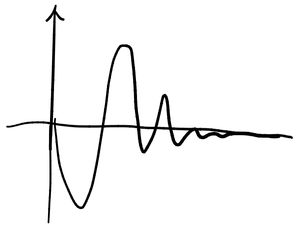
Vibration

1. Types of vibration
- free / natural
- forced 受迫振动
- damped 阻尼振动

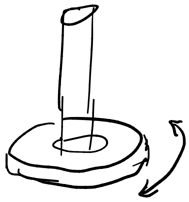


2. 横波：波传递与振动同向

纵波：垂直

弹性变形 → 振动

转轴振动



单摆振动周期. $T = 2\pi\sqrt{\frac{L}{g}}$

3. kinematics of vibration

$f = \frac{1}{T}$

$y = A \sin \omega t$ $\quad T = \frac{2\pi}{\omega}$ seconds

$f = \frac{\omega}{2\pi}$ cycles per seconds

4.
$$\left.\begin{aligned} x &= x_0 \sin \omega t \\ \dot{x} &= x_0 \omega \cos \omega t \\ \ddot{x} &= -x_0 \omega^2 \sin \omega t \end{aligned}\right\} \Rightarrow \ddot{x} + \omega^2 x = 0$$

$$E = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{动}} + \underbrace{\frac{1}{2} k x^2}_{\text{弹势}} = \text{constant}$$

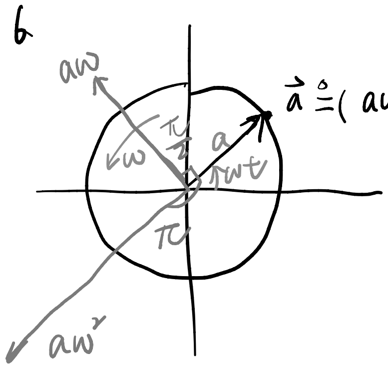
$$\dot{E} = \dot{x}(m\ddot{x} + kx) = 0 \Rightarrow m\ddot{x} + kx = 0$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

5. $x_1 = a\sin t$

$x_2 = b\sin(t+\phi)$ → 图左移. phase lead ϕ

6.



$\vec{a} \hat{=} (a\cos\omega t,\ a\sin\omega t)$

$x = x_0 \sin\omega t$

$\dot{x} = x_0\omega\cos\omega t = x_0\omega\sin(\omega t+\frac{\pi}{2})$

$\ddot{x} = -x_0\omega^2\sin\omega t = x_0\omega^2\sin(\omega t+\pi)$

7. 频率相同合成

不同频率合成

8. 复平面

$\vec{a} = a\cos\omega t + i a\sin\omega t = ae^{i\omega t}$

$\dot{\vec{a}} = a(i\omega)e^{i\omega t} = i\omega\vec{a}$

$\ddot{\vec{a}} = a(i\omega)^2 e^{i\omega t} = (i\omega)^2\vec{a}$

9. 力 $p = P_0 \sin(\omega t + \phi)$

$$\int_0^{\frac{2\pi}{\omega}} p \frac{dx}{dt} dt$$

反复振动

⋮

$W = \pi P_0 x_0 \sin\phi$

10. 傅立叶级数

级数中的每一项 和 $\sin n\omega t$ 相乘 积分

$m \neq n$ 时 积分都为0

如何找傅立叶级数的函数

用几项去估量 有效 项 (幅值)

每量精确时即可

到达一定比例即可截断

Work done on:

$$x = x_0 \sin(m\omega t + \phi) \qquad \dot{x} = x_0 m\omega \cdot \cos(m\omega t + \phi)$$

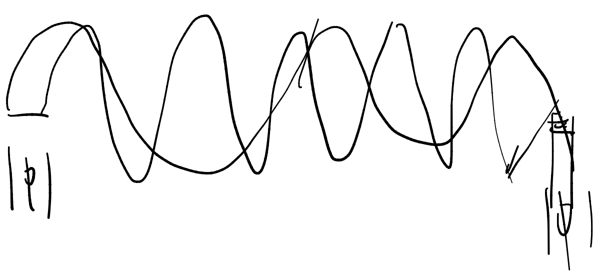
Harmonic force:

$$P = P_0 \sin(n\omega t)$$

Work done on x by P:

$$W = \int P\,dx = \int_0^T p \frac{dx}{dt} dt = \int_0^T P_0 \sin(n\omega t) \cdot x_0 m\omega \cos(m\omega t + \phi) dt$$

$$= m\omega P_0 x_0 \int_0^T \sin(n\omega t) \cos(m\omega t + \phi) dt$$



Fourier Series & Transform

一个周期函数 → 如何展开成三角函数形式？系数怎么求？

Part 1. 三角函数的正交性

1. 三角函数系 → 集合

$$\{0,\ 1,\ \sin x,\ \cos x,\ \sin 2x,\ \cos 2x \cdots \sin nx,\ \cos nx\} = \sin nx,\ \cos nx \quad n=0,1,2\ldots$$

2. 三角函数系正交性（内积为0）

性质：从三角函数系中任取2个三角函数，系数不相等的情况下，从$-\pi$到π积分值为0

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m)$$

内积： 向量内积 $\vec{a} = (a_1, a_2 \cdots a_n)$, $\vec{b} = (b_1, b_2 \cdots b_n)$ $\vec{a}$ $\vec{b}$ 正交则有：$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i = 0$

函数内积 $a = f(x)$, $b = g(x)$ $f(x)$ $g(x)$ 正交则有：$a \cdot b = \int_{x_0}^{x_1} f(x) g(x) \, dx = 0$

证明：三角函数正交性

积化和差

$$\int_{-\pi}^{\pi} \cos nx \cdot \cos mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(n-m)x + \cos(n+m)x\right] dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \, dx = \frac{1}{2}\left[\frac{1}{n-m} \cdot \sin(n-m)x \Big|_{-\pi}^{\pi} + \frac{1}{n+m} \sin(n+m)x \Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2}\left[\frac{1}{n-m} \cdot 0 + \frac{1}{n+m} \cdot 0\right] = 0 \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin nx \cdot \sin mx \, dx = 0 \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin mx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx \, dx = 0$$

当 $m=n$ 时：

$$\int_{-\pi}^{\pi} (\cos mx)^2 \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1+\cos 2mx) \, dx = \int_{-\pi}^{\pi} \frac{1}{2} dx + \int_{-\pi}^{\pi} \cos 2mx \, dx$$

$$= \frac{1}{2} + \underbrace{\int_{-\pi}^{\pi} \cos 0x \cdot \cos 2mx \, dx}_{\text{正交性}} = \frac{1}{2} x \Big|_{-\pi}^{\pi} = \pi$$

Part 2. 周期为 2π 的函数展开为 Fourier Series

$T=2\pi \quad f(x)=f(x+2\pi)$

1>

$$\boxed{f(x)=\sum_{n=0}^{\infty} a_n\cdot\cos nx+\sum_{n=0}^{\infty} b_n \sin nx}$$

2>

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)}$$

$$f(x)=\underbrace{a_0'}_{\frac{a_0}{2}}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

? 为什么一边是 a_0' 一边是 $\frac{a_0}{2}$?

求 a_0': 两边从 $-\pi$ 到 π 对 x 积分

正交性, $=0$

$$\int_{-\pi}^{\pi} f(x)\,dx=\int_{-\pi}^{\pi} a_0'\,dx+\underbrace{\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n\cos nx \cdot \cos 0x\,dx}+\underbrace{\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n\sin nx \cdot \cos 0x\,dx}$$

$$\int_{-\pi}^{\pi} f(x)\,dx=2\pi a_0' \Rightarrow a_0'=\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(x)\,dx \quad \text{令 } a_0'=\frac{a_0}{2}$$

$$\boxed{a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

求 a_n: 两边乘 $\cos mx$, 积分

$$\int_{-\pi}^{\pi} f(x)\cos mx\cdot dx=\underbrace{\int_{-\pi}^{\pi}\frac{a_0}{2}\cos mx\,dx}_{m\neq 0 \text{ 积分为0}}+\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n\cos nx\cos mx\,dx+\underbrace{\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n\sin nx\cos mx\,dx}$$

$m=n$ 积分为0

$m\neq n$ 积分为0

当 $n=m$ 时:

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\int_{-\pi}^{\pi} a_n\cos^2 nx\,dx=a_n\int_{-\pi}^{\pi}\cos^2 nx\,dx=a_n\cdot\pi$$

$$\boxed{a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

同理, 两侧乘 $\sin mx$, 得到: $\boxed{b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$

$T=2\pi \quad f(x)=f(x+2\pi)$

Fourier Series 展开: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx$

其中 $a_n=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos nx\,dx \quad n=1,2\cdots$

$a_0=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\,dx$

$b_n=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin nx\,dx \quad n=1,2\cdots$

Part 3：周期为2L的Fourier Series展开

$f(t)=f(t+2L)$

令 $x=\frac{t}{L}\cdot\pi$

则 $t=\frac{xL}{\pi}$，有 $f(t)=f(\frac{L}{\pi}x)\triangleq g(x)$ $\quad g(x)=g(x+2\pi)$

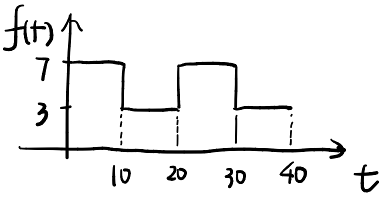
$$\boxed{\begin{aligned} f(t)&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}t+b_n\sin\frac{n\pi}{L}t\right)\\ a_n&=\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{n\pi}{L}t\,dt \qquad a_0=\frac{1}{L}\int_{-L}^{L}f(t)\,dt\\ b_n&=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi}{L}t\,dt \end{aligned}}$$

$\downarrow$ $T=2L$ $\quad \omega=\frac{2\pi}{T}=\frac{\pi}{L}$ $\quad L=\frac{T}{2}$ $\quad \int_{-L}^{L}dt\rightarrow\int_{0}^{2L}dt\rightarrow\int_{0}^{T}dt$

☆
$$\boxed{\begin{aligned} f(t)&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega t+b_n\sin n\omega t\right)\\ a_n&=\frac{2}{T}\int_{0}^{T}f(t)\cos n\omega t\,dt \qquad a_0=\frac{2}{T}\int_{0}^{T}f(t)\,dt\\ b_n&=\frac{2}{T}\int_{0}^{T}f(t)\sin n\omega t\,dt \end{aligned}}$$

？$T\to\infty$，$f(t)$不再为周期函数，如何展开？ $\rightarrow$ Fourier Transform

eg. 写出以下周期函数的Fourier Series



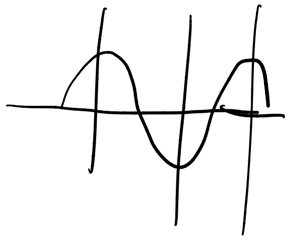
$T=20 \quad \omega=\frac{2\pi}{T}=\frac{\pi}{10}$

$$\begin{cases} f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\frac{\pi}{10}t+b_n\sin n\frac{\pi}{10}t\right)\\ a_n=\frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt=\frac{1}{10}\int_0^{20}f(t)\cos n\frac{\pi}{10}t\,dt \qquad a_0=\frac{1}{10}\int_0^{20}f(t)\,dt\\ b_n=\frac{1}{10}\int_0^{20}f(t)\sin n\frac{\pi}{10}t\,dt \end{cases}$$

$$a_0=\frac{1}{10}\int_0^{20}f(t)\,dt=\frac{1}{10}\left[\int_0^{10}7\,dt+\int_{10}^{20}3\,dt\right]=\frac{1}{10}(70+30)=10$$

$$a_n=\frac{1}{10}\cdot\left[\int_0^{10}7\cos\frac{n\pi}{10}t\,dt+\int_{10}^{20}3\cos\frac{n\pi}{10}t\,dt\right]=\frac{1}{10}\left(7\cdot\frac{10}{n\pi}\sin\frac{n\pi}{10}t\Big|_0^{10}+3\cdot\frac{10}{n\pi}\sin\frac{n\pi}{10}t\Big|_{10}^{20}\right)=0$$

$$b_n=\frac{1}{10}\left(\int_0^{10}7\sin n\frac{\pi}{10}t\,dt+\int_{10}^{20}3\sin\frac{n\pi}{10}t\,dt\right)$$



$$=\frac{1}{10}\left(-7\cdot\frac{10}{n\pi}\cos\frac{n\pi}{10}t\Big|_0^{10}-3\cdot\frac{10}{n\pi}\cos\frac{n\pi}{10}t\Big|_{10}^{20}\right)$$

$$=-\frac{7}{n\pi}(\cos n\pi-\cos 0)-\frac{3}{n\pi}(\cos 2n\pi-\cos n\pi)$$

n为偶数时： $=-\frac{7}{n\pi}\cdot 0-\frac{3}{n\pi}\cdot 0=0$

n为奇数时 $=-\frac{7}{n\pi}\cdot(-2)-\frac{3}{n\pi}(1-(-1))=\frac{14}{n\pi}-\frac{6}{n\pi}=\frac{8}{n\pi}$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega t+b_n\sin n\omega t$$

$$f(t)=5+b_1\sin\omega t+b_3\sin 3\omega t+b_5\sin 5\omega t+\cdots+b_n\sin n\omega t$$ n为奇数

$n=1,3,5\cdots$

$$f(t)=5+\frac{8}{\pi}\sin\frac{\pi}{10}t+\frac{8}{3\pi}\cdot\sin\frac{3\pi}{10}t+\frac{8}{5\pi}\sin\frac{5\pi}{10}t+\frac{8}{7\pi}\sin\frac{7\pi}{10}t+\cdots+\frac{8}{n\pi}\cdot\sin\frac{n\pi}{10}t$$

$n\uparrow\quad\frac{8}{n\pi}\to 0$ 低频正弦波为主要成分

n取1、3、5

$$f(t)\approx 5+\frac{8}{\pi}\sin\frac{\pi}{10}t+\frac{8}{3\pi}\cdot\sin\frac{3\pi}{10}t+\frac{8}{5\pi}\sin\frac{5\pi}{10}t$$

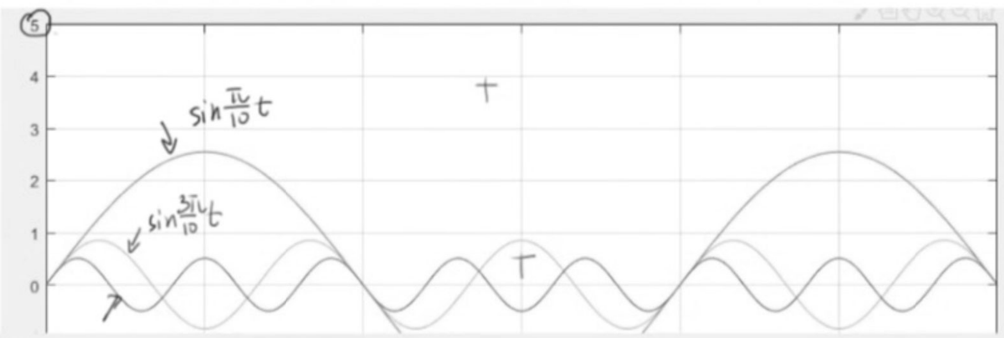
$$\Rightarrow\quad f(t)=5+\sum_{n=1}^{\infty}\frac{8}{n\pi}\cdot\sin\frac{n\pi}{10}t,\qquad n=1,3,5,7,\cdots\cdots$$

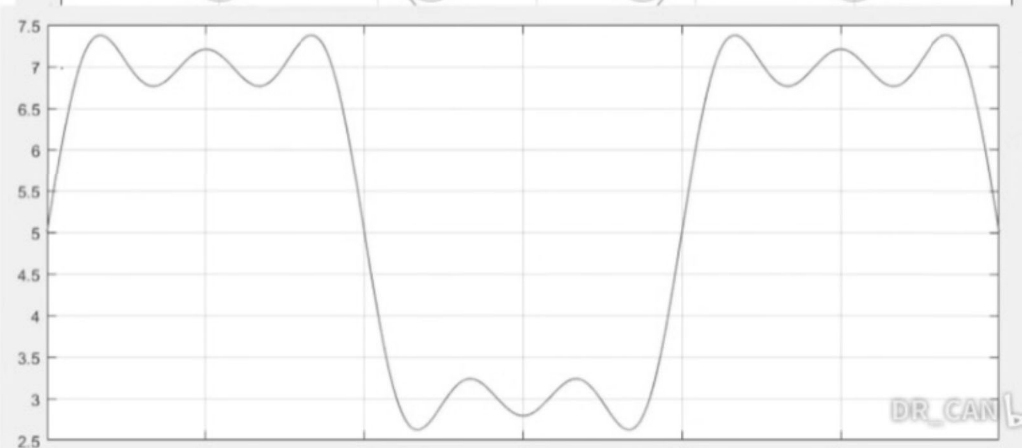
$n\uparrow\quad\frac{8}{n\pi}\to 0$

$n=1,\quad\frac{8}{\pi}\sin\frac{\pi}{10}t$

$n=3\quad\frac{8}{3\pi}\sin\frac{3\pi}{10}t$

$n=5\quad\frac{8}{5\pi}\sin\frac{5\pi}{10}t$





Part 4. ☆ Fourier Series 的复数形式 (Fourier Transform 的基础)

1. Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = -\frac{1}{2}i(e^{i\theta} - e^{-i\theta})$$

2. Euler's Formula 代入 $f(t)$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cdot\frac{1}{2}(e^{in\omega t} + e^{-in\omega t}) + b_n\left(-\frac{1}{2}i\right)(e^{in\omega t} - e^{-in\omega t})\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{a_n - ib_n}{2}e^{in\omega t} + \sum_{n=1}^{\infty}\frac{a_n + ib_n}{2}e^{-in\omega t}$$

$e^0 = 1$ ↓令"$n=-n$"

$$= \sum_{n=0}^{0}\frac{a_0}{2}\cdot e^{in\omega t} + \sum_{n=1}^{\infty}\frac{a_n - ib_n}{2}e^{in\omega t} + \sum_{n=-\infty}^{-1}\frac{a_{-n} + ib_{-n}}{2}e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{in\omega t}$$

$$C_n = \begin{cases} \dfrac{a_0}{2} & n = 0 \\ \dfrac{a_n - ib_n}{2} & n = 1, 2, 3 \ldots \\ \dfrac{a_{-n} + ib_{-n}}{2} & n = -1, -2 \ldots \end{cases}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt$$

$$C_0 = \frac{a_0}{2} = \frac{1}{T}\int_0^T f(t)\,dt = \boxed{\frac{1}{T}\int_0^T f(t)e^{-in\omega t}\,dt}$$

$n > 0$:

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\cdot\left(\frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt - i\frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt\right)$$

$$= \frac{1}{T}\int_0^T f(t)(\cos n\omega t - i\sin n\omega t)\,dt$$

$$\left(= \cos(-n\omega t) + i\sin(-n\omega t) = e^{-in\omega t}\right)$$

$$= \boxed{\frac{1}{T}\int_0^T f(t)e^{-in\omega t}\,dt}$$

$n < 0$:

$$C_n = \frac{a_{-n} + ib_{-n}}{2} = \frac{1}{2}\left(\frac{2}{T}\int_0^T f(t)\cos(-n)\omega t\,dt + i\frac{2}{T}\int_0^T f(t)\sin(-n)\omega t\,dt\right]$$

$$= \frac{1}{T}\int_0^T f(t)(\cos n\omega t - i\sin n\omega t)\,dt = \boxed{\frac{1}{T}\int_0^T f(t)e^{-in\omega t}\,dt}$$

☆ $f_T(t)=f_T(t+T)$

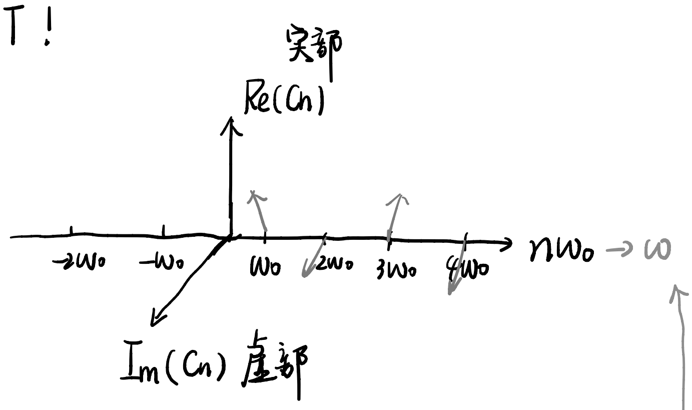
$$\boxed{\begin{aligned} f_T(t) &= \sum_{-\infty}^{\infty} C_n e^{in\omega t} \\ C_n &= \frac{1}{T}\int_0^T f(t)e^{-in\omega t}\,dt \end{aligned}}$$

→ 可令 $T\to\infty$ 非周期函数，Fourier Transform

Part5. Fourier Transform FT!

$$\boxed{\begin{aligned} f_T(t) &= \sum_{-\infty}^{\infty} C_n e^{in\omega_0 t} \quad ① \\ C_n &= \frac{1}{T}\int_0^T f_T(t)e^{-in\omega_0 t}\,dt \quad ② \end{aligned}}$$

$\omega_0=\frac{2\pi}{T}$ 基频率



频域（频谱）：频率为 $n\omega_0$ 的量的系数 (C_n)

Fourier Series：离散情况 $n=-\infty,\ldots\ldots,-1,0,1,2\cdots$ $\Delta\omega=\omega_0=\frac{2\pi}{T}$

当 $T\to\infty$，$\Delta\omega=\omega_0=\frac{2\pi}{T}\to 0$ 离散频谱变成连续曲线

将 C_n的表达式②代入 $f(t)$ 的 Fourier Series ①中：

$$f_T(t)=\sum_{n=-\infty}^{\infty}\frac{1}{T}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-in\omega_0 t}\,dt\right]e^{in\omega_0 t} \qquad \left(\frac{1}{T}=\frac{\Delta\omega}{2\pi}\right)$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Delta\omega}{2\pi}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-in\omega_0 t}\,dt\right]e^{in\omega_0 t}$$

当 $T\to\infty$ 时：

$$\left.\begin{aligned} \int_{-\frac{T}{2}}^{\frac{T}{2}}dt &\to \int_{-\infty}^{\infty}dt \\ n\omega_0 &\to \omega \\ \Delta\omega &\to d\omega \\ \sum_{n=-\infty}^{\infty} &\to \int_{-\infty}^{\infty} \end{aligned}\right\}$$

$$f(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\right)e^{i\omega t}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\right)e^{i\omega t}\,d\omega$$

令
$$\boxed{\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \end{aligned}}$$

→ Fourier Transform

→ inverse FT

Laplace Transform

$f(t) \longrightarrow F(s) \quad (s=\sigma+j\omega \quad j=\sqrt{-1})$

时域　　s域

$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt \qquad s=\sigma+j\omega$

特别地：当 $\sigma=0$ 时 $s=j\omega$

$F(s) = F(j\omega) = \int_0^{\infty} f(t) e^{-j\omega t} dt \rightarrow$ Fouries Transform　傅立叶变换是拉普拉斯的特殊情况

重要的 Laplace Transform

$L[e^{-at}] = \frac{1}{a+s}$

$L[1] = \lim_{\tau\to\infty} \int_0^{\tau} e^{-st} dt = \lim_{\tau\to\infty} \left(\frac{e^{-st}}{-s} \Big|_0^{\tau} \right) = \lim_{\tau\to\infty} \left(\frac{1}{s} + \frac{e^{-s\tau}}{-s} \right) = \frac{1}{s}$

$L[t] = \frac{1}{s^2} = F(s) \xrightarrow{s平移} F(s-a) = \frac{1}{(s-a)^2} = L[t \cdot e^{at}]$

$L[te^{-at}] = \int_0^{\infty} te^{-at} e^{-st} dt = \int_0^{\infty} t e^{-(a+s)t} dt = \frac{1}{(s+a)^2}$

$L[\sin \omega t] = L\left[\frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})\right] = \frac{1}{2i} \cdot \left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega} \right) = \frac{\omega}{s^2+\omega^2}$

$L[\cos \omega t] = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right) = \frac{s}{s^2+\omega^2}$

$L[\cosh \omega t] = \frac{s}{s^2-\omega^2}$

$L[\sinh \omega t] = \frac{\omega}{s^2-\omega^2}$

$\Downarrow$ S域平移

$\frac{\omega}{(s-a)^2+\omega^2} = L[e^{at} \sin \omega t] \qquad \frac{s-a}{(s-a)^2+\omega^2} = L[e^{at} \cos \omega t]$

$\frac{s-a}{(s-a)^2-\omega^2} = L[e^{at} \cosh \omega t] \qquad \frac{\omega}{(s-a)^2-\omega^2} = L[e^{at} \sinh \omega t]$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta+\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

① $t\geqslant 0$；$t<0$ 时，$f(t)=0$。

重要の Laplace Transform 性质

1、线性（叠加原理）　$L[af(t)+bg(t)] = aF(s)+bG(s)$

2、First Translation Theorem（频域平移性质）

对于 $F(s) = L[f(t)]$　$(Re(s) > 0)$

有 $F(s+a) = L[e^{-at}f(t)]$　$(Re(s) > a)$

Proof: $F(s+a) = \int_0^\infty e^{-(s+a)t} f(t)\,dt = \int_0^\infty e^{-at}\cdot e^{-st} f(t)\,dt = \int_0^\infty e^{-st}(e^{-at}f(t))\,dt = L(e^{-at}f(t))$

3. Second Translation Theorem（时域平移性质）

$$L[u_a(t)f(t-a)] = e^{-as}F(s) \quad (a \geq 0)$$

Proof: $\int_0^\infty e^{-st}[u_a(t)f(t-a)]\,dt = \int_a^\infty e^{-st}f(t-a)\,dt \xrightarrow{令\tau=t-a} \int_0^\infty e^{-s(a+\tau)}f(\tau)\,d\tau$

$= e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau$

$= e^{-as}\cdot L[f(\tau)]$

$= e^{-as}\cdot F(s)$

4. Differentiation Theorem for F(s)

频域微分性质

$$\frac{d^n}{ds^n}[F(s)] = L[(-1)^n t^n f(t)] \quad n=1,2,3\cdots$$

Proof: $\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}(e^{-st}f(t)\,dt)$

$= \int_0^\infty -te^{-st}f(t)\,dt = L[-tf(t)]$

eg. $L[\cos\omega t] = F(s) = \frac{s}{s^2+\omega^2}$

$L[-t\cos\omega t] = \frac{d}{ds}F(s) = \frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$L[\sin\omega t] = F(s) = \frac{\omega}{s^2+\omega^2}$

$L[-t\sin\omega t] = \frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2}$

5、Integration Theorem for F(s) 频域积分性质

$$\int_s^\infty F(s)\,ds = L\left[\frac{f(t)}{t}\right]$$

6、时域微分性质☆ 求解微分方程

"S"理解为求导

$L[f'(t)] = sF(s) - f(0^-)$

$L[f''(t)] = s^2F(s) - sf(0^-) - f'(0^-)$

$L[f^{(3)}(t)] = s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$

$L[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0^-) - \cdots - f^{n-1}(0^-)$

7、卷积性质 $L[f_1(t) * f_2(t)] = F_1(s)\cdot F_2(s)$

性质	$f(t)$	$F(S)$
线性性质	$a_1f_1(t)+a_2f_2(t)$	$a_1F_1(s)+a_2F_2(s)$
尺度变换性质	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
时域平移性质	$f(t-a)u(t-a)$	$e^{-as}F(s)$
频域平移性质	$e^{-at}f(t)$	$F(s+a)$
	$\frac{df}{dt}$	$sF(s)-f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s)-sf(0^-)-f'(0^-)$
时域微分性质	$\frac{d^3f}{dt^3}$	$s^3F(s)-s^2f(0^-)-sf'(0^-)-\cdots-f^n(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-)-sf'(0^-)-\cdots-f^{n-1}(0^-)$
时域积分性质	$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$
频域微分性质	$tf(t)$	$-\frac{d}{ds}F(s)$
频域积分性质	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
时域周期性质	$f(t)=f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
初值定理	$f(0)$	$\lim_{s\to\infty} sF(s)$
终值定理	$f(\infty)$	$\lim_{s\to 0} sF(s)$
卷积性质	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$

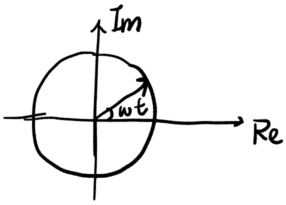
Laplace Transform 存在的条件：积分收敛（值不为∞）

ROC：Region of Convergence 收敛域

令 $s=\sigma+j\omega$，求 $L[e^{-at}]$ 存在的条件

$$F(s)=\int_0^{+\infty} e^{-at}e^{-st}dt=\int_0^{\infty} e^{-at}\cdot e^{-(\sigma+j\omega)t}dt=\int_0^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}dt$$

由于 $e^{-j\omega t}=\cos\omega t-i\sin\omega t$　　故不会对积分的收敛性产生影响。



$|e^{-j\omega t}|=1$

要使积分收敛，必须有 $-(a+\sigma)<0$

$$-\frac{1}{a+\sigma}\cdot e^{-(a+\sigma)t}\Big|_0^{\infty}=-\frac{1}{a+\sigma}\left(e^{-(a+\sigma)\cdot\infty}-1\right)$$

综上：$L[e^{-at}]$ 的收敛域为 $\sigma>-a$ 即 $Re(s)>-a$

Inverse Laplace Transform

Partial Fractions 因式分解

$$F(s)=\frac{P(s)}{Q(s)}$$ → s的多项式　（P阶数 ≤ Q阶数）

1、$\dfrac{A}{as+b}$

2、$\dfrac{A}{(as+b)^n}=\dfrac{A_1}{as+b}+\dfrac{A_2}{(as+b)^2}+\cdots+\dfrac{A_n}{(as+b)^n}$

3、$\dfrac{As+B}{as^2+bs+c}\rightarrow$ $\sin\omega t$　$\cos\omega t$　$\sinh\omega t$　$\cosh\omega t$

4、$\dfrac{As+B}{(as^2+bs+c)^n}=\dfrac{A_1s+B_1}{as^2+bs+c}+\dfrac{A_2s+B_2}{(\quad)^2}+\cdots+\dfrac{A_ns+B_n}{(\quad)^n}$

eg. $F(s)=\dfrac{s+1}{s^2(s-1)}=\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s-1}$

求解系数

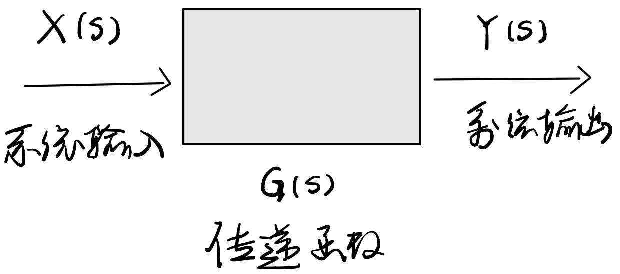
$C=(s-1)F(s)\Big|_{s=1}$　　$B=s^2F(s)\Big|_{s=0}$　　☆ $\boxed{A=\dfrac{d(s^2F(s))}{ds}\Big|_{s=0}}$

传递函数 Transfer Function 与 Laplace Transform

LT

(卷积 ⇒ 乘积)

$X(S)\cdot G(S)=Y(S)$



用 matlab simulink 观察 F(s) 在时域的图像

当 $\underbrace{X(s)=1}_{x(t)=\delta(t)}$ 时　$Y(s)=G(s)$，令 $G(s)=F(s)$ 即可