Calculating Vladimirov derivatives

Ziming Ji

October 23, 2018

1 Derivative of the χ character

Calculating Vladimirov derivative of $\chi(k_n y)$ where $k_n = p^{-n}$:

$$
\begin{aligned}
D_y^s \chi(k_n y) &= \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-n}y)}{|x-y|_p^{1+s}} \\
&= \frac{1}{\Gamma_p(-s)} \left(\int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}(u+y))}{|u|_p^{1+s}} - \int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}y)}{|u|_p^{1+s}} \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(\int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}u)}{|u|_p^{1+s}} - \int\limits_{\mathbb{Q}_p} du \frac{1}{|u|_p^{1+s}} \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(-p^{(n-1)(s+1)-n} + \sum_n^\infty p^{ls}(1-\frac{1}{p}) - \sum_{-\infty}^\infty p^{ls}(1-\frac{1}{p}) \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(p^{ns}\Gamma_p(-s) \right) = p^{ns}\chi(k_n y)
\end{aligned}
\tag{1}
$$

We notice that two ways of regulating the integral are equivalent:

- Including $(-\chi(p^{-n}y))$ in the numerator;
- "Analytic continuation" of $\Gamma_p(-s) = \int\limits_{\mathbb{Q}_p} \frac{dx}{|x|_p} \chi(x)|x|_p^{-s}$ to the region $-s > 0$ by $\Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(1-s)}$.

It is really like defining all the quantity modulo $\sum\limits_{-\infty}^{\infty} p^{ls}$.

2 Check shift invariance of a general p-adic integral

2.1 Integrand depending only on the p-adic norm

Define a general integral over p-adic number field as:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} dx f(p^{-l}) = \sum_{l=-\infty}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l} \tag{2}$$

A shift of the integral variable gives:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \int_{\mathbb{Q}_p} d(u-y) f(|u-y|_p) \quad \text{where} \quad u = x+y \tag{3}$$

Suppose that $d(u-y) = du$ when y is constant. Then we can proceed by splitting $\mathbb{Q}_p$:

$$\int_{\mathbb{Q}_p} d(u) f(|u-y|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u-y|_p) \tag{4}$$

Note that we cannot write $|u-y|_p$ as p^{-l} in each block any more. Suppose that $y = p^m(a_0 + a_1 p + a_2 p^2 + ...)$, we have:

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u-y|_p) &= \sum_{l=-\infty}^{m-1} \int_{p^l \mathbb{U}_p} du f(p^{-l}) + \sum_{l=m+1}^{\infty} \int_{p^l \mathbb{U}_p} du f(p^{-m}) + (l=m) \text{ term} \\ &= \sum_{l=-\infty}^{m-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + f(p^{-m})p^{-(m+1)} + (l=m) \text{ term} \end{aligned} \tag{5}$$

This "contact term" can be written as:

$$\begin{aligned} &\int_{p^m \mathbb{U}_p} du f(|u-y|_p) = \sum_{b_0=1, \neq a_0}^{p-1} \int_{p^{m+1}\mathbb{Z}_p} du f(p^{-m}) + (a_0 = b_0) \text{ term} \\ &= f(p^{-m})(p-2)p^{-(m+1)} + \sum_{b_1=0, \neq a_1}^{p-1} \int_{p^{m+2}\mathbb{Z}_p} du f(p^{-(m+1)}) + (a_1 = b_1) \text{ term} \\ &= f(p^{-m})(p-2)p^{-(m+1)} + \sum_{l=m+1}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l} \end{aligned} \tag{6}$$

In the end, all these terms add up to the integral after shift:

$$f(p^{-m})p^{-(m+1)}(p-1) + \sum_{l=-\infty}^{m-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + \sum_{l=m+1}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l} \tag{7}$$

This is exactly $\sum_{l=-\infty}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l}$, the integral before shifting.

2.2 Integrand depending also on p-adic digits