Minimum Spanning Tree

Kazi Lutful Kabir

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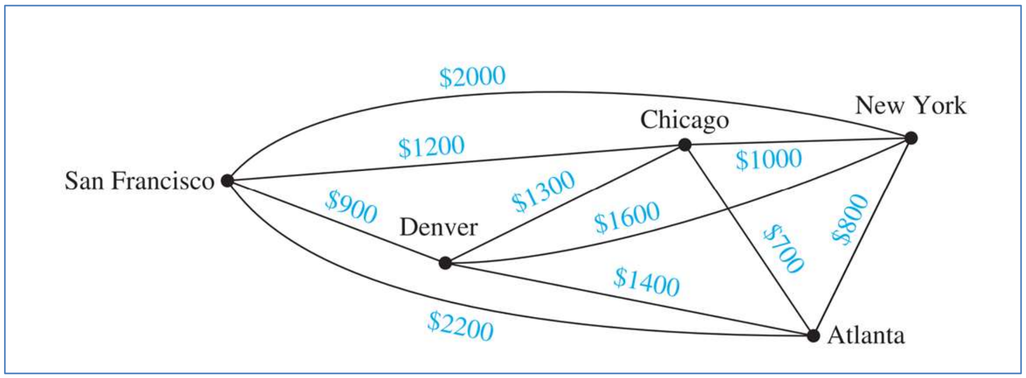
Spanning Tree

A company plans to build a communications network connecting its five computer centers. Any pair of these centers can be linked with a leased telephone line. Which links should be made to ensure that there is a path between any two computer centers so that the total cost of the network is minimized?



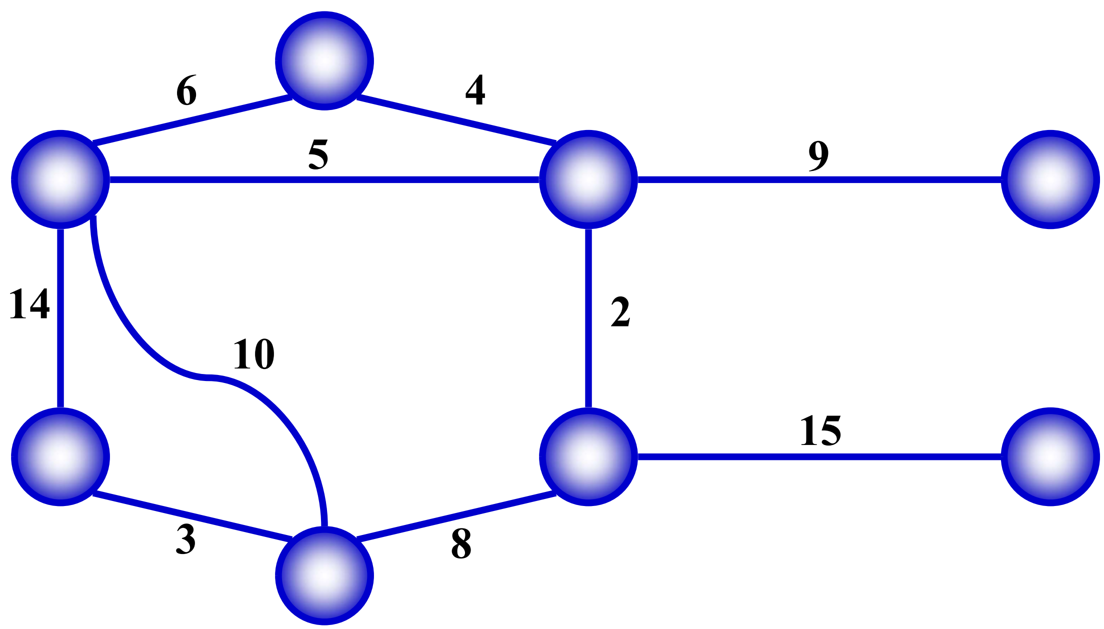
1	{Chicago, Atlanta}		\$ 700
2	{Atlanta, New York}		\$ 800
3	{Chicago, San Francisco}		\$1200
4	{San Francisco, Denver}		\$ 900
		Total:	\$3600

- A **spanning tree** T of an undirected graph G is a subgraph that is a tree which includes all the vertices of G

A *minimum spanning tree* in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

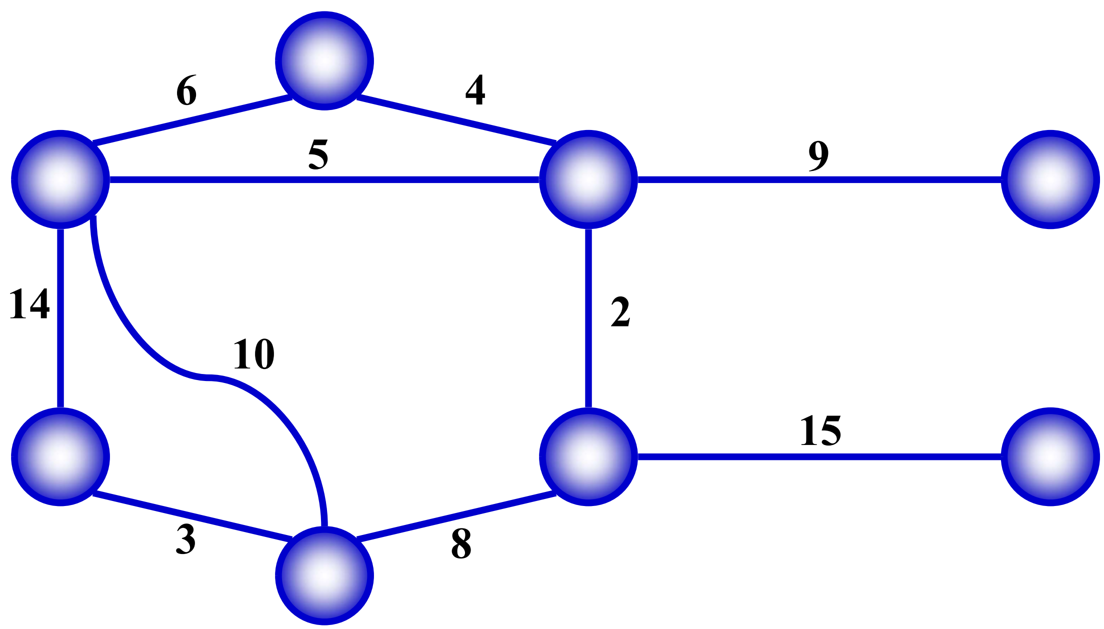
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



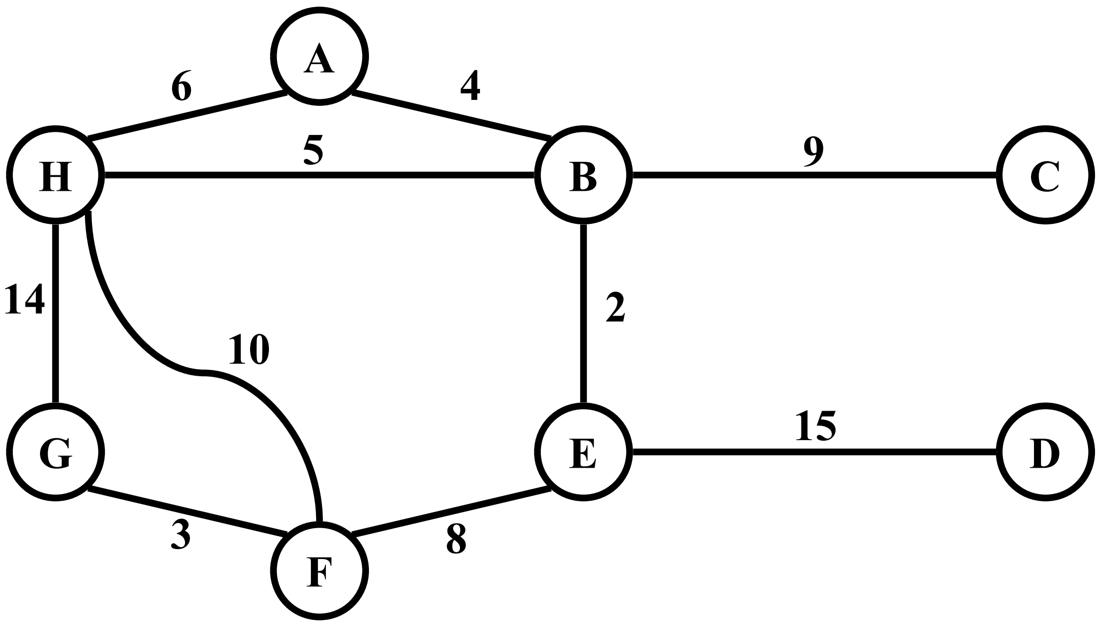
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



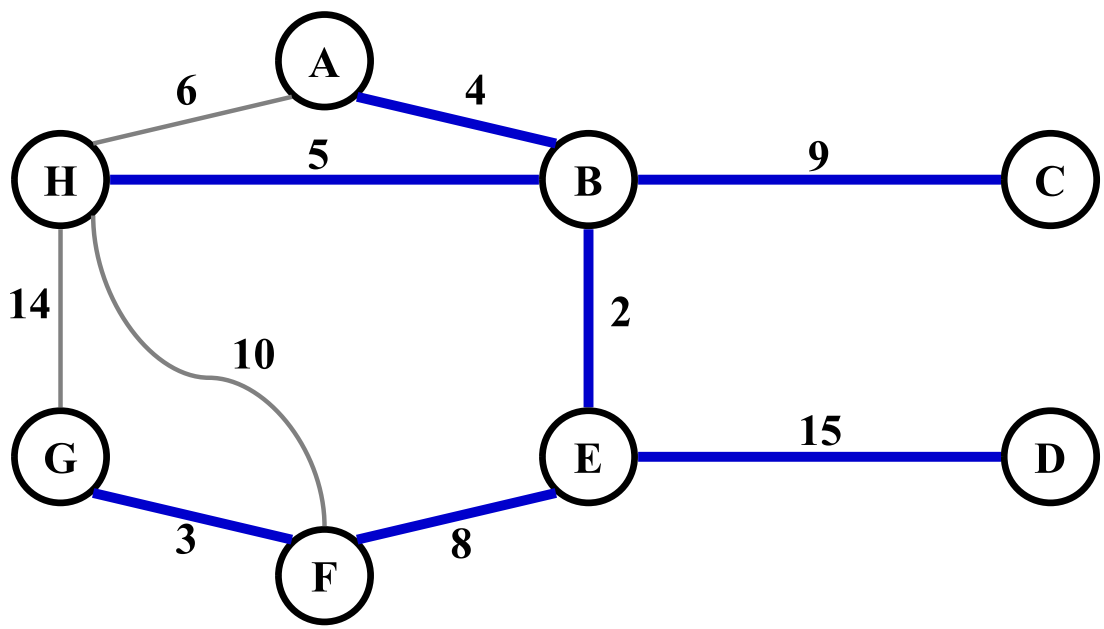
Minimum Spanning Tree

- Which edges form the minimum spanning tree (MST) of the graph as shown below?



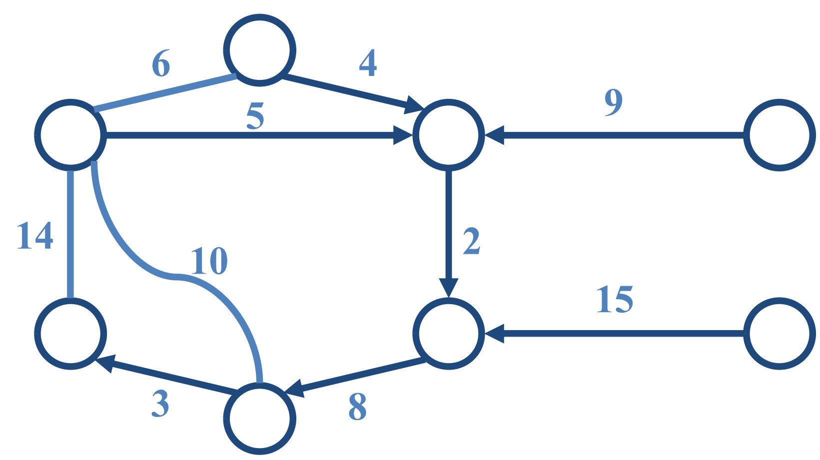
Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- *Optimal substructure* property: an optimal minimum spanning tree is composed of optimal minimum spanning subtrees



Prim's Algorithm

MST-PRIM(G, w, r)

```
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
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        if v ∈ Q and w(u, v) < v.key
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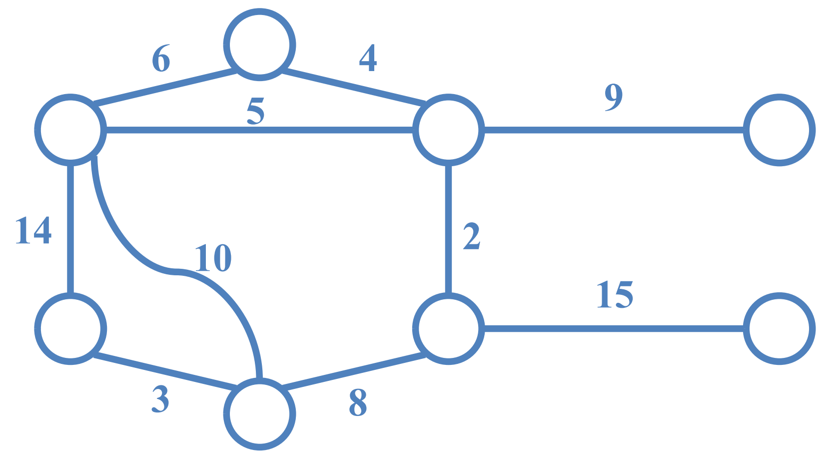

Run on example graph

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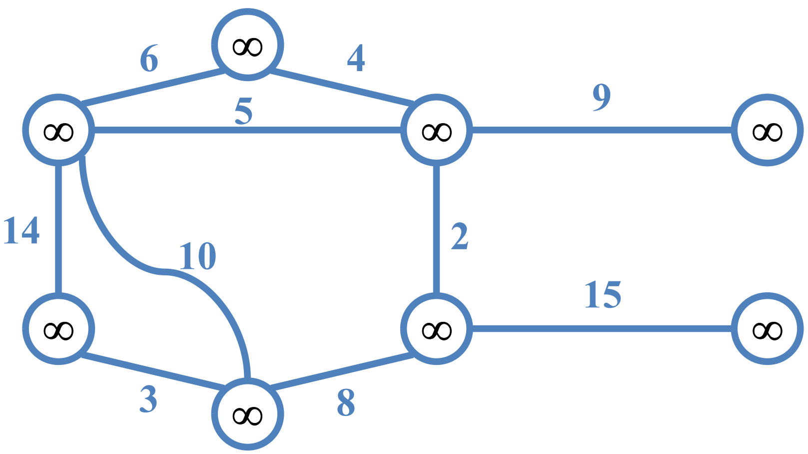

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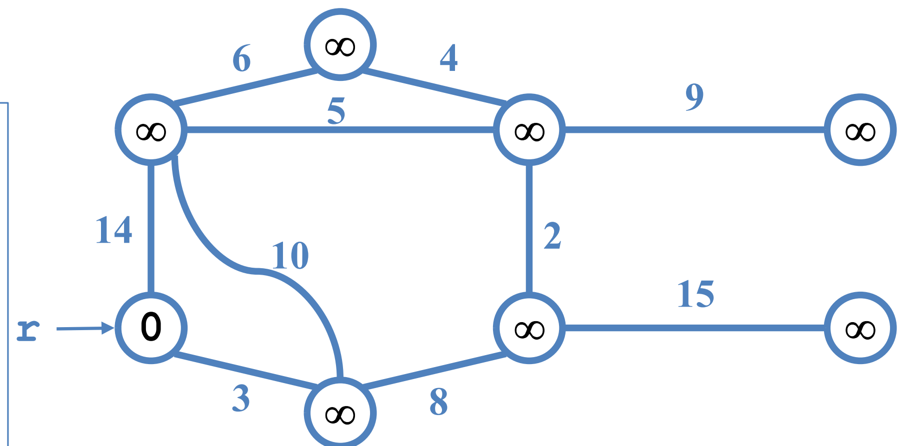

Pick a start vertex r

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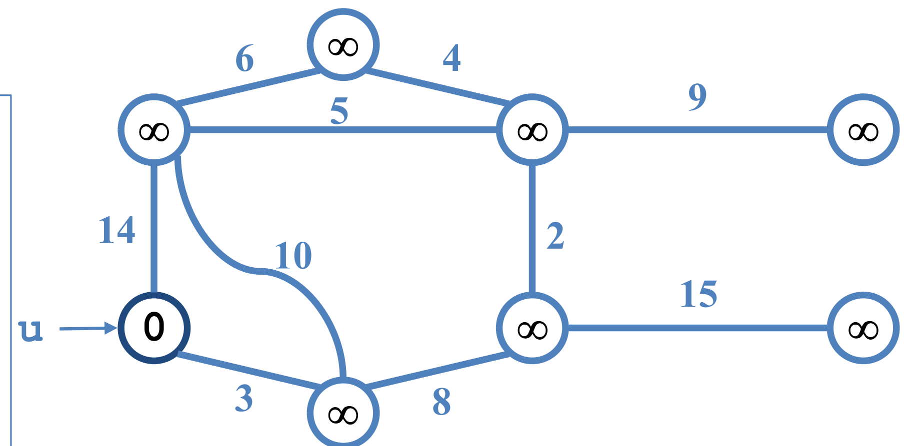

Dark Blue vertices have been removed from Q

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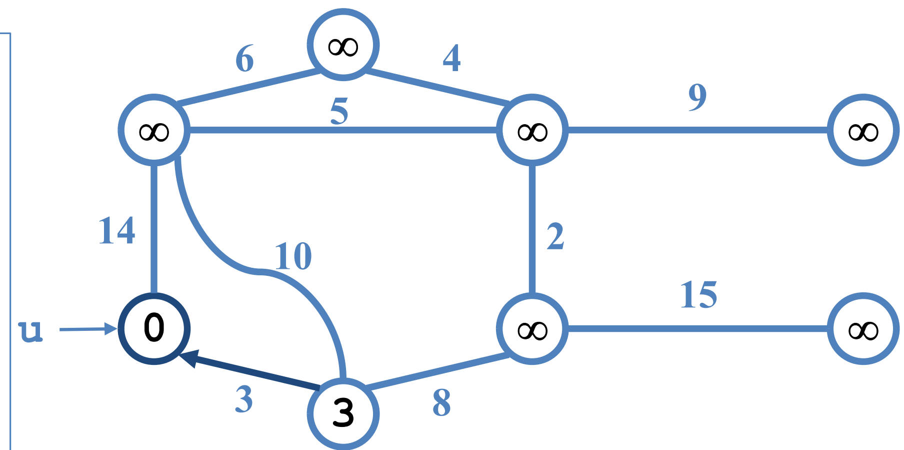

Dark Blue arrows indicate parent pointers

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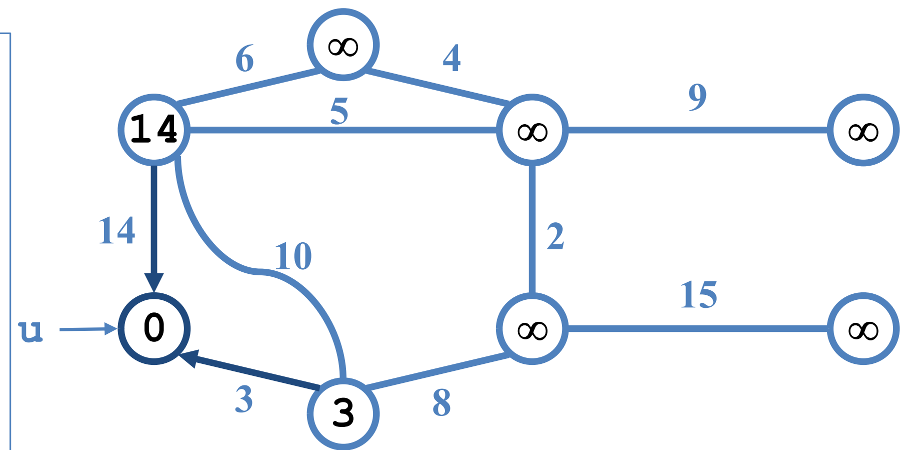

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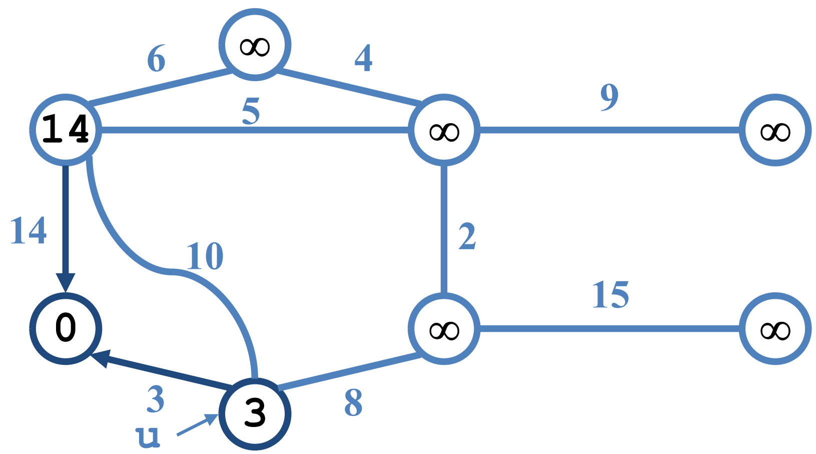

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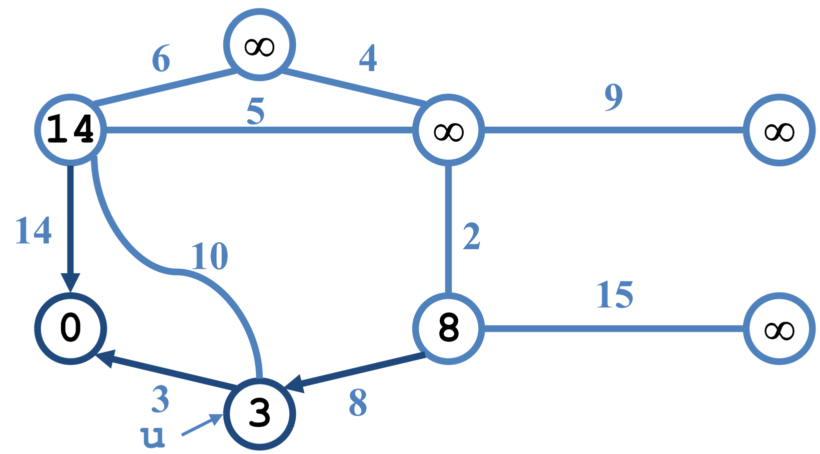

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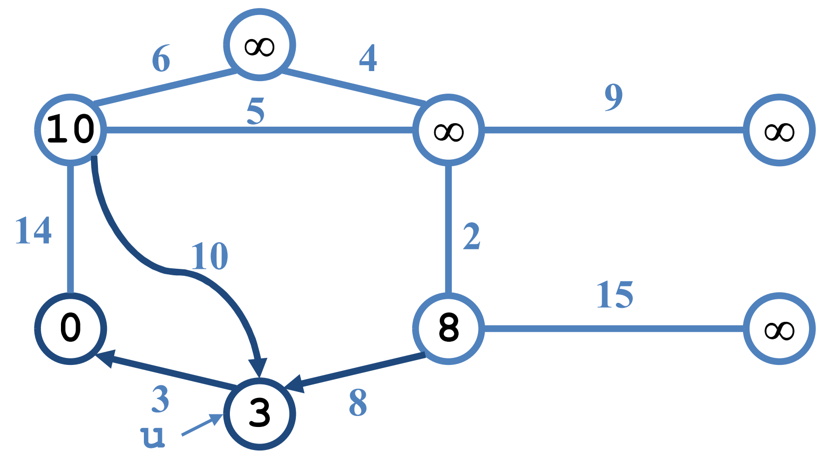

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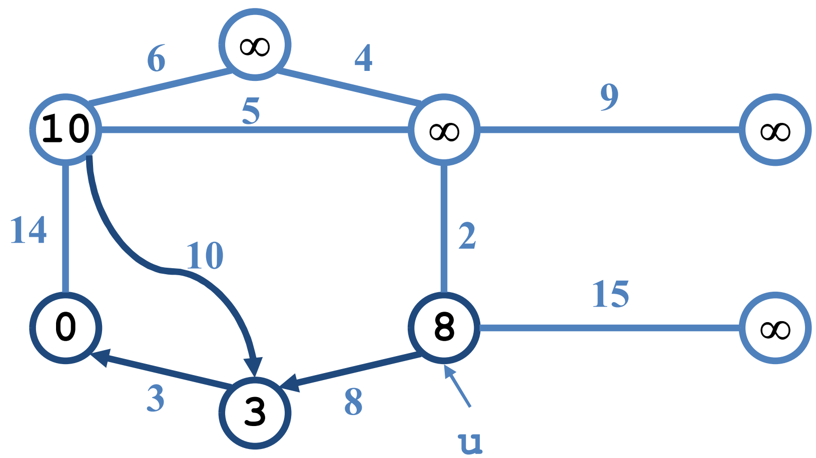

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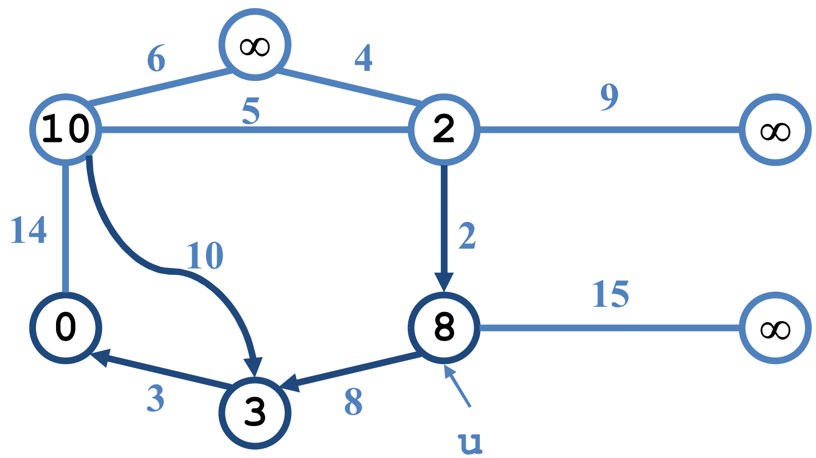

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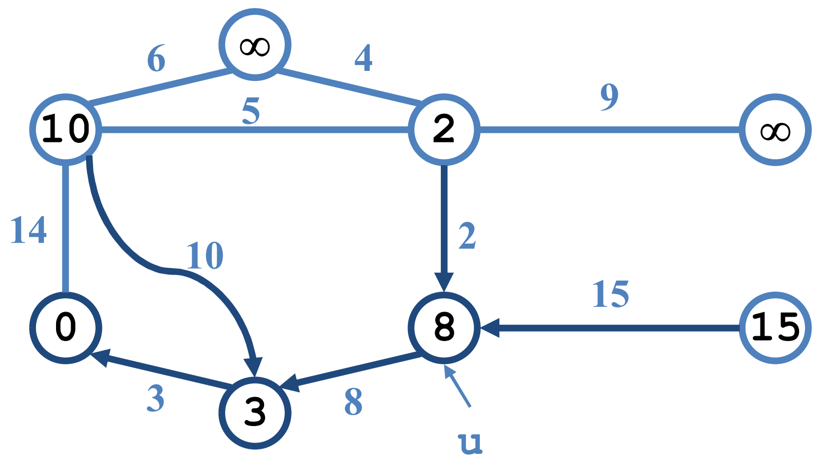

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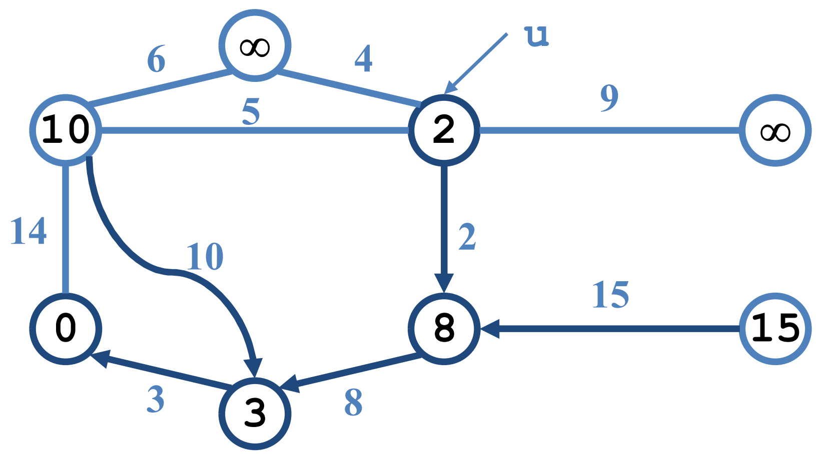

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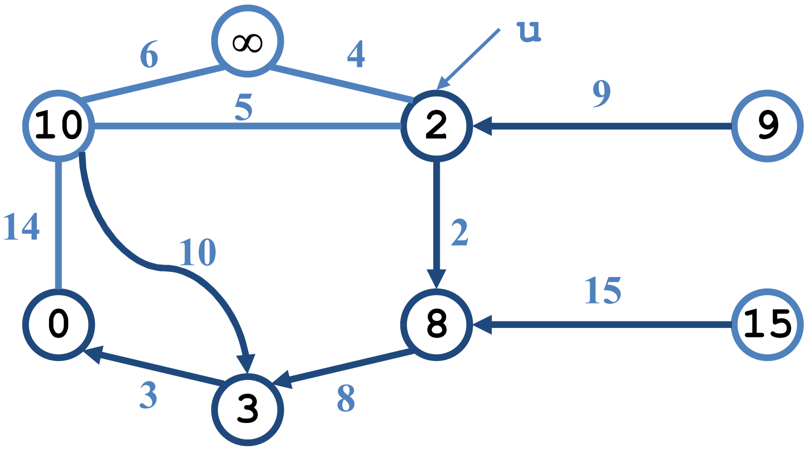

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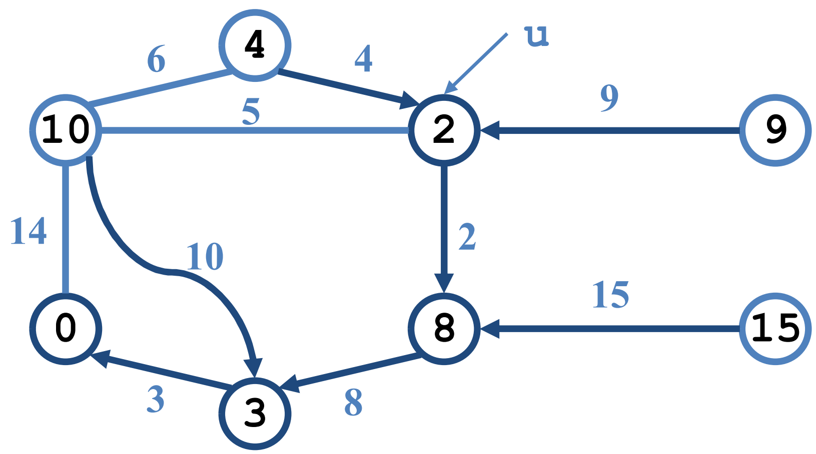

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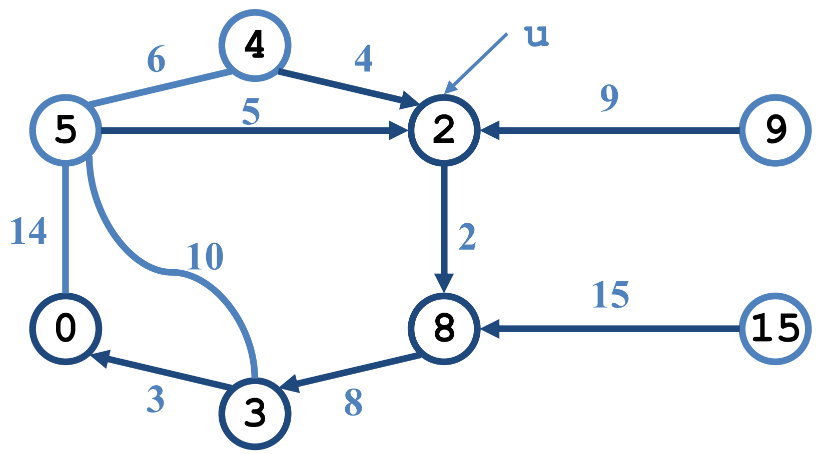

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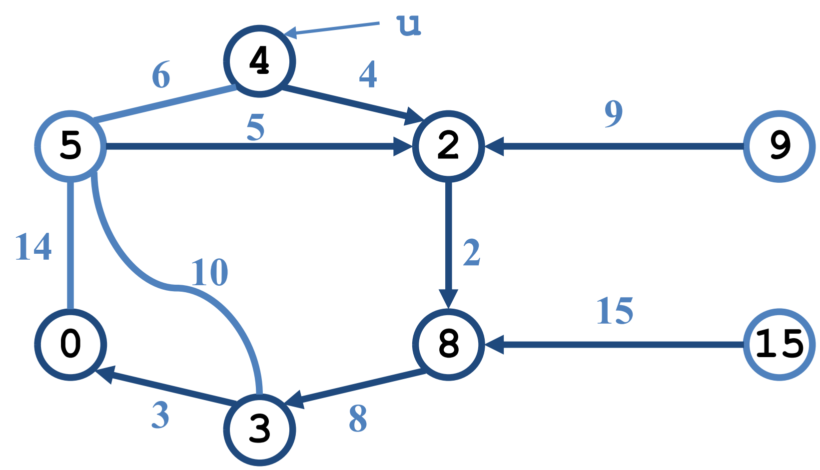

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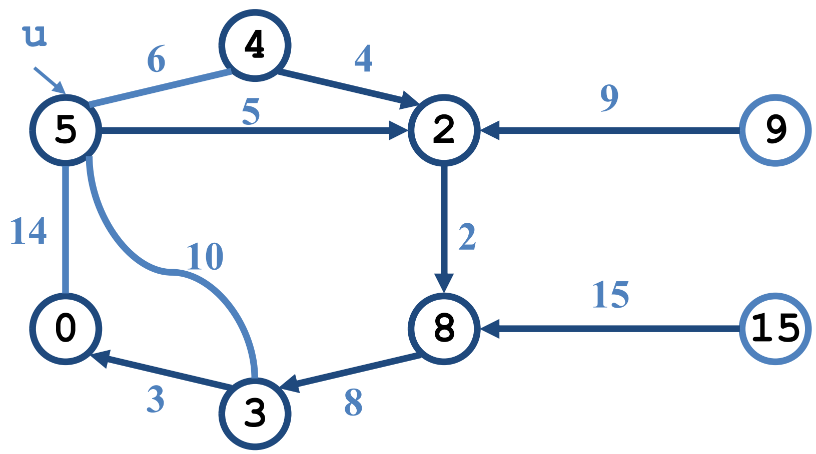

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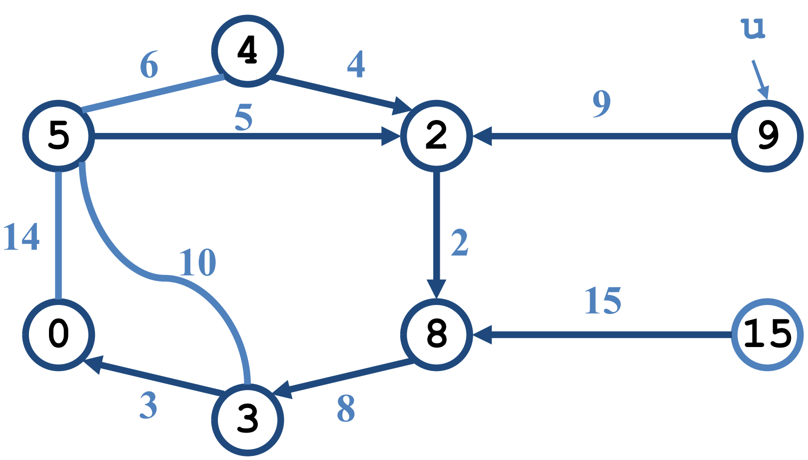

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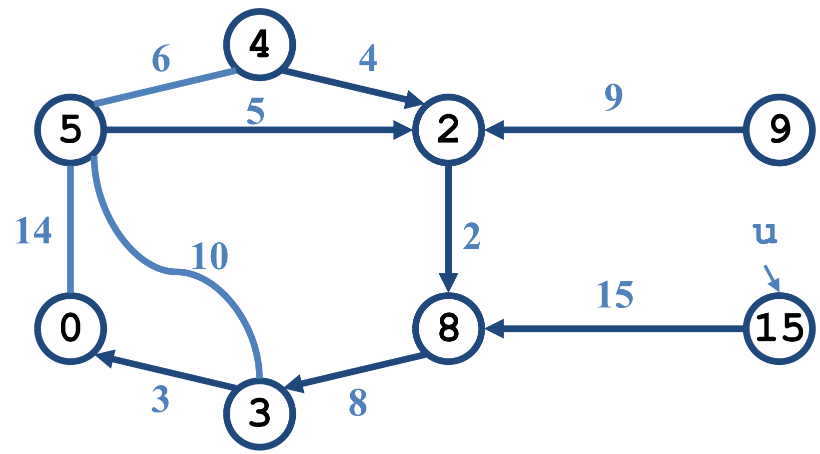

Prim’s Running Time

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```
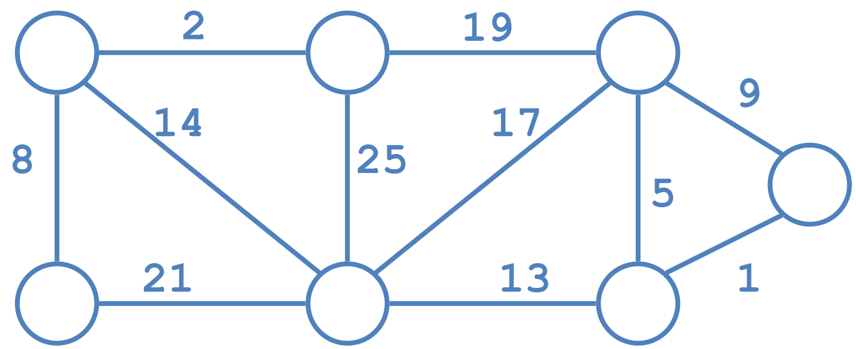

A simple implementation of Prim's, using an adjacency matrix or an adjacency list graph representation and linearly searching an array of weights to find the minimum weight edge to add, requires **$O(|V|^2)$** running time.

Kruskal's Algorithm

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

Kruskal's Algorithm



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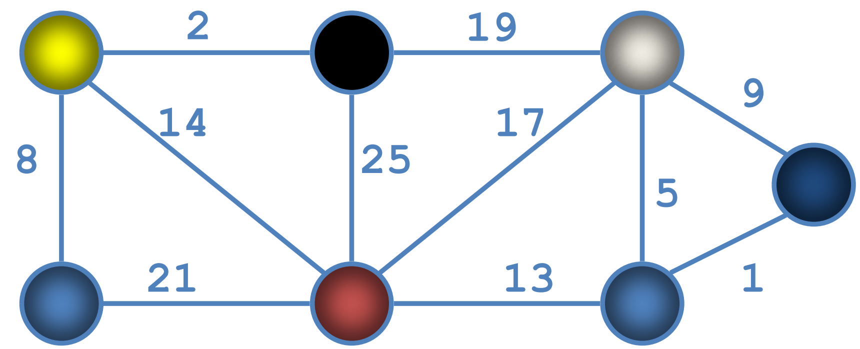

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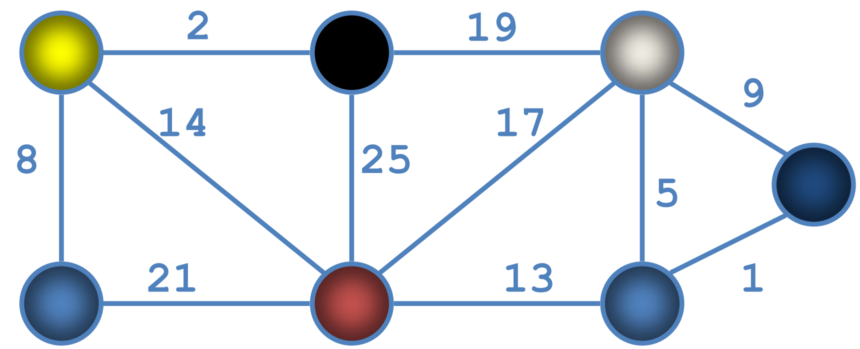

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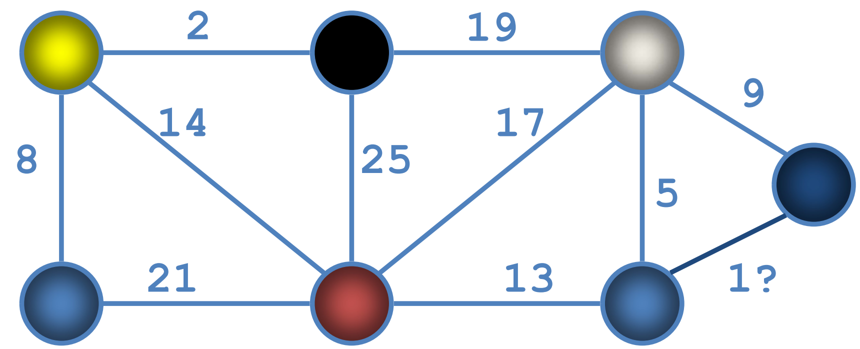

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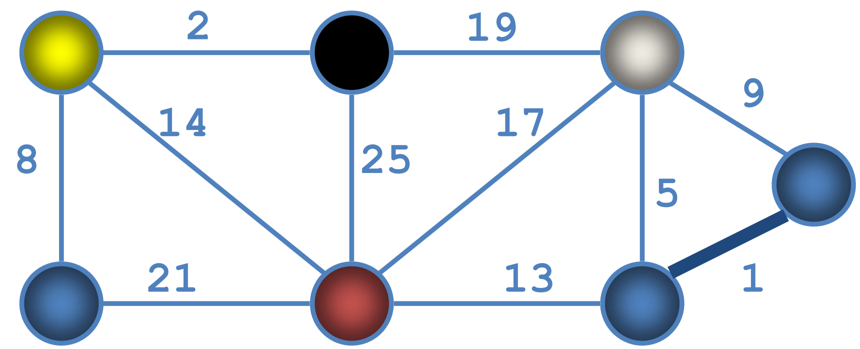

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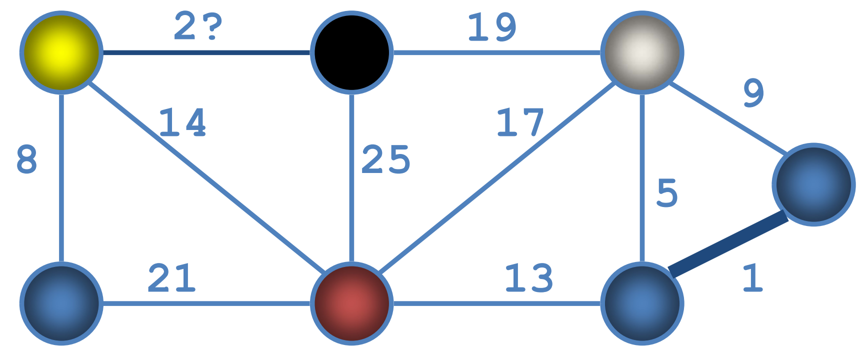

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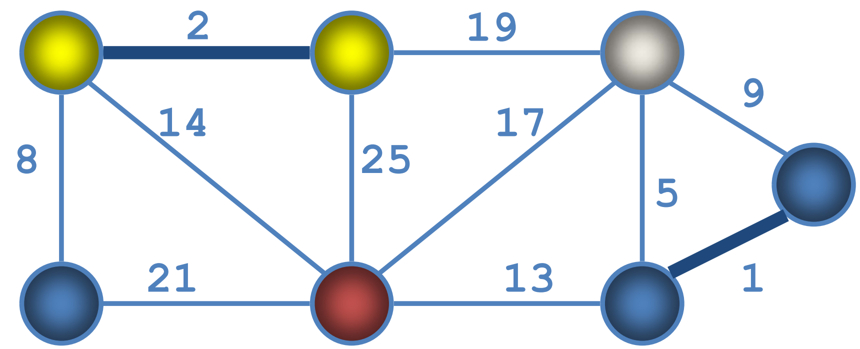

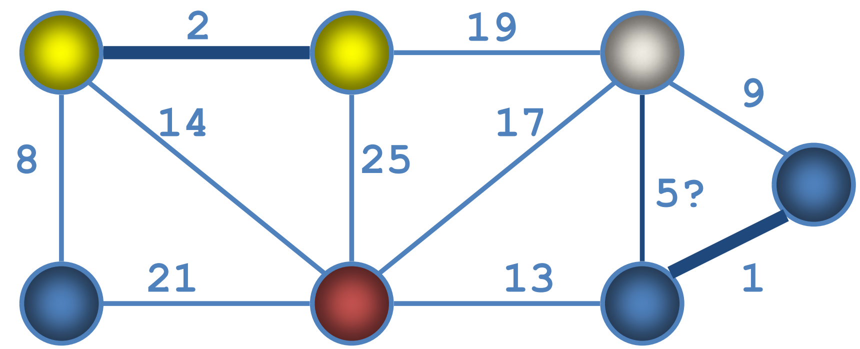
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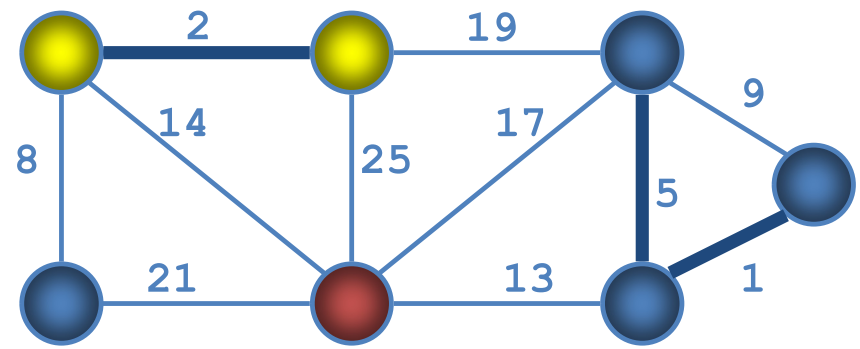
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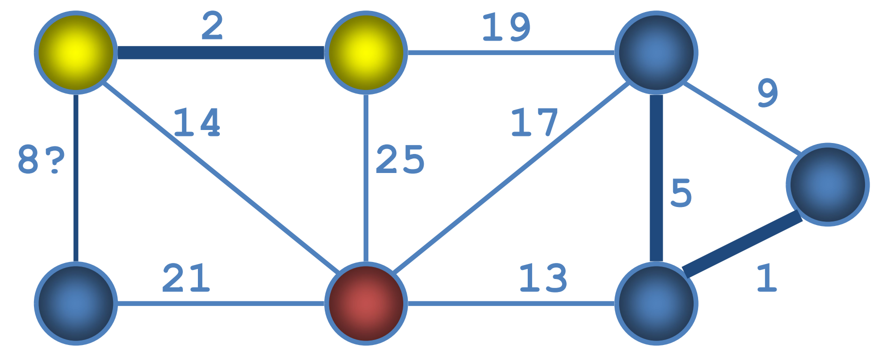
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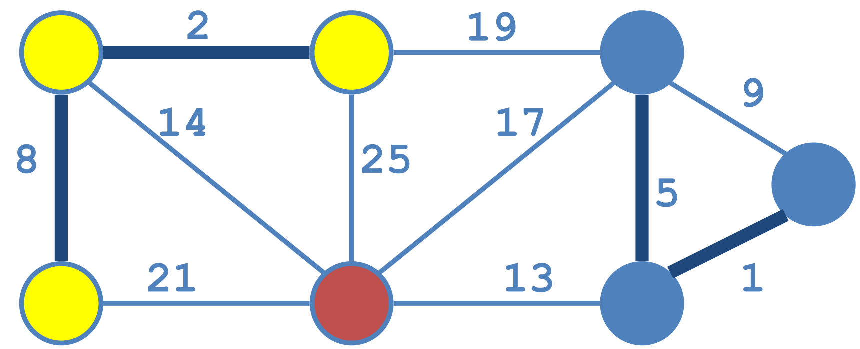
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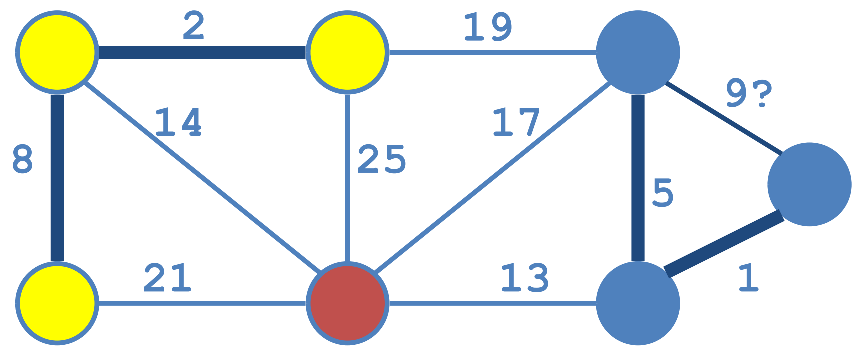

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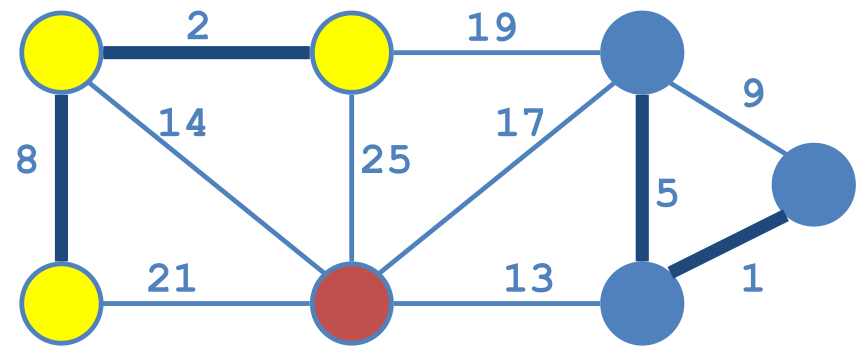
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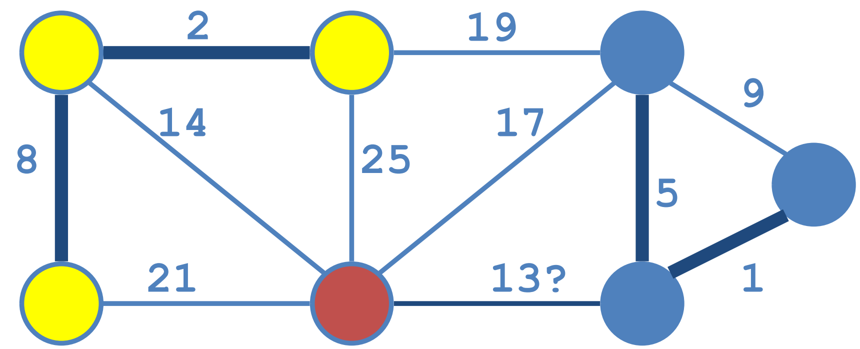

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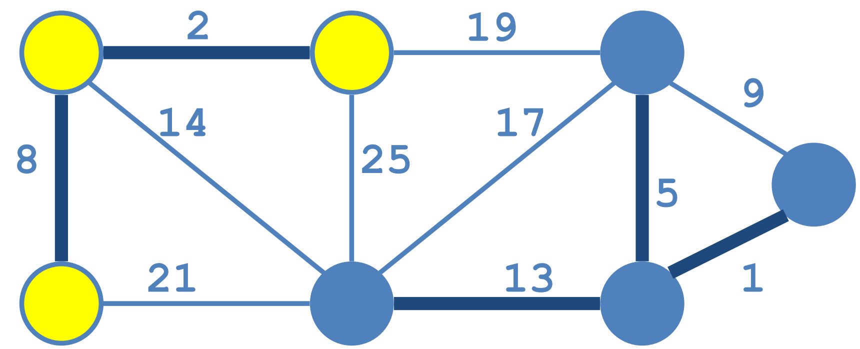
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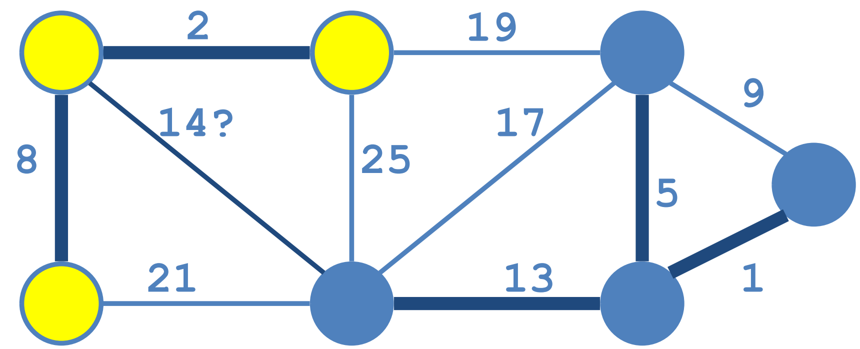
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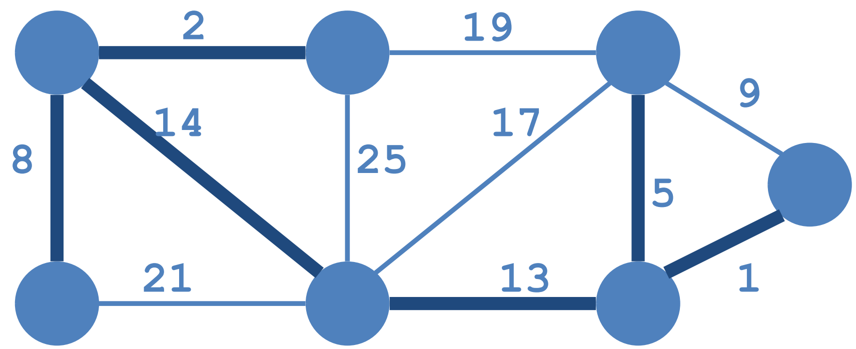

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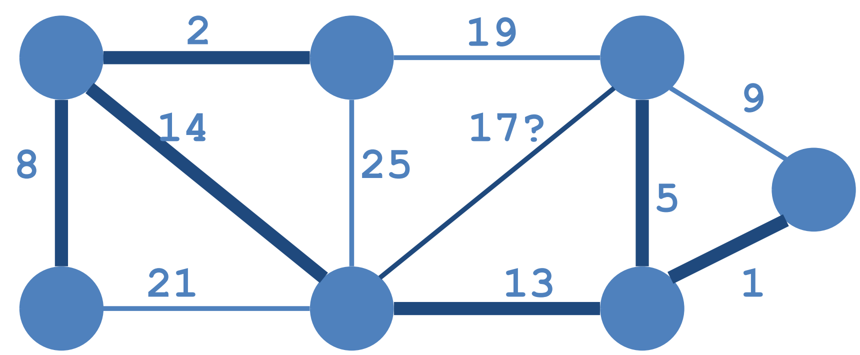
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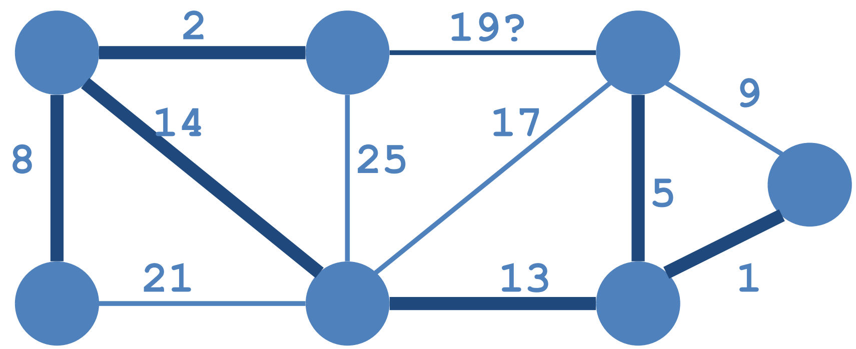
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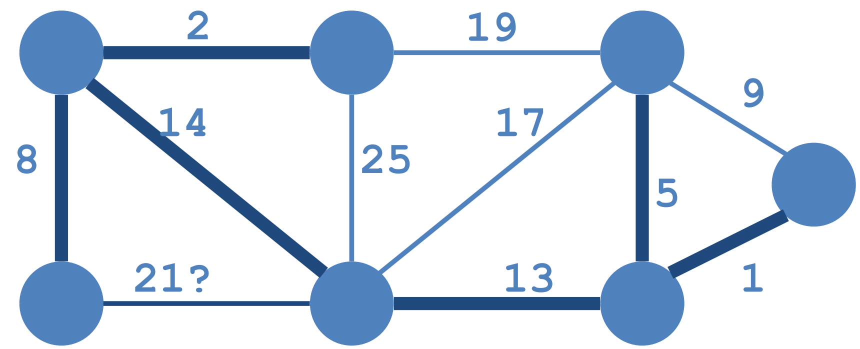
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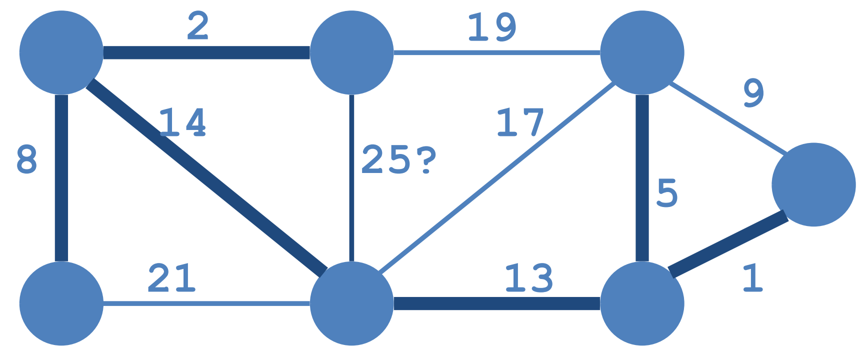
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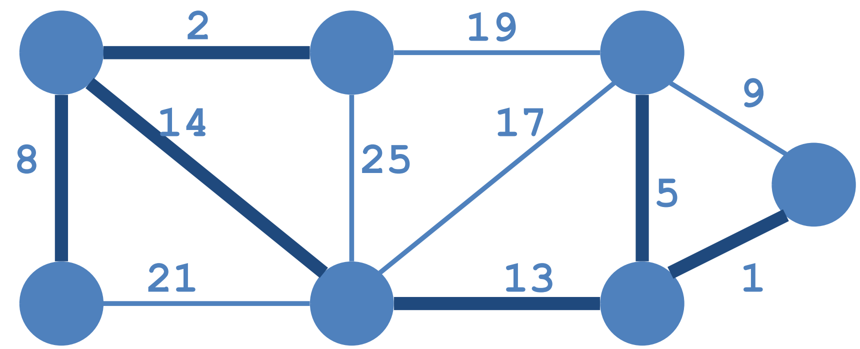
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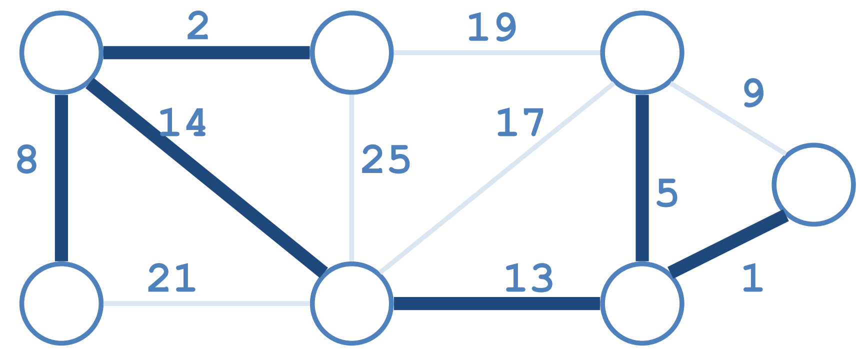
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Kruskal's Algorithm: Running Time

- To summarize:
  - Sort edges: O(E lg E)
  - O(V) for Make-Set()
  - O(E) for Find-Set()
  - O(E) for Union()
  - overall time: **O(E lg E)**

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

Reference

Chap-23, Introduction to Algorithms (3rd Ed.) by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein

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