離散數學期末考

2022/1/12

注意事項:

1. 禁止使用計算機、翻譯機，手機請關機。禁止攜帶計算紙。
2. 當然不可以作弊。
3. 計算與證明題需有計算過程方予計分。
4. 請於答案卷左上角填上題號以方便閱卷。
5. 使用兩張答案卷的同學記得兩張都要寫名字，並將之合併在一起交回。
6. 滿分 110 分，請努力作答。

一 簡答題 (40%)

1. (5-2 no.3, 11, 5-3 no.7, 5-4 no.5, 6, Ch5 no.24, 29) If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 5\}$, and $C = \{v, w, x, y, z\}$, determine the following: (2% × 13)
   (a) How many functions $f: B \to C$ are there?
   (b) How many functions $f: B \to C$ are one-to-one?
   (c) How many functions $f: B \to C$ satisfy $f(3) = x$?
   (d) Let $f: B \to C$. In how many way can f be extended to a function $g: A \to C$?
   (e) Determine the number of functions $f: A \to C$ where $f(A) = \{w, x, y\}$.
   (f) Determine the number of functions $f: A \to C$ where $|f(A)| = 3$.
   (g) How many functions $f: A \times A \to A$ are there?
   (h) How many closed binary operation on A are commutative?
   (i) How many closed binary operations f on C satisfy $f(x, y) = f(w, v) = z$?
   (j) How many of the functions f in part (i) have an identity?
   (k) How many of the functions f in part (j) are commutative?
   (l) A closed ternary (3-ary) operation on A is a function $f: A \times A \times A \to A$. How many closed ternary operations are there on A?
   (m) How many functions $f: A \to A$ (simultaneously) satisfy $f^{-1}(\{1, 2\}) = \phi$, $f^{-1}(\{3, 4\}) = \{1, 3\}$, and $f^{-1}(6) = \{6\}$?

2. (5-5 no.20) How many times must we roll a single die in order to get the same score:
   (a) At least twice? (2%)
   (b) At least three times? (2%)

3. (5-6 no.17) Let $f, g: \mathbf{Z}^+ \to \mathbf{Z}^+$ where for all $x \in \mathbf{Z}^+$, $f(x) = x + 2$ and $g(x) = \max\{2x, x + 3\}$, the maximum of $2x$ and $x + 3$. (10%)
   (a) (1%) What is the range of f?
   (b) (1%) Is f an onto function?
   (c) (1%) Is the function f one-to-one?
   (d) (1%) What is the range of g?
   (e) (1%) Is g an onto function?
   (d) (1%) Is the function g one-to-one?

(e) (2%) Determine $(f \circ g)(x)$ for $x = 1$, and 7.

(f) (2%) Determine $(g \circ f)(x)$ for $x = 1$, and 7.

二 計算與證明(須有計算過程或詳細證明，否則不予計分) (70%)

4. (5-1 no.11) For $A, B, C \subseteq \mathcal{U}$, prove that
$$(B - C) \times A = (B \times A) - (C \times A).$$ (10%)

5. (5-2 no.11)
(a) Find all real numbers x where $\lfloor 5x \rfloor = 5\lfloor x \rfloor$. (5%)
(b) Let $n \in \mathbf{Z}^+$ where $n > 1$. Determine all $x \in \mathbf{R}$ such that $\lfloor nx \rfloor = n\lfloor x \rfloor$. (5%)

6. (5-3 no.12)
(a) In how many ways can 31,100,905 be factored into three factors, each greater than 1, if the order of the factors is not relevant? (5%)
(b) In how many ways can one factor 31,100,905 into two or more factors where each factor is greater than 1 and no regard is paid to the order of the factors? (5%)

7. (5-5 no.12) Let $A \subset \{1, 2, 3, \ldots, 50\}$ with $|A| = 10$. For any subset B of A let s_B denote the sum of the elements in B. Prove that there are distinct subsets C, D of A such that $|C| = |D| = 4$ and $s_C = s_D$. (10%)

8. (5-5 Ex.5.44) Let $S = \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$.
(a) If six numbers are selected from S, prove that two of them must have the product 1024. (10%)
(b) If seven numbers are selected from S, prove that three of them must have the product 2^{15}. (8%)
(c) In (b), can seven be replace by six? (2%)

9. (Chap 5 no.23) Given a nonempty set A, let $f: A \to A$ and $g: A \to A$ where
$$f(a) = g(f(f(a))) \text{ and } g(a) = f(g(f(a)))$$
for all a in A. Prove that $f = g$. (10%)