CS590/CPE590

Graph Algorithms | DFS

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Graph Searching

- Given: a graph G = (V, E), directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree

- There are two standard graph traversal techniques:
  - Breadth-First Search (BFS)
  - **Depth-First Search (DFS)**

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
  - When all of *v*'s edges have been explored, backtrack to the vertex from which *v* was discovered

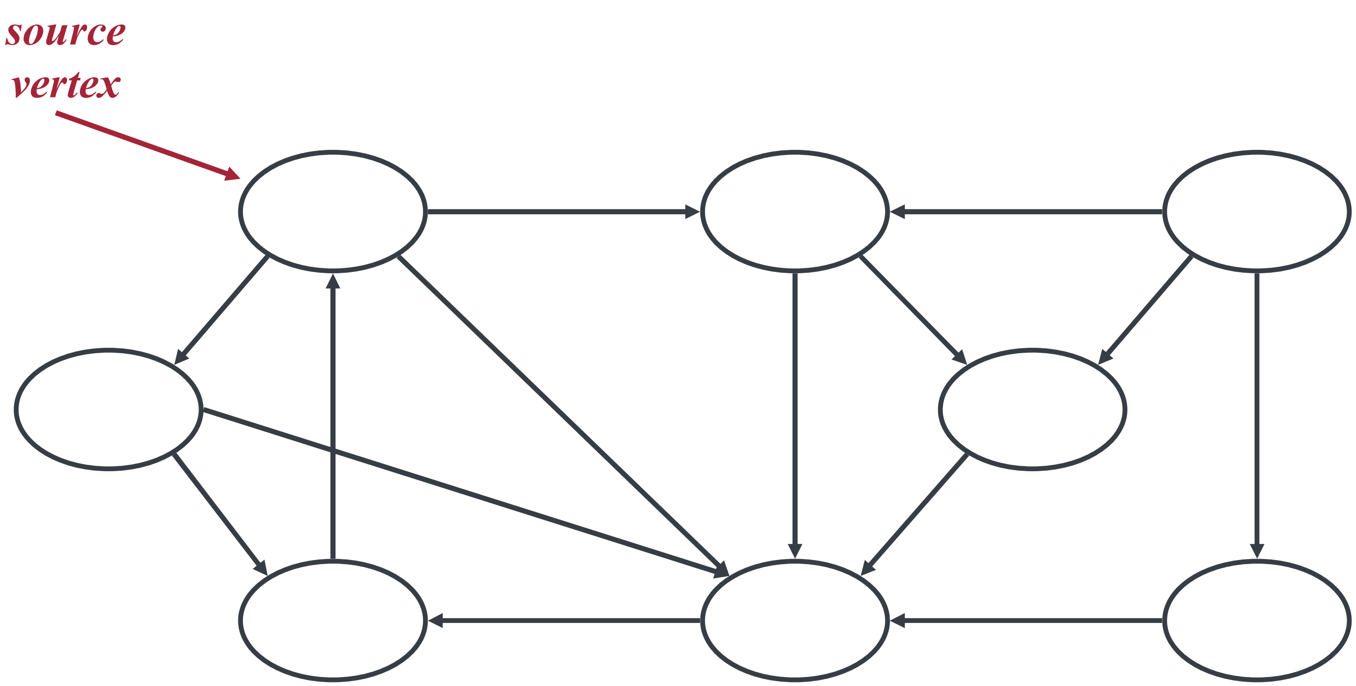
- Vertices initially colored white
- Then colored grey when discovered
- Then black when finished

Depth-First Search: The Code

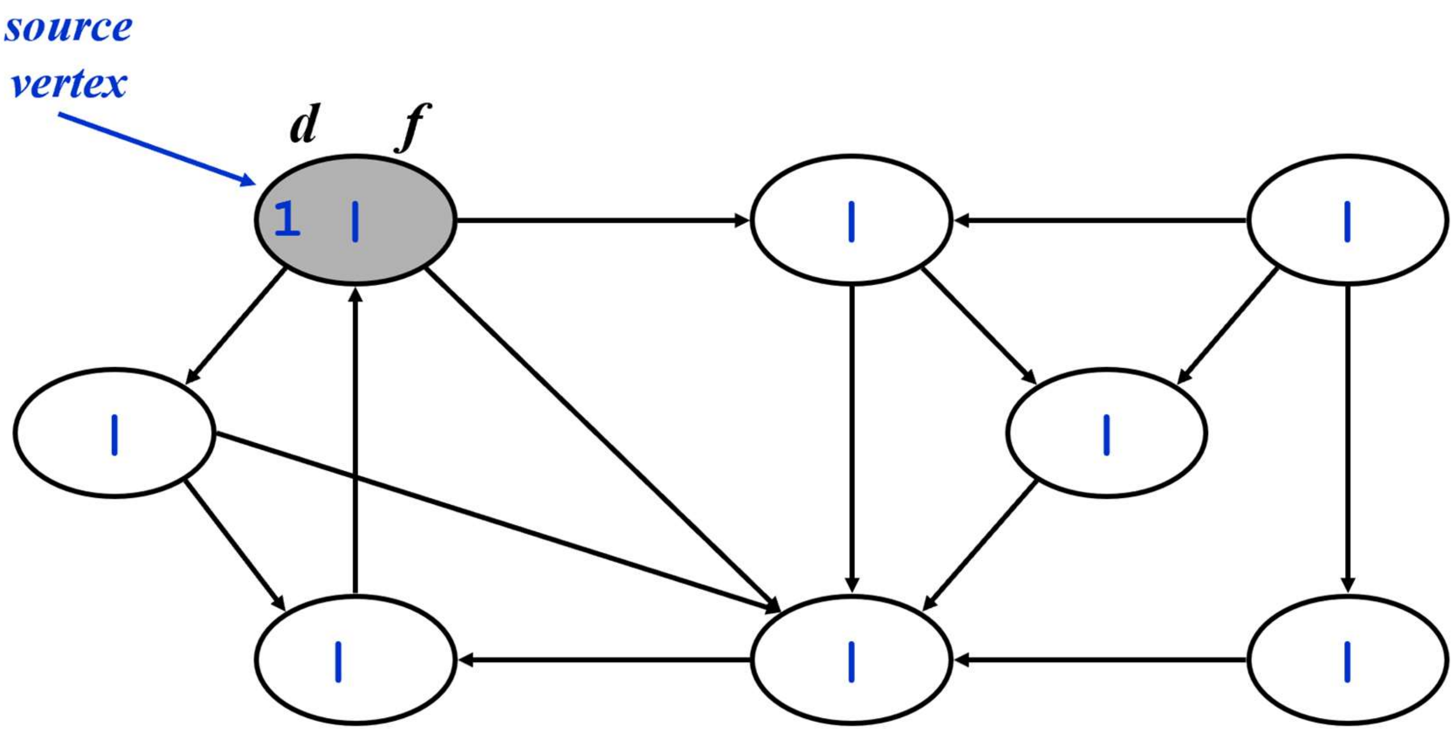
```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

DFS Example



DFS Example



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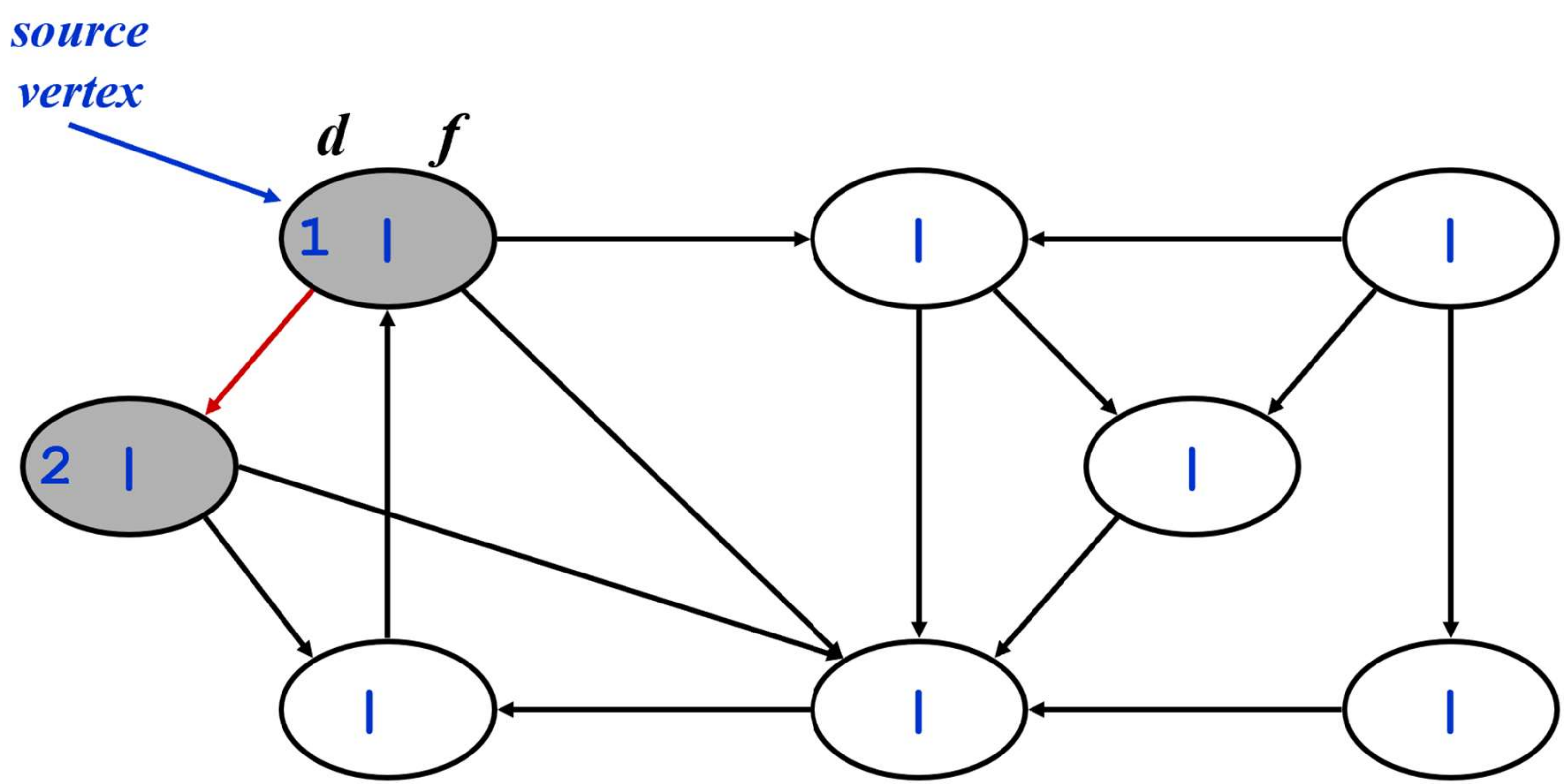

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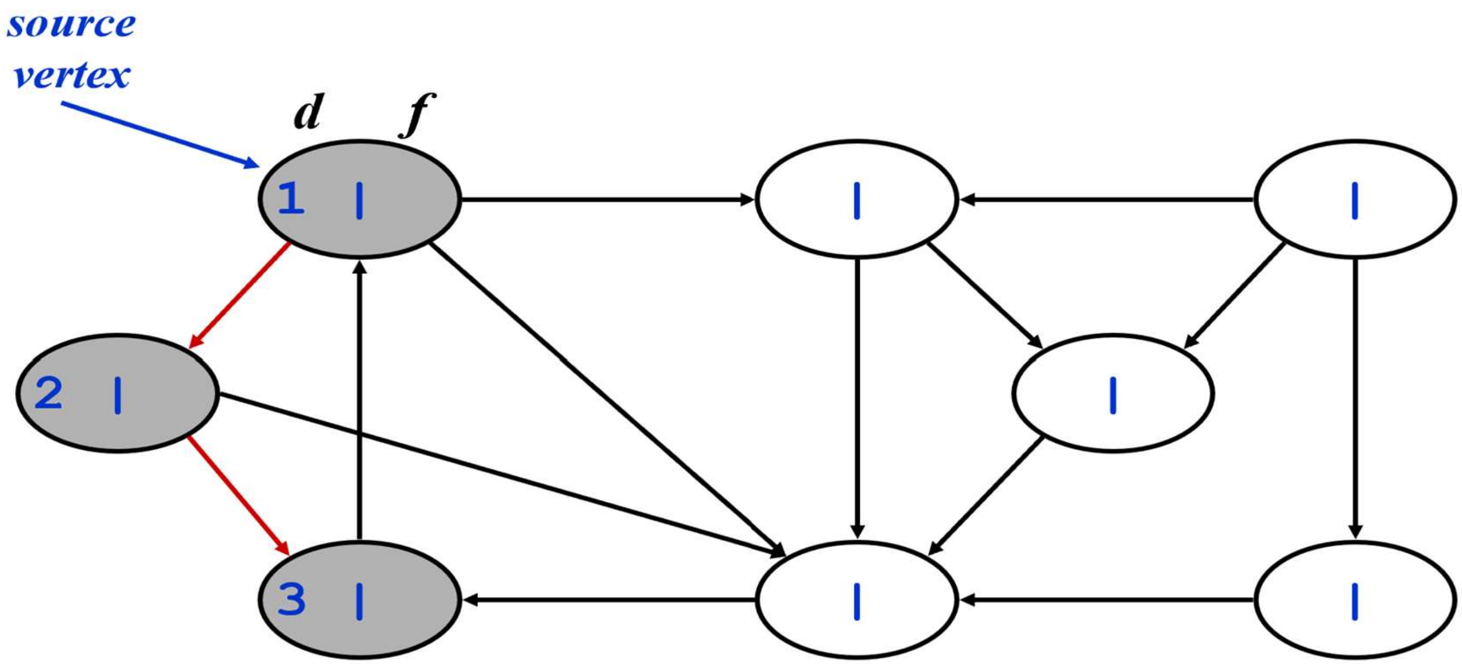

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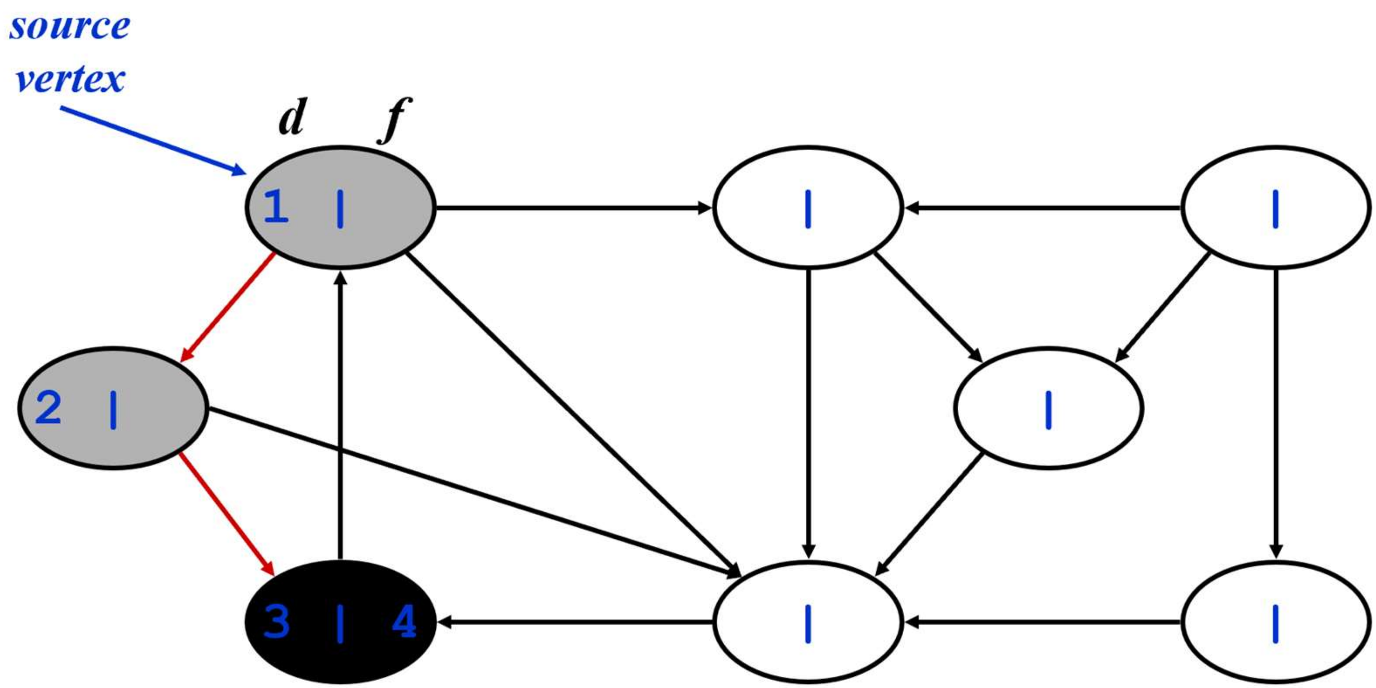

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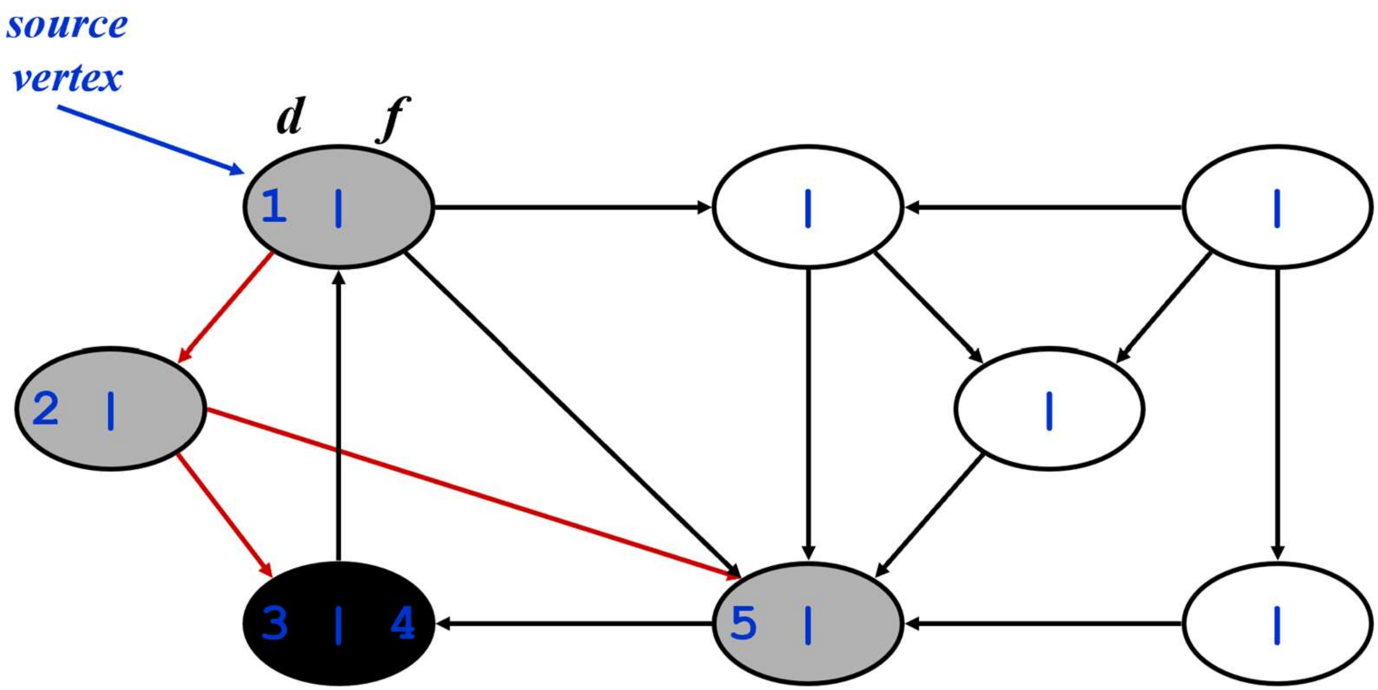

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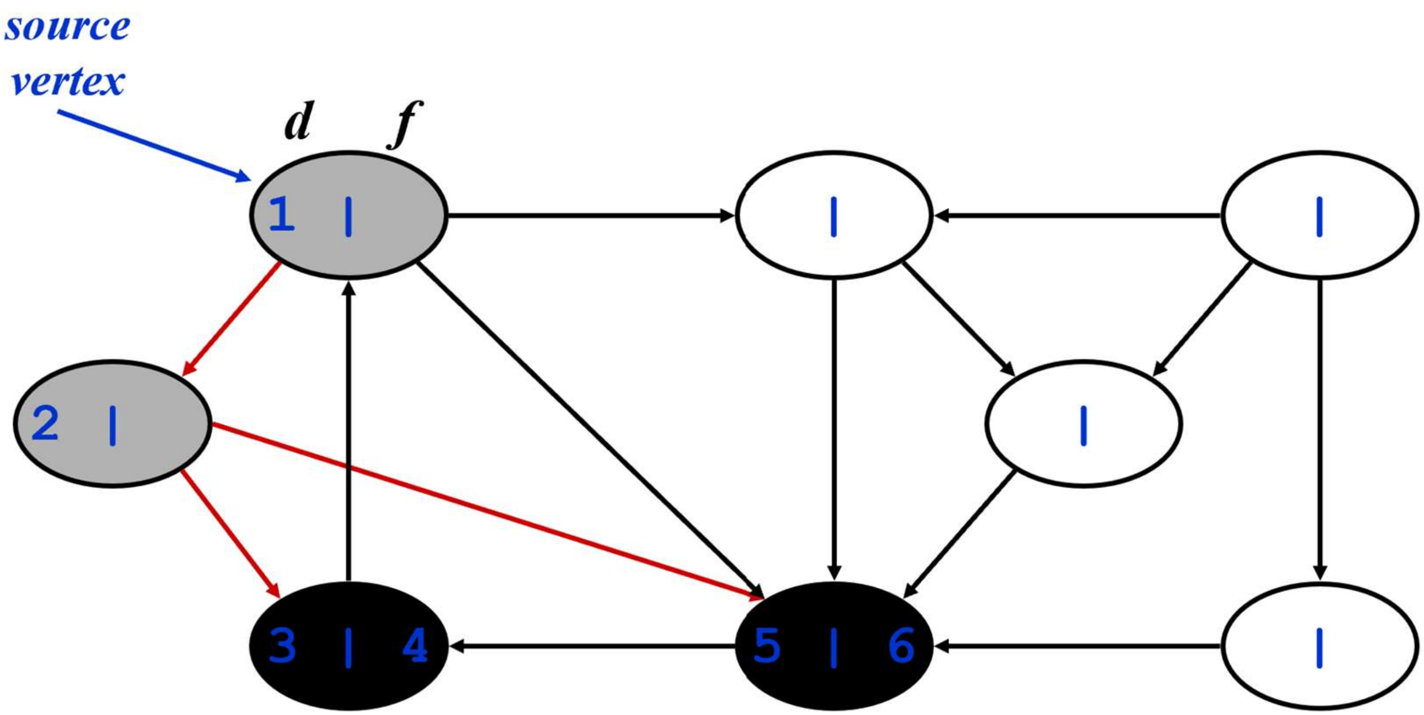

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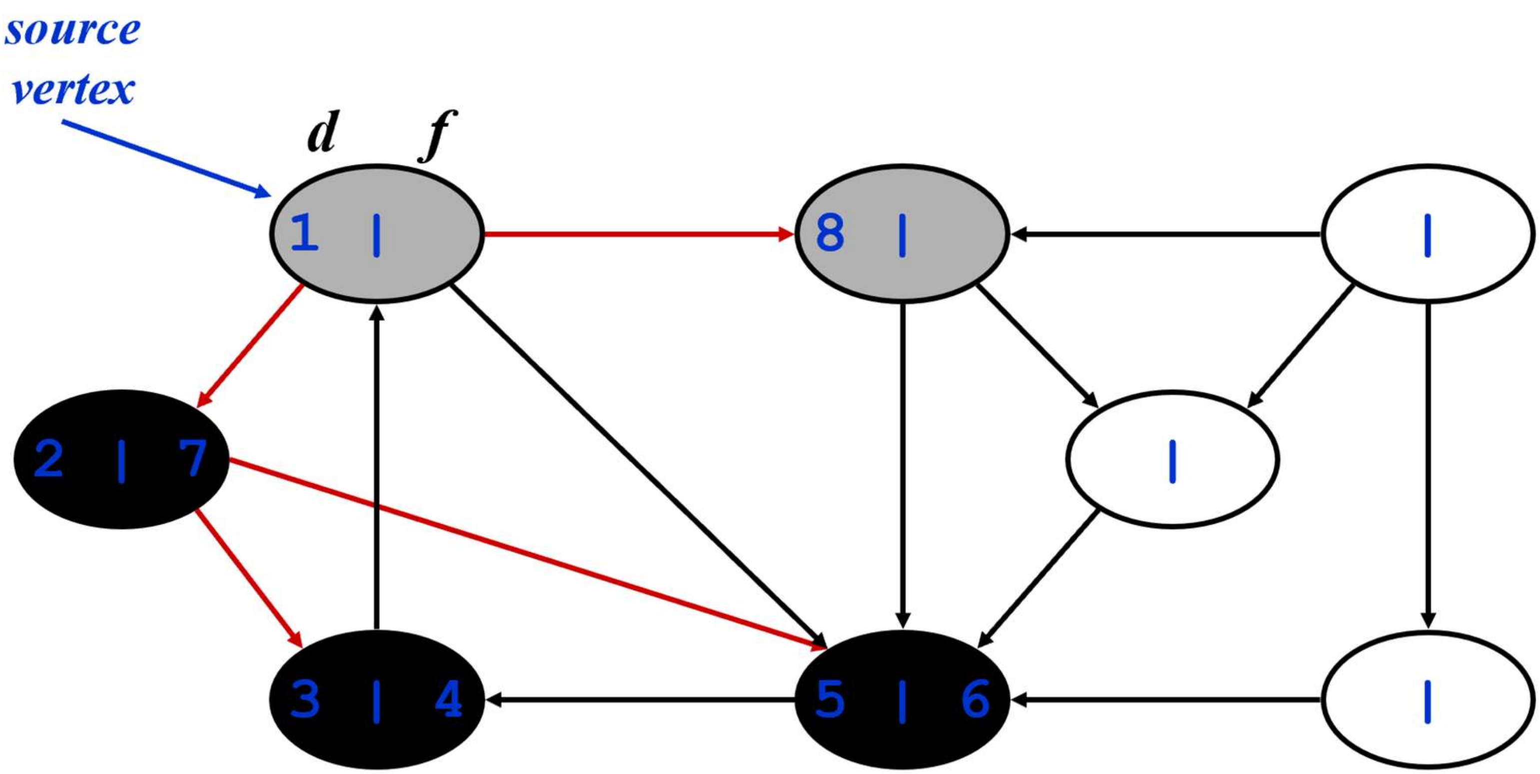

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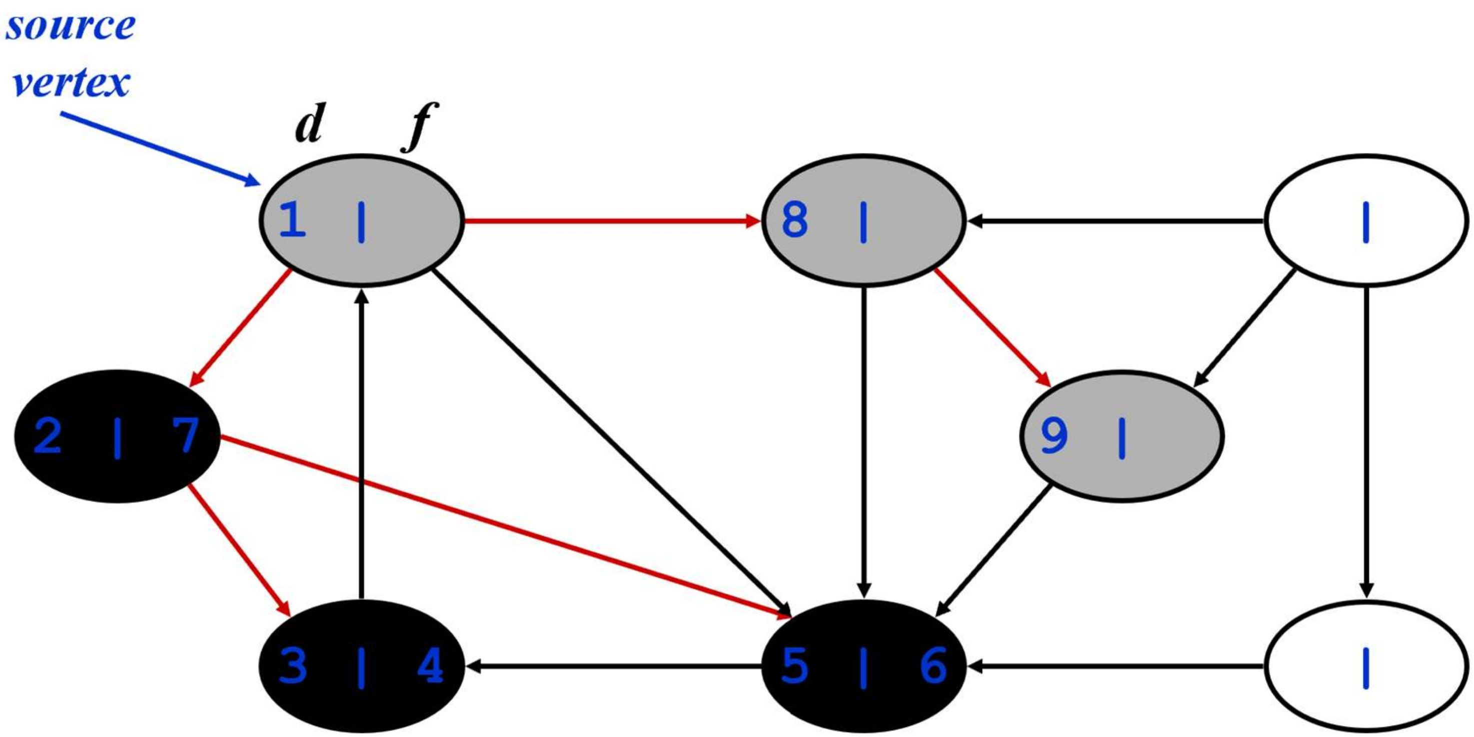

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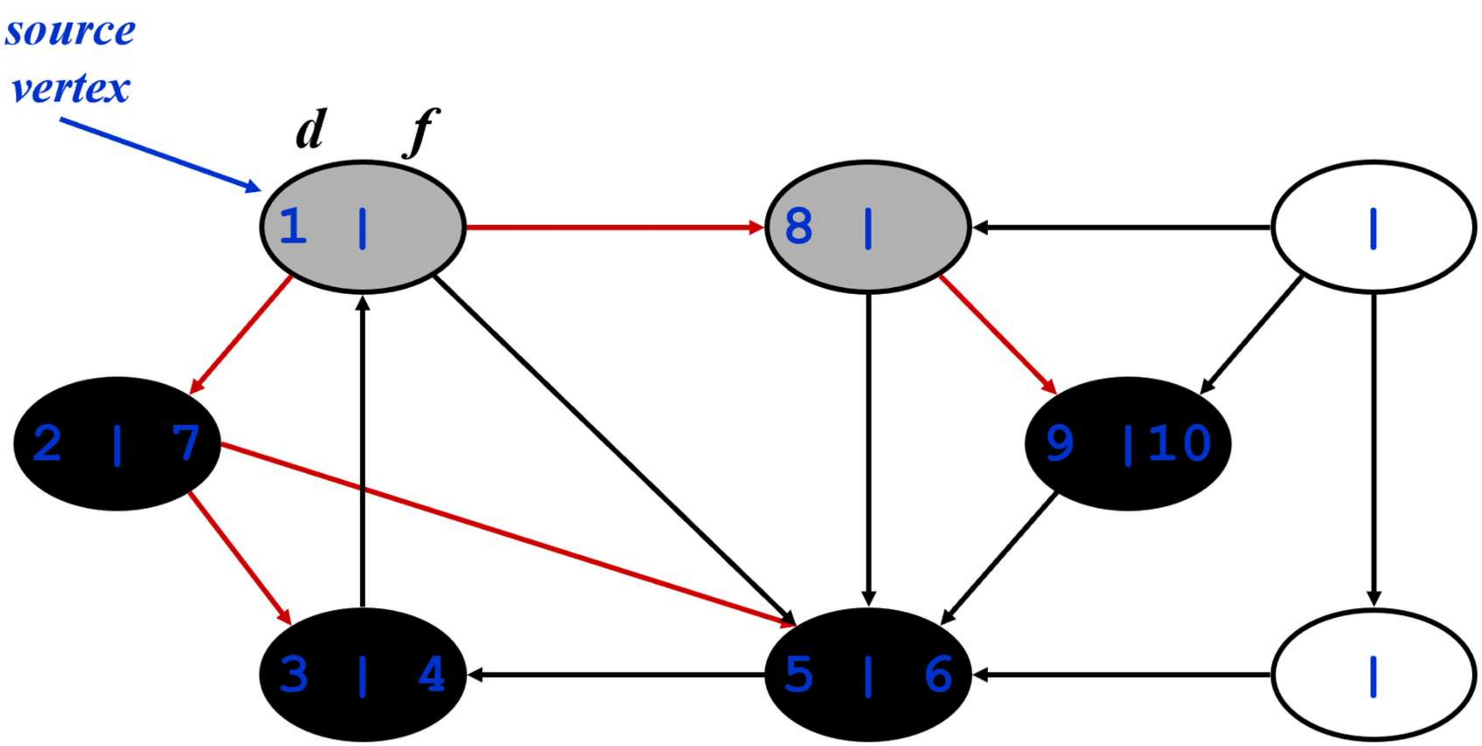

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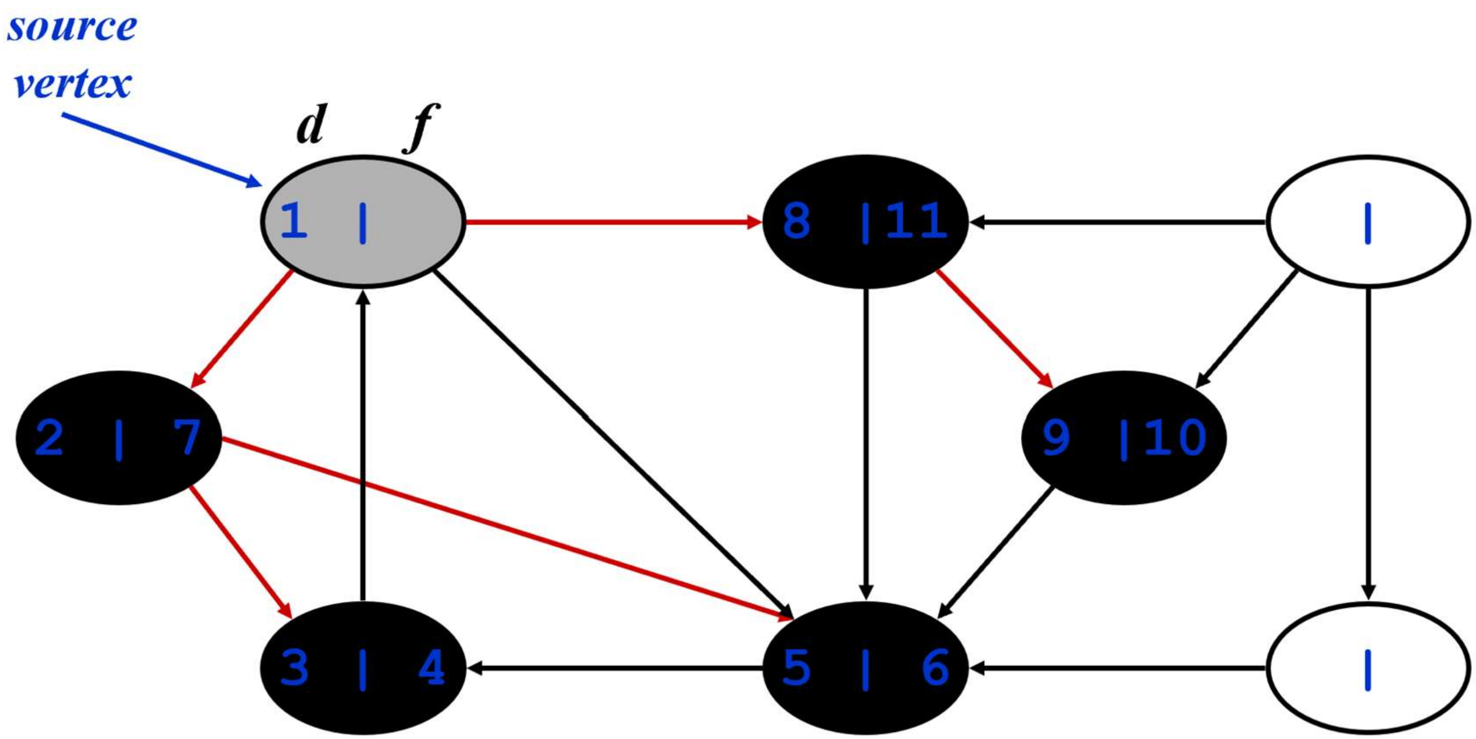

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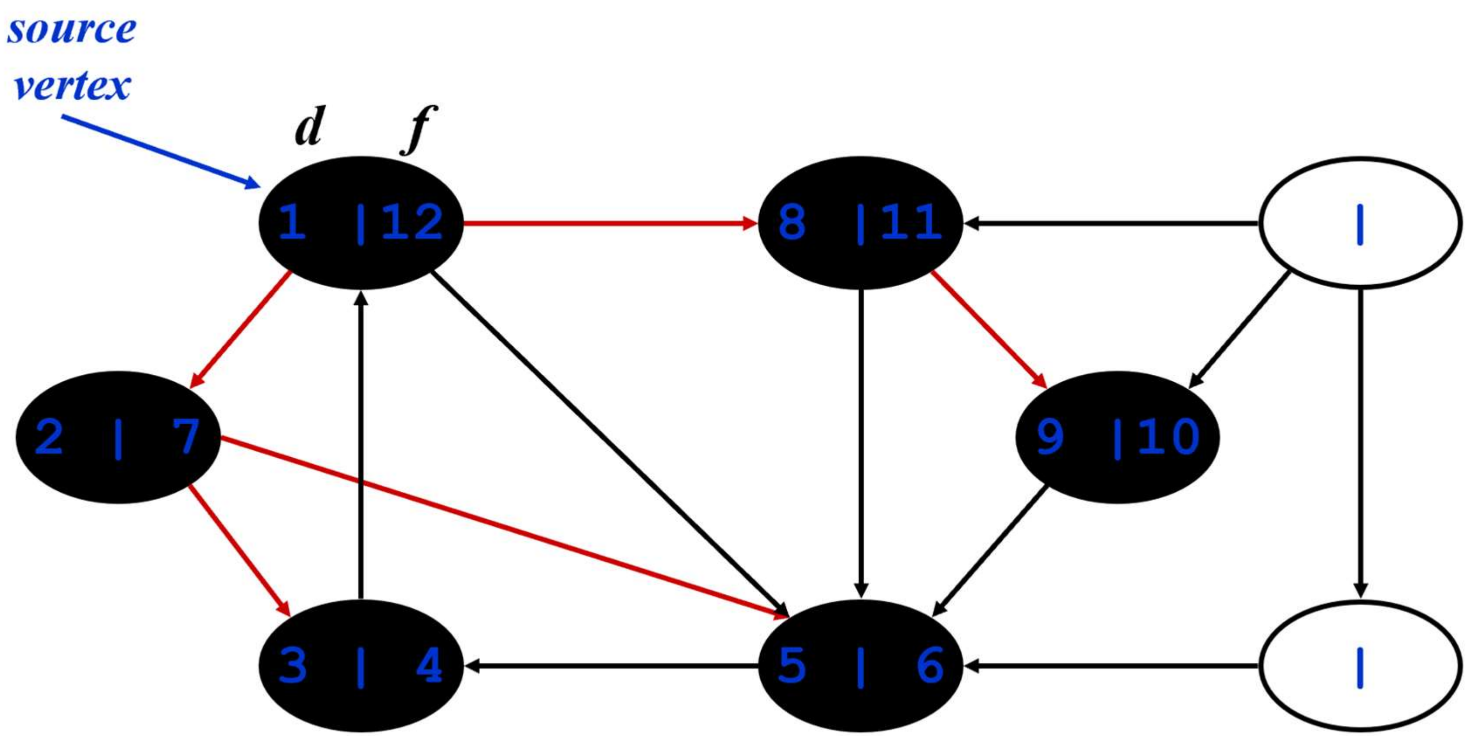

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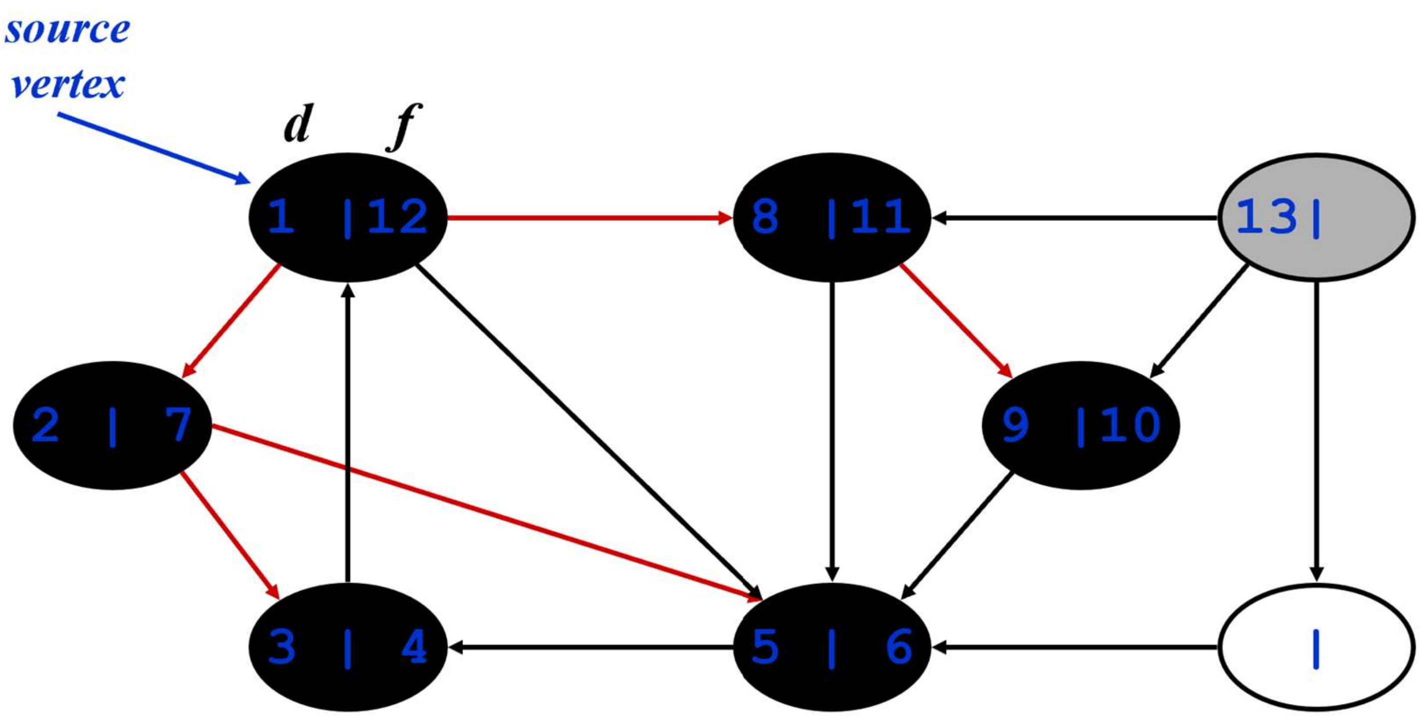

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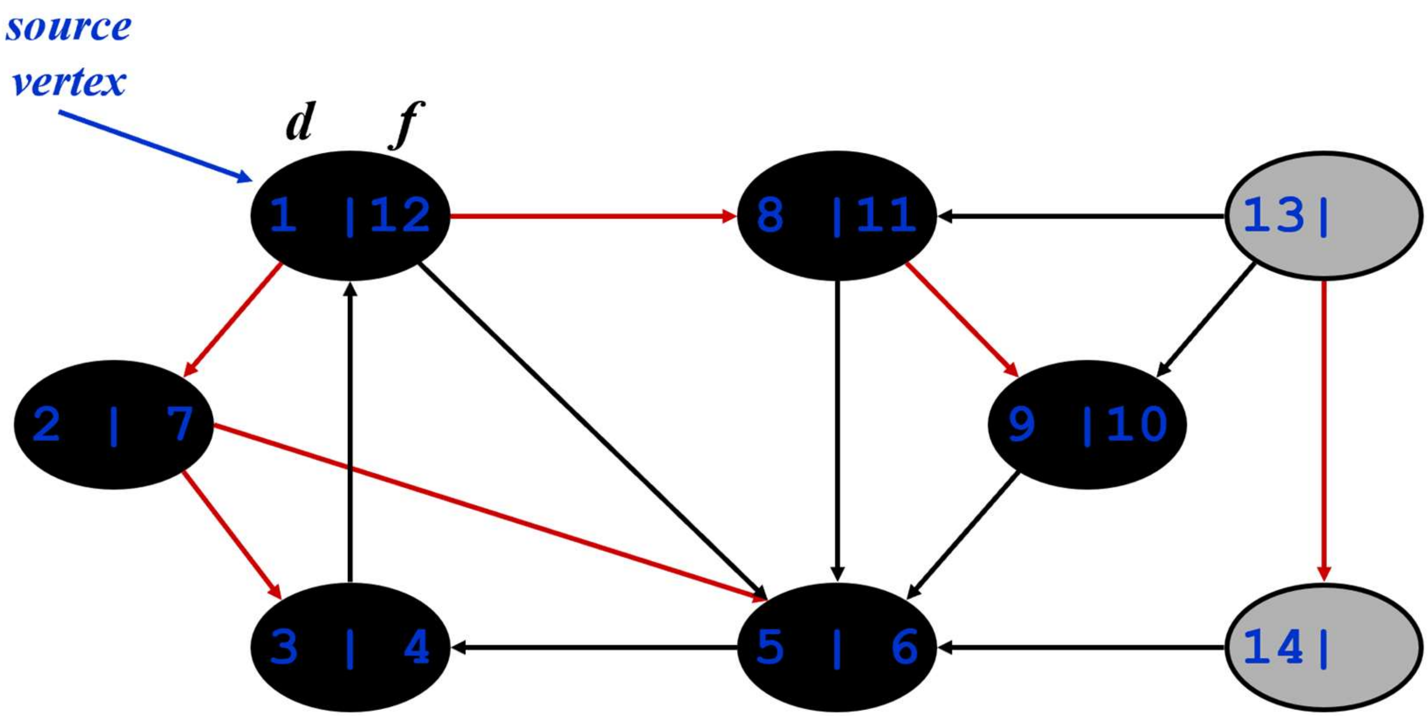

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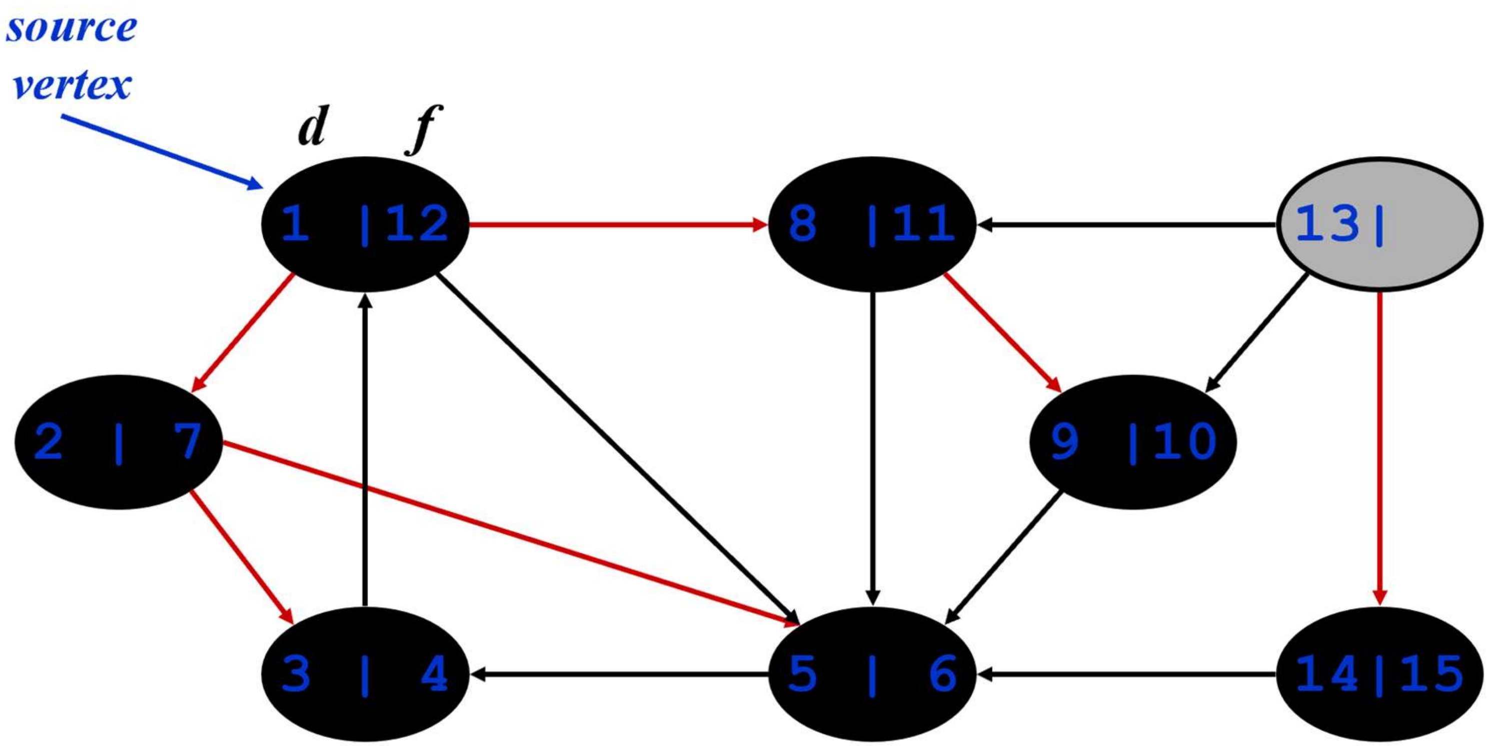

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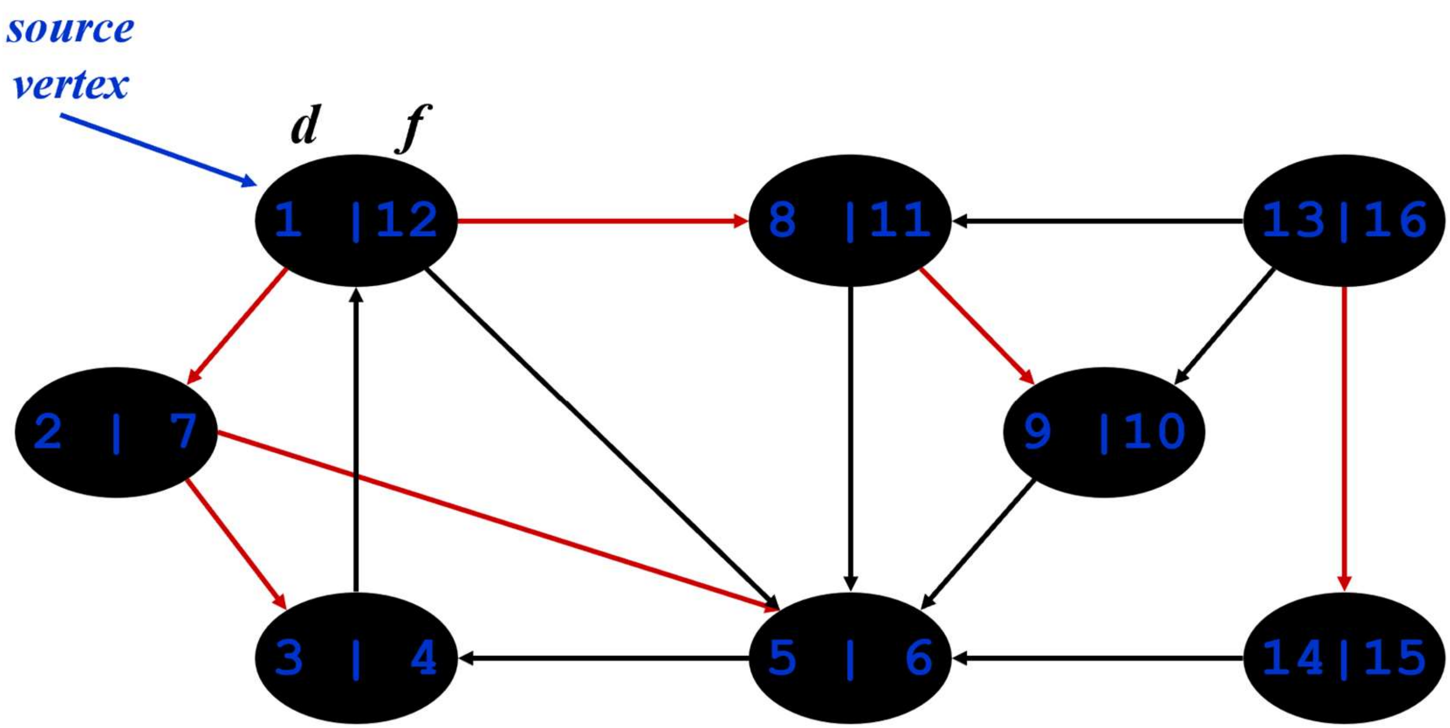


Depth-First Search Analysis

- This running time argument is an example of informal analysis
  - Consider the exploration of edge to the edge:
    - Each loop in **DFS_Visit** can be attributed to an edge in the graph
    - Runs once/edge if directed graph, twice if undirected
    - Thus, loop will run in $\theta(E)$ time, algorithm $\theta(V + E)$

DFS: Kinds of edges

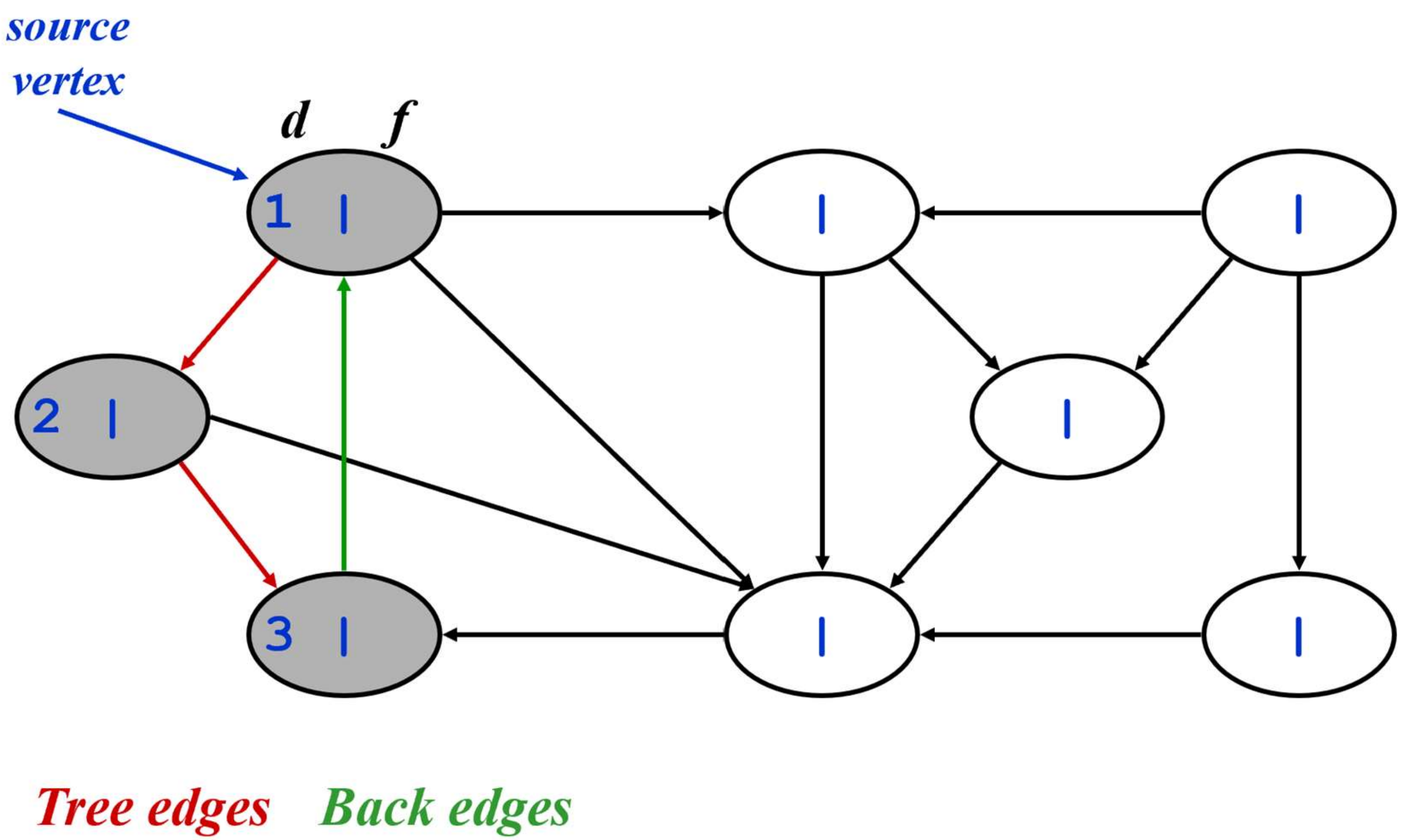
- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex

    -- Can tree edges form cycles? Why or why not?

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
    - Encounter a grey vertex (grey to grey)

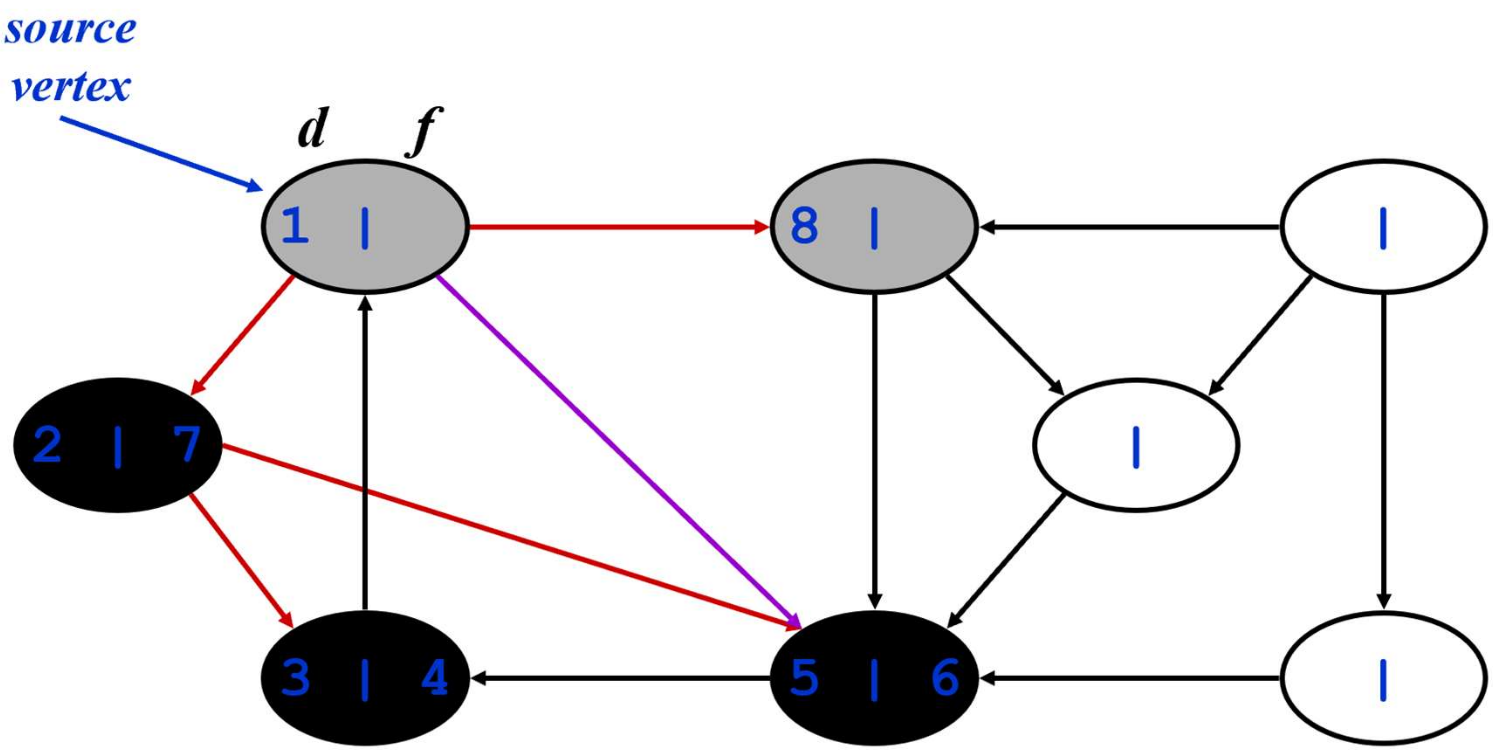
DFS: Kinds of edges



DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
    - Not a tree edge, though
    - From grey node to black node

DFS: Kinds of edges

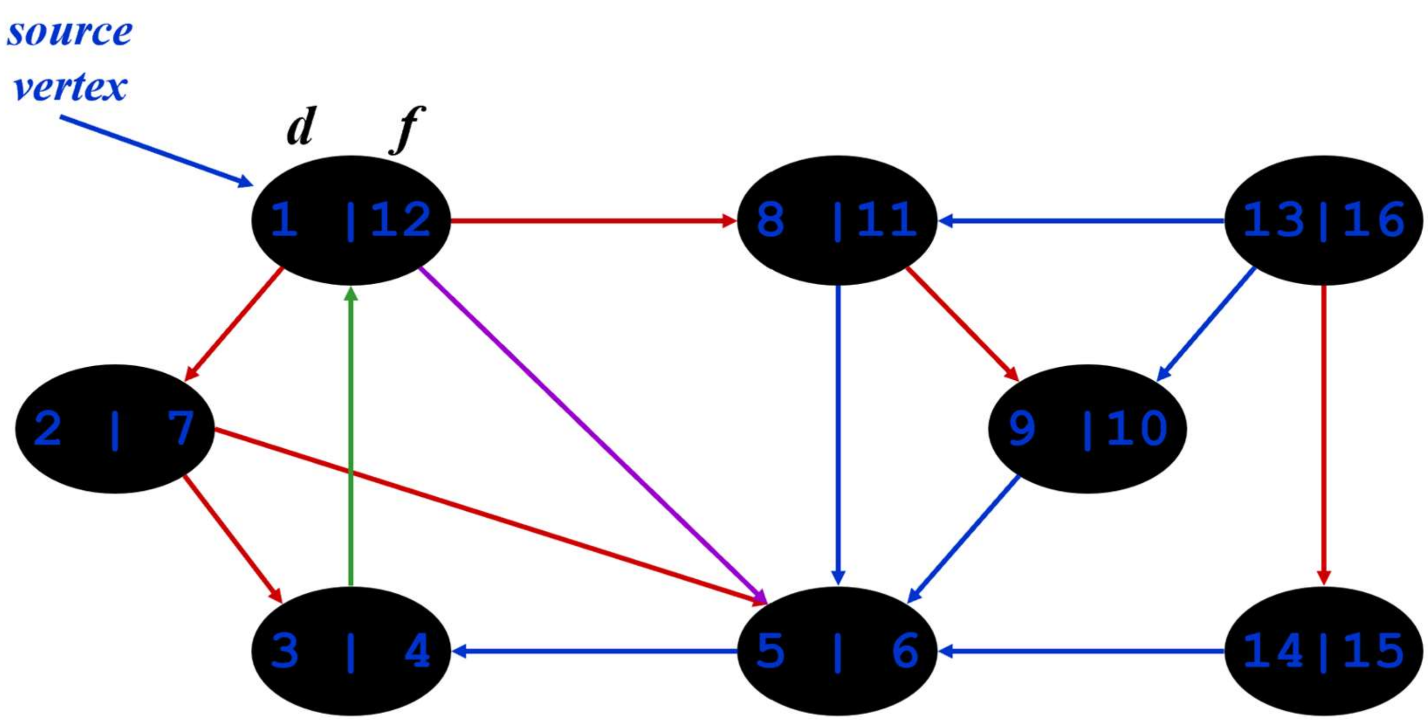


Tree edges *Back edges* *Forward edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
    - From a grey node to a black node

DFS: Kinds of edges



Tree edges *Back edges* *Forward edges* *Cross edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees

- Note: tree and back edges are very important
  -- some algorithms use forward and cross edges

DFS: Kinds of edges

- In a DFS of an undirected graph G, every edge is either a tree edge or a back edge.

  Proof: Theorem 22.10

- A directed graph G is *acyclic* if and only if a DFS of G yields no back edges.

  Proof: Theorem 22.11

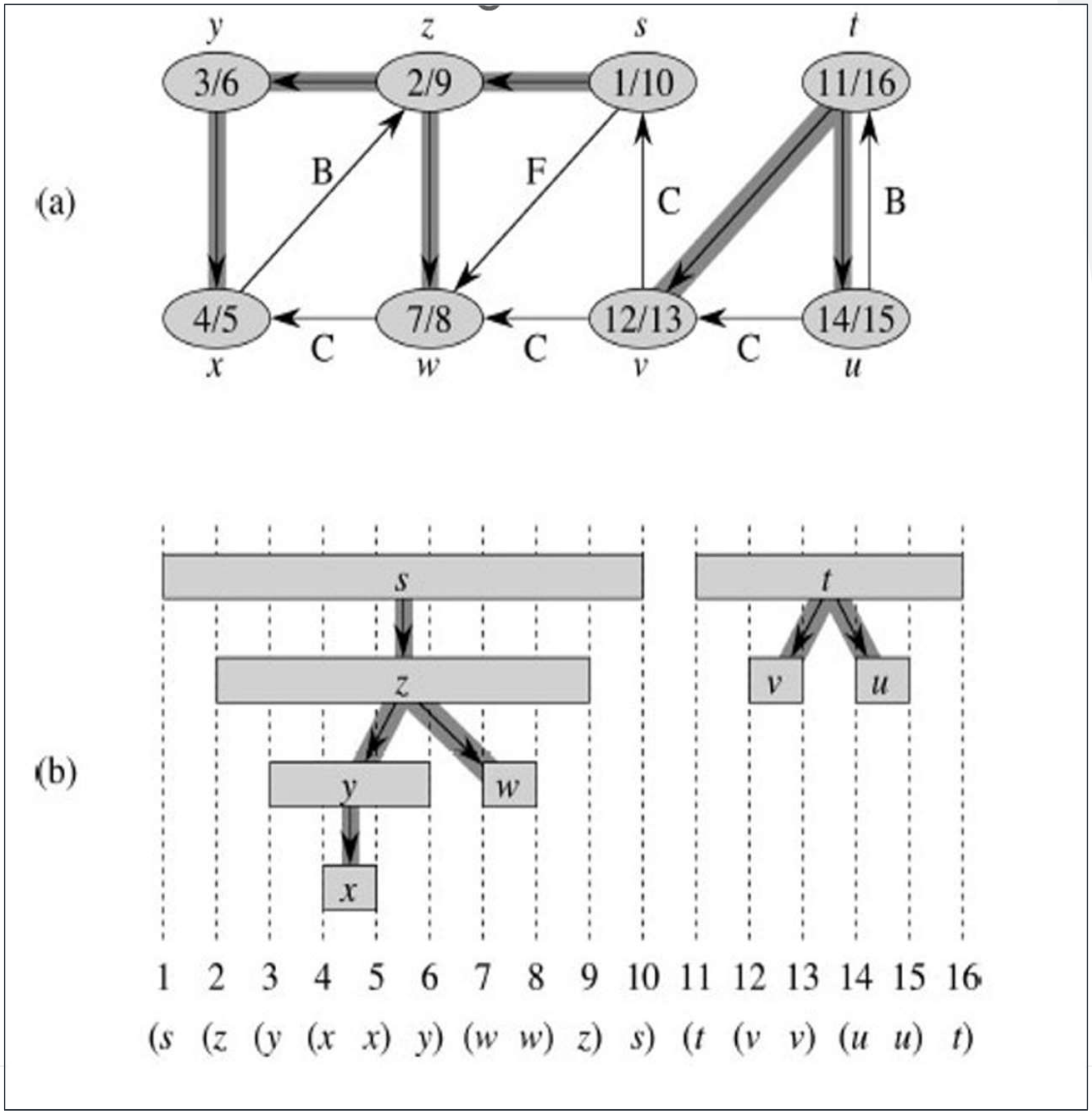
- DFS can be utilized to find cycles

Properties of DFS: Parenthesis Structure

- The discovery and finishing times of vertices have parenthesis structure.

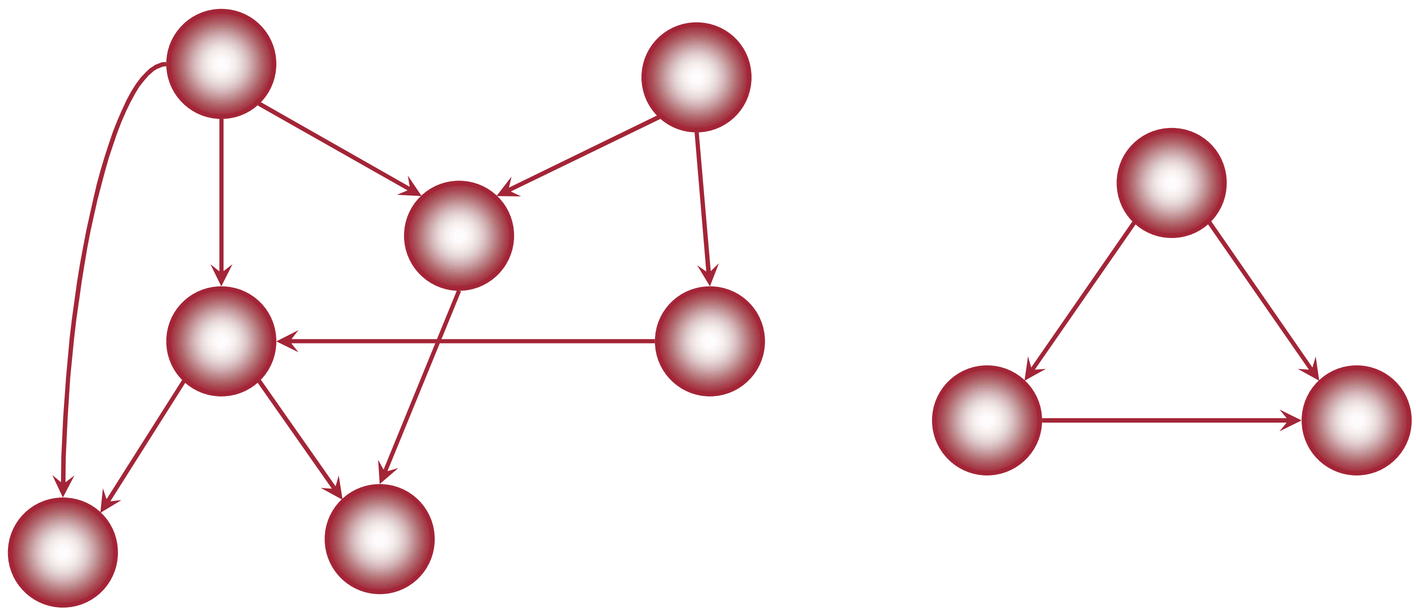
- In any DFS of a graph G, for any two vertices u and v, one of the following three conditions holds: [Theorem 22.7]
  - The intervals $[d(u), f(u)]$ and $[d(v), f(v)]$ are entirely disjoint, and neither u nor v is a descendent of the other in the DFS forest,
  - The interval $[d(u), f(u)]$ is contained entirely within the interval $[d(v), f(v)]$, and u is a descendent of v in a DFS tree, or
  - The interval $[d(v), f(v)]$ is contained entirely within the interval $[d(u), f(u)]$, and v is a descendent of u in a DFS tree.

Properties of DFS: Parenthesis Structure



Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles



Topological Sort

- A *topological sort* of a DAG is
  - A linear ordering of all vertices of the graph G such that vertex u comes before vertex v if (u, v) is an edge in G.

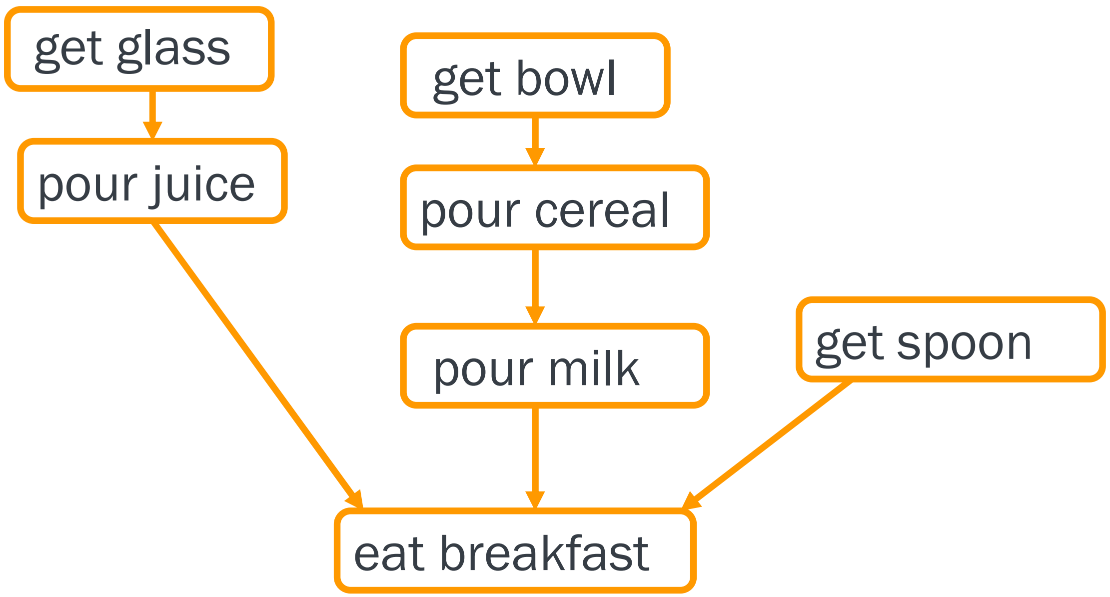
- DAG indicates precedence among events:
  - Events are graph vertices, edge from u to v means event u has precedence over event v

- Real-world example:
  - Getting dressed
  - Course registration
  - Tasks for eating meal

Precedence Example

- Tasks that have to be done to eat breakfast:
  - get glass, pour juice, get bowl, pour cereal, pour milk, get spoon, eat.
- Certain events must happen in a certain order (ex: get bowl before pouring milk)
- For other events, it doesn't matter (ex: get bowl and get spoon)

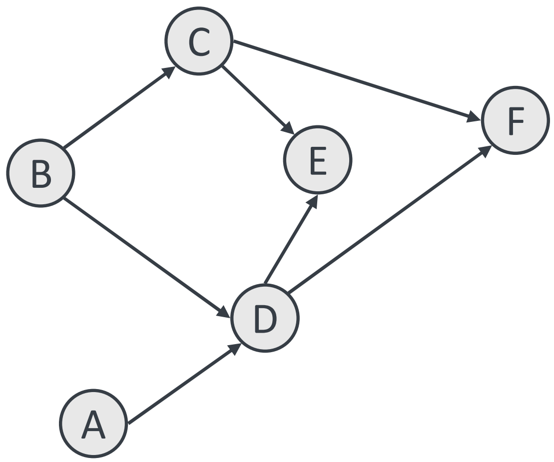
Precedence Example



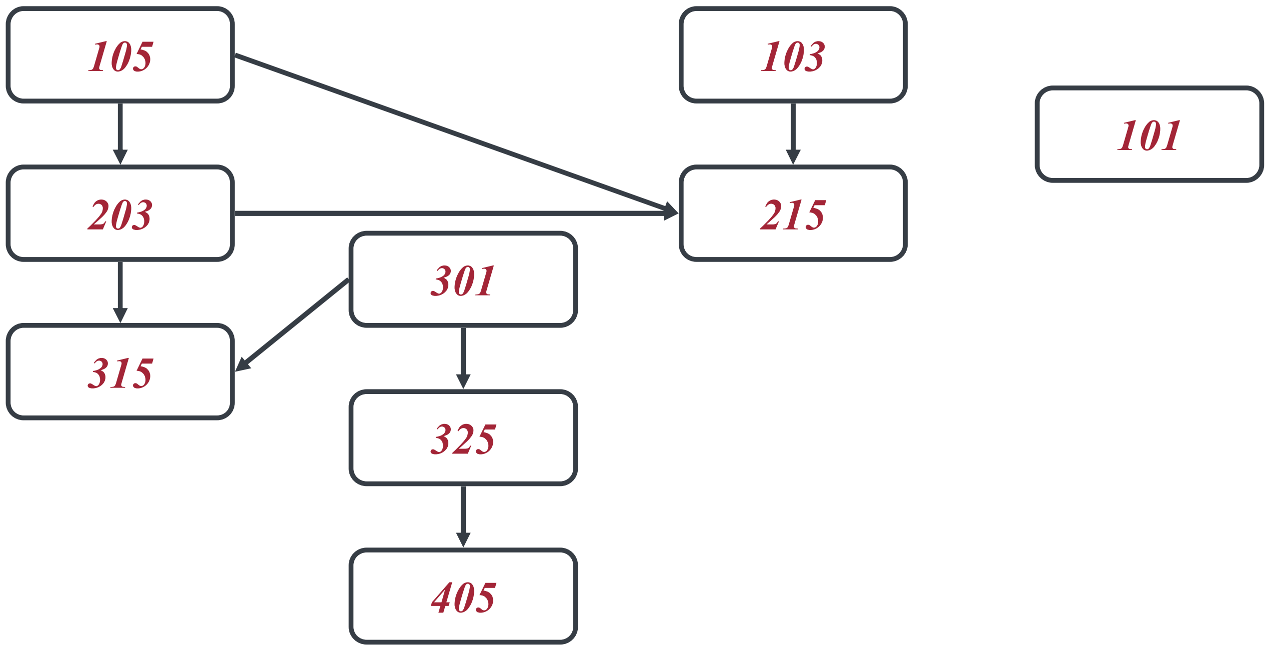
Order: glass, juice, bowl, cereal, milk, spoon, eat.

More Example …

- How many valid topological sort orderings can you find for the vertices in the graph below?
  - [A, B, C, D, E, F]
  - [A, B, C, D, F, E]
  - [A, B, D, C, E, F]
  - [A, B, D, C, F, E]
  - [B, A, C, D, E, F]
  - [B, A, C, D, F, E]
  - [B, A, D, C, E, F]
  - [B, A, D, C, F, E]
  - [B, C, A, D, E, F]
  - [B, C, A, D, F, E]
  - ....



Another Example: Course Registration



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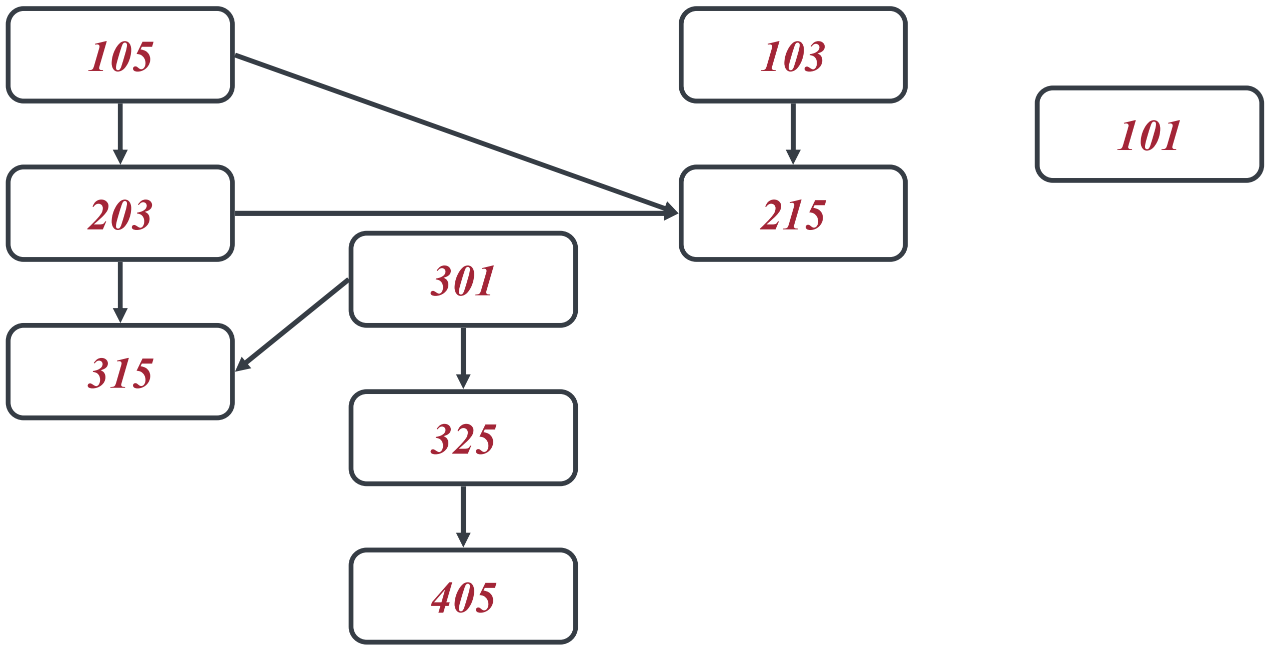

Why Acyclic?

- Why must directed graph be acyclic for the topological sort problem?
- Otherwise, no way to order events linearly without violating a precedence constraint.

Topological Sort Algorithm

TOPOLOGICAL-SORT(G)

1 call DFS(G) to compute finishing times $v.f$ for each vertex v
2 as each vertex is finished, insert it onto the front of a linked list
3 **return** the linked list of vertices

- Time Complexity: $\theta(V + E)$

Contents of this presentation are based on
Book Chapter- 22, Introduction to Algorithms by *Cormen, Leiserson, Rivest,* & *Stein*

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THANK **YOU**

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