Calculating Vladimirov derivatives and a little more

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1 Derivative of the χ character

Calculating Vladimirov derivative of $\chi(k_n y)$ where $k_n = p^{-n}$:

$$\begin{aligned}
D_y^s \chi(k_n y) &= \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-n}y)}{|x-y|_p^{1+s}} \\
&= \frac{1}{\Gamma_p(-s)} \left(\int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}(u+y))}{|u|_p^{1+s}} - \int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}y)}{|u|_p^{1+s}} \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(\int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}u)}{|u|_p^{1+s}} - \int\limits_{\mathbb{Q}_p} du \frac{1}{|u|_p^{1+s}} \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(-p^{(n-1)(s+1)-n} + \sum_n^\infty p^{ls}(1-\frac{1}{p}) - \sum_{-\infty}^{\infty} p^{ls}(1-\frac{1}{p}) \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(p^{ns}\Gamma_p(-s) \right) = p^{ns}\chi(k_n y)
\end{aligned} \tag{1}$$

We notice that two ways of regulating the integral are equivalent:

- Including $(-\chi(p^{-n}y))$ in the numerator;
- "Analytic continuation" of $\Gamma_p(-s) = \int\limits_{\mathbb{Q}_p} \frac{dx}{|x|_p} \chi(x)|x|_p^{-s}$ to the region $-s > 0$ by $\Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(1-s)}$.

It is really like defining all the quantities modulo $\sum\limits_{-\infty}^{\infty} p^{ls}$.

2 Check shift invariance of a general p-adic integral

2.1 Integrand depending only on the p-adic norm

Define a general integral over p-adic number field as:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} dx f(p^{-l}) = \sum_{l=-\infty}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l} \tag{2}$$

A shift of the integral variable gives:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \int_{\mathbb{Q}_p} d(u-y) f(|u-y|_p) \quad \text{where} \quad u = x+y \tag{3}$$

Suppose that $d(u-y) = du$ when y is constant. Then we can proceed by splitting $\mathbb{Q}_p$:

$$\int_{\mathbb{Q}_p} d(u) f(|u-y|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u-y|_p) \tag{4}$$

Note that we cannot write $|u-y|_p$ as p^{-l} in each block any more. Suppose that $y = p^m(a_0 + a_1 p + a_2 p^2 + ...)$, we have:

$$\begin{aligned}\sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u-y|_p) &= \sum_{l=-\infty}^{m-1} \int_{p^l \mathbb{U}_p} du f(p^{-l}) + \sum_{l=m+1}^{\infty} \int_{p^l \mathbb{U}_p} du f(p^{-m}) + (l=m) \text{ term} \\ &= \sum_{l=-\infty}^{m-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + f(p^{-m})p^{-(m+1)} + (l=m) \text{ term}\end{aligned} \tag{5}$$

This "contact term" can be written as:

$$\begin{aligned}\int_{p^m \mathbb{U}_p} du f(|u-y|_p) &= \sum_{b_0=1,\neq a_0}^{p-1} \int_{p^{m+1}\mathbb{Z}_p} du f(p^{-m}) + (a_0 = b_0) \text{ term} \\ &= f(p^{-m})(p-2)p^{-(m+1)} + \sum_{b_1=0,\neq a_1}^{p-1} \int_{p^{m+2}\mathbb{Z}_p} du f(p^{-(m+1)}) + (a_1 = b_1) \text{ term} \\ &= f(p^{-m})(p-2)p^{-(m+1)} + \sum_{l=m+1}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l}\end{aligned} \tag{6}$$

In the end, all these terms add up to the integral after shift:

$$f(p^{-m})p^{-(m+1)}(p-1) + \sum_{l=-\infty}^{m-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + \sum_{l=m+1}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l} \tag{7}$$

This is exactly $\sum_{l=-\infty}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l}$, the integral before shifting.

2.2 Integrand depending also on p-adic digits

3 Useful Fourier(and maybe Mellin) transforms

A most common one is the Fourier transform of $|k|_p^s$ or a Mellin transform of $\chi(kx)$:

$$\int_{\mathbb{Q}_p} dk \chi(kx)|k|_p^s = \Gamma_p(s+1)\frac{1}{|x|_p^{s+1}}, \tag{8}$$

recalling that the definition of Γ_p is:

$$\Gamma_p(s) = \int_{\mathbb{Q}_p} \frac{du}{|u|}\chi(u)|u|^s. \tag{9}$$

A slight variation is:

$$\int_{\mathbb{Q}_p} dk \chi(kx)|k|_p^s \left(\frac{k}{p}\right)_L = up^{s+\frac{1}{2}}\frac{\left(\frac{x}{p}\right)_L}{|x|_p^{s+1}}, \tag{10}$$

where $\left(\frac{k}{p}\right)_L$ is the Legendre symbol, u equals 1 for $p = 1 \bmod 4$ while i for $p = 3 \bmod 4$. A useful Gauss sum here is

$$\sum_{\alpha \in \mathbb{F}_p} \left(\frac{k}{p}\right)_L \chi\left(-\frac{k_0\alpha}{p}\right) = \frac{\sqrt{p}}{u}\left(\frac{k_0}{p}\right)_L. \tag{11}$$

We want to know about the Fourier transform of the characteristic function itself:

$$\int_{\mathbb{Q}_p} dk \chi(kx)\chi(k) = 0. \tag{12}$$

3.1 A simple example of Adelic product relation

We need to know the Mellin transform of $e^{-\pi x^2}$ in $\mathbb{R}$ and $\gamma_0(x)$ the indicator(or characteristic) function of $\mathbb{Z}_p$ in their respective fields:

$$\int_{\mathbb{R}} \frac{dx}{|x|}|x|^s e^{-\pi x^2} = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \equiv \zeta_\infty(s), \quad \int_{\mathbb{Q}_p} \frac{dx}{|x|_p}|x|_p^s \gamma_0(x) = (1-\frac{1}{p})(\frac{1}{1-p^{-s}}) \equiv (1-\frac{1}{p})\zeta_p(s) \tag{13}$$

and the Mellin transform of respective character functions are definitions of the Gelfan-Graev gamma functions:

$$\int_{\mathbb{R}} dx|x|^s e^{2\pi i x} \equiv \Gamma_\infty(s), \quad \int_{\mathbb{Q}_p} \frac{dx}{|x|_p}|x|_p^s \chi(x) \equiv \Gamma_p(s). \tag{14}$$

Interestingly, we have

$$\Gamma_\infty(s) = \frac{\zeta_\infty(s)}{\zeta_\infty(1-s)}, \quad \Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(1-s)}. \tag{15}$$

And remember that the definition of completed Riemann Zeta function $\xi(s)$ is an Adelic Mellin transform of the indicator functions($e^{-\pi x^2}$ is an analogue in Reals?) over Haar measures, which cancels the $(1-\frac{1}{p})$ factor in the integral over $\mathbb{Q}_p$. This is

$$\xi(s) = \zeta_\infty(s)\prod_p \zeta_p(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s). \tag{16}$$

This completed Riemann Zeta function solves the functional equation

$$\xi(s) = \xi(1-s). \tag{17}$$

Then

$$\Gamma_\infty(s)\prod_p \Gamma_p(s) = \frac{\zeta_\infty(s)}{\zeta_\infty(1-s)}\prod_p \frac{\zeta_p(s)}{\zeta_p(1-s)} = \frac{\xi(s)}{\xi(1-s)} = 1. \tag{18}$$

Note that indicator functions($e^{-\pi x^2}$ is an analogue in Reals?) are their own Fourier transforms.