

Calculating Vladimirov derivatives

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1 Derivative of the χ character

Calculating Vladimirov derivative of $\chi(k_n y)$ where $k_n = p^{-n}$:

$$\begin{aligned} D_y^s \chi(k_n y) &= \frac{1}{\Gamma_p(-s)} \int_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-n}y)}{|x - y|_p^{1+s}} \\ &= \frac{1}{\Gamma_p(-s)} \left(\int_{\mathbb{Q}_p} du \frac{\chi(p^{-n}(u+y))}{|u|_p^{1+s}} - \int_{\mathbb{Q}_p} du \frac{\chi(p^{-n}y)}{|u|_p^{1+s}} \right) \\ &= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(\int_{\mathbb{Q}_p} du \frac{\chi(p^{-n}u)}{|u|_p^{1+s}} - \int_{\mathbb{Q}_p} du \frac{1}{|u|_p^{1+s}} \right) \\ &= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(-p^{(n-1)(s+1)-n} + \sum_n^\infty p^{ls} \left(1 - \frac{1}{p}\right) - \sum_{-\infty}^\infty p^{ls} \left(1 - \frac{1}{p}\right) \right) \\ &= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) (p^{ns} \Gamma_p(-s)) = p^{ns} \chi(k_n y) \end{aligned} \tag{1}$$

2 Check shift invariance of a general p -adic integral

2.1 Integrand depending only on the p -adic norm

Define a general integral over p -adic number field as:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \sum_{l=-\infty}^\infty \int_{p^l \mathbb{U}_p} dx f(p^{-l}) = \sum_{l=-\infty}^\infty f(p^{-l}) \left(1 - \frac{1}{p}\right) p^{-l} \tag{2}$$

A shift of the integral variable gives:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \int_{\mathbb{Q}_p} d(u-y) f(|u-y|_p) \quad \text{where} \quad u = x + y \tag{3}$$

Suppose that $d(u - y) = du$ when y is constant. Then we can proceed by splitting \mathbb{Q}_p :

$$\int_{\mathbb{Q}_p} d(u) f(|u - y|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u - y|_p) \quad (4)$$

Note that we cannot write $|u - y|_p$ as p^{-l} in each block any more. Suppose that $y = p^m(a_0 + a_1p + a_2p^2 + \dots)$, we have:

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u - y|_p) &= \sum_{l=-\infty}^{m-1} \int_{p^l \mathbb{U}_p} du f(p^{-l}) + \sum_{l=m+1}^{\infty} \int_{p^l \mathbb{U}_p} du f(p^{-m}) + (l = m) \text{ term} \\ &= \sum_{l=-\infty}^{m-1} f(p^{-l}) \left(1 - \frac{1}{p}\right) p^{-l} + f(p^{-m}) p^{-(m+1)} + (l = m) \text{ term} \end{aligned} \quad (5)$$

This “contact term” can be written as:

$$\begin{aligned} \int_{p^m \mathbb{U}_p} du f(|u - y|_p) &= \sum_{b_0=1, \neq a_0}^{p-1} \int_{p^{m+1} \mathbb{Z}_p} du f(p^{-m}) + (a_0 = b_0) \text{ term} \\ &= f(p^{-m}) (p - 2) p^{-(m+1)} + \sum_{b_1=0, \neq a_1}^{p-1} \int_{p^{m+2} \mathbb{Z}_p} du f(p^{-(m+1)}) + (a_1 = b_1) \text{ term} \\ &= f(p^{-m}) (p - 2) p^{-(m+1)} + \sum_{l=m+1}^{\infty} f(p^{-l}) \left(1 - \frac{1}{p}\right) p^{-l} \end{aligned} \quad (6)$$

In the end, all these terms add up to the integral after shift:

$$f(p^{-m}) p^{-(m+1)} (p - 1) + \sum_{l=-\infty}^{m-1} f(p^{-l}) \left(1 - \frac{1}{p}\right) p^{-l} + \sum_{l=m+1}^{\infty} f(p^{-l}) \left(1 - \frac{1}{p}\right) p^{-l} \quad (7)$$

This is exactly $\sum_{l=-\infty}^{\infty} f(p^{-l}) \left(1 - \frac{1}{p}\right) p^{-l}$, the integral before shifting.

2.2 Integrand depending also on p -adic digits