CSIE Probability Exam I Solution

1. Two archers X and Y are shooting the target. Archer X shoots the bull's eye with 20% accuracy, and Archer Y shoots with 80% accuracy. Pick one of the two archers randomly and let the archer shoot the target twice. B_1 is the event that the first shot hits the bull's eye. B_2 is the event that the second shot hits the bull's eye. Verify if B_1 and B_2 are independent. (15pts)

   $\mathrm{P}[B_1] = \mathrm{P}[X]\mathrm{P}[B_1|X] + \mathrm{P}[Y]\mathrm{P}[B_1|Y] = \frac{1}{2}(0.2) + \frac{1}{2}(0.8) = 0.5.$

   $\mathrm{P}[B_2] = \mathrm{P}[X]\mathrm{P}[B_2|X] + \mathrm{P}[Y]\mathrm{P}[B_2|Y] = \frac{1}{2}(0.2) + \frac{1}{2}(0.8) = 0.5.$

   $\mathrm{P}[B_1B_2] = \mathrm{P}[X]\mathrm{P}[B_1B_2|X] + \mathrm{P}[Y]\mathrm{P}[B_1B_2|Y] = \frac{1}{2}(0.2)(0.2) + \frac{1}{2}(0.8)(0.8) = 0.34 \neq \mathrm{P}[B_1]\ \mathrm{P}[B_2]$. B_1 and B_2 are NOT independent.

2. At NCNU, 60% of the students are undergrads(U), and among the graduate students, the master students(M) is seven times the doctoral students(D), $\mathrm{P}[M] = 7\mathrm{P}[D]$. 60% of the students at NCNU have no computer(C_0). Two-thirds of the undergrads have no computer, $\mathrm{P}[C_0|U] = \frac{2}{3}$. Among the students with one or more computers(C_1), Three-eighths of them are master students, $\mathrm{P}[M|C_1] = \frac{3}{8}$. Every doctoral student possesses one or more computers, $\mathrm{P}[C_1|D] = 1$. Finish the following table, and find the probability when you meet a graduate student, she or he has no computer $\mathrm{P}[C_0|M \cup P]$. (20pts)

	U	M	D
C_0	0.4	0.2	0
C_1	0.2	0.15	0.05

$$P[C_0|M \cup P] = \frac{P[MC_0] + P[PC_0]}{P[M] + P[P]} = \frac{0.2 + 0}{0.35 + 0.05} = 0.5.$$

3. Roll a fair die twice. The outcome is independent from roll to roll. Let X_1 and X_2 be the first and the second outcomes. I is an indicator random variable such that $I = 1$ when $\max\{X_1, X_2\} = 6$; otherwise $I = 0$. Find $\mathrm{P}[I = 0]$, $\mathrm{P}[I = 1]$, and $\mathrm{E}[I]$. (15pts)

   $\mathrm{P}[I = 1] = \mathrm{P}[\max\{X_1, X_2\} = 6] = \mathrm{P}[\max\{X_1, X_2\} \leq 6]$ - $\mathrm{P}[\max\{X_1, X_2\} \leq 5] = \mathrm{P}[X_1 \leq 6, X_2 \leq 6]$ - $\mathrm{P}[X_1 \leq 5, X_2 \leq 5] = 1 - (\frac{5}{6})^2 = \frac{11}{36}.$

   $\mathrm{P}[I = 0] = 1$ - $\mathrm{P}[I = 1] = \frac{25}{36}.$

   $\mathrm{E}[I] = 1(\mathrm{P}[I = 1]) + 0(\mathrm{P}[I = 0]) = \frac{11}{36}.$

4. You roll two fair dice until you get doubles. (a) X is the number of the rolls. Find P$[X > 3]$. (15pts)

   X is geometric with $p = 1/6$. P$[X > 3] = (1 - p)^3 = (5/6)^3 = \frac{125}{216} \simeq$ 0.5787037..

   (b) You continue rolling the dice until you get doubles for the fifth time. Y is the number of the total rolls. Find E$[Y]$. (10pts)

   Y is Pascal with $p = 1/6, k = 5$. E$[Y] = k/p = 30$.

5. At a base station, the number X of the messages it receives during 6:00-6:20am is a Poisson random variable with E$[X] = 2$. Find the probability that one or more messages show up during 6:00-6:05am. (15pts)

   E$[X] = 2 = 20\lambda,\ \lambda = 0.1$. The number Y of the messages in the first 5 minutes is a Poisson random variable with parameter $\alpha = 5\lambda = 0.5$. Thus, P$[Y \geq 1] = 1$ - P$[Y = 0] = 1 - e^{-0.5} \simeq 0.3934693$.