Liability Matching with Defaultable Securities:

A Stochastic Dominance Approach

Ziming Huang, JC Borman, Nicholas Colona, Aisha Koyas

Advisor:

Prof. Darinka Dentcheva

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Abstract

We propose a new framework of the interest-rate-immunized liability matching problem in which defaultable securities and stochastic liability streams are present. Liability-matching refers to the problem when an investor aims to cover predictable future payments by trading in credit or fixed income securities. Interest-rate immunization aims at constructing a portfolio which is indifferent to small interest rate changes. Immunization is achieved by imposing two constraints. The first one imposes a stochastic order relation in which the random cash-flows are required to stochastically dominate the random stream of liability obligations. The second constraint requires equal interest-rate sensitivity (in expectation) of the cash-flow and liability streams. We subsequently develop and analyze a method for numerical optimization method based on the methods proposed in (Dentcheva and Ruszczyński 2010). As a result, we compare our stochastic dominance model with other models such as the standard "no-default" model, a risk-neutral model (expectation), and a risk-averse model (mean semi-deviation). Stress-testing shows that our stochastic dominance model outperforms all aforementioned models in different interest-rate and credit market regimes.

1 Introduction

Asset-Liability Matching with bond portfolios is a well-studied topic in finance, and many studies generalize the initial immunization framework by (Redington 1952). (FONG and VASICEK 1984) relaxes the assumption of a parallel interest rate change as in the duration-matching technique by proposing a new measure of interest risk–M^2–which considers the term structure of the investment portfolio. (Iyengar and Ma 2009) incorporate short-maturity bonds and interest-rate models into the cash-flow matching technique by using a conditional tail expectation constraint of the shortfalls at each time point. (Hiller and Eckstein 1993) uses a stochastic programming approach with risk-measures to account for randomness in both the liability and the investment cash-flows.

Our problem framework uses duration matching, meaning we match the net-present-value and the interest sensitivity of our portfolio to those of the liability stream. Our model extends the duration matching model by incorporating the probability of default and random liability streams. We formulate the problem with a second-order stochastic dominance relation, and numerical solution is implemented using the "quantile cutting plane method" as described in (Dentcheva and Ruszczyński 2010). Using methods based on the inverse formulation of stochastic dominance constraints using Lorenz functions, the paper provides numerical portfolio optimization experiments to confirm the efficiency of this method.

The paper is organized as follows:

1. Brief discussion of stochastic orders and relevant properties;

2. Mathematical formulation of the duration-matching problem;

3. Matrix formulation and numerical solution of the optimization problem;

4. Interpretation of model output;

5. Analysis of model performance

While our paper will discuss some aspects of simulation of liability streams and probability of bond defaults, the emphasis of this research primarily focuses on the formulation and numerical treatment of the portfolio optimization. In an industry production-level model system, the model user would typically develop a separate joint-probability-of-default model, a liability model, an interest rate model etc.. Our model would still fit into this pipeline, as the input to the model can be a simulated sample of default events and liability streams.

2 Stochastic orders

Consider a random variable $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and its distribution function

$$F(X;\eta) = P[X \le \eta] \quad \text{for } \eta \in \mathbb{R}.$$

Define the function $F^{(2)}(X;\cdot)$ as

$$F^{(2)}(X;\eta) = \int_{-\infty}^{\eta} F(X;\alpha)\ d\alpha \quad \text{for } \eta \in \mathbb{R}. \tag{1}$$

As an integral of a nondecreasing function, it is a convex function of η.

Furthermore, for $X \in \mathcal{L}_m(\Omega, \mathcal{F}, P)$ we can define recursively the functions

$$F^{(k)}(X;\eta) = \int_{-\infty}^{\eta} F^{(k-1)}(X;\alpha)\ d\alpha \quad \text{for } \eta \in \mathbb{R}, \quad k = 3, \dots, m+1. \tag{2}$$

They are also convex and nondecreasing functions of the second argument.

Definition 2.1. A random variable $X \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$ *dominates* in the k-th order another random variable $Y \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$ if

$$F^{(k)}(X;\eta) \le F^{(k)}(Y;\eta) \text{ for all } \eta \in \mathbb{R}. \tag{3}$$

We shall denote relation (3) as $X \succeq_{(k)} Y$. It introduces a partial order among random variables in $\mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$. The k-th order dominance relation implies the $(k+1)$-st order dominance by definition, if the random variables in question have finite k-th moments.

Changing the order of integration in (1), we get

$$\begin{aligned} F^{(2)}(X;\eta) &= \int_{-\infty}^{\eta} (\eta - \xi)\ P_X(d\xi) \\ &= \mathbb{E}\{\max(\eta - X, 0)\} = P\{X \le \eta\}\, \mathbb{E}\{\eta - X | X \le \eta\}. \end{aligned} \tag{4}$$

Therefore, an equivalent representation of the second order stochastic dominance relation is:

$$\mathbb{E}[(\eta - X)_+] \le \mathbb{E}[(\eta - Y)_+] \quad \text{for all } \eta \in \mathbb{R}. \tag{5}$$

2.1 Quantile representations

For a real random variable X, we define the left-continuous inverse of the cumulative distribution function $F^{(1)}(X;\cdot)$ as follows:

$$F^{(-1)}(X;p) = \inf\ \{\eta : F(X;\eta) \geq p\} \quad \text{for} \quad 0 < p < 1.$$

Given $p \in [0,1]$, the number $q = q(X;p)$ is called a p-quantile of the random variable X if

$$P(X < q) \leq p \leq P(X \leq q).$$

For $p \in (0,1)$ the set of p-quantiles is a closed interval and $F^{(-1)}(X;p)$ represents its left end.

Directly from the definition of the first order dominance we see that

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F^{(-1)}(X;p) \geq F^{(-1)}(Y;p) \quad \text{for all} \quad 0 < p < 1. \tag{6}$$

The first order dominance constraint can be interpreted as a continuum of probabilistic (chance) constraints, studied in stochastic optimization. We consider the *second quantile function* $F_X^{(-2)} : \mathbb{R} \to \bar{\mathbb{R}}$, defined as

$$F^{(-2)}(X;p) = \int_0^p F^{(-1)}(X;\alpha)d\alpha \quad \text{for} \quad 0 < p \leq 1, \tag{7}$$

$F^{(-2)}(X;0) = 0$. For completeness, we also set $F^{(-2)}(X:p) = +\infty$ for $p \notin [0,1]$.

Similarly to $F^{(2)}(X;\cdot)$, the function $F^{(-2)}(X;\cdot)$ is well defined for any random variable X satisfying the condition $\mathbb{E}\,|X| < \infty$. It is convex as an integral of a non-decreasing function. The function $F^{(-2)}(X;\cdot)$ is called the *absolute Lorenz function* and its graph is called the *absolute Lorenz curve.*

The Lorenz curves are widely used in economics for inequality ordering of positive random variables, relative to their expectations: $p \mapsto \frac{1}{\mathbb{E}(X)}F^{(-2)}(X;p)$. The second-order stochastic-dominance relation is equivalent to the following relation between the absolute Lorenz functions:

Theorem 2.1.

$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F^{(-2)}(X;p) \geq F^{(-2)}(Y;p) \quad \forall p \in [0,1]$$

Furthermore, if η *is such that* $P\{X \leq \eta\} = p$, *then* $F^{(-2)}(X;p) = p\,\mathbb{E}\,\{X|X \leq \eta\}$

Therefore, the properties of $F^{(-2)}$ are of profound importance for stochastic dominance relations. We also infer that the relation between two Lorenz curse is equivalent to a continuum of conditional value at risk inequalities (also called average value-at-risk).

2.2 Expected Utility functions

Univariate stochastic orders can be characterized by families of utility functions. Let us consider the set $\mathcal{U}$ of concave non-decreasing functions $u : \mathbb{R} \to \mathbb{R}$ satisfying the following linear growth condition:

$$\lim_{t \to -\infty} u(t)/t < \infty. \tag{8}$$

For every random variable $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and for every $u \in \mathcal{U}$ the expectation $\mathbb{E}[u(X)]$ is well-defined and finite.

Proposition 1. *For each $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ the relation $X \succeq_{(2)} Y$ is equivalent to*

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)] \quad \textit{for all } u \in \mathcal{U}_2. \tag{9}$$

This shows that second-order stochastic dominance is consistent with the preferences of every risk-averse investor.

3 Model Representation

An investor initially chooses to start with some capital to invest in corporate bonds for companies denoted $i = 1 \dots N$. We choose the time unit and the time horizon T for the decision problem; $t = 1 \dots T$ signifies discrete points in time until the ultimate time horizon T. Once a bond has defaulted it will pay a predetermined recovery rate times face value, and remain in default until time T. The investor chooses the allocation with the objective of matching future liability streams, while being indifferent to small interest changes.

The point in time liabilities to be paid are represented by L_t. Conversely, since the investor allocates capital into bonds, she will also be receiving cash flows at future points in time. We denote the cumulative cash flows plus excess cash received up until time t by the variable X_t.

The cash-flow matching problem finds the minimal initial capital to invest in each bond:

$$\min_{u \in \mathbb{R}} \sum_{i=1}^{N} a_i u_i$$

Our decision variables are u_i, the nominal allocation of capital into corporate bond i, $i = 1 \dots N$. The discount/premium to purchase the bond is a_i. In addition, the cash flow received from bond i at time t, normalized by face value, is $C_{i,t}$ which takes either 0 (after default), some positive recovery value, or the coupon payment per dollar face value. If the bond does not default, at time T, the face value is repaid, and hence $C_{i,T}$ increases by 1. Essentially, $C_{i,t}$ is the "source" of the randomness in the portfolio cash-flows.

The following constraints apply:

1. Incoming Cash Flows – The cash received at time t is:

$$X_t = \sum_{i=1}^{N} C_{i,t} u_i; \quad X_0 = 0$$

2. Stochastic Dominance – Each cash flows received X_t must stochastically dominate the liabilities to be paid L_t. Here we have preference to large outcomes and the proper stochastic comparison would be the second-order stochastic dominance. It has two advantages as already mentioned: it is consistent with risk-averse preferences and it allows an effective numerical solution algorithm. One possibility to impose a relation between the available cash at any period and the liability obligations in that period, that is:

$$X_t \succeq_{(2)} L_t, \quad t = 1, \dots T. \tag{10}$$

Formulation (10) is too restrictive and would lead to very expensive portfolio. Additionally, it does not facilitate the formulation of the requirement for equal sensitivity to interest rate changes. To simplify this model, we assume further that there is some unknown rate $\tilde{\gamma} > 0$ that the investor can borrow and reinvest at any time t. Then, as long as the investors can match the liability in terms of net present value, they can match the entire stream. Whenever there is a deficit/surplus, the investor automatically borrow/reinvest at a discount rate, therefore the cash-flows can be compared in terms of NPV.

An additional requirement comes from the fact that the discount rate can change throughout the time horizon. The portfolio cash-flow net present value must, therefore, match the liability NPV for all rate $\tilde{\gamma}$ in a neighborhood of some initial estimate γ.

We formulate the optimization problem:

$$\begin{aligned} &\min_u \sum_{i=1}^{N} a_i u_i \\ \text{s.t.} \quad & X_t = \sum_{i=1}^{N} C_{i,t} u_i \\ & \sum_{t=1}^{T} \frac{1}{\gamma^t} X_t \succeq_{(2)} \sum_{t=1}^{T} \frac{1}{\gamma^t} L_t \end{aligned} \tag{11}$$

Problem (11) still does not account for changes in interest rates. We would like to consider

$$E[\sum_{t=1}^{T} \frac{X_t}{\tilde{\gamma}^t}] \geq E[\sum_{t=1}^{T} \frac{l_t}{\tilde{\gamma}^t}], \quad \forall \tilde{\gamma} \in (\gamma - \delta, \gamma + \delta) \tag{12}$$

The following lemma allows further simplification.

Theorem 3.1. *Let $X_r(\omega), Y_r(\omega)$ be random variables defined on a common probability space (Ω, F, P) the mappings $r \mapsto X_r(\omega)$, $r \mapsto Y_r(\omega)$ are continuously differentiable for P-almost all ω. Furthermore assume that their derivatives are Lipschitz-continuous on I for P-almost all ω with an integrable Lipschitz constant. If $E[\frac{dX_r}{dr}] = E[\frac{dX_r}{dr}]$ and $X_r \succeq_{(2)} Y_r$ for $r \in \mathrm{I}$, then a constant C exists, such that*

$$E[X_{\tilde{r}}] + C(\tilde{r} - r)^2 \geq E[Y_{\tilde{r}}], \quad \forall \tilde{r} \in I.$$

Proof. Using mean-value theorem, we obtain:

$$\begin{aligned} X_{\tilde{r}}(\omega) &= X_r(\omega) + (\tilde{r} - r)\frac{dX_r}{dr}(\bar{r}_X, \omega) \\ Y_{\tilde{r}}(\omega) &= Y_r(\omega) + (\tilde{r} - r)\frac{dX_r}{dr}(\bar{r}_Y, \omega) \end{aligned}$$

Subtracting both equations, we have:

$$X_{\tilde{r}}(\omega) - Y_{\tilde{r}}(\omega) = X_r(\omega) - Y_r(\omega) + (\tilde{r} - r)\Big[\frac{dX_r}{dr}(\bar{r}_X, \omega) - \frac{dX_r}{dr}(\bar{r}_Y, \omega)\Big]$$

We evaluate the right hand site of the last displayed equation using the Lipschitz continuity of the derivatives.

$$\begin{aligned}(\tilde r - r)\Big(\frac{dX_r}{dr}(\bar r_X,\omega) - \frac{dX_r}{dr}(\bar r_Y,\omega)\Big) &= (\tilde r - r)\Big(\frac{dX_r}{dr}(r,\omega) - \frac{dY_r}{dr}(r,\omega)\Big)\\ &+ (\tilde r - r)\Big(\frac{dX_r}{dr}(\bar r_X,\omega)\Big) - \frac{dX_r}{dr}(r,\omega)\Big) + (\tilde r - r)\Big(\frac{dY_r}{dr}(r,\omega) - \frac{dY_r}{dr}(\bar r_Y,\omega)\Big)\\ &\geq (\tilde r - r)\Big(\frac{dX_r}{dr}(r,\omega) - \frac{dY_r}{dr}(r,\omega)\Big) - c_X(\omega)(\tilde r - r)^2 - c_Y(\omega)(\tilde r - r)^2\end{aligned}$$

At the right-hand side, $c_X(\omega)$ and $c_Y(\omega)$ stand for the Lipschitz-constants of the mappings $r \mapsto \frac{dX_r}{dr}(r,\omega)$ and $r \mapsto \frac{dY_r}{dr}(r,\omega)$, respectively. Taking expectations, we obtain

$$E\big[X_{\tilde r} - Y_{\tilde r}\big] \geq E[X_r - Y_r] + (\tilde r - r)E\Big[\frac{dX_r}{dr} - \frac{dY_r}{dr}\Big] - E\big[c_X + c_Y\big](\tilde r - r)^2. \tag{13}$$

Recall that $X_r \succeq_{(2)} Y_r$ implies $E[X_{\tilde r} - Y_{\tilde r}] \geq 0$. Using the assumption $E[\frac{dX_r}{dr} - \frac{dY_r}{dr}] = 0$, we obtain

$$E[X_{\tilde r} - Y_{\tilde r}] \geq E\big[c_X + c_Y\big](\tilde r - r)^2$$

Reorganizing, we obtain the result. □

The theorem applied to the net-present value of the cash-flows and of the obligations as functions of the interest rates justifies the requirement that the expectation of the two have to be matched. Now, $\frac{dX_\gamma}{d\gamma}$ ($\frac{dL_\gamma}{d\gamma}$, analogously) is easily obtained:

$$\frac{dX_\gamma}{d\gamma} = \frac{d}{d\gamma}\sum_{t=1}^{T}\frac{X_t}{\gamma^t} = (-\gamma)\sum_{t=1}^{T} t\frac{X_t}{\gamma^t}$$

Our final problem formulations takes on the following form:

$$\min_u \sum_{i=1}^{N} a_i u_i \tag{14}$$

$$\text{s.t.} \quad X_t = \sum_{i=1}^{N} C_{i,t} u_i \tag{15}$$

$$\sum_{t=1}^{T}\frac{1}{\gamma^t}X_t \succeq_{(2)} \sum_{t=1}^{T}\frac{1}{\gamma^t}L_t \tag{16}$$

$$E\left[\sum_{t=1}^{T}\frac{t}{\gamma^t}X_t\right] = E\left[\sum_{t=1}^{T}\frac{t}{\gamma^t}L_t\right] \tag{17}$$

4 Solution Scheme

The starting point of our model is the assumption that:

- A scenario generator black-box for the cash-flows of each bond is available;

  Without assuming how the scenarios are generated, the model can take realizations of complex econometric models for the defaults. The simplest form of this scenario generator can be a random sample from a geometric distribution with parameter given by the CDS implied default probability, which is available to almost every market participant.

- An "acceptable" estimate of discount rate is available;

  An "acceptable" estimate of discount rate can be the current benchmark rate. Since our model is already immune to small changes, this estimate is not required to be an accurate prediction as long as it is a sensible reference.

- A distribution of the liabilities is available.

  A distribution of the liabilities should also be readily available to anyone that has predictable payouts. This would depend on the actual use case of our model: for example, for a pension fund with a steady payout, its liability structure would be different from that of a ultra-high-net-worth individual.

Consequently, the following are given:

- $C^s_{i,t}$; the realizations of random cash-flow streams for each asset. s is the index of the scenario.
- γ; the estimated discount rate.
- $\mathbf{P}_L : L \sim \mathbf{P}_L$ the distribution of liability streams. This can be realizations of some scenario generation process as well.

4.1 The Scaled Quantile Cutting Plane Method

(Dentcheva and Ruszczyński 2010) established efficient methods for optimization with second-order stochastic dominance constraints. We use the "scaled quantile cutting plane method" and only reiterate some key ideas.

For simplicity of notation, let $X = \sum_{t=1}^{T} \frac{1}{\gamma^t} X_t$ and $L = \sum_{t=1}^{T} \frac{1}{\gamma^t} L_t$. (16) becomes $X \succeq_{(2)} L$. By Theorem 2.1, we write:

$$\frac{1}{p}F^{(-2)}(X;p) \geq \frac{1}{p}F^{(-2)}(L;p) \quad \forall p \in (0,1)$$

Apply the equivalent formulation of the Lorenz function to the left hand side (Dentcheva and Ruszczyński 2010), we have:

$$\sup_{\eta}\{\eta - E((\eta - X)_+)\} \geq \frac{1}{p}F^{(-2)}(L;p) \quad \forall p \in (0,1)$$

Due to our i.i.d. sample from equally-likely scenarios:

$$\eta_i - \frac{1}{i}\sum_{s=1}^{S}(\eta_i - x_s)_+ \geq \frac{i}{S}F^{(-2)}(L;i/S) \quad \forall i = 1,2,...,S$$

The right-hand side is easily calculated by sorting the realizations of L. Here η_i's are the i/S-quantiles of X, which are obtained through optimization.

We do not need to impose this constraint on all probability levels i/S. The "Scaled quantile cutting plane method" iteratively adds constraints for the probability level where the above constraint is violated the most.

The algorithm is as follows:

Step 0: Initialize

set $k = 0 \quad \Lambda_0 = \{1,2,...,S\} \quad i_0 = 0 \quad \delta_0 = 0$

Step 1: Solve sub-problem

find solution (u^k, η^k) of:

$$\begin{aligned}
&\min_{u,\eta} \sum_{i=1}^{N} a_i u_i \\
&\text{s.t.} \quad x_{s,t} = \sum_{i=1}^{N} C_{s,i,t} u_i \\
&\qquad \eta_{i_j} - \frac{1}{i_j}\sum_{s\in\Lambda_j}(\eta_{i_j} - \sum_{t=1}^{T}\frac{t}{\gamma^t}x_{s,t}) \geq \frac{i_j}{S}F^{(-2)}(\frac{t}{\gamma^t}L_t; i_j/S) \quad \forall j = 0,1,...,k-1 \\
&\qquad \sum_{i=1}^{S}\sum_{t=1}^{T}\frac{t}{\gamma^t}x_{s,t} = E\left[\sum_{t=1}^{T}\frac{t}{\gamma^t}L_t\right]
\end{aligned} \tag{18}$$

Step 2: Find cut

$$\delta_k = \max_i\{\frac{i}{S}F^{(-2)}(L;i/S) - \eta_i^k + \frac{1}{i}\sum_{s=1}^{S}(\eta_i^k - \sum_{t=1}^{T}\frac{t}{\gamma^t}x_{s,t})_+\}$$

$$i_k = \text{argmax}_i \{ \frac{i}{S} F^{(-2)}(L; i/S) - \eta_i^k + \frac{1}{i} \sum_{s=1}^{S} (\eta_i^k - \sum_{t=1}^{T} \frac{t}{\gamma^t} x_{s,t})_+ \}$$

$$\Lambda_k = \{ s : \eta_{i^k}^k - x_s > 0 \}$$

Step 3: Check stopping criteria:

If $\delta_k \leq 0$ then stop and output u^k. Otherwise increment k and go to step 1.

Our problem is numerically difficult due to the low probability of defaults. This makes gradient methods undesirable. We further employ matrix manipulation to transform all sub-problems to the standard form and solve each of them using the simplex method, which greatly improves numerical efficiency in our case.

4.2 Numerical Implementation

To turn each sub-problem (18) into $\{u | A_k u \leq b_k\}$, we first need two matrices (here N is the number of assets and T is the time horizon):

- $M \in R^{S*N} : u \to \sum_{t=1}^{T} \frac{t}{\gamma^t} x_{s,t}$

  This matrix maps an allocation to the vector of realizations of the net-present-value of the portfolio.

- $W \in R^{1*N} : u \to \sum_{i=1}^{S} \sum_{t=1}^{T} \frac{t}{\gamma^t} x_{s,t}$

  This matrix maps an allocation to the expected "duration" of the portfolio.

Define the following matrices:

$CF_s \in R^{N*T}$: realized cash flow matrix for scenario s.

$$CF_s = \begin{bmatrix} C_{1,1} & C_{1,2} & ... & C_{1,T} \\ C_{2,1} & C_{2,2} & ... & C_{2,T} \\ ... & & & \\ C_{N,1} & C_{N,2} & ... & C_{N,T} \end{bmatrix}$$

Then $CF_s^T u$ gives the realized cash-flow stream for scenario s.

$d \in R^T$: discount curve.

$$d = [1/\gamma \quad 1/\gamma^2 \quad ... \quad 1/\gamma^T]^T$$

Then, for each scenario, we have that the discounted value of cash flows is:

$$d^T CF_s^T u = (CF_s d)^T u$$

Let our block matrix M be such that:

$$M = \begin{bmatrix} (CF_1 d)^T \\ (CF_2 d)^T \\ ... \\ (CF_S d)^T \end{bmatrix} \tag{19}$$

Then we have:

$$Mu = \begin{bmatrix} \sum_{t=1}^T \frac{t}{\gamma^t} x_{1,t} \\ \sum_{t=1}^T \frac{t}{\gamma^t} x_{2,t} \\ ... \\ \sum_{t=1}^T \frac{t}{\gamma^t} x_{S,t} \end{bmatrix}$$

Which is a mapping from our decision space $u \in R^N$ to all S realizations of the discounted cash flow in R^S.

As described in the previous section, the conjugate function method works by iteratively adding constraints. Define:

$e_{\Lambda_k} \in R^S : (e_{\Lambda_k})_j = 1 \iff j \in \Lambda_k$

Then we have a convenient formula:

$$\eta_{i_j} - \frac{1}{i_j} \sum_{s \in \Lambda_j} (\eta_{i_j} - \sum_{t=1}^{T} \frac{t}{\gamma^t} x_{s,t}) = \eta_{i_j} - (\eta_{i_j} - \frac{1}{i_j} e_{\Lambda_j} Mu) = \frac{1}{i_j} e_{\Lambda_j} Mu$$

.

For the k'th iteration, increment the following matrices:

$$A_k = \begin{bmatrix} \frac{1}{i_1} e_{\Lambda_1} M \\ \frac{1}{i_2} e_{\Lambda_2} M \\ ... \\ \frac{1}{i_k} e_{\Lambda_k} M \end{bmatrix}$$

$$b_k = \begin{bmatrix} \frac{i_1}{S} F^{(-2)}(\frac{t}{\gamma^t} L_t; i_1/S) \\ \frac{i_2}{S} F^{(-2)}(\frac{t}{\gamma^t} L_t; i_2/S) \\ ... \\ \frac{i_k}{S} F^{(-2)}(\frac{t}{\gamma^t} L_t; i_k/S)] \end{bmatrix}$$

.

Each sub-problem in (18) therefore becomes:

$$\min a^T u$$

$$A_k u \geq b_k$$

$$\sum_{i=1}^{S} \sum_{t=1}^{T} \frac{t}{\gamma^t} x_{s,t} = E\left[\sum_{t=1}^{T} \frac{t}{\gamma^t} L_t\right]$$

It remains to further simplify the last equality. Similar to previous manipulations, we define:

$$\tau = \begin{bmatrix} 1/\gamma & 2/\gamma^2 & ... & T/\gamma^T \end{bmatrix}^T$$

;

$$H = \begin{bmatrix} (CF_1\tau)^T \\ (CF_2\tau)^T \\ ... \\ (CF_T\tau)^T \end{bmatrix}$$

Easy to check that:

$$\mathbf{1}^T H u = \sum_{i=1}^{S} \sum_{t=1}^{T} \frac{t}{\gamma^t} x_{s,t}$$

Let $W = \mathbf{1}^T H$, then eventually we have:

$$\min a^T u$$

$$\begin{bmatrix} A_k \\ W \end{bmatrix} u = \begin{bmatrix} b_k \\ E\left[\sum_{t=1}^{T} \frac{t}{\gamma^t} L_t\right] \end{bmatrix}$$

This can be efficiently solved by any simplex solver.

5 Results

We simulate a portfolio of 100 bonds, each having different specifications. The annual default probability of each bond draw uniformly from range $[3\%, 40\%]$. The maturity, coupon rate, and recovery rate of the bonds are also drawn uniformly from: $[6, 36]$ (months), $[0, 15\%]$ (annual coupon), $[20\%, 60\%]$ (recovery rate). The price of the bond is then calculated as the break-even required return plus some credit risk premium, specified as $2.5\% + \epsilon$, where ϵ is some small random noise in credit risk premium. The numbers are loosely based on historical observed credit events.

We model the credit event as a geometric random variable. We draw the liabilities from a inverted log-normal distribution.

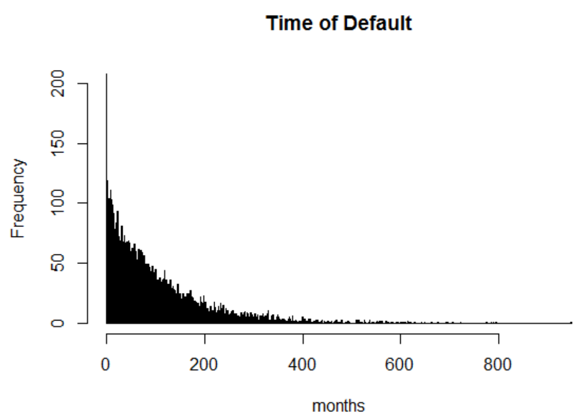

Figure 1: Simulated Time of Defaults

5.1 Optimization

Taking the simulated default events as inputs, we solve problem (14) using the Cutting Plane Method. The resulting CDF's of the net-present-value of the optimal portfolio and the liabilities are shown below. The difference of the integrated CDF's are also shown, and we see that indeed the dominance constraint is satisfied.

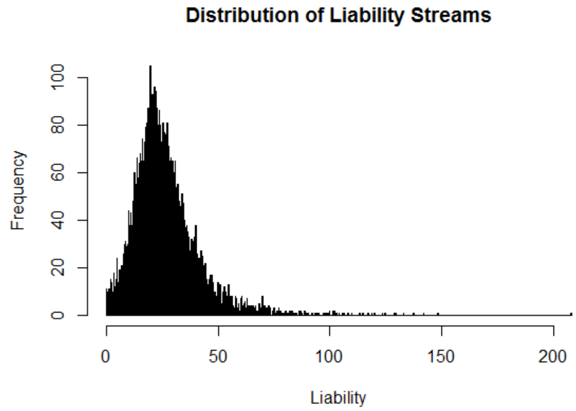

Figure 2: Simulated Liability Stream at Time t

The algorithm ran for 30 iterations.

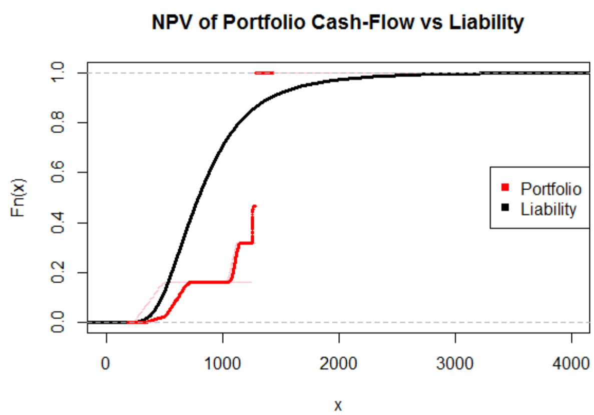

Figure 3: CDF of NPV of Portfolio vs Liabilities

5.2 Interest Rate Stress Testing

We stress our model's results under falling, stagnant and rising interest rate regimes, and we compare the result with the model assuming no defaults. The test proceed in the following steps:

1. We first run the optimization based on some constant estimate of interest rate and obtain both the optimal stochastic-dominance portfolio (the stochastic model) and the optimal portfolio assuming no defaults (the baseline model).

2. We simulated interest rate changes based on an AR(1) historical interest rate changes. An AR(1)

model would correspond to a discrete time Vasicek interest rate model. We also generate new scenarios for the liabilities and the cash flows based on the same parameters used in the optimization.

3. We run our and apply the corresponding rate changes to the cash flows produced by the stochastic model and the non-random baseline model.

The result assuming that interest rate moves sideways is shown in 4:

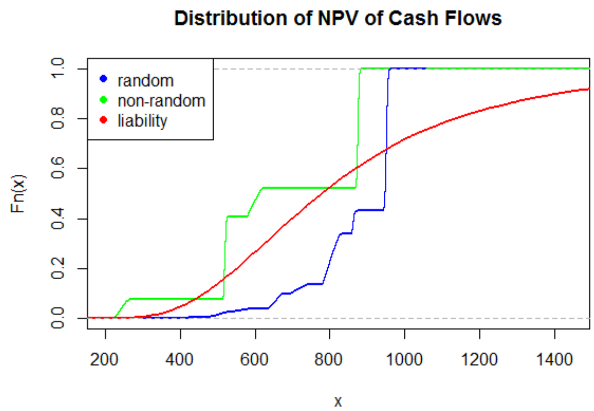

Figure 4: "Out-of-sample" Performance: Stochastic vs. Baseline Portfolio

The other interest regimes show similar results. We observe that the CDF of the stochastic portfolio is below that of the baseline portfolio, indicating that at any probability level, the loss of the stochastic portfolio will be smaller than that of the baseline portfolio. The cost of the stochastic portfolio is about 15% more expensive than the baseline model, but even when initial cost is subtracted, the stochastic portfolio still outperforms the baseline model as in 7.

5.3 Default Probability Stress Testing

Analogous to the previous section, we test model performance when the true default probability is different from the estimated default probability.

1. We first obtain both the optimal stochastic-dominance portfolio and the optimal portfolio assuming no defaults.

2. We generate new scenarios when the model input default probabilities consistently under/over estimate the true default probabilities.

3. We run our model and obtain the out-of-sample performance.

When the model input underestimates true default risk, which is perhaps the case of more practical interest, the result is 5.

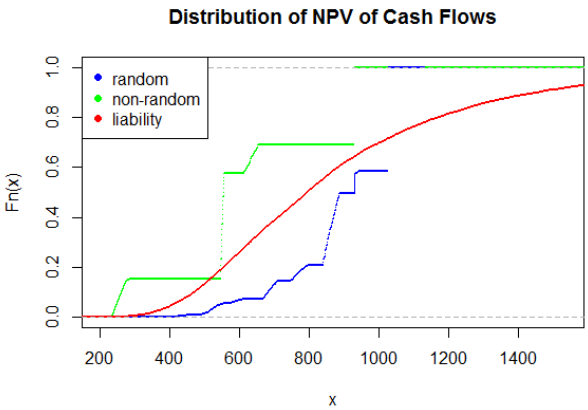

Figure 5: "Out-of-sample" Performance: Stochastic vs. Other Risk Models

When the model overestimates true default risk, the result is 6

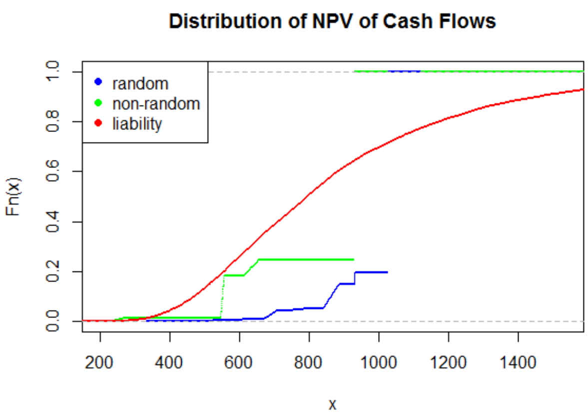

Figure 6: "Out-of-sample" Performance: Stochastic vs. Other Risk Models

Note that in both cases the stochastic model outperforms the baseline model. In our numerical example, the stochastic model is still able to generate cash flows that secon- order-stochastically dominate the liability, but in general there is no such guarantee.

5.4 Comparison with Other Models

When default risk is high, it is natural to consider other "risk-aware" models. We repeat the previous stress test procedure for the following models.

1. Risk-Neutral

$$
\begin{aligned}
&\min_u \sum_{i=1}^{N} a_i u_i \\
\text{s.t.}\quad & X_t = \sum_{i=1}^{N} C_{i,t} u_i \\
& E[\sum_{t=1}^{T} \frac{1}{\gamma^t} X_t] \geq E[\sum_{t=1}^{T} \frac{1}{\gamma^t} L_t] \\
& E\left[\sum_{t=1}^{T} \frac{t}{\gamma^t} X_t\right] = E\left[\sum_{t=1}^{T} \frac{t}{\gamma^t} L_t\right]
\end{aligned}
$$

2. Risk-Averse

$$
\begin{aligned}
&\min_u \sum_{i=1}^{N} a_i u_i \\
\text{s.t.}\quad & X_t = \sum_{i=1}^{N} C_{i,t} u_i \\
& \rho(\sum_{t=1}^{T} \frac{1}{\gamma^t} X_t) \leq \rho(\sum_{t=1}^{T} \frac{1}{\gamma^t} L_t) \\
& E\left[\sum_{t=1}^{T} \frac{t}{\gamma^t} X_t\right] = E\left[\sum_{t=1}^{T} \frac{t}{\gamma^t} L_t\right]
\end{aligned}
$$

Where ρ is some coherent risk-measure. We use the mean semi-deviation.

There are many well-known numerical algorithms for solving the above problems, so we shall not discuss that in detail. Note that we can always use inverse cutting-plane method again, since the above are necessary conditions for the second-order dominance constraint.

The result assuming interest rate moves sideways is shown in 7

The other interest rate regimes yields similar results.

The mean semi-deviation model and the expectation model both seems to outperform the stochastic dominance mode. We note that the use of the stochastic dominance constraint is not to generate higher expected income, but to control the tail-risk of portfolio loss. In 7, although the mean semi-deviation model and the expectation model outperforms in terms of expectation, there is significantly more tail risk when the

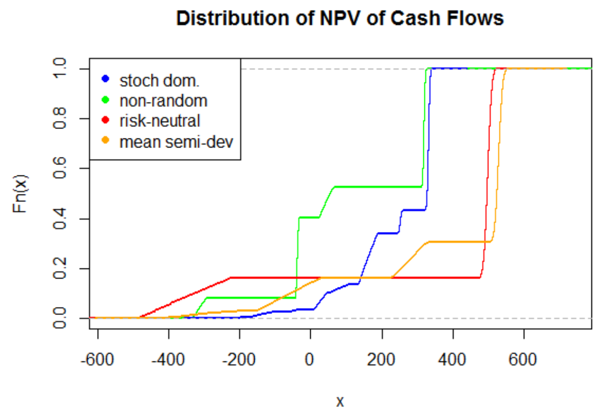

Figure 7: "Out-of-sample" Performance: Stochastic vs. Other Risk Models

incurred loss is below the 20% quantile. Therefore, we conclude that the stochastic dominance portfolio is a better model when the investor is risk-averse.

5.5 Re-balancing Cost in a Two-Stage Problem

When time horizon is long, interest rate may gradually drift significantly away from the starting reference point. Bond price and discount rate has a non-linear relationship, and when interest rate is far from the starting point, duration matching breaks down as linear approximation becomes inaccurate. We evaluate the re-balancing cost of the stochastic against the baseline model by looking at the cost of re-balancing at some time $0 < \tau \leq T$ when the new interest rate is $\Delta r + r$.

We adopt a scenario-based approach analogous to a two-stage problem. We assume that if a company defaults, it will remain default. Otherwise, default probabilities and recovery rates remain constant through time. In addition, there is no transaction cost. Let $\tau, \Delta r$ be arbitrary and fixed.

1. With information at time 0, simulate cash flow and liability scenarios for the entirety of the time horizon. Obtain the optimal stochastic-dominance portfolio and the baseline portfolio based on these initial scenarios.

2. For each scenario s, gather all information up to time τ. We also have a new reference discount rate $\Delta r + r$, and some bonds may have defaulted before τ. The value of our initial portfolios will change accordingly. Based on information up to time τ, we generate new scenarios, and obtain a new stochastic-dominance portfolio and baseline portfolio. The re-balancing cost for scenario s is the difference between the new portfolio and the value of the initial portfolio at time τ.

3. Repeat 2 for each initial scenario, and obtain the expected re-balancing cost for each portfolio by averaging.

The above steps are repeated again for a range of $\tau, \Delta r$'s, and the result is shown below:

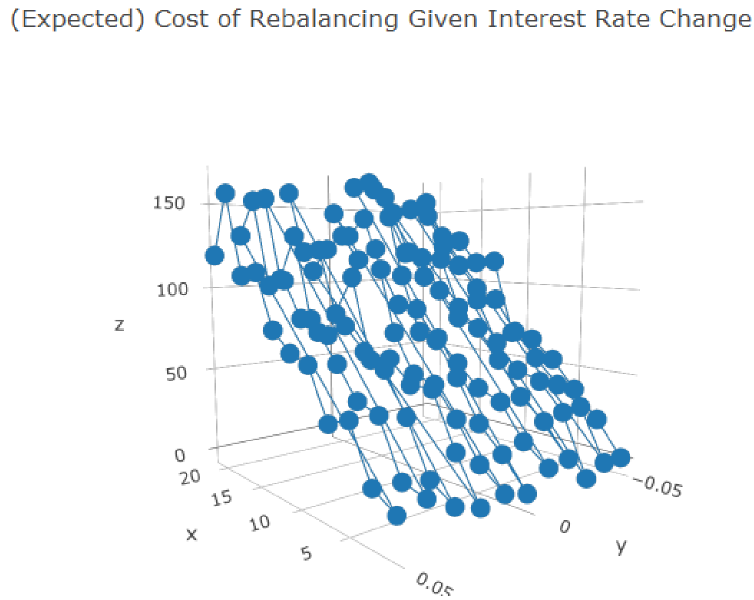

Figure 8: Rebalancing Cost: Stochastic Portfolio Minus Baseline Portfolio

We observe than the rebalancing cost of the stochastic portfolio is uniformly less than the baseline portfolio.

6 Conclusion & Future Research

6.1 Limitations & Assumptions

This model inherently has its own limitations and assumptions. First, duration-matching assumes only parallel shift in the yield curve, which in reality is rarely the case. (Nawalkha and Chambers 1996) and (FONG and VASICEK 1984) defines a different measure of interest rate to account for non-parallel interest rate changes. We do not use their immunization method because the goal of this paper is to use an "all-else-remain-equal" approach to introduce a new way of incorporating credit risk to asset-liability-matching problems.

Second, the default events were assumed to be independent of one another when generating the simulations. The nature of defaults in financial markets is systemic and events such as defaults have a rippling effect across the economy. The study of copula methods can give rise to a more meaningful dependence structure of the credit events; however, such is out of the scope of this paper. Nevertheless, our method does not explicitly depend on independence: if the model user uses a very complex copula model to generate the cash-flow scenarios, the model only need the scenarios to run.

Third, we assume in the formulation of our model that the investor's distribution of liabilities and cash flows are independent from one another. In the real world, these two distributions are likely to be dependent on one another particularly for larger institutional investors such as banks, pension funds and insurance companies.

6.2 Future Research

There is still much to be researched when considering the model itself, the inputs and the stress testing of the results. In regards to expanding the portfolio model, implementing a convexity constraint would help reduce risk of larger interest rate shocks that often occur amidst major economic recessions. While the duration constraint worked as intended, it only reduces the impact of minor fluctuations in interest rates. Earlier, we discussed various limitations and assumptions regarding the default probabilities used here. As the saying goes: "garbage in, garbage out" that is surely the case here. This model is dependent on default probability and liability distribution data by construction. Therefore, future works should look to ensure that default probabilities simulated account for elements of systemic risk and contagions of defaults. Finally, future research should explore the efficacy of this model across different asset universes, particularly those consisting of risky securities that are more liable to defaults.

6.3 Conclusion

We believe the intuition behind our problem framework, model and methodology to be robust. There is a clear need to fill the gaps where other methods of portfolio optimization fail to incorporate probabilities of default. We believe both the model and its implementation to be novel. As demonstrated in the results section, our stochastic model outperforms in terms of both higher net-present-value and lower rebalancing cost. We also showed that our model is robust to changes in interest rate and default probability estimation errors. The numerical implementation of this model via a linear re-formulation allows it to be solved quickly through the standard Simplex method and allows for future works to readily scale the number of securities and companies involved in executing this model.

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