1. (20 points)

a) Derive the vibration equation for cantilever beam in Fig. 1 (Ignore gravity). Assuming that only the first mode is considered, that is

$$y(x,t) = \phi(x)q(t) = \frac{3x^2 l - x^3}{2l^3} q(t)$$

the parameters of the material are density: ρ, cross-section area A, Young's modulus: E, length: l, area moment: I_z and deflection of free-end: $q(t)$.

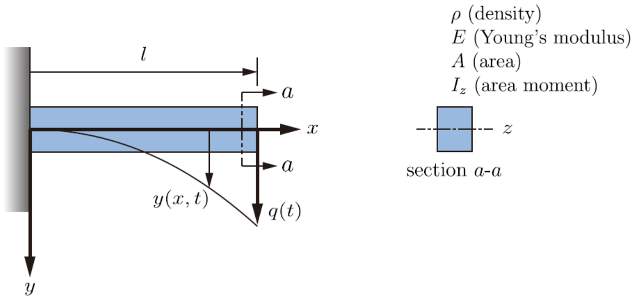

图 1: Figure for Exercise 1.

b) Release the cantilever beam with +10 mm static deflection at the end. Determine the end-point deflection after 1s. ($\rho = 2(10^3)$ kg/m^3, $E = 200(10^9)$ N/m^2, $A = 1.2(10^{-5})$ m^2, $I_z = 1.2(10^{-11})$ m^4 and $l = 2$ m)

a)

$$\phi(x):\ shape\ function \quad q(t):\ amplitude\ at\ free\ end$$

$$y(x,t) = \phi(x)q(t) = \frac{3x^2 l - x^3}{2l^3} q(t)$$

$$\dot{y} = \frac{dy}{dt} = \phi(x) * \frac{dq}{dt} = \phi(x) * \dot{q}$$

$$y'' = \frac{d^2 y}{dx^2} = \frac{d^2 \phi}{dx^2} * q(t) = \phi'' * q(t)$$

$$T = \frac{1}{2}\int_0^l \rho A\, \dot{y}^2 dx = \frac{1}{2}\int_0^l \rho A\,(\phi(x) * \dot{q})^2 dx = \frac{1}{2}\dot{q}^2 \int_0^l \rho A\, \phi^2(x) dx = \frac{1}{2} m_e \dot{q}^2$$

$$其中\ m_e = \int_0^l \rho A\, \phi^2(x) dx = \frac{33}{140}\rho A l$$

$$V = \frac{1}{2}\int_V \sigma_x \epsilon_x d\,V = \frac{1}{2}\int_0^l \int_A \frac{\sigma_x^2}{E} d\,A dx = \frac{1}{2}\int_0^l E \int_A y^2 d\,A (y'')^2 dx = \frac{1}{2}\int_0^l E\, I_z (y'')^2 dx$$

$$= \frac{1}{2} q^2(t) \int_0^l E\, I_z \phi''^2 dx = \frac{1}{2} k_e q^2(t)$$

$$k_e = \int_0^l E\, I_z (\phi'')^2 dx = \frac{3EI_z}{l^3}$$

$$equation\ of\ motion:\ \dot{E} = \dot{T} + \dot{V} = 0$$

$$m_e \dot{q}\ddot{q} + k_e q \dot{q} = 0$$

$$m_e \ddot{q} + k_e q = 0$$

$$let\ \omega_n = \sqrt{\frac{k_e}{m_e}} \qquad \ddot{q} + \omega_n^2 q = 0$$

$$Laplace\ Transform:\ s^2 Q(s) - sq(0) - \dot{q}(0) + \omega_n^2 Q(s) = 0$$

$$(s^2 + \omega_n^2)Q(s) = sq(0) + \dot{q}(0)$$

$$Q(s) = \frac{sq(0) + \dot{q}(0)}{s^2 + \omega_n^2} = q(0)\frac{s}{s^2 + \omega_n^2} + \frac{\dot{q}(0)}{\omega_n}\frac{\omega_n}{s^2 + \omega_n^2}$$

$$q(t) = q(0)\cos(\omega_n t) + \frac{\dot{q}(0)}{\omega_n}\sin(\omega_n t)$$

$$\omega_n = \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{\frac{3EI_z}{l^3}}{\frac{33}{140}\rho A l}} = \frac{\sqrt{\frac{3*140}{33}}}{l^2}\sqrt{\frac{EI_z}{\rho A}} = \frac{3.5675}{l^2}\sqrt{\frac{EI_z}{\rho A}}$$

$$y(x,t) = \phi(x)q(t) = \frac{3x^2 l - x^3}{2l^3}\left(q(0)\cos(\omega_n t) + \frac{\dot{q}(0)}{\omega_n}\sin(\omega_n t)\right) \quad \omega_n = \frac{3.5675}{l^2}\sqrt{\frac{EI_z}{\rho A}}$$

b)

$$q(0) = 10^{-2}m \quad \dot{q}(0) = 0\ m/s$$

$$\omega_n = \frac{3.5675}{l^2}\sqrt{\frac{EI_z}{\rho A}} = 8.92$$

$$y(l,1) = q(0)\cos(\omega_n) = -0.008752818770834643\ m$$

2. (20 points)

a) Derive the equation of motion for free damped vibration in Fig. 2.

b) Let the object fall freely at the original length of the spring, determine the position at $t = 2s$. (spring stiffness $k = 1$ N/m, mass $m = 1$ kg,damping $c = 1$ N/ms^{-1} and acceleration of gravity $g = 10$ m/s^{-2})

□

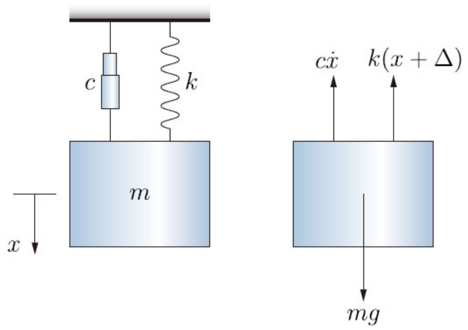

图 2: Figure for Exercise 2.

a)

$$mg - c\dot{x} - k(x+\Delta) = m\ddot{x}$$

$$Initial\ condition:\ x = 0\ \ \dot{x} = 0\ \ k\Delta = mg$$

$$Equation\ of\ motion:\ \ m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \dot{x} + x = 0$$

$$Laplace\ transform:\ \ s^2L(x) - sx(0) - \dot{x}(0) + sL(x) - x(0) + L(x) = 0$$

$$(s^2 + s + 1)L(x) - (s+1)x(0) - \dot{x}(0) = 0$$

$$L(x) = \frac{(s+1)x(0) + \dot{x}(0)}{s^2 + s + 1} = \frac{\left(s + \frac{1}{2}\right)x(0) + \frac{1}{2}x(0) + \dot{x}(0)}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= x(0)\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{\frac{1}{2}x(0) + \dot{x}(0)}{\frac{\sqrt{3}}{2}}\frac{\frac{\sqrt{3}}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$x(t) = x(0)e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{x(0) + 2\dot{x}(0)}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

b) x=0 时 弹簧拉力与重力恰好平衡，是物块的平衡点。

以弹簧原长状态作为初始条件，物块由静止释放，则:

$$x(0) = -\Delta = -\frac{mg}{k} = -10$$

$$\dot{x}(0) = 0$$

则运动方程变为

$$x(t) = -10e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{-10}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Position at t=2s:

$$x(2) = -10e^{-1}\cos(\sqrt{3}) + \frac{-10}{\sqrt{3}}e^{-1}\sin(\sqrt{3}) = -1.5057436514588765$$

3. (20 points)

a) Derive the equation of motion for forced damped vibration in Fig. 3. The force has the form of $F(t) = F_0 \cos \omega t$.

b) Let the object start to move from the equilibrium point under force F, determine the position at $t = 2s$. (spring stiffness $k = 400(10^3)$ N/m, mass $m = 1200$ kg,damping

ratio $\zeta = 0.5$, maximum amplitude of the force $F_0 = 6(10^3)$ N, acceleration of gravity $g = 10$ m/s^{-2} and frequency of the force $\omega = 5.8$ rad/s)

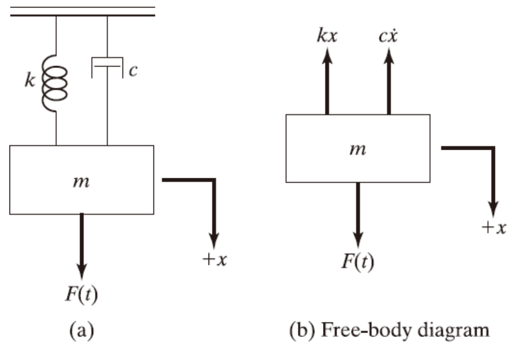

图 3: Figure for Exercise 3.

a)

$$F(t) - kx - c\dot{x} = m\ddot{x}$$
$$F_0 cos\omega t = kx + c\dot{x} + m\ddot{x}$$

b)

initial condition: $x(0) = 0 \quad \dot{x}(0) = 0$

$$laplace\ transform:$$

$$F_0 \frac{s}{s^2 + \omega^2} = m\big(s^2 L(x) - sx(0) - \dot{x}(0)\big) + c\big(sL(x) - x(0)\big) + kL(x)$$

$$(ms^2 + cs + k)L(x) = F_0 \frac{s}{s^2 + \omega^2} + (ms + c)x(0) + m\dot{x}(0)$$

$$L(x) = \frac{F_0 \frac{s}{s^2 + \omega^2} + (ms + c)x(0) + m\dot{x}(0)}{ms^2 + cs + k} = \frac{\frac{F_0}{m} \frac{s}{s^2 + \omega^2} + \left(s + \frac{c}{m}\right)x(0) + \dot{x}(0)}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

$$= \frac{\frac{F_0}{m} \frac{s}{s^2 + \omega^2} + \left(s + \frac{c}{m}\right)x(0) + \dot{x}(0)}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}}$$

$$= \frac{\frac{F_0}{m} \frac{s}{s^2 + \omega^2}}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}} + \frac{\left(s + \frac{c}{2m}\right)x(0)}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}} + \frac{\dot{x}(0) + \frac{c}{2m}x(0)}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}}$$

$$= \frac{\frac{F_0}{m} \frac{s}{s^2 + \omega^2}}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}} + x(0)\frac{\left(s + \frac{c}{2m}\right)}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}}$$

$$+ \frac{\dot{x}(0) + \frac{c}{2m}x(0)}{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}} \frac{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}}{\left(s + \frac{c}{2m}\right)^2 + \frac{k}{m} - \frac{c^2}{4m^2}}$$

$$\omega_n=\sqrt{\frac{k}{m}}=\frac{10\sqrt{30}}{3}\quad damping\ ratio\ \ \zeta=\frac{c}{2\sqrt{km}}\quad \zeta\omega_n=\frac{c}{2m}$$

代入初始条件:

$$L(x)=\frac{\frac{F_0}{m}\frac{s}{s^2+\omega^2}}{\left(s+\frac{c}{2m}\right)^2+\frac{k}{m}-\frac{c^2}{4m^2}}=$$

$$=\frac{F_0}{m}\frac{\frac{s}{s^2+\omega^2}}{(s+\zeta\omega_n)^2+(1-\zeta^2)\omega_n^2}=\frac{6*10^3}{1200}\frac{\frac{s}{s^2+\omega^2}}{\left(s+0.5*\sqrt{4*\frac{10^5}{1200}}\right)^2+0.75*4*\frac{10^5}{1200}}$$

$$=5\frac{s}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)}=5\frac{s}{(s^2+\omega^2)(s^2+\omega_n s+\omega_n^2)}$$

$$=5(\frac{As+B}{s^2+\omega^2}+\frac{Cs+D}{s^2+\omega_n s+\omega_n^2})$$

其中

$$A=\frac{\omega_n^2-\omega^2}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}\qquad B=\frac{2\zeta\omega_n\omega^2}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}$$

$$C=-\frac{\omega_n^2-\omega^2}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}\qquad D=-\frac{2\zeta\omega_n^3}{(2\zeta\omega_n)^2\omega^2+(\omega_n^2-\omega^2)^2}$$

$$A=\frac{\frac{10\sqrt{30}}{3}^2-5.8^2}{\frac{10\sqrt{30}}{3}^2*5.8^2+\left(\frac{10\sqrt{30}}{3}^2-5.8^2\right)^2}=\frac{\frac{22477}{75}}{\frac{33640}{3}+\frac{22477}{75}^2}=3*10^{-3}$$

$$B=\frac{\frac{10\sqrt{30}}{3}*5.8^2}{\frac{33640}{3}+\frac{22477}{75}^2}=6.08*10^{-3}$$

$$C=-3*10^{-3}$$

$$D=\frac{\frac{10\sqrt{30}}{3}^3}{\frac{33640}{3}+\frac{22477}{75}^2}=9.02*10^{-4}$$

$$L(x)=5(\frac{As+B}{s^2+\omega^2}+\frac{Cs+D}{s^2+\omega_n s+\omega_n^2})=5(A\frac{s}{s^2+\omega^2}+\frac{B}{\omega}\frac{\omega}{s^2+\omega^2}+\frac{Cs+D}{\left(s+\frac{1}{2}\omega_n\right)^2+\frac{3}{4}\omega_n^2})$$

$$=5(A\frac{s}{s^2+\omega^2}+\frac{B}{\omega}\frac{\omega}{s^2+\omega^2}+C\frac{\left(s+\frac{1}{2}\omega_n\right)}{\left(s+\frac{1}{2}\omega_n\right)^2+\frac{3}{4}\omega_n^2}$$

$$-\frac{D+\frac{1}{2}C\omega_n}{\frac{\sqrt{3}}{2}\omega_n}\frac{\frac{\sqrt{3}}{2}\omega_n}{\left(s+\frac{1}{2}\omega_n\right)^2+\frac{3}{4}\omega_n^2})$$

$$x(t) = 5\left(Acos(\omega t) + \frac{B}{\omega}\sin(\omega t) + Ce^{-\frac{1}{2}\omega_n t}cos\left(\frac{\sqrt{3}}{2}\omega_n t\right)\right.$$

$$\left. - \frac{D + \frac{1}{2}C\omega_n}{\frac{\sqrt{3}}{2}\omega_n} e^{-\frac{1}{2}\omega_n t} sin\left(\frac{\sqrt{3}}{2}\omega_n t\right)\right)$$

$$x(2) = 0.004116661958302297$$

4. (20 points)

a) Derive the equation of motion for forced damped vibration in Fig. 4. The force is expressed as $F(t)$ in time domain. Set the position $x(t)$ as output and the force $F(t)$ as input, determine $X(s)/F(s)$ in s-domain.

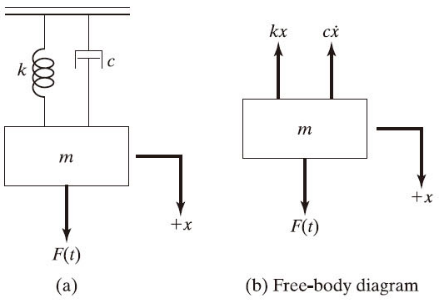

图 4: Figure for Exercise 4.

b) In fact we call $X(s)/F(s)$ the **Transfer Function** when all initial conditions are zero ($x(0) = 0$, $\dot{x}(0) = 0$, $\ddot{x}(0) = 0$, ...). What kind of input can make the output has the same expression as Transfer Function?

a)

$$F(t) - kx - c\dot{x} = m\ddot{x}$$

$$F(t) = kx + c\dot{x} + m\ddot{x}$$

$$laplace\ transform:$$

$$F(s) = m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + c\left(sX(s) - x(0)\right) + kX(s)$$

$$F(s) = (ms^2 + cs + k)X(s) - (ms + 1)x(0) - m\dot{x}(0)$$

$$1 = (ms^2 + cs + k)\frac{X(s)}{F(s)} - \frac{(ms + 1)x(0)}{F(s)} - \frac{m\dot{x}(0)}{F(s)}$$

$$\frac{X(s)}{F(s)} = \frac{1}{(ms^2 + cs + k)} + \frac{(ms + 1)x(0)}{(ms^2 + cs + k)F(s)} + \frac{m\dot{x}(0)}{(ms^2 + cs + k)F(s)}$$

b)

transfer function: initial condition all equal to 0

$$TransFunc = \frac{X(s)}{F(s)} = \frac{1}{(ms^2 + cs + k)}$$

When input $F(s) = 1$, output $X(s) = TransFunc = \frac{1}{(ms^2+cs+k)}$

5. (20 points) Fig. 5 shows a simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road. The vehicle has a mass of 1200 kg. The suspension system has a spring constant of 400 kN/m and a damping ratio of $\zeta = 0.5$. If the vehicle speed is 20 km/h, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of Y = 0.05 m and a wavelength of 6 m.

Hint: the equation of motion is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

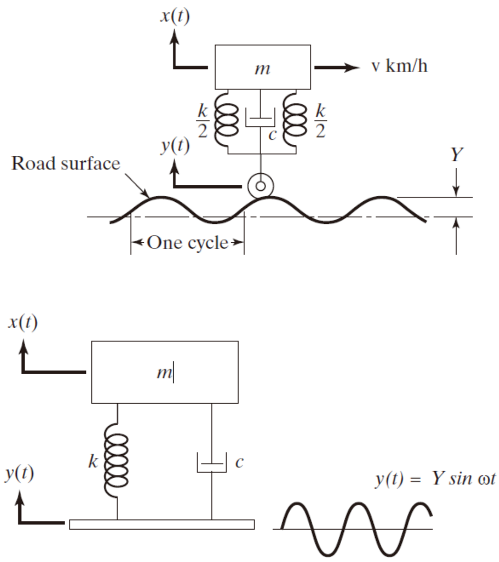

图 5: Figure for Exercise 5.

$$\frac{20km}{h} = \frac{20000}{3600} = \frac{50}{9}m/s$$

$$T = \frac{\lambda}{v} = \frac{6}{\frac{50}{9}} = 1.08s$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.08}$$

$$y(t) = 0.05\sin\left(\frac{2\pi}{1.08}t\right)$$

$$Y(s) = Y\frac{s}{s^2 + \omega^2}$$

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

$$m(s^2X(s) - sx_0 - \dot{x}_0) + c\big(sX(s) - x_0 - sY(s) + y(0)\big) + k\big(X(s) - Y(s)\big) = 0$$

$$(ms^2 + cs + k)X(s) - (ms + c)x_0 - m\dot{x}_0 - (cs + k)Y(s) = 0$$

$$X(s)=\frac{\left(s+\frac{c}{m}\right)x_0+\dot{x_0}+\left(\frac{c}{m}s+\frac{k}{m}\right)Y(s)}{\left(s^2+\frac{c}{m}s+\frac{k}{m}\right)}=\frac{(s+2\zeta\omega_n)x_0+\dot{x_0}+(2\zeta\omega_n s+\omega_n^2)Y(s)}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$=\frac{(s+2\zeta\omega_n)x_0+\dot{x_0}}{s^2+2\zeta\omega_n s+\omega_n^2}+\frac{Y(2\zeta\omega_n s+\omega_n^2)}{s^2+2\zeta\omega_n s+\omega_n^2}*\frac{s}{s^2+\omega^2}$$

$$=\frac{(s+2\zeta\omega_n)x_0+\dot{x_0}}{s^2+2\zeta\omega_n s+\omega_n^2}+(\frac{as+b}{s^2+2\zeta\omega_n s+\omega_n^2}+\frac{cs+d}{s^2+\omega^2})$$

$$(\omega_n=\sqrt{\frac{k}{m}}\quad \zeta=\frac{c}{2\sqrt{mk}})$$

$$(as+b)(s^2+\omega^2)+(cs+d)(s^2+2\zeta\omega_n s+\omega_n^2)=\ Y(2\zeta\omega_n s^2+\omega_n^2 s)$$

$$[a+c+2c\zeta\omega_n\quad b+d+2d\zeta\omega_n\quad a\omega^2+c\omega_n^2\quad b\omega^2+d\omega_n^2]\begin{bmatrix}s^3\\s^2\\s\\1\end{bmatrix}$$

$$=[0\quad 2Y\zeta\omega_n\quad Y\omega_n^2\quad 0]\begin{bmatrix}s^3\\s^2\\s\\1\end{bmatrix}$$

$$[a+c+2c\zeta\omega_n\quad b+d+2d\zeta\omega_n\quad a\omega^2+c\omega_n^2\quad b\omega^2+d\omega_n^2]=[0\quad 2Y\zeta\omega_n\quad Y\omega_n^2\quad 0]$$

$$\begin{bmatrix}1&0&1+2\zeta\omega_n&0\\0&1&0&1+2\zeta\omega_n\\\omega^2&0&\omega_n^2&0\\0&\omega^2&0&\omega_n^2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}0\\2Y\zeta\omega_n\\Y\omega_n^2\\0\end{bmatrix}$$

$$solution{:}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}\frac{Y\mathrm{\omega n}^2(1+2\zeta\mathrm{\omega n})}{-\mathrm{\omega n}^2+\omega^2(1+2\zeta\mathrm{\omega n})}\\-\frac{2Y\zeta\mathrm{\omega n}^3}{-\mathrm{\omega n}^2+\omega^2(1+2\zeta\mathrm{\omega n})}\\\frac{Y\mathrm{\omega n}^2}{\mathrm{\omega n}^2-\omega^2(1+2\zeta\mathrm{\omega n})}\\\frac{2Y\zeta\omega^2\mathrm{\omega n}}{-\mathrm{\omega n}^2+\omega^2(1+2\zeta\mathrm{\omega n})}\end{bmatrix}$$

$$steady-state\ solution$$

$$X_s(s)=\frac{cs+d}{s^2+\omega^2}=c\frac{s}{s^2+\omega^2}+\frac{d}{\omega}\frac{\omega}{s^2+\omega^2}$$

$$x_s(t)=c*\cos(\omega t)+\frac{d}{\omega}\sin(\omega t)$$

$$Amplitude=\sqrt{c^2+\left(\frac{d}{\omega}\right)^2}=\sqrt{\frac{Y^2\mathrm{\omega n}^2(4\zeta^2\omega^2+\mathrm{\omega n}^2)}{(\mathrm{\omega n}^2-\omega^2(1+2\zeta\mathrm{\omega n}))^2}}$$

$$Y=0.05\quad \omega=\frac{2\pi}{1.08}\quad \omega_n=\sqrt{\frac{400000}{1200}}=\frac{10\sqrt{30}}{3}\quad \zeta=0.5$$

$$Amplitude=0.05492790607337223$$