatrix}$$

$$\begin{bmatrix} mI\dot{V}^s_{ab} + mJc_a\ddot{\theta} \\ (mJc_a)^T \cdot \dot{V}^s_{ab} + m_1 q_{1a}^T q_{1a}\ddot{\theta} + m_2 q_{2a}^T q_{2a}\ddot{\theta} \end{bmatrix} + \begin{bmatrix} mI\dot{V}^s_{ab} + mJ\dot{c}_a\dot{\theta} \\ (mJ\dot{c}_a)^T \dot{V}^s_{ab} + (2m_1 q_{1a}^T \dot{q}_{1a} + 2m_2 q_{2a}^T \dot{q}_{2a})\dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} mI\dot{V}^s_{ab} + mJc_a\ddot{\theta} \qquad + mJ\dot{c}_a\dot{\theta} \\ (mJc_a)^T \dot{V}^s_{ab} + m_2 q_{2a}^T q_{2a}\ddot{\theta} + (mJ\dot{c}_a)^T V^s_{ab} + (2m_1 q_{1a}^T \dot{q}_{1a} + 2m_2 q_{2a}^T \dot{q}_{2a})\dot{\theta} \end{bmatrix}$$

$$= mI\dot{V}^s_{ab} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} m_1x_1+m_2x_2 \\ m_1y_1+m_2y_2 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} m_1\dot{x}_1 + m_2\dot{x}_2 \\ m_1\dot{y}_1 + m_2\dot{y}_2 \end{bmatrix}\dot{\theta}$$

$\boxed{\dot{V}^s_{ab} \neq 0}$

$$m_2 q_{2a}^T q_{2a}\ddot{\theta} + (2m_1 q_{1a}^T \dot{q}_{1a} + 2m_2 q_{2a}^T \dot{q}_{2a})\dot{\theta}$$

$$C_a = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2} \qquad V_{ab}^s = \dot{q}_{1a} - \dot{\theta} J q_{1a} = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} - \dot{\theta} \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 + y_1 \dot{\theta} \\ \dot{y}_1 - x_1 \dot{\theta} \end{bmatrix}$$

$$\dot{C}_a = \frac{m_1 \dot{q}_1 + m_2 \dot{q}_2}{m_1 + m_2} \qquad \dot{V}_{ab}^s = \ddot{q}_{1a} - \ddot{\theta} J q_{1a} - \dot{\theta} J \dot{q}_{1a} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{bmatrix} - \ddot{\theta} \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} - \dot{\theta} \begin{bmatrix} -\dot{y}_1 \\ \dot{x}_1 \end{bmatrix}$$

$$\ddot{C}_a = \frac{m_1 \ddot{q}_1 + m_2 \ddot{q}_2}{m_1 + m_2} \qquad = \begin{bmatrix} \ddot{x}_1 + y_1 \ddot{\theta} + \dot{y}_1 \dot{\theta} \\ \ddot{y}_1 - x_1 \ddot{\theta} - \dot{x}_1 \dot{\theta} \end{bmatrix}$$

第一行：

$$mI \dot{V}_{ab}^s + \begin{bmatrix} -m_1 y_1 - m_2 y_2 \\ m_1 x_1 + m_2 x_2 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} -m_1 \dot{y}_1 - m_2 \dot{y}_2 \\ m_1 \dot{x}_1 + m_2 \dot{x}_2 \end{bmatrix} \dot{\theta}$$

$$m \begin{bmatrix} \ddot{x}_1 + y_1 \ddot{\theta} + \dot{y}_1 \dot{\theta} \\ \ddot{y}_1 - x_1 \ddot{\theta} - \dot{x}_1 \dot{\theta} \end{bmatrix} + \begin{bmatrix} -m_1 y_1 \ddot{\theta} - m_2 y_2 \ddot{\theta} \\ m_1 x_1 \ddot{\theta} + m_2 x_2 \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -m_1 \dot{y}_1 \dot{\theta} - m_2 \dot{y}_2 \dot{\theta} \\ m_1 \dot{x}_1 \dot{\theta} + m_2 \dot{x}_2 \dot{\theta} \end{bmatrix}$$

$$\approx (m_1+m_2)\ddot{x}_1 + (\cancel{m_1}+m_2)y_1\ddot{\theta} + (\cancel{m_1}+m_2)\dot{y}_1\dot{\theta} - m_1\cancel{y_1}\ddot{\theta} - m_2 y_2\ddot{\theta} + (m_1+m_2)\dot{x}_1$$

$$+(m_1+m_2)y_1\dot{\theta} - m_1\cancel{\dot{y}_1}\dot{\theta} - m_2\dot{y}_2\dot{\theta}$$

$$y_2 = y_1 + h\sin\theta$$

$$\dot{y}_2 = \dot{y}_1 + h\cos\theta\,\dot{\theta}$$

$$(m_1+m_2)\ddot{x}_1 + m_2\cancel{y_1}\ddot{\theta} + m_2\cancel{\dot{y}_1}\dot{\theta} - m_2(\cancel{y_1} + h\sin\theta)\ddot{\theta}$$

$$- m_2(\cancel{\dot{y}_1} + h\cos\theta\,\dot{\theta})\cdot\dot{\theta}$$

Euler-Lagrange equation

$$m_1\ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) = \cdots$$
$$f_{1,x} + f_{2,x}$$
$$m_1\ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) = \cdots$$
$$f_{1,y} + f_{2,y} - (m_1+m_2)g$$
$$m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) = \cdots$$
$$-f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2 gh\cos\theta$$

$$m \cdot \begin{bmatrix} \\ \ddot{y}_1 - x_1\ddot{\theta} - \dot{x}_1\dot{\theta} \end{bmatrix} \quad \begin{bmatrix} \\ m_1 x_1 \ddot{\theta} + m_2 x_2 \ddot{\theta} \end{bmatrix} \quad + \quad \begin{bmatrix} \\ m_1 \dot{x}_1 \dot{\theta} + m_2 \dot{x}_2 \dot{\theta} \end{bmatrix}$$

$$(m_1+m_2)\ddot{y}_1 - (\cancel{m_1}+m_2)x_1\ddot{\theta} - (\cancel{m_1}+m_2)\dot{x}_1\dot{\theta} + m_1\cancel{x_1}\ddot{\theta} + m_2x_2\ddot{\theta}$$

$$+(m_1+m_2)\dot{y}_1 - (m_1+m_2)x_1\dot{\theta} + m_1\cancel{\dot{x}_1}\dot{\theta} + m_2\dot{x}_2\dot{\theta} =$$

$$\underline{(m_1+m_2)\ddot{y}_1} - m_2\cancel{x_1}\ddot{\theta} - m_2\cancel{\dot{x}_1}\dot{\theta} + m_2(\cancel{x_1} + \underline{h\cos\theta})\ddot{\theta}$$

$$+ m_2 \cdot (\cancel{\dot{x}_1} - \underline{h\sin\theta\,\dot{\theta}})\dot{\theta}$$

$$V^s_{ab} = \dot{q}_{1a} - \dot{\theta} J q_{1a} = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} - \dot{\theta} \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 + y_1 \dot{\theta} \\ \dot{y}_1 - x_1 \dot{\theta} \end{bmatrix}$$

$$\dot{V}^s_{ab} = \ddot{q}_{1a} - \ddot{\theta} J q_{1a} - \dot{\theta} J \dot{q}_{1a} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{bmatrix} - \ddot{\theta} \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} - \dot{\theta} \begin{bmatrix} -\dot{y}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} \ddot{x}_1 + y_1 \ddot{\theta} + \dot{y}_1 \dot{\theta} \\ \ddot{y}_1 - x_1 \ddot{\theta} - \dot{x}_1 \dot{\theta} \end{bmatrix}$$

$$(f_{1x}, f_{1y}) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (f_{2x} \; f_{2y}) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_1 g \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -m_2 g \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$f[\vec{i}] \cdot q =$ 求力矩 叉乘 $q \times f = \ldots = x_1 y_2 - x_2 y_1$

空间力矩 $= q \times f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ q_x & q_y & q_z \\ f_x & f_y & f_z \end{vmatrix}$

$$-m_2 h \sin\theta(\ddot{x}_1 + \cancel{y_1\ddot{\theta}} + \cancel{\dot{y}_1\dot{\theta}})$$

第三行：$-m_1 y_1(\ddot{x}_1 + \cancel{y_1\ddot{\theta}} + \cancel{\dot{y}_1\dot{\theta}}) - m_2 y_1(\ddot{x}_1 + \cancel{y_1\ddot{\theta}} + \cancel{\dot{y}_1\dot{\theta}}) + m_1 x_1(\ddot{y}_1 - \cancel{x_1\ddot{\theta}} - \cancel{\dot{x}_1\dot{\theta}})$

$$(m)[Ca]^T \dot{V}^s_{ab} = [-m_1 y_1 - m_2 y_2,\ m_1 x_1 + m_2 x_2] \cdot \begin{bmatrix} \ddot{x}_1 + y_1\ddot{\theta} + \dot{y}_1\dot{\theta} \\ \ddot{y}_1 - x_1\ddot{\theta} - \dot{x}_1\dot{\theta} \end{bmatrix} + m_2 x_1(\ddot{y}_1 - \cancel{x_1\ddot{\theta}} - \cancel{\dot{x}_1\dot{\theta}}) + m_2 h\cos\theta(\ddot{y}_1 - \cancel{x_1\ddot{\theta}} - \cancel{\dot{x}_1\dot{\theta}})$$

$$m_1 q_{1a}^T q_{1a} \ddot{\theta} = \boxed{m_1(\cancel{x_1^2} + \cancel{y_1^2})\ddot{\theta}}$$

$$m_2 q_{2a}^T q_{2a} \ddot{\theta} = m_2(x_2^2 + y_2^2)\ddot{\theta} = \boxed{m_2(\cancel{x_1^2} + \cancel{y_1^2} + h^2 + \cancel{2h x_1\cos\theta} + \cancel{2h y_1 \sin\theta})\ddot{\theta}}$$

$$(m)[\dot{C}a]^T V^s_{ab} = [-m_1 \dot{y}_1 - m_2 \dot{y}_2,\ m_1 \dot{x}_1 + m_2 \dot{x}_2]\begin{bmatrix} \dot{x}_1 + y_1\dot{\theta} \\ \dot{y}_1 - x_1\dot{\theta} \end{bmatrix}$$

$$\boxed{\begin{aligned}-m_1\dot{y}_1(\cancel{\dot{x}_1}+\cancel{y_1}\cancel{\dot{\theta}})-m_2\dot{y}_1(\cancel{\dot{x}_1}+\cancel{y_1}\cancel{\dot{\theta}})-m_2h\cos\theta\,\dot{\theta}(\cancel{\dot{x}_1}+\cancel{y_1}\cancel{\dot{\theta}})+m_1\dot{x}_1(\cancel{\dot{y}_1}-\cancel{x_1}\cancel{\dot{\theta}})\\ +m_2\dot{x}_1(\cancel{\dot{y}_1}-\cancel{x_1}\cancel{\dot{\theta}})-m_2h\sin\theta\,\dot{\theta}(\cancel{\dot{y}_1}-\cancel{x_1}\cancel{\dot{\theta}})\end{aligned}}$$

$$2m_1 q_{1a}^T\dot{q}_{1a}\dot{\theta}=\boxed{2m_1\cdot(\cancel{x_1\dot{x}_1}+\cancel{y_1\dot{y}_1})\dot{\theta}}$$

$$2m_2 q_{2a}^T\dot{q}_{2a}\dot{\theta}=2m_2(x_2\dot{x}_2+y_2\dot{y}_2)\dot{\theta}$$

$$=2m_2\left[(x_1+h\cos\theta)(\dot{x}_1-h\sin\theta\dot{\theta})+(y_1+h\sin\theta)(\dot{y}_1+h\cos\theta\dot{\theta})\right]\dot{\theta}$$

$$\boxed{\begin{aligned}=2m_2\Big[&x_1(\cancel{\dot{x}_1\dot{\theta}}-h\cancel{\sin\theta}\dot{\theta}^2)+h\cos\theta(\cancel{\dot{x}_1\dot{\theta}}-\cancel{h\sin\theta\dot{\theta}^2})+y_1(\cancel{\dot{y}_1\dot{\theta}}+h\cancel{\cos\theta}\dot{\theta}^2)\\ &+h\sin\theta(\cancel{\dot{y}_1\dot{\theta}}+\cancel{h\cos\theta\dot{\theta}^2})\Big]\end{aligned}}$$

Euler-Lagrange equation

$$m_1\ddot{x}_1 + m_2(\ddot{x}_1 - h\cos\theta\dot{\theta}^2 - h\sin\theta\ddot{\theta}) = \cdots$$
$$f_{1,x} + f_{2,x}$$
$$m_1\ddot{y}_1 + m_2(\ddot{y}_1 - h\sin\theta\dot{\theta}^2 + h\cos\theta\ddot{\theta}) = \cdots$$
$$f_{1,y} + f_{2,y} - (m_1 + m_2)g$$
$$m_2(-\ddot{x}_1 h\sin\theta + \ddot{y}_1 h\cos\theta + h^2\ddot{\theta}) = \cdots$$
$$-f_{2,x}h\sin\theta + f_{2,y}h\cos\theta - m_2 gh\cos\theta$$

$-m_1 y_1 \ddot{x}_1 - m_2 y_1 \ddot{x}_1 + m_1 x_1 \ddot{y}_1 + m_2 x_1 \ddot{y}_1$

$-m_2 x_1 h \sin\theta\, \dot{\theta}^2$

$-m_2 y_1 h \cos\theta\, \dot{\theta}^2$

$-y_1 \ddot{x}_1 m + x_1 \ddot{y}_1 m - m_2 h \dot{\theta}^2 (x_1 h \sin\theta$

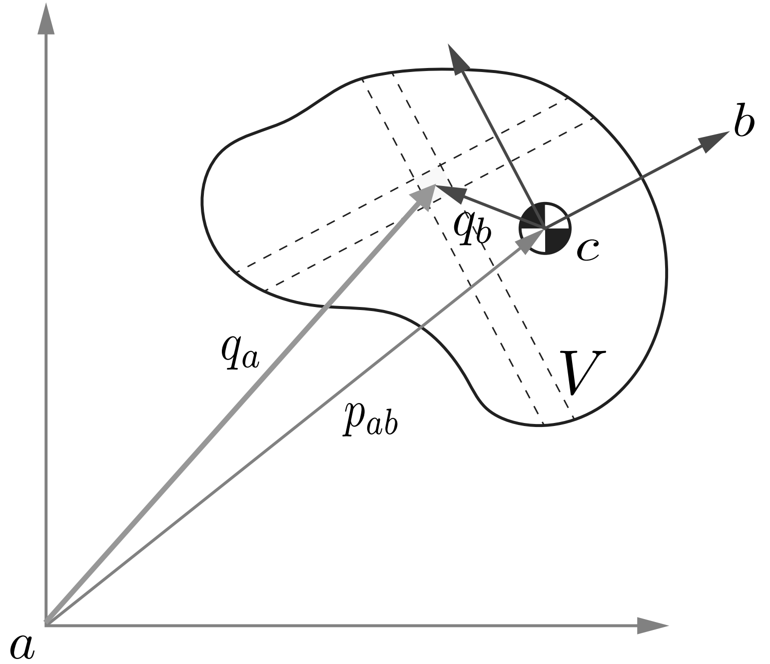
$y_1 h \cos\theta)$

The generalized inertia matrix M^s is not constant; it is easier to work with b-frame in this case.

$$T = \frac{1}{2}\int_V \rho \dot{q}_a^T \dot{q}_a dV = \frac{1}{2}\int_V \rho \dot{q}_b^T R_{ab}^T R_{ab} \dot{q}_b dV = \frac{1}{2}\int_V \rho \dot{q}_b^T \dot{q}_b dV$$

$$= \frac{1}{2}\int_V \rho(\hat{\omega}_{ab}^b q_b + v_{ab}^b)^T (\hat{\omega}_{ab}^b q_b + v_{ab}^b) dV$$

$$= \frac{1}{2} m {v_{ab}^b}^T v_{ab}^b + \frac{1}{2}{\omega_{ab}^b}^T \underbrace{\int_V -\rho \hat{q}_b^2 dV}_{\mathcal{I}^b} \omega_{ab}^b - {v_{ab}^b}^T \int_V \rho \hat{q}_b dV \omega_{ab}^b$$



To proceed, we first define the *center of mass*:

$$c_a = \frac{1}{m}\int_V \rho q_a dV, \quad m = \int_V \rho q_a dV$$

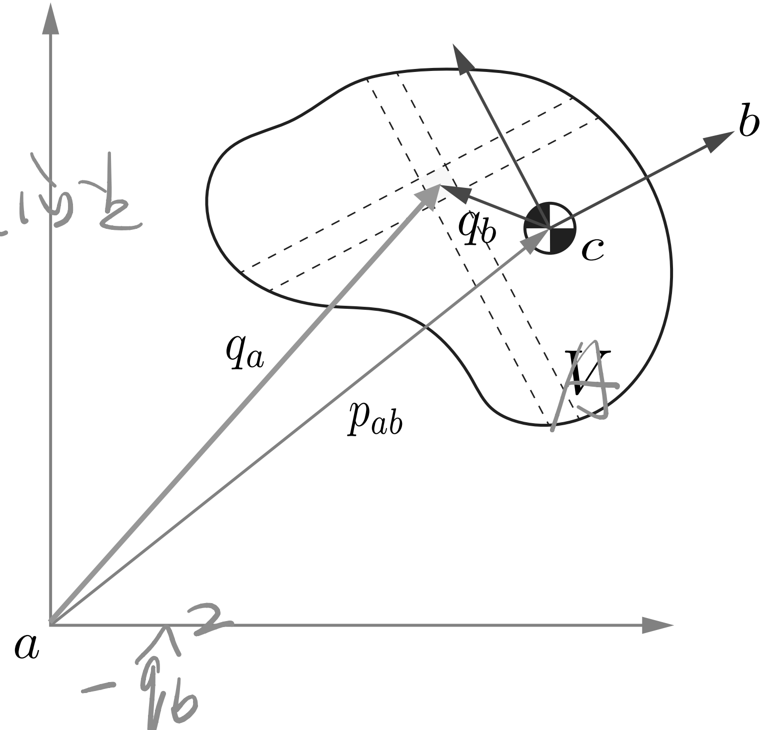
If we attach b-frame to the center of mass, we have $c_b = 0$:

惯量矩阵

$$m\hat{c}_b = \int_V \rho \hat{q}_b dV = 0$$

⇒ 物体坐标系设在质心上 以消除耦合项

$$T = \frac{1}{2}\underbrace{\begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix}^T}_{V_{ab}^{b\,T}} \underbrace{\begin{bmatrix} mI_{3\times 3} & 0 \\ 0 & \mathcal{I}^b \end{bmatrix}}_{M^b} \underbrace{\begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix}}_{V_{ab}^b} = \frac{1}{2} V_{ab}^{b\,T} M^b V_{ab}^b$$



Since $V_{ab}^s = Ad_{g_{ab}} V_{ab}^b$, we have:

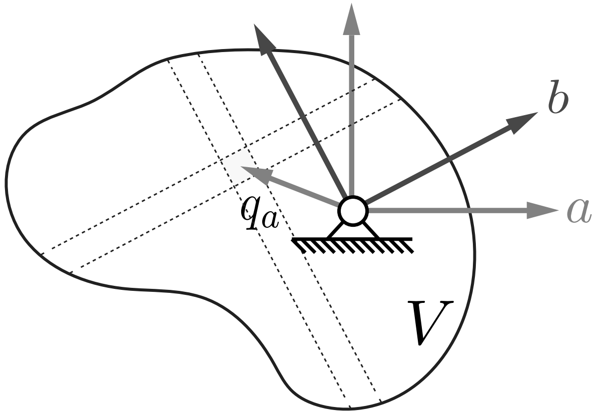
$$\boxed{M^s = Ad_{g_{ab}}^{-T} M^b Ad_{g_{ab}}^{-1}}$$

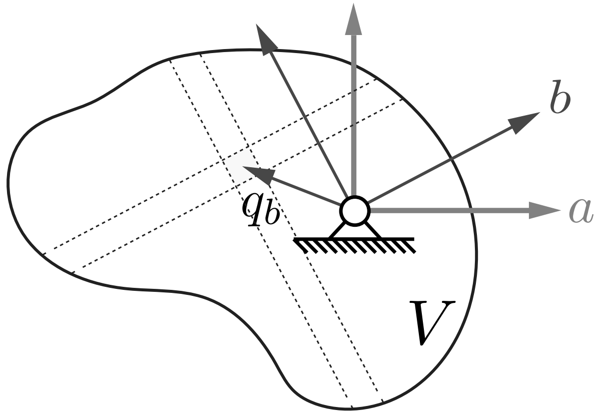
$$\mathcal{I}^b = \int_V -\rho \hat{q}_b^2 dV = \int_V \rho \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dxdydz$$

This is the usual inertia matrix you see in your physics class.

Newton-Euler equation – computing inertia matrix – Planar example

The kinetic energy of a planar rigid body:





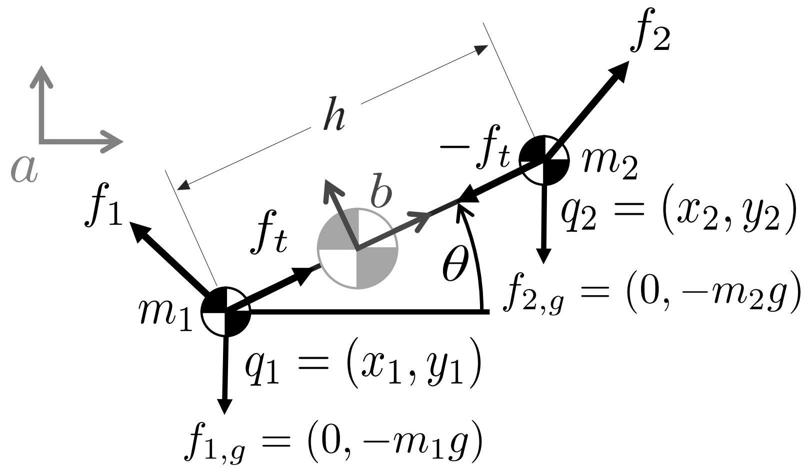
$$T = \frac{1}{2}\int_V \rho \dot{q}_a^T \dot{q}_a dV = \frac{1}{2}\int_V \rho (J q_a)^T J q_a dV \dot{\theta}^2$$
$$= \frac{1}{2}\int_V \rho q_a^T q_a dV \dot{\theta}^2 = \frac{1}{2}\mathcal{I}^s \dot{\theta}^2$$

$$T = \frac{1}{2}\int_V \rho \dot{q}_a^T \dot{q}_a dV = \frac{1}{2}\int_V \rho (J q_a)^T J q_a dV \dot{\theta}^2$$
$$= \frac{1}{2}\int_V \rho q_a^T q_a dV \dot{\theta}^2 = \frac{1}{2}\int_V \rho q_b^T q_b dV \dot{\theta}^2$$
$$= \frac{1}{2}\mathcal{I}^b \dot{\theta}^2$$

$\mathcal{I}^s = \mathcal{I}^b!$ But this is a very special case, and in general does not hold in general!

Newton-Euler equation – Planar example

Attach the body frame b to the center of mass c, align the axis as in the figure:



$$V_{ab}^s = \begin{bmatrix} R(\theta) & -Jc_a \\ 0 & 1 \end{bmatrix} V_{ab}^b, \quad V_{ab}^b = \begin{bmatrix} R(-\theta)\dot{c}_a \\ \dot{\theta} \end{bmatrix}$$

$$\mathcal{I}^b = m_1 \frac{m_2^2 h^2}{(m_1+m_2)^2} + m_2 \frac{m_1^2 h^2}{(m_1+m_2)^2} = \frac{m_1 m_2 h^2}{m_1+m_2}$$

$$\begin{aligned} M^s &= \begin{bmatrix} R(\theta) & -Jc_a \\ 0 & 1 \end{bmatrix}^{-T} \begin{bmatrix} mI & 0 \\ 0 & \frac{m_1 m_2 h^2}{m} \end{bmatrix} \begin{bmatrix} R(\theta) & -Jc_a \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} R(\theta) & 0 \\ (JR(-\theta)c_a)^T & 1 \end{bmatrix} \begin{bmatrix} mI & 0 \\ 0 & \frac{m_1 m_2 h^2}{m} \end{bmatrix} \begin{bmatrix} R(-\theta) & JR(-\theta)c_a \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & \frac{m_1 m_2 h^2}{m} + mc^T c_a \end{bmatrix} \end{aligned}$$

Question: Is M^s the same as in the following equation?

$$T = \frac{1}{2} V_{ab}^{s\,T} \underbrace{\begin{bmatrix} mI & mJc_a \\ (mJc_a)^T & m_1 q_{1a}^T q_{1a} + m_2 q_{2a}^T q_{2a} \end{bmatrix}}_{M^s} V_{ab}^s$$

References – Part II

Text reference:

Ch.1-5 of Cornelius, *The Variational Principles of Mechanics*

Ch.4 of Murray, *A Mathematical Introduction to Robotic Manipulation*

Blanding, *Exact Constraint: Machine Design Using Kinematic Processing* (for using wrench geometrically in mechanical design)

力旋量 专书无公式。

Video reference:

https://www.bilibili.com/video/BV1K54y1v7Kn (full course on Goldstein's book)

https://www.bilibili.com/video/BV1Mt411B72Y (Leonard Susskind's Classical Mechanics)

MATLAB tutorial:

https://www.cyclismo.org/tutorial/matlab/index.html

https://www.coursera.org/learn/matlab

Calculus of Variation

March 16, 2021

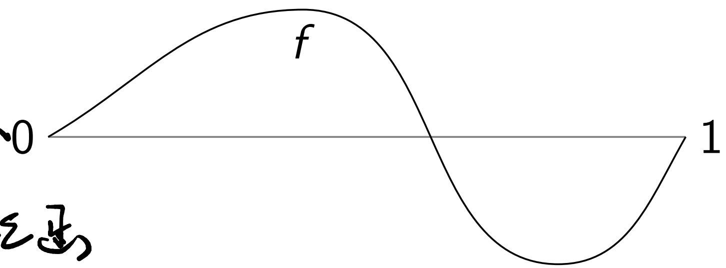
Calculus of Variation

Calculus of Variation: Shortest Path

Find shortest path between two points.

$y=f(x)$ 轨迹

$L(f) \Rightarrow$ 轨迹的长度



L 是函数的函数 $\Rightarrow$ 泛函

求 L 的极小值 $\Rightarrow$ 依然类似于求函数极值 线性逼近

The path is given by $y = f(x)$, $x \in [0,1]$, $f(0) = f(1) = 0$.
The length of the path is a function of f

$$\boxed{\mathcal{L}(f) = \int_0^1 \sqrt{1+f'^2}\,dx.}$$

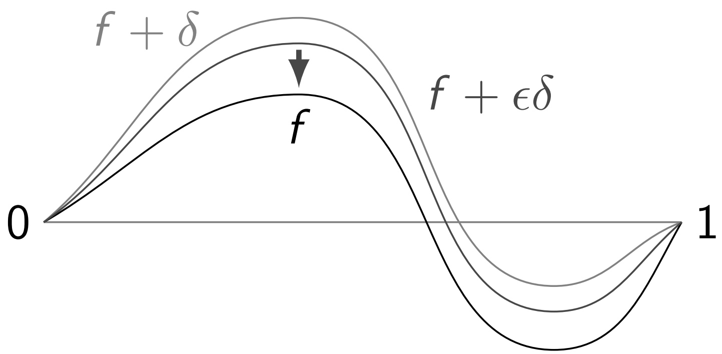
$$\int_0^1 \sqrt{dx^2+dy^2}$$

By shortest path, we mean

$$\boxed{\mathcal{L}(g) \geq \mathcal{L}(f) \quad \text{for any function } g \text{ close to } f.}$$

Calculus of Variation: Shortest Path

Take nearby $g(x) = f(x) + \delta(x)$, where $\delta(x)$ is a small function satisfying $\delta(0) = \delta(1) = 0$. 初末位置函数重合、差值 $\delta(x)=0$



For each $\delta(x)$, further consider the family of nearby $g(x) = f(x) + \epsilon\delta(x)$. Change the problem to

$\epsilon \in (-\infty, +\infty)$

$$\mathcal{L}(f(x) + \epsilon\delta(x)) \geq \mathcal{L}(f(x)).$$

Then the function $\mathcal{L}(f(x) + \epsilon\delta(x))$ of ϵ minimizes at $\epsilon = 0$.

真实轨迹是 $f(x)$

$\epsilon \to 0$，逼近真实轨迹

Calculus of Variation: Shortest Path

固定一个 δ 函

考虑 ε

?

$\mathcal{L}(f(x)+\epsilon\delta(x))$ minimises at $\epsilon \Rightarrow 0$.

$$
\begin{aligned}
\mathcal{L}(f(x)+\epsilon\delta(x)) &= \int_0^1 \sqrt{1+(f'(x)+\epsilon\delta'(x))^2}dx \\
&= \int_0^1 \sqrt{1+f'(x)^2+2\epsilon f'(x)\delta'(x)+o(\epsilon)}dx \\
&= \int_0^1 \sqrt{1+f'(x)^2}\left(1+\epsilon\frac{2f'(x)\delta'(x)}{1+f'(x)^2}+o(\epsilon)\right)^{\frac{1}{2}} dx \\
&= \int_0^1 \sqrt{1+f'(x)^2}\left(1+\frac{1}{2}\epsilon\frac{2f'(x)\delta'(x)}{1+f'(x)^2}+o(\epsilon)\right) dx \\
&= \int_0^1 \left(\sqrt{1+f'(x)^2}+\epsilon\frac{f'(x)\delta'(x)}{\sqrt{1+f'(x)^2}}+o(\epsilon)\right) dx \\
&= \mathcal{L}(f(x))+\epsilon\int_0^1 \frac{f'(x)\delta'(x)}{\sqrt{1+f'(x)^2}}dx+o(\epsilon).
\end{aligned}
$$

$\rightarrow$ 泰勒展开

$(1+x)^p = 1+px+\cdots o(x)$

The red term should vanish. 一次项系数 $\Rightarrow$ 导数 $=0$ $\Rightarrow$ 极值

Calculus of Variation: Shortest Path

$\delta(x)$ 任取

$\mathcal{L}(f(x)+\epsilon\delta(x))$ minimises at $\epsilon = 0$.

$$\int_0^1 \frac{f'(x)\delta'(x)}{\sqrt{1+f'(x)^2}}dx = \int_0^1 \frac{f'(x)}{\sqrt{1+f'(x)^2}}d\delta(x)$$

$$= \frac{f'(x)}{\sqrt{1+f'(x)^2}}\delta(x)\Bigg|_{x=0}^{x=1} - \int_0^1 \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)\delta(x)dx$$

$$= -\int_0^1 \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)\delta(x)dx. \quad \text{(by } \underline{\delta(0)=\delta(1)=0}\text{)}$$

$\Downarrow$ 对于所有 $\delta(x)$ 都成立

积分为0，则 ( ) =0

The above should vanishes *for all* $\delta(x)$

$$\Longrightarrow \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right) = 0 \Longrightarrow \frac{f'(x)}{\sqrt{1+f'(x)^2}} = \text{constant}$$

$\Longrightarrow f'(x) = \text{constant}$, i.e., straight line.

Calculus of Variation of Single Function

Problem. For all $f(a) = A$ and $f(b) = B$, minimise

曲线f

$$\mathcal{L}(f) = \int_a^b L(t, f, f')dt.$$

This becomes minimising the following at $\epsilon = 0$

f、δ固定的函数

ε 极小的数值

$$\mathcal{L}(f + \epsilon\delta) = \int_a^b L(t, f + \epsilon\delta, f' + \epsilon\delta')dt$$

多元函数

线性逼近 ?

$$= \underline{\int_a^b \left[L(t, f, f') + \epsilon \frac{\partial L}{\partial f}(t, f, f')\delta + \epsilon \frac{\partial L}{\partial f'}(t, f, f')\delta' + o(\epsilon) \right] dt}$$

$$= \mathcal{L}(f) + \epsilon \int_a^b \frac{\partial L}{\partial f}(t, f, f')\delta dt + \epsilon \int_a^b \frac{\partial L}{\partial f'}(t, f, f')d\delta + o(\epsilon)$$

$$= \mathcal{L}(f) + \epsilon \int_a^b \frac{\partial L}{\partial f}\delta dt + \epsilon \left. \frac{\partial L}{\partial f'}\delta \right|_{t=a}^{t=b} - \epsilon \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right)\delta dt + o(\epsilon)$$

$$= \mathcal{L}(f) + \epsilon \int_a^b \left(\frac{\partial L}{\partial f} - \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right)\right)\delta dt + o(\epsilon).$$

Calculus of Variation of Single Function

限定 $\delta(t)$ 满足 $\begin{cases} \delta(a) = 0 \\ \delta(b) = 0 \end{cases}$

Necessary condition: For all $\delta(t)$, we have

$$\int_a^b \left(\frac{\partial L}{\partial f} - \frac{d}{dt} \left(\frac{\partial L}{\partial f'} \right) \right) \delta(t) dt = 0.$$

This is equivalent to Euler-Lagrange equation

$$\frac{\partial L}{\partial f} - \frac{d}{dt} \left(\frac{\partial L}{\partial f'} \right) = 0.$$

Calculus of Variation of Several Functions

Problem. For $f_1(a) = A_1, f_1(b) = B_1, f_2(a) = A_2, f_2(b) = B_2$, minimise

$$\mathcal{L}(f_1, f_2) = \int_a^b L(t, f_1, f_2, f_1', f_2')dt.$$

This becomes minimising the following at $\epsilon = 0$

$$\mathcal{L}(f_1 + \epsilon\delta_1, f_2 + \epsilon\delta_2) \;= \int_a^b L(t, f_1 + \epsilon\delta_1, f_2 + \epsilon\delta_2, f_1' + \epsilon\delta_1', f_2' + \epsilon\delta_2')dt$$

线性逼近

$$\Rightarrow \quad = \int_a^b \Big[L(t, f_1, f_2, f_1', f_2') + \epsilon \frac{\partial L}{\partial f_1}(t, f_1, f_2.f_1', f_2')\delta_1 + \epsilon \frac{\partial L}{\partial f_2}(t, f_1, f_2.f_1', f_2')\delta_2$$

$$+ \epsilon \frac{\partial L}{\partial f_1'}(t, f_1, f_2.f_1', f_2')\delta_1' + \epsilon \frac{\partial L}{\partial f_2'}(t, f_1, f_2.f_1', f_2')\delta_2' + o(\epsilon) \Big] dt$$

$$= \mathcal{L}(f_1, f_2) + \epsilon \int_a^b \left(\frac{\partial L}{\partial f_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_1'} \right) \right) \delta_1 dt$$

无关，任取

$$+ \epsilon \int_a^b \left(\frac{\partial L}{\partial f_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_2'} \right) \right) \delta_2 dt + o(\epsilon).$$

Calculus of Variation of Several Functions

The necessary condition for (f_1, f_2) to be a local extreme is the Euler-Lagrange equation

$$\frac{\partial L}{\partial f_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_1'}\right) = 0, \quad \frac{\partial L}{\partial f_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial f_2'}\right) = 0.$$

Or

$$\frac{\partial L}{\partial (f_1, f_2)} - \frac{d}{dt}\left(\frac{\partial L}{\partial (f_1', f_2')}\right) = \vec{0}.$$

Least Action Path

least action principle

Change $(f_1(t), f_2(t))$ to a path $\vec{x}(t) = (x_1(t), \dots, x_n(t))$ in $\mathbb{R}^n$, connecting $\vec{x}(a) = \vec{A}$ and $\vec{x}(b) = \vec{B}$. Minimising the action

$$\mathcal{L}(\vec{x}(t)) = \int_a^b L(t, \vec{x}(t), \vec{x}\,'(t))dt.$$

Euler-Lagrange $\dfrac{\partial L}{\partial x_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial x_i'}\right) = 0.$

Example. For $L = t(x_1x_2' - x_2x_1')$, $= L(t, x_1, x_2, x_1', x_2')$

$$\frac{\partial L}{\partial x_1} = tx_2', \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial x_1'}\right) = \frac{d}{dt}(\overset{-tx_2(t)}{-tx_2}) = -x_2 - tx_2',$$

$$\frac{\partial L}{\partial x_2} = -tx_1', \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial x_2'}\right) = \frac{d}{dt}(tx_1) = x_1 + tx_1'.$$

Euler-Lagrange: $2tx_2' + x_2 = 0$ and $-2tx_1' - x_1 = 0$. Solution $(x_1, x_2) = \frac{1}{\sqrt{t}}(c_1, c_2)$, straight line passing through the origin.

Calculus of Variation in General Coordinates

Use example $x = r\cos\theta$ and $y = r\sin\theta$.
An expression of L in x, y can then be rewritten in terms of r, θ

$$\begin{aligned} L(t, x, y, x', y') &= L(t, r\cos\theta, r\sin\theta, r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta) \\ &= \tilde{L}(t, r, \theta, r', \theta'). \end{aligned}$$

坐标任选，怎么方便怎么算

Then

$$\int_a^b L(t, x(t), y(t), x'(t), y'(t))dt = \int_a^b \tilde{L}(t, r(t), \theta(t), r'(t), \theta'(t))dt.$$

Minimising left $\iff$ minimising the right. Therefore solution of

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = 0, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) = 0,$$

is related to the solution of

$$\frac{\partial \tilde{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial r'}\right) = 0, \quad \frac{\partial \tilde{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \theta'}\right) = 0,$$

by $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$.

Calculus of Variation in General Coordinates

In general, change of general coordinates $\vec{y} = \phi(\vec{x})$. Then $\vec{y}\,' = \phi'(\vec{x})\vec{x}\,'$, and

$$L(t, \vec{y}, \vec{y}\,') = L(t, \phi(\vec{x}), \phi'(\vec{x})\vec{x}\,') = \tilde{L}(t, \vec{x}, \vec{x}\,').$$

Then solutions of

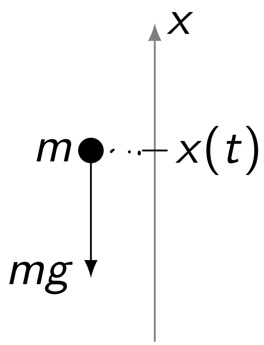
$$\frac{\partial L}{\partial \vec{y}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{y}\,'}\right) = 0, \quad \frac{\partial \tilde{L}}{\partial \vec{x}} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \vec{x}\,'}\right) = 0,$$

are related by $\vec{y}(t) = \phi(\vec{x}(t))$.

This means the calculus of variations can be done over manifolds.

Mechanics

Gravity: First Method



Newton's second law of motion

$$mx''(t) = F = -mg.$$

This is $x''(t) = -g$, and we get

$$x(t) = a + vt - gt^2.$$

$a = x(0)$ is the initial height of the mass.
$v = x'(0)$ is the initial velocity of the mass.

Gravity: Second Method

Kinetic energy

$$K = \frac{1}{2}mv^2.$$

Potential energy

$$V = mgx.$$

Conservation of energy: The total energy is constant

$$T = K + V = \frac{1}{2}mv^2 + mgx = \frac{1}{2}mx'^2 + mgx$$

能量守恒

This is the same as the equation by the first method

$$T' = (mx'' + mg)x' = 0.$$

强为 0

$$\Rightarrow \quad mx'' + mg = 0$$

Gravity: Third Method

The dynamics is the path $x(t)$ that minimises the action

$$A = \mathcal{L}(x(t)) = \int_0^b L(x, x')dt, \quad L = K - V = \frac{1}{2}mx'^2 - mgx.$$

We have

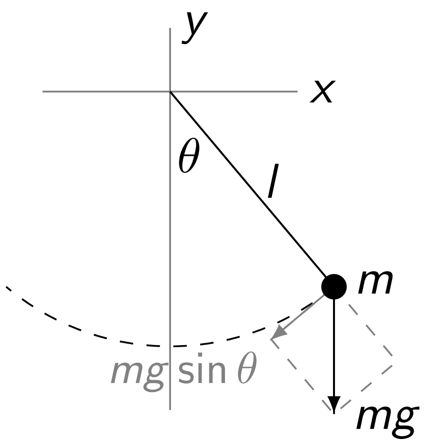
$$\frac{\partial L}{\partial x} = -mg, \quad \frac{\partial L}{\partial x'} = mx', \quad \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = mx''.$$

Euler-Lagrange is the same as the equation by the first method

$$-mg - mx'' = 0.$$

Pendulum: First Method

钟摆



Newton's second law of motion: The acceleration is $l\theta''$, and the tangential force is the $\sin\theta$ component of the gravity mg

$$ml\theta'' = -mg\sin\theta.$$

Gravity: Second Method

Kinetic energy: The speed is $v = l\theta'$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\theta'^2.$$

Potential energy: Height $y = -l\cos\theta$

$$V = -mgl\cos\theta.$$

Conservation of energy: The total energy is constant

$$T = K + V = \frac{1}{2}ml^2\theta'^2 - mgl\cos\theta$$

This is the same as the equation by the first method

$$T' = (ml\theta'' + mg\sin\theta)l\theta' = 0.$$

Gravity: Third Method

The dynamics is the path $\theta(t)$ that minimises the action

$$\mathcal{L}(x(t)) = \int_0^b L(x, x')dt, \quad L = K - V = \frac{1}{2}ml^2\theta'^2 + mgl\cos\theta.$$

We have

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \quad \frac{\partial L}{\partial \theta'} = ml^2\theta', \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) = ml^2\theta''.$$

Euler-Lagrange is the same as the equation by the first method

$$-mgl\sin\theta - ml^2\theta'' = 0.$$

Thank You