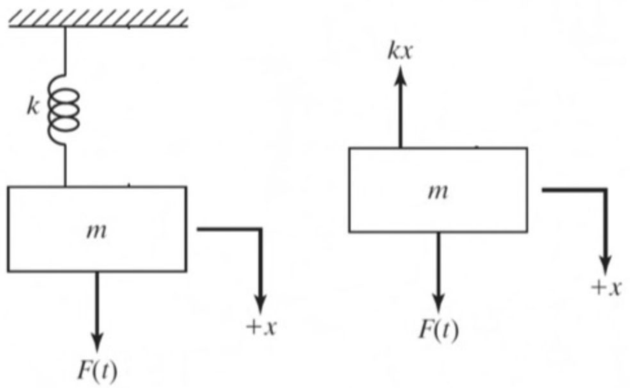
Forced Vibration 单自由度受迫振动



(a) (b) Free-body diagram

equation of motion : $F(t)+mg-k(x+\Delta)=m\ddot{x}$

习惯上定义物体力平衡点为x原点

Laplace Transform :

$$\mathcal{L}(x)=\frac{F_0}{m}\frac{s}{(s^2+\omega^2)(s^2+\omega_n^2)}+\frac{x_0 s+\dot{x}_0}{s^2+\omega_n^2},\quad \omega_n=\sqrt{\frac{k}{m}}$$

$$\mathcal{L}(x)=\frac{F_0}{m(\omega_n^2-\omega^2)}\left(\frac{s}{s^2+\omega^2}-\frac{s}{s^2+\omega_n^2}\right)+x_0\frac{s}{s^2+\omega_n^2}+\frac{\dot{x}_0}{\omega_n}\frac{\omega_n}{s^2+\omega_n^2}$$

Define static deflection $\delta_{st}=F_0/k$

$$x(t)=\left(x_0-\frac{\delta_{st}}{m(\omega_n^2-\omega^2)/k}\right)\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t+\frac{\delta_{st}}{m(\omega_n^2-\omega^2)/k}\cos\omega t$$

$$=\boxed{\left(x_0-\frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t}+\boxed{\underbrace{\frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}}_{X}\cos\omega t}$$

Homogeneous solution x_h (transient)

Particular solution x_p (steady-state)

plitude ratio X/δ_{st} and frequency ratio $r=\omega/\omega_n$

X 1

受迫振动中，频率为自然频率 ω_n 的部分为瞬态 transient
(振动物体)
频率为施加力的频率 ω 的部分为稳态 steady state

振动图像为瞬态与稳态的叠加. (见Matlab Session)

存在阻尼时，瞬态部分会逐渐衰减，最终只剩下稳态部分

frequency ratio: $r = \omega / \omega_n$

amplitude ratio: $X/\delta_{st} = \dfrac{1-\left(\frac{\omega}{\omega_n}\right)^2}{F_0/k}$

δ_{st} ↓ Static deflection

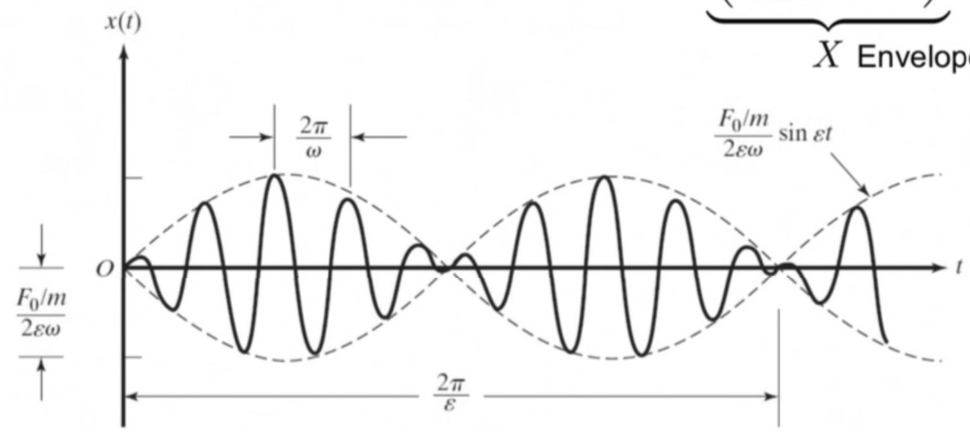
Beating $\omega = \omega_n$

Beating phenomenon (let $x_0 = \dot{x}_0 = 0$):

$$x(t) = \frac{\delta_{st}}{1-r^2}\left[2\sin\frac{\omega+\omega_n}{2}t\ \sin\frac{\omega_n-\omega}{2}t\right]$$

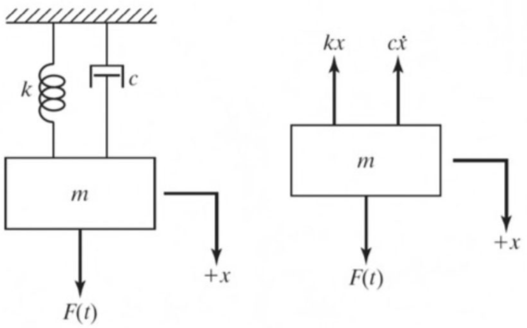
If $\omega_n - \omega = 2\varepsilon$ is a small positive quantity, then $\omega_n \approx \omega$ and

$$\omega + \omega_n \approx 2\omega, \quad \omega_n^2 - \omega^2 \approx 4\varepsilon\omega \quad \Rightarrow \quad x(t) = \underbrace{\left(\frac{F_0/m}{2\varepsilon\omega}\sin\varepsilon t\right)}_{X \text{ Envelope (dashed line)}}\sin\omega t$$



单自由度阻尼受迫振动

Response of a damped system under harmonic force



(a) (b) Free-body diagram

$$m\ddot{x} + c\dot{x} + kx = F_0\cos\omega t$$

⇓ Laplace transform

$$\mathcal{L}(x) = \frac{F_0}{m}\frac{s}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)} + \frac{x_0(s+2\zeta\omega_n)+\dot{x}_0}{s^2+2\zeta\omega_n s+\omega_n^2}, \quad \omega_n = \sqrt{\frac{k}{m}}, \zeta = \frac{c}{2\sqrt{mk}}$$

Partial fraction of steady-state part: $F(s) = \mathcal{L}(x_p(t))$

求解太困难了

$$F(s) = \frac{F_0}{m}\left(\frac{a_1 s + a_2}{s^2 + \omega^2} + \frac{a_3 s + a_4}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

$$a_1 = \frac{\omega_n^2 - \omega^2}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2} \qquad a_3 = -\frac{\omega_n^2 - \omega^2}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2}$$

$$a_2 = \frac{2\zeta\omega_n\omega^2}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2} \qquad a_4 = -\frac{2\zeta\omega_n^3 \ \cancel{\omega_n^2 - \omega^2}}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2}$$

Partial fraction of complete solution: $X(s) = \mathcal{L}(x(t))$

力的频率 steady

$$X(s) = \frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2\omega^2 + (\omega_n^2 - \omega^2)^2}\left[(\omega_n^2 - \omega^2)\left(\frac{s}{s^2 + \omega^2}\right) + (2\zeta\omega_n\omega)\left(\frac{\omega}{s^2 + \omega^2}\right)\right.$$
$$\left. - (\omega_n^2 - \omega^2)\left(\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) - (2\zeta\omega_n)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\right]$$

自然频率 transient

Total solution:

$$x(t) = \boxed{\frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2 + (\omega_n^2 - \omega^2)^2}[(\omega_n^2 - \omega^2)\cos\omega t + 2\zeta\omega_n\omega\sin\omega t \quad \text{Steady-state } x_p(t)}$$

$$\boxed{\begin{aligned}&+\frac{(\omega_n^2 - \omega^2)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi) \qquad \phi = \tan^{-1}\left(\frac{1-\zeta^2}{\zeta}\right)\\ &-\frac{(2\zeta\omega_n^2)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t)] \qquad \text{Transient } x_h(t) \text{ (decays to 0 if } \zeta > 0)\end{aligned}}$$

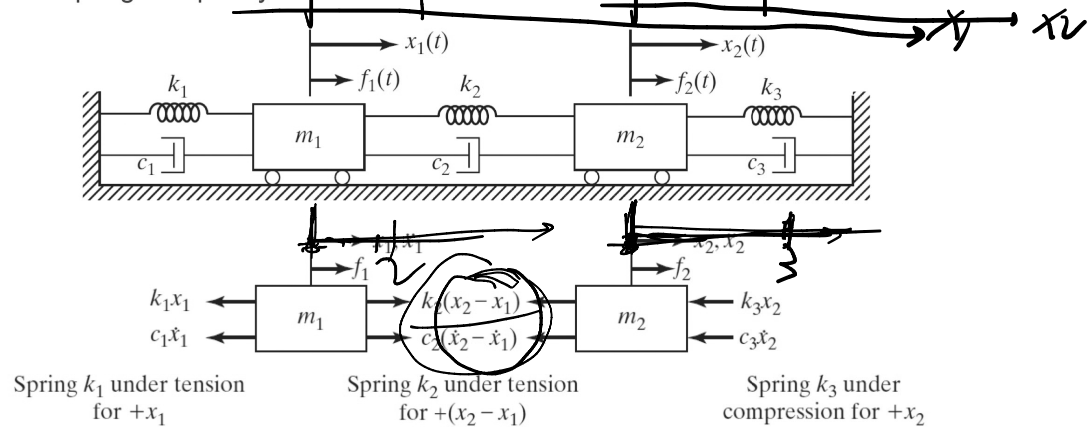
$$x_p(t) = \frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2 + (\omega_n^2 - \omega^2)^2}[(\omega_n^2 - \omega^2)\cos\omega t + 2\zeta\omega_n\omega\sin\omega t]$$
$$= \underbrace{\frac{F_0}{\sqrt{c^2\omega^2 + (k - m\omega^2)^2}}}_{X}\cos(\omega t - \phi) \qquad \phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

对比(有/无阻尼受迫振动

```matlab
%% Transfer Function 1
    clc; close all;
    x0=1;x0_dot=1;  %Initial condition
    L1=TransFunc(0,1);
    L2=TransFunc(0.5,1);
    L3=TransFunc(1,1);
    L4=TransFunc(2,1);
    figure
    impulse(L1)
    xlim([0 20])
    hold on
    impulse(L2)
    impulse(L3)
    impulse(L4)
    legend("ksi = 0","ksi = 0.5","ksi = 1","ksi = 2")
    title("Impulse Response when omega = 1")
    hold off

%% Transfer Function 2 无阻尼受迫振动
    x0=0.1;x0_dot=0;      %Initial condition
    F0 = 10;
    m = 4;
    w = 5; %稳态频率（力的频率）
    wn = 2; %瞬态频率（物体自然频率）

    num_static=[1 0];
    den_static=[1 0 w^2];
    Ls=F0/(m*(wn^2-w^2))*tf(num_static,den_static)

    num_transient1=[1 0];
    den_transient1=[1 0 wn^2];
    num_transient2=[wn];
    den_transient2=[1 0 wn^2];
    Lt=(-F0/(m*(wn^2-w^2))+x0)*tf(num_transient1,den_transient1)+x0_dot/wn*tf(num_transient2,den_transient2)

    L=Ls+Lt
    figure
    hold on
    impulse(Ls)
    impulse(Lt)
    impulse(L)
    xlim([0 20])
    ylim([-0.5 0.5])
    legend("Ls","Lt","L")


%% Transfer Function3 阻尼受迫振动
    x0=0.1;x0_dot=0;      %Initial condition
    F0 = 10;
    m = 4;
    w = 5; %稳态频率（力的频率）
    wn = 2; %瞬态频率（物体自然频率）
    zeta = 0.1; %阻尼系数

    num_s1=[1 0];
    den_s1=[1 0 w^2];
    num_s2=[w];
    den_s2=[1 0 w^2];
    Ls=F0/m*((wn^2-w^2)/((2*zeta*wn)^2*w^2+(wn^2-w^2)^2))*tf(num_s1,den_s1)+...
    F0/m*((2*zeta*wn*w)/((2*zeta*wn)^2*w^2+(wn^2-w^2)^2))*tf(num_s2,den_s2);

    num_t1=[1 0];
    den_t1=[1 2*zeta*wn wn^2];
    num_t2=[wn^2];
    den_t2=[1 2*zeta*wn wn^2];
    Lt=-F0/m*((wn^2-w^2)/((2*zeta*wn)^2*w^2+(wn^2-w^2)^2))*tf(num_t1,den_t1)-...
    F0/m*((2*zeta*wn)/((2*zeta*wn)^2*w^2+(wn^2-w^2)^2))*tf(num_t2,den_t2);

    L=Ls+Lt
    figure
    hold on
    impulse(Ls)
    impulse(Lt)
    impulse(L)
    xlim([0 20])
    ylim([-0.5 0.5])
    legend("Ls","Lt","L")

function [L] = TransFunc(xi,w)
    x0=1;x0_dot=1;      %Initial condition
    num=[x0 x0_dot+2*xi*w*x0];
    den=[1 2*xi*w  w^2];
    L=tf(num,den);
end
```

2 DoF vibration

Two-DoF mass-spring-damper systems:



m1与m2的运动方程:

m1: $f_1 + k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1) - k_1x_1 - c_1\dot{x}_1 = m_1\ddot{x}_1$

$$f_1 = m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2$$

m2: $f_2 - k_3x_2 - c_3\dot{x}_2 - k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) = m_2\ddot{x}_2$

$$f_2 = m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 + (c_2+c_3)\dot{x}_2 - c_2\dot{x}_1$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$M\ddot{\vec{x}}(t) + C\dot{\vec{x}}(t) + K\vec{x}(t) = \vec{f}(t)$$

$M \downarrow$ 惯量矩阵　$C \downarrow$ 阻尼矩阵　$K \downarrow$ 刚度矩阵

dimension of Matrix = degree of freedom

1> 2DoF 无阻尼自由振动

$C=0 \quad \vec{f}(t)=0 \qquad M\ddot{\vec{x}}(t) + K\vec{x}(t) = 0$

$$M[s^2X(s) - s\vec{x}(0) - \dot{\vec{x}}(0)] + KX(s) = 0$$

$$(Ms^2+K)X(s) = sM\vec{x}(0) + M\dot{\vec{x}}(0)$$

$$X(s) = (s^2M+K)^{-1}\cdot(sM\vec{x}(0) + M\dot{\vec{x}}(0))$$

其中 $(s^2M+K)^{-1} = \dfrac{\text{adj}(s^2M+K)}{\det(s^2M+K)}$

adj → adjoint Matrix 伴随矩阵

det → determinant 行列式

$$X(s) = \frac{\operatorname{adj}(s^2M+K)\cdot(sM\vec{x}(0)+M\dot{\vec{x}}(0))}{\det(s^2M+K)}$$

characteristic equation : $\det(s^2M+K)=0$

$$s^2M+K = s^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} = \begin{bmatrix} s^2m_1+k_1+k_2 & -k_2 \\ -k_2 & s^2m_2+k_2+k_3 \end{bmatrix}$$

特征方程: $(s^2m_1+k_1+k_2)(s^2m_2+k_2+k_3)-k_2^2=0$

$$s^4m_1m_2 + s^2\left(m_1(k_2+k_3)+m_2(k_1+k_2)\right) + (k_1k_2+k_1k_3+k_2k_3)=0$$

求根: $$s^2 = \frac{-[(k_1+k_2)m_2+(k_2+k_3)m_1]\pm\sqrt{[(k_1+k_2)m_2-(k_2+k_3)m_1]^2+4k_2^2m_1m_2}}{2m_1m_2}$$

记作 $s_1^2=-\omega_1^2$ $\quad s_2^2=-\omega_2^2$

$$\det(s^2M+K) = (s^2+\omega_1^2)(s^2+\omega_2^2)$$

$$X(s) = \frac{1}{(s^2+\omega_1^2)(s^2+\omega_2^2)}\begin{bmatrix} a_0s^3+a_1s^2+a_2s+a_3 \\ b_0s^3+b_1s^2+b_2s+b_3 \end{bmatrix}$$

参数

$\mathcal{L}^{-1} \downarrow$

$$x_1(t) = X_1^{(1)}\cos(\omega_1t+\boxed{\phi_1}) + X_1^{(2)}\cos(\omega_2t+\boxed{\phi_2})$$

$$x_2(t) = X_2^{(1)}\cos(\omega_1t+\boxed{\phi_1'}) + X_2^{(2)}\cos(\omega_2t+\boxed{\phi_2'})$$

$\phi_1=\phi_1'$ $\quad \phi_2=\phi_2'$

Prove : Diagonalization of the differential equations

对于 2DoF 的无阻尼自由振动：

$$M\ddot{\vec{x}}(t) + K\vec{x}(t) = 0 \qquad M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

$\boxed{\text{假设存在矩阵}A}$ (constant invertible matrix)

使得：

$$A^T M A = I_{2\times 2}$$

$$A^T K A = D = \mathrm{diag}(d_1, d_2) = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}$$

定义 $\vec{x}(t) = A \cdot \vec{y}(t)$

有：

$$M\cdot\ddot{\vec{x}}(t) + K\vec{x}(t) = 0$$

$$M\cdot A\ddot{\vec{y}}(t) + KA\vec{y}(t) = 0$$

$$A^T\cdot MA\ddot{\vec{y}}(t) + A^TKA\vec{y}(t) = 0$$

$$I_{2\times 2}\ddot{\vec{y}}(t) + D\vec{y}(t) = 0$$

$\downarrow$ Laplace Transform

$$I\cdot\left[s^2\vec{Y}(s) - s\vec{y}(0) - \dot{\vec{y}}(0)\right] + D\cdot\vec{Y}(s) = 0$$

$$(s^2 I + D)\vec{Y}(s) = s\vec{y}(0) + \dot{\vec{y}}(0)$$

$$\begin{bmatrix} s^2+d_1 & \\ & s^2+d_2 \end{bmatrix}\vec{Y}(s) = \begin{bmatrix} s\vec{y}_1(0) + \dot{\vec{y}}_1(0) \\ s\vec{y}_2(0) + \dot{\vec{y}}_2(0) \end{bmatrix}$$

$$\vec{Y}(s) = \begin{bmatrix} \frac{1}{s^2+d_1} & \\ & \frac{1}{s^2+d_2} \end{bmatrix}\begin{bmatrix} s\vec{y}_1(0) + \dot{\vec{y}}_1(0) \\ s\vec{y}_2(0) + \dot{\vec{y}}_2(0) \end{bmatrix} = \begin{bmatrix} \frac{s\vec{y}_1(0)+\dot{\vec{y}}_1(0)}{s^2+d_1} \\ \frac{s\vec{y}_2(0)+\dot{\vec{y}}_2(0)}{s^2+d_2} \end{bmatrix}$$

$\downarrow$

$$\vec{y}(t) = \begin{bmatrix} Y_1\cos(\omega_1 t + \phi_1) \\ Y_2\cos(\omega_2 t + \phi_2) \end{bmatrix}$$

其中 $\omega_1^2 = d_1$

$\omega_2^2 = d_2$

$$\vec{x}(t) = A\vec{y}(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} Y_1\cos(\omega_1 t+\phi_1) \\ Y_2\cos(\omega_2 t+\phi_2) \end{bmatrix} = \begin{bmatrix} a_{11}Y_1\cos(\omega_1 t+\phi_1)+a_{12}Y_2\cos(\omega_2 t+\phi_2) \\ a_{21}Y_1\cos(\omega_1 t+\phi_1)+a_{22}Y_2\cos(\omega_2 t+\phi_2) \end{bmatrix}$$

证明矩阵 A 存在（A 可对角化 K，单位化 M）

正定矩阵：positive definite

给定大小为 $n\times n$ 的实对称矩阵 A，若对于任意非0向量 $x\in R^n$（n维向量）有 $x^T A x > 0$ 恒成立，则 A 为正定矩阵。

M 正定. $M^{\frac{1}{2}} = \begin{bmatrix} m_1^{\frac{1}{2}} & 0 \\ 0 & m_2^{\frac{1}{2}} \end{bmatrix}$

K 对称 $M^{-\frac{1}{2}} K M^{-\frac{1}{2}} = \begin{bmatrix} m_1^{-1}\cdot(k_1+k_2) & -(m_1 m_2)^{-\frac{1}{2}}k_2 \\ -(m_1 m_2)^{-\frac{1}{2}}k_2 & m_2^{-1}(k_2+k_3) \end{bmatrix}$ 对称阵

$\underbrace{\text{实对称矩阵}}$可被$\underbrace{\text{正交矩阵}}$对角化

$$(M^{-\frac{1}{2}} K M^{-\frac{1}{2}}) = Q D Q^T \quad (Q^T Q = I,\ D = \mathrm{diag}(d_1, d_2))$$

$$Q^T M^{-\frac{1}{2}} K M^{-\frac{1}{2}} Q = Q^T Q D Q^T Q = D$$

$$(M^{-\frac{1}{2}} Q)^T K (M^{-\frac{1}{2}} Q) = D$$

let $\boxed{A = M^{-\frac{1}{2}} Q}$，有 $A^T K A = D$

$A^T M A = Q^T M^{-\frac{1}{2}} M M^{-\frac{1}{2}} Q = Q^T Q = I$ 得证.

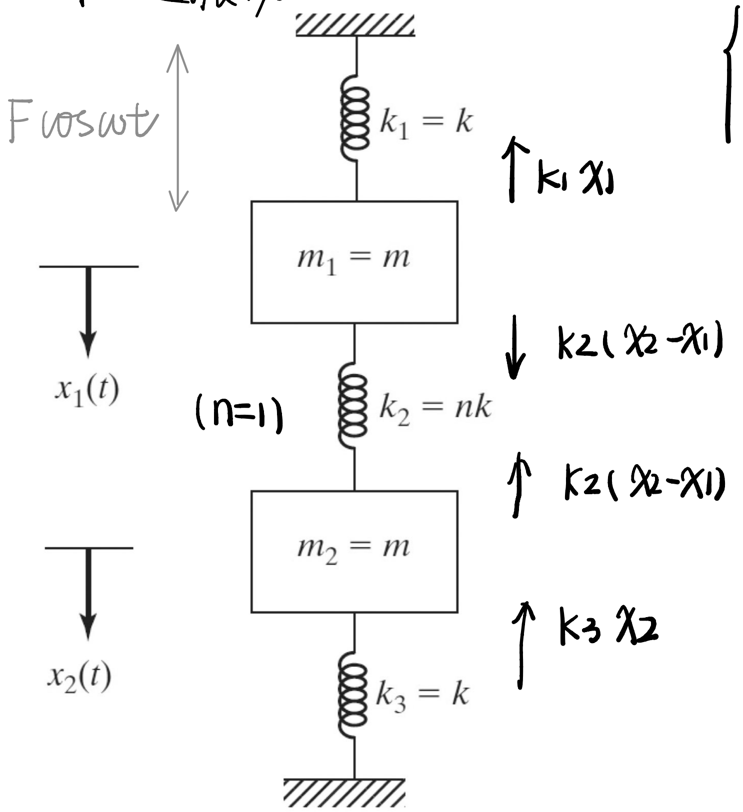
$$\vec{x}(t)=\begin{bmatrix} a_{11}Y_1\cos(\omega_1 t+\phi_1)+a_{12}Y_2\cos(\omega_2 t+\phi_2) \\ \\ a_{21}Y_1\cos(\omega_1 t+\phi_1)+a_{22}Y_2\cos(\omega_2 t+\phi_2)\end{bmatrix}=\begin{bmatrix}X_{11}\\ \\ X_{21}\end{bmatrix}\cos(\omega_1 t+\phi_1)+\begin{bmatrix}X_{12}\\ \\ X_{22}\end{bmatrix}\cos(\omega_2 t+\phi_2)$$

First Mode　　Second Mode

$X_{11}=a_{11}Y_1$　　$X_{12}=a_{12}Y_2$

$X_{21}=a_{21}Y_1$　　$X_{22}=a_{22}Y_2$

2DoF受迫振动



$F\cos\omega t+$

$$\begin{cases} k_2(x_2-x_1)-k_1x_1=m_1\ddot{x}_1 \\ -k_3x_2-k_2(x_2-x_1)=m_2\ddot{x}_2\end{cases}$$

$$\begin{cases} m_1\ddot{x}_1+(k_1+k_2)x_1+(-k_2)x_2=F\cos\omega t \\ m_2\ddot{x}_2+(-k_2)x_1+(k_2+k_3)x_2=0\end{cases}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2\end{bmatrix}\ddot{\vec{x}}(t)+\begin{bmatrix}k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3\end{bmatrix}\vec{x}(t)=\begin{bmatrix}F\cos\omega t \\ 0\end{bmatrix}$$

$$m I_{2\times2}\ddot{\vec{x}}(t)+\begin{bmatrix}2k & -k \\ -k & 2k\end{bmatrix}\vec{x}(t)=\begin{bmatrix}F\cos\omega t \\ 0\end{bmatrix}$$

$$mI s^2\vec{X}(s)+K\vec{X}(s)=\vec{F}(s)$$

2DoF阻尼受迫振动

$$M\ddot{\vec{x}}(t)+C\dot{\vec{x}}(t)+K\vec{x}(t)=\vec{f}(t)$$

$$M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2\end{bmatrix}\quad C=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3\end{bmatrix}\quad K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3\end{bmatrix}$$

$$M s^2\vec{X}(s)+C s\vec{X}(s)+K\vec{X}(s)=\vec{F}(s)$$

$$\begin{bmatrix} s^2m_1+(c_1+c_2)s+(k_1+k_2) & -c_2s-k_2 \\ \\ -c_2-k_2 & m_2s^2+(c_2+c_3)s+(k_2+k_3)\end{bmatrix}\vec{X}(s)=\vec{F}(s)$$

$\Downarrow$

求逆

$$\Delta = \begin{vmatrix} s^2 m_1 + (c_1+c_2)s + (k_1+k_2) & -c_2 s - k_2 \\ -c_2 - k_2 & m_2 s^2 + (c_2+c_3)s + (k_2+k_3) \end{vmatrix} = 0$$

解得 $(s^2 + 2\zeta_1 \omega_1 s + \omega_1^2)(s^2 + 2\zeta_2 \omega_2 s + \omega_2^2) = 0$

$$x_1 = e^{-\zeta_1\omega_1 t} X_1^{(1)} \cos(\sqrt{1-\zeta_1^2}\omega_1 t + \phi_1) + e^{-\zeta_2\omega_2 t} X_1^{(2)} \cos(\sqrt{1-\zeta_2^2}\omega_2 t + \phi_2)$$

$$x_2 = e^{-\zeta_1\omega_1 t} X_2^{(1)} \cos(\sqrt{1-\zeta_1^2}\omega_1 t + \phi_1') + e^{-\zeta_2\omega_2 t} X_2^{(2)} \cos(\sqrt{1-\zeta_2^2}\omega_2 t + \phi_2')$$

$$x_1 = e^{-\zeta_1\omega_1 t} X_1^{(1)} \cos(\sqrt{1-\zeta_1^2}\omega_1 t + \phi_1) + e^{-\zeta_2\omega_2 t} X_1^{(2)} \cos(\sqrt{1-\zeta_2^2}\omega_2 t + \phi_2)$$

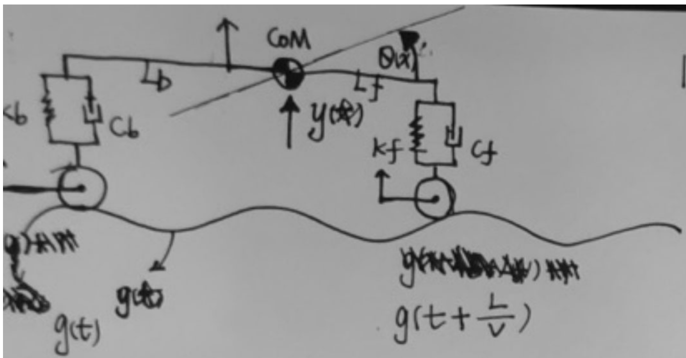
$$x_2 = e^{-\zeta_1\omega_1 t} X_2^{(1)} \cos(\sqrt{1-\zeta_1^2}\omega_1 t + \phi_1) + e^{-\zeta_2\omega_2 t} X_2^{(2)} \cos(\sqrt{1-\zeta_2^2}\omega_2 t + \phi_2)$$

proportional damping condition 对 C M. K 对角化

当且仅当 $CM^{-1}K = KM^{-1}C$ 时

$$CM^{-1}K - KM^{-1}C = \begin{bmatrix} 0 & m_1^{-1}(k_1c_2 - k_2c_1) + m_2^{-1}(k_2c_3 - k_3c_2) \\ -m_1^{-1}(k_1c_2 - k_2c_1) - m_2^{-1}(k_2c_3 - k_3c_2) & 0 \end{bmatrix}$$

2DoF (half car suspension system)



$|\theta| < 4^\circ \Rightarrow \begin{cases} \sin\theta = \theta \\ \cos\theta = 1 \end{cases}$

$\Delta y_b = y(t) - L_b \cdot \sin(\theta(t)) \approx y - L_b\theta - g(t)$

$\Delta y_f = y(t) + L_f \sin(\theta(t)) - g(t + \frac{L}{v}) \approx y + L_f \cdot \theta - g(t + \frac{L}{v})$

$T = \frac{1}{2} m \cdot \dot{y}^2 + \frac{1}{2} J \cdot \dot{\theta}^2$

$V = \frac{1}{2} k_b \cdot [y - L_b \cdot \theta - g(t)]^2 + \frac{1}{2} k_f \cdot [y + L_f \cdot \theta - g(t + \frac{L}{v})]^2$

$E_D = \frac{1}{2} c_b \cdot [\dot{y} - L_b \cdot \dot{\theta} - \dot{g}(t)]^2 + \frac{1}{2} c_f \cdot [\dot{y} + L_f \dot{\theta} - \dot{g}(t + \frac{L}{v})]^2$

$L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial E_D}{\partial \dot{q}_i} = Q_i$$

$$y: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} + \frac{\partial E_D}{\partial \dot{y}} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + 0 - 0 - 0$$

$$\frac{\partial E_D}{\partial \dot{q}_i} = + C_b\cdot[\dot{y} - L_b\dot{\theta} - \dot{g}(t)] + C_f\cdot[\dot{y} + L_f\dot{\theta} - \dot{g}(t+\tfrac{L}{v})]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \quad \cancel{C_b[\dot{y} - L_b\dot{\theta} - \dot{g}(t)] - C_f[\dot{y} + L_f\dot{\theta} - \dot{g}(t+\tfrac{L}{v})]}$$

$$\frac{\partial L}{\partial y} = 0 + 0 - K_b\cdot[y - L_b\theta - g(t)] - K_f[y + L_f\theta - g(t+\tfrac{L}{v})]$$

EOM1: $$m\ddot{y} + K_b[y - L_b\theta - g(t)] + K_f[y + L_f\cdot\theta - g(t+\tfrac{L}{v})] + C_b[\dot{y} - L_b\dot{\theta} - \dot{g}(t)] + C_f[\dot{y} + L_f\dot{\theta} - \dot{g}(t+\tfrac{L}{v})] = 0$$

$$\theta: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial E_D}{\partial \dot{\theta}} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = J\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -K_b\cdot[y - L_b\cdot\theta - g(t)]\cdot(-L_b) - K_f[y + L_f\theta - g(t+\tfrac{L}{v})]\cdot(L_f) = K_b L_b[y - L_b\theta - g(t)] - K_f L_f[y + L_f\theta - g(t+\tfrac{L}{v})]$$

$$\frac{\partial E_D}{\partial \dot{\theta}} = C_b\cdot[\dot{y} - L_b\dot{\theta} - \dot{g}(t)]\cdot(-L_b) + C_f\cdot[\dot{y} + L_f\dot{\theta} - \dot{g}(t+\tfrac{L}{v})]\cdot(L_f)$$
$$= -L_b C_b\cdot[\dot{y} - L_b\dot{\theta} - \dot{g}(t)] + L_f C_f[\dot{y} + L_f\dot{\theta} - \dot{g}(t+\tfrac{L}{v})]$$

EOM2:

$$J\ddot{\theta} - K_b L_b[y - L_b\theta - g(t)] + K_f L_f(y + L_f\theta - g(t+\tfrac{L}{v})] - C_b L_b[\dot{y} - L_b\dot{\theta} - \dot{g}(t)] + C_f L_f(\dot{y} + L_f\dot{\theta} - \dot{g}(t+\tfrac{L}{v})] = 0$$

EOM:

$$m\ddot{y} + (C_b + C_f)\dot{y} + (L_f C_f - L_b C_b)\dot{\theta} + (K_b + K_f)y + (L_f K_f - L_b K_b)\theta = C_b\cdot\dot{g}(t) + C_f\,\dot{g}(t+\tfrac{L}{v}) + K_b\cdot g(t) + K_f\,g(t+\tfrac{L}{v})$$

$$J\ddot{\theta} + (L_f C_f - L_b C_b)\dot{y} + (L_f^2 C_f + L_b^2 C_b)\dot{\theta} + (L_f K_f - L_b K_b)y + (L_f^2 K_f + L_b^2 K_b)\theta = -L_b C_b\,\dot{g}(t) + L_f C_f\,\dot{g}(t+\tfrac{L}{v}) - L_b K_b\,g(t) + L_f K_f\,g(t+\tfrac{L}{v})$$

Matrix form:

$$\begin{bmatrix} m & \\ & J \end{bmatrix}\begin{bmatrix}\ddot{y} \\ \ddot{\theta}\end{bmatrix} + \begin{bmatrix} C_b + C_f & L_f C_f - L_b C_b \\ L_f C_f - L_b C_b & L_f^2 C_f + L_b^2 C_b \end{bmatrix}\begin{bmatrix}\dot{y} \\ \dot{\theta}\end{bmatrix} + \begin{bmatrix} K_b + K_f & L_f K_f - L_b K_b \\ L_f K_f - L_b K_b & L_f^2 K_f + L_b^2 K_b \end{bmatrix}\begin{bmatrix} y \\ \theta \end{bmatrix}$$

$$= \begin{bmatrix} C_b & C_f & K_b & K_f \\ -L_b C_b & L_f C_f & -L_b K_b & L_f K_f \end{bmatrix}\begin{bmatrix} \dot{g}(t) \\ \dot{g}(t+\tfrac{L}{v}) \\ g(t) \\ g(t+\tfrac{L}{v}) \end{bmatrix}$$