Assignment7

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1. (20 points) Determine $\mathcal{L}(\sin^2 \omega t)$ and $\mathcal{L}(\cos^2 \omega t)$ using the formulas

$$\sin^2 \omega t = \frac{1}{2} - \frac{1}{2}\cos 2\omega t, \qquad \cos^2 \omega t = 1 - \sin^2 \omega t$$

respectively.

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$
$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$
$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$
$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$L(\sin^2 \omega t) = L\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega t\right) = L\left(\frac{1}{2}u(t)\right) - L\left(\frac{1}{2}\cos 2\omega t\right) = \frac{1}{2}L\big(u(t)\big) - \frac{1}{2}L(cos2\omega t)$$
$$= \frac{1}{2} * \frac{1}{s} - \frac{1}{2}L\left(\frac{1}{2}\left(e^{i2\omega t} + e^{-i2\omega t}\right)\right) = \frac{1}{2s} - \frac{1}{4}L\left(e^{i2\omega t} + e^{-i2\omega t}\right)$$
$$= \frac{1}{2s} - \frac{1}{4}\left(L\left(e^{i2\omega t}\right) + L\left(e^{-i2\omega t}\right)\right) = \frac{1}{2s} - \frac{1}{4}\left(\frac{1}{s - 2i\omega} + \frac{1}{s + 2i\omega}\right)$$
$$= \frac{1}{2s} - \frac{1}{4}\left(\frac{2s}{s^2 + 4\omega^2}\right) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 4\omega^2}\right) = \frac{1}{2}\left(\frac{4\omega^2}{s(s^2 + 4\omega^2)}\right) = \frac{2\omega^2}{s(s^2 + 4\omega^2)}$$

$$L(\cos^2 \omega t) = L(1 - \sin^2 \omega t) = L\big(u(t)\big) - L(\sin^2 \omega t) = \frac{1}{s} - \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 4\omega^2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2 + 4\omega^2}\right) = \frac{1}{2}\left(\frac{2s^2 + 4\omega^2}{s(s^2 + 4\omega^2)}\right) = \frac{s^2 + 2\omega^2}{s(s^2 + 4\omega^2)}$$

2. (20 points) Determine

a) $\mathcal{L}(e^{7t} \sinh \sqrt{2}t)$

b) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2 + 6s + 1}\right)$

a)

双曲函数定义:

$$sinht = \frac{e^t - e^{-t}}{2}$$
$$cosht = \frac{e^t + e^{-t}}{2}$$
$$f(t) = sinh\sqrt{2}t$$

$$F(s) = L(sinh\sqrt{2}t) = L\left(\frac{e^{\sqrt{2}t} - e^{-\sqrt{2}t}}{2}\right) = \frac{1}{2}\left(L\left(e^{\sqrt{2}t}\right) - L\left(e^{-\sqrt{2}t}\right)\right) = \frac{1}{2}\left(\frac{1}{s-\sqrt{2}} - \frac{1}{s+\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}}{s^2 - 2}$$

根据频域平移性质:

$$L(e^{7t}f(t)) = F(s-7) = \frac{\sqrt{2}}{(s-7)^2 - 2}$$

b)

$$L^{-1}\left(\frac{s}{s^2+6s+1}\right) = L^{-1}\left(\frac{s}{(s+3)^2 - 8}\right)$$

$$= L^{-1}\left(\frac{s+3}{(s+3)^2 - \left(2\sqrt{2}\right)^2} - \frac{3}{2\sqrt{2}}\frac{2\sqrt{2}}{(s+3)^2 - \left(2\sqrt{2}\right)^2}\right)$$

$$= L^{-1}\left(L\left(e^{-3t}\cosh\left(2\sqrt{2}t\right)\right) - \frac{3}{2\sqrt{2}}L\left(e^{-3t}\sinh\left(2\sqrt{2}t\right)\right)\right)$$

$$= e^{-3t}\cosh\left(2\sqrt{2}t\right) - \frac{3}{2\sqrt{2}}e^{-3t}\sinh\left(2\sqrt{2}t\right)$$

$$= e^{-3t}\frac{e^{2\sqrt{2}t} + e^{-2\sqrt{2}t}}{2} - \frac{3}{2\sqrt{2}}e^{-3t}\frac{e^{2\sqrt{2}t} - e^{-2\sqrt{2}t}}{2}$$

$$= \frac{e^{2\sqrt{2}t-3t} + e^{-2\sqrt{2}t-3t}}{2} - \frac{3}{2\sqrt{2}}\frac{e^{2\sqrt{2}t-3t} - e^{-2\sqrt{2}t-3t}}{2}$$

$$= \left(\frac{1}{2} - \frac{3}{4\sqrt{2}}\right)e^{2\sqrt{2}t-3t} + \left(\frac{1}{2} + \frac{3}{4\sqrt{2}}\right)e^{-2\sqrt{2}t-3t}$$

3. (20 points) Determine

a) $\mathcal{L}(t^2 \sin \omega t)$

b) $\mathcal{L}\left(\frac{1-\cosh \omega t}{t}\right)$

a)

$$f(t) = sin\omega t$$

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

由频域微分性质（二阶）

$$L(t^2 sin\omega t) = -\frac{d^2}{ds^2}F(s) = -\frac{d}{ds}\left(\frac{2s\omega}{(s^2+\omega^2)^2}\right)$$

$$= -\frac{(-2\omega)(s^2+\omega^2)^2 + 2s\omega * 2(s^2+\omega^2) * 2s}{(s^2+\omega^2)^4}$$

$$= \frac{2\omega(s^2+\omega^2)^2 - 8s^2\omega(s^2+\omega^2)}{(s^2+\omega^2)^4} = \frac{2\omega(s^2+\omega^2) - 8s^2\omega}{(s^2+\omega^2)^3} = \frac{2\omega^3 - 6s^2\omega}{(s^2+\omega^2)^3}$$

b)

$$f(t) = 1 - cosh\omega t$$

$$F(s) = \frac{1}{s} - \frac{s}{s^2 - \omega^2}$$

由频域积分性质

$$L\left(\frac{1 - cosh\omega t}{t}\right) = \int_s^{\infty} F(s)ds = \int_s^{\infty} \left(\frac{1}{s} - \frac{s}{s^2 - \omega^2}\right) ds = \int_s^{\infty} \frac{1}{s} ds - \int_s^{\infty} \frac{s}{s^2 - \omega^2} ds$$

$$= \lim_{a \to \infty} \left(lns - \frac{1}{2}\ln(s^2 - \omega^2)\right)_s^a = \lim_{a \to \infty} \left(\ln\left(\frac{s}{\sqrt{s^2 - \omega^2}}\right)\right)_s^a$$

$$= \ln\left(\lim_{a \to \infty} \frac{a}{\sqrt{a^2 - \omega^2}}\right) - \ln\left(\frac{s}{\sqrt{s^2 - \omega^2}}\right)$$

$$= \ln\left(\lim_{a \to \infty} \frac{1}{\sqrt{1 - \frac{\omega^2}{a^2}}}\right) - \ln\left(\frac{s}{\sqrt{s^2 - \omega^2}}\right) = -\ln\left(\frac{s}{\sqrt{s^2 - \omega^2}}\right)$$

4. (20 points) Determine

a) $\mathcal{L}^{-1}\left(\frac{2s^2}{(s^2+1)(s-1)^2}\right)$

b) $\mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)(s^2+b^2)}\right) \qquad (a \neq b)$

a)

$$F(s) = \frac{2s^2}{(s^2 + 1)(s - 1)^2}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{As + D}{s^2 + 1} + \frac{B}{s - 1} + \frac{C}{(s - 1)^2}\right)$$

$$C = F(s) * (s - 1)^2{}_{s=1} = \frac{2s^2}{s^2 + 1}_{s=1} = 1$$

$$\frac{2s^2}{(s^2 + 1)(s - 1)^2} = \frac{(As + D)(s - 1)^2 + B(s^2 + 1)(s - 1) + (s^2 + 1)}{(s^2 + 1)(s - 1)^2}$$

$$(As + D)(s^2 + 1 - 2s) + B(s^3 - s^2 + s - 1) + s^2 + 1 = 2s^2$$

$$As^3 + As - 2As^2 + Ds^2 + D - 2Ds + B(s^3 - s^2 + s - 1) + s^2 + 1 = 2s^2$$

$$(A + B)s^3 + (-2A + D - B + 1)s^2 + (A - 2D + B)s + (D - B + 1) = 2s^2$$

$$A + B = 0$$

$$-2A + D - B + 1 = 2$$

$$A - 2D + B = 0$$

$$D - B + 1 = 0$$

解得

$$A = -1 \quad B = 1 \quad C = 1 \quad D = 0$$

$$F(s) = \frac{-s}{s^2 + 1} + \frac{1}{s - 1} + \frac{1}{(s - 1)^2}$$

$$f(t) = -\cos t + e^t + te^t$$

b)

$$F(s) = \frac{s}{(s^2+a^2)(s^2+b^2)}$$

$$F(s) = \frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$As+B = F(s)*(s^2+a^2) \quad let\ s = -ai$$

$$Cs+D = F(s)*(s^2+b^2) \quad let\ s = -bi$$

$$-Aai + B = \frac{-ai}{-a^2+b^2}$$

$$-Cbi + D = \frac{-bi}{-b^2+a^2}$$

Solution

$$B = 0 \quad D = 0 \quad A = \frac{1}{b^2-a^2} \quad C = \frac{1}{a^2-b^2}$$

5. (20 points) Consider

$$\ddot{y} + y = e^{-t}$$

and let $y(0) = y_0, \dot{y}(0) = y_1$ be **unspecified.** Determine the general solution of this ordinary differential equation.

$$s^2Y(s) - sy(0-) - \dot{y}(0-) + Y(s) = \frac{1}{s+1}$$

$$(s^2+1)Y(s) - y_0s - y_1 = \frac{1}{s+1}$$

$$Y(s) = \frac{\frac{1}{s+1} + y_0s + y_1}{s^2+1} = \frac{1}{(s^2+1)(s+1)} + y_0\frac{s}{s^2+1} + y_1\frac{1}{s^2+1}$$

$$\frac{1}{(s^2+1)(s+1)} = \frac{As+C}{s^2+1} + \frac{B}{s+1}$$

$$A = -\frac{1}{2} \quad B = \frac{1}{2} \quad C = \frac{1}{2}$$

$$Y(s) = -\frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{1}{s+1} + y_0\frac{s}{s^2+1} + y_1\frac{1}{s^2+1}$$

$$y(t) = -\frac{1}{2}\cos t + \frac{1}{2}sint + \frac{1}{2}e^{-t} + y_0cost + y_1sint = \left(y_0 - \frac{1}{2}\right)cost + \left(y_1 + \frac{1}{2}\right)sint + \frac{1}{2}e^{-t}$$