Calculating Vladimirov derivatives

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1 Derivative of the χ character

It seems to me that there are two ways to do the calculation: one is to split $\mathbb{Q}_p$ into different domains($\xi \mathbb{U}_p$) and do the integral of χ; the other is to decompose the χ character function into piece-wise constant functions γ_n and apply the V-derivatives on them.

a) Splitting $\mathbb{Q}_p$

$$D_y^s \chi(k_n y) = \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|x - p^{n-m}y_0|_p^{1+s}} \tag{1}$$

Where $k_n = p^{-n}$ and $y = p^{n-m}y_0$ with $m \geq 1$(other wise χ will be a constant function and the derivative is zero).
Useful integrals are:

$$\int\limits_{\xi \mathbb{U}_p} dx \chi(x) = |\xi|_p(\gamma_0(\xi) - \frac{1}{p}\gamma_0(p\xi)) = \begin{cases} |\xi|_p(1 - \frac{1}{p}), & v_p(\xi) \geq 0 \\ -1, & v_p(\xi) = -1 \\ 0, & v_p(\xi) < -1 \end{cases} \tag{2}$$

Consider the integral in each $x \in p^l \mathbb{U}_p$ block($x = p^l x_0$ where $x_0 \in \mathbb{U}_p$) and we have 3 parts:
First, for $l < n - m$, we have $|x - y|_p = |x|_p = p^{-l}$ and the integral becomes:

$$\frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int\limits_{p^l \mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{p^{-l(1+s)}}$$

$$= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int\limits_{p^{l-n}\mathbb{U}_p} dz|p^n|_p \frac{\chi(z) - \chi(p^{-m}y_0)}{p^{-l(1+s)}}$$

$$= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} p^{-n} \frac{0 - \chi(p^{-m}y_0)}{p^{-l(1+s)}} (1 - \frac{1}{p})|p^{l+n}|_p \tag{3}$$

$$= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} -\chi(p^{-m}y_0)(1 - \frac{1}{p})p^{ls}$$

$$= \frac{-\chi(p^{-m}y_0)}{\Gamma_p(-s)} \frac{p-1}{p^s - 1} p^{s(n-m)-1}.$$

The second part, $l = n - m$, is more complicated. Note that $\mathbb{U}_p = \{a_0\} + p\mathbb{Z}_p$, where a_0 is the first digit of numbers in $\mathbb{U}_p$. So we can do integrals in this way as well:

$$\int_{\mathbb{U}_p} dy = |\{a_0\}| \times \int_{p\mathbb{Z}_p} dx = (p-1)|p|_p = 1 - \frac{1}{p}. \tag{4}$$

In this sense we can do the splitting infinitely as $p^{n-m}(\{a_0\} + \{a_1\} + \{a_2\} + \{a_3\} + ... + \{a_{n-1}\} + p^n\mathbb{Z}_p)$, and the integral is equal to$(p-1)p^{n-1}|p^n|_p = 1 - \frac{1}{p}$ correctly.

Thus the integral region $p^{n-m}\mathbb{U}_p$ can be split into $p^{n-m}(\{a_0\} + \{a_1\} + \{a_2\} + \{a_3\} + ... + \{a_{n-1}\} + p^n\mathbb{Z}_p)$. For a given $y_0 \in \mathbb{U}_p$ and all $x_0 \in \mathbb{U}_p$, there is 1 case where they have the same first digit(where $|x_0 - y_0|_p$ is determined by following digits) and $p - 2$ cases where they have different first digits(where $|x_0 - y_0|_p = 1$) ; there is 1 case where they have the same second digit(where $|x_0 - y_0|_p$ is determined by following digits) and $p - 1$ cases where they have different second digits(where $|x_0 - y_0|_p = p^{-1}$); there is 1 case where they have the same third digit(where $|x_0 - y_0|_p$ is determined by following digits) and $p - 1$ cases where they have different third digits(where $|x_0 - y_0|_p = p^{-2}$);...Continue this spiting till $n \to \infty$ and we should obtain the summation form of the integral.

Note that here different digits are weighted by $\chi(p^{-n}x) = \chi(p^{-m}x_0)$ and the summation is trickier. Consider for example for a **particular** first digit of x_0, $a_0 \neq b_0$, which is the first digit of y_0. Without any pre-factor and constant term, the integral is:

$$\int_{p\mathbb{Z}_p} dx_0 \frac{\chi(p^{-m}(a_0 + a_1p + ...))}{|x_0 - y_0|_p^{1+s}}$$

$$= \int_{\mathbb{Z}_p} dx_1 |p|_p \chi(p^{-m}a_0)\chi(p^{-m+1}x_1) \tag{5}$$

$$= \frac{S_0}{p}\chi(p^{-m}a_0).$$

We define S_0 to be the remaining integral of $\chi(p^{-m+1}x_1)$. Then we do the summation over all $a_0 \neq b_0$:

$$\sum_{a_0=1, \neq b_0}^{p-1} \frac{S_0}{p}\chi(p^{-m}a_0)$$

$$= \frac{S_0}{p}\left(\frac{e^{2\pi i p^{1-m}} - 1}{e^{2\pi i p^{-m}} - 1} - 1 - \chi(p^{-m}b_0)\right) \tag{6}$$

$$= \frac{S_0}{p}(\Sigma_p(m) - \chi(p^{-m}b_0) - 1).$$

This discussion is valid for all digits, except that for not the first digits we have summations that start from $a_i = 0$ and the summation of χ is $(\Sigma_p(m) - \chi(p^{-m}b_i))$ for $i \geq 1$. Then

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including all the factors and constant terms, our integral is:

$$\frac{1}{\Gamma_p(-s)} \int\limits_{p^{n-m}\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}|_p^{1+s}|x_0 - y_0|_p^{1+s}}$$

$$= \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{U}_p} dx_0 \frac{|p^{n-m}|_p}{|p^{n-m}|_p^{1+s}} \frac{\chi(p^{-m}x_0) - \chi(p^{-m}y_0)}{|x_0 - y_0|_p^{1+s}}$$

$$= \frac{p^{(n-m)s}}{\Gamma_p(-s)} \int\limits_{\mathbb{U}_p} dx_0 \frac{\chi(p^{-m}x_0) - \chi(p^{-m}y_0)}{|x_0 - y_0|_p^{1+s}}$$

$$= \frac{p^{(n-m)s}}{\Gamma_p(-s)} \left(\sum_{a_0=1,\neq b_0}^{p-1} \int\limits_{p\mathbb{Z}_p} dx_0 \frac{\chi(p^{-m}a_0)\chi(p^{-m+1}x_1) - \chi(p^{-m}y_0)}{|p^0|_p^{1+s}} \right.$$

$$+ (a_0 = b_0) \text{ terms}$$

$$= \frac{p^{(n-m)s}}{\Gamma_p(-s)} \left(\sum_{a_0=1,\neq b_0}^{p-1} \int\limits_{\mathbb{Z}_p} dx_1 |p|_p \frac{\chi(p^{-m}a_0)\chi(p^{-m+1}x_1) - \chi(p^{-m}y_0)}{|p^0|_p^{1+s}} \right.$$

$$+ (a_0 = b_0) \text{ terms}$$

$$= \frac{p^{(n-m)s}}{\Gamma_p(-s)} \left(\frac{\Sigma_p(m) - \chi(p^{-m}b_0) - 1}{p} \int\limits_{\mathbb{Z}_p} dx_1 \chi(p^{-m+1}x_1) \right.$$

$$- \frac{p-2}{p} \int\limits_{\mathbb{Z}_p} dx_1 \chi(p^{-m}y_0) + (a_0 = b_0) \text{ terms} .$$

The first integral depends on the value of m and the result is:

$$\frac{p^{(n-m)s}}{\Gamma_p(-s)} ((\Sigma_p(m) - \chi(p^{-m}b_0) - 1)p^{-m} \times \left(\begin{cases} p^{m-1}(1 - \frac{1}{p}), & \text{for } m = 1 \\ -1, & \text{for } m = 2 \\ 0, & \text{for } m > 2 \end{cases} \right)$$

$$- \frac{p-2}{p}\chi(p^{-m}y_0) + (a_0 = b_0) \text{ terms}$$

Similarly, this $(a_0 = b_0)$ terms equals to:

$$\sum_{a_1=0,\neq b_1}^{p-1} \int_{p^2\mathbb{Z}_p} dx_0 \frac{\chi(p^{-m}(b_0 + pa_1))\chi(p^{-m+2}x_2) - \chi(p^{-m}y_0)}{|p^1|_p^{1+s}}$$

$$+ (a_1 = b_1) \text{ terms}$$

$$= \chi(p^{-m}b_0)(\Sigma_p(m-1) - \chi(p^{-m+1}b_1)) \int_{\mathbb{Z}_p} dx_2 |p^2|_p \frac{\chi(p^{-m+2}x_2)}{p^{-(1+s)}}$$

$$- (p-1) \int_{\mathbb{Z}_p} dx_2 |p^2|_p \frac{\chi(p^{-m}y_0)}{p^{-(1+s)}} + (a_1 = b_1) \text{ terms}$$

(9)

$$= \chi(p^{-m}b_0)(\Sigma_p(m-1) - \chi(p^{-m+1}b_1))p^{s-m+1} \times \left(\begin{cases} p^{m-2}(1 - \frac{1}{p}), & \text{for } ,1 \leq m \leq 2 \\ -1, & \text{for } m = 3 \\ 0, & \text{for } m > 3 \end{cases} \right)$$

$$- (p-1)p^{s-1}\chi(p^{-m}y_0) + (a_1 = b_1) \text{ terms.}$$

From here it is clear that the $(a_i = b_i)$ term for arbitrary i is:

$$\chi(p^{-m}(b_0 + b_1 p + ... + b_i p^i))(\Sigma_p(m-1-i) - \chi(p^{-m+1+i}b_{i+1}))p^{(i+1)(s+1)-m}$$

$$\times \left(\begin{cases} p^{m-2-i}(1 - \frac{1}{p}), & \text{for } ,1 \leq m \leq 2+i \\ -1, & \text{for } m = 3+i \\ 0, & \text{for } m > 3+i \end{cases} \right) - (p-1)p^{s(i+1)-1}\chi(p^{-m}y_0) + (a_{i+1} = b_{i+1}) \text{ terms.}$$

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Now let's do the summation for all digits(over the whole integral region) term by term:

- First, the simple ones, $\chi(p^{-m}y_0)$ terms:

$$- \frac{p^{(n-m)s}}{\Gamma_p(-s)}\chi(p^{-m}y_0)(\frac{p-2}{p} + \sum_{i=0}^{\infty}(p-1)p^{s(i+1)-1})$$

$$= -\frac{p^{(n-m)s}}{\Gamma_p(-s)}\chi(p^{-m}y_0)\frac{2 - p - p^s + (p-1)p^{(2+\infty)s}}{p(p^s - 1)}.$$

(11)

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The last term, $l > n - m$, is:

$$\frac{1}{\Gamma_p(-s)} \sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}y_0|_p^{1+s}}$$

$$= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx (\chi(p^{-n}x) - \chi(p^{-m}y_0))$$

$$= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \Big(\sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \chi(p^{-n}x) - \sum_{l=n-m+1}^{\infty} p^{-l}(1 - \frac{1}{p})\chi(p^{-m}y_0) \Big)$$

$$= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \Big(\sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \chi(p^{-n}x) - p^{-1+m-n}\chi(p^{-m}y_0) \Big)$$

$$(12)$$

While this integral $\sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \chi(p^{-n}x)$ should be considered in two cases:

- When $m = 1$, this is $\sum_{l=n}^{\infty} p^l(1 - \frac{1}{p}) = p^{-n}$;

- When $m \geq 2$, this is $-p^{-n} + \sum_{l=n}^{\infty} p^l(1 - \frac{1}{p}) = 0$.

So in this region, the integral becomes:

$$\begin{cases} \frac{1}{\Gamma_p(-s)} p^{(n-m)s-1}(p^{1-m} - \chi(p^{-m}y_0)), & m = 1 \\ \frac{-1}{\Gamma_p(-s)} p^{(n-m)s-1}\chi(p^{-m}y_0), & m \geq 2 \end{cases} \qquad (13)$$