Computergestuurde Regeltechniek
Solution exercise session 1

Analyzing a MIMO system

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In this document you can find the correct methods, matlab commands and some remarks for exercise 1.

1a. The poles of the system are the eigenvalues of the A-matrix (course p. 76). Therefore, type in matlab: **eig(A)**.

1b. The system is a linear system in discrete time. This system is thus stable when all poles are placed inside the unit circle. Therefore, check the cryptical definition:

$$\exists ? |eig(A)| > 1 \rightarrow onstabiel \tag{1}$$

1c. Calculate the transmission zeros using the command **tzero(A,B,C,D)**

1d. Choose a transmission zero, call it ζ and solve the corresponding homogeneous system (course p. 84):

$$\underbrace{\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}}_{z} = 0 \tag{2}$$

The theory (course p. 82) teaches us that if ζ is a transmission zero, then M has to be rank-deficient. Therefore, a non-trivial solution z ($z \neq 0$) can be found as solution of the aforementioned homogeneous system. To become this solution, look for the vectors in the right zero space of M which can be done with the following commands:

```
zz=tzero(A,B,C,D)
zeta=zz(1); % choose one of the transmission zeros
M=[zeta*eye(length(A))-A -B; C D];
z=null(M);
x0=z(1:length(A),1);
u0=z(length(A)+1:length(z),1);
```

When the zero space is higher dimensional, then z contains multiple columns. Each linear combination of the columns of z is a solution for the system.
The asked initial state and initial signal which make the output identical to zero are x_0 and $u[k] = u_0\zeta^k$. Then it seems that $x[k] = x_0\zeta^k$. You can check this easily by filling in $x[k] = x_0\zeta^k$ and $u[k] = u_0\zeta^k$ in the system equations and by taking into account the fact that $Mz = 0$.
When ζ is a complex number, then both the initial state x_0 and the input $u[k]$ are complex which is mathematically possible but physically impossible. You can verify that in this case also the complex conjugate ζ^* is a transmission zero. Therefore, combine both transmission zeros and take $\Re_e x_0$ as initial state and $u[k] = |u_0||\zeta|^k \times \cos(\angle_\zeta k + \angle_{u_0})$ as input. $\Re_e x_0$ is the real part of x_0, $u_0 = |u_0| \times e^{\angle_{u_0}}$, $\zeta = |\zeta|e^{\angle_\zeta}$, "$\times$" represents an element-wise multiplication and "cos()" and "$e^{()}$" element-wise operators.
Realize that the command **tzero** calculates the solution for the matrix-vector product at the bottom of page 82 in the course, which does not correspond completely with the more severe definition of transmission zero, namely the decrease in the rank of $G(\epsilon)$as shown at the top of the same page. **tzero** finds in this way a number of zeros which are not a real transmission zero according to the severe definition at the top of page 82. Notice that these ghost-solutions correspond to non-observable and non-controllable modes.

1e. The transfer function from input 1 to output 1 can be calculated as follows:

```
[num,den]=ss2tf(A,B,C,D,1);
```

$num(1,:)$ corresponds in this case to the numerator of the unknown transfer function and den with the denumerator. The command **roots** allows to calculate the zeros and poles of this SISO-subsystem. The poles should correspond to the poles of the global system as calculated in question 1a. The relation between the zeros of the SISO-subsystem and the transmission zeros of the MIMO-system is not that clear. Not all transmission zeros have to be zeros of $num(1,:)$ and also reverse, not all SISO-zeros have to be transmission zeros. The moral of the story: watch out for hasty conclusions w.r.t. transmission zeros!

2a. Check first if rank($\mathbf{C}$) > length(A) (rank-test, see course p. 65). You can calculate $\mathbf{C}$ in matlab with the command **ctrb(A,B)**. If the rank of the controllability matrix is smaller than the order of the system, the system is not controllable.

To do the PBH-test, you need – according to the course p. 68 – the left eigenvectors of A. Well, the left eigenvectors of A are the right eigenvectors of A^T and those you can calculate with the matlab command **eig**:

```
[V,d]=eig(A');
```

Then, check if the matrix $B' \times V$ contains columns equal to zero. If this is the case, the eigenvalues corresponding to these zero-columns (for them look to the corresponding elements in the diagonal matrix d) refer to the uncontrollable modes.

2b. For this, use analogue commands and see course p. 67 and 69:

```
OO=obsv(A,C);
rank(OO) % < length(A) ?
[V,d]=eig(A);
C*V
```

Also here, look for the columns that contain only zeros and look at the corresponding eigenvalues.

2c. The answer to this question is already given in items 2a and 2b. The stability of the uncontrollable/unobservable modes is determined by the eigenvalues that correspond to the zero columns in the PBH test. Alternatively, the Kalman decomposition also allows you to find the controllable and/or observable modes. You can split $[A_k, B_k, C_k, D_k]$ into 4 subsystems as shown on pages 72 to 74 in the course. Afterwards, calculate the eigenvalues of A_{co}, $A_{c\bar{o}}$, $A_{\bar{c}o}$ and $A_{\bar{c}\bar{o}}$ to know the controllable and/or observable modes.
If there exist unstable uncontrollable modes, the system is not stabilizable. If there are unstable unobservable modes, then the system is not detectable.

2d. Type:

```
sys=ss(A,B,C,D,-1);
sysm=minreal(sys);
[Am,Bm,Cm,Dm]=ssdata(sysm);
```

Now, only the controllable and observable subsystem remains. You can get the same solution up to a state transformation (see course p. 75) from the Kalman decomposition, namely $[A_{co}, B_{co}, C_{co}, D]$.

3a. The control canonical form is given by:

$$
\begin{aligned}
A_c &= \begin{pmatrix} -7 & -12 \\ 1 & 0 \end{pmatrix} \\
B_c &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
C_c &= \begin{pmatrix} 1 & a \end{pmatrix}.
\end{aligned}
$$

The observer canonical form is given by:

$$
\begin{aligned}
A_o &= \begin{pmatrix} -7 & 1 \\ -12 & 0 \end{pmatrix} \\
B_o &= \begin{pmatrix} 1 \\ a \end{pmatrix} \\
C_o &= \begin{pmatrix} 1 & 0 \end{pmatrix}.
\end{aligned}
$$

3b. The control canonical form is controllable for all values of a. However, it is unobservable for $a = 3$ and $a = 4$ (pole zero cancellation in the transfer function). The observer canonical form is observable for all values of a. However, it is uncontrollable for $a = 3$ and $a = 4$.

3c. From the previous observations we learn that it is in general not possible to decide observability/controllability from a transfer function, it is a function of the state of the system.

3d. The following statements are true:

- If a system is controllable, its state space representation can always be converted to control canonical form (via a nonsingular linear transformation).
- If a system is observable, its state space representation can always be converted to observer canonical form (via a nonsingular linear transformation).
- Controllability and observability are invariant under nonsingular linear transformations.

In the above example, it is not possible to transform the observer canonical form into the control canonical form for $a = 3$ or $a = 4$ (or vice versa) via a nonsingular transformation because the systems are not observable/controllable. Hence, these systems are not equivalent for those values of a. On the other hand, if you would investigate any other value of parameter a, you would easily find that there the linear nonsingular transformation between both realizations does exist.