

# Calculating Vladimirov derivatives and a little more

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## 1 Derivative of the $\chi$ character

Calculating Vladimirov derivative of  $\chi(k_n y)$  where  $k_n = p^{-n}$ :

$$\begin{aligned} D_y^s \chi(k_n y) &= \frac{1}{\Gamma_p(-s)} \int_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-n}y)}{|x - y|_p^{1+s}} \\ &= \frac{1}{\Gamma_p(-s)} \left( \int_{\mathbb{Q}_p} du \frac{\chi(p^{-n}(u + y))}{|u|_p^{1+s}} - \int_{\mathbb{Q}_p} du \frac{\chi(p^{-n}y)}{|u|_p^{1+s}} \right) \\ &= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left( \int_{\mathbb{Q}_p} du \frac{\chi(p^{-n}u)}{|u|_p^{1+s}} - \int_{\mathbb{Q}_p} du \frac{1}{|u|_p^{1+s}} \right) \\ &= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left( -p^{(n-1)(s+1)-n} + \sum_n^\infty p^{ls} \left(1 - \frac{1}{p}\right) - \sum_{-\infty}^\infty p^{ls} \left(1 - \frac{1}{p}\right) \right) \\ &= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) (p^{ns} \Gamma_p(-s)) = p^{ns} \chi(k_n y) \end{aligned} \tag{1}$$

We notice that two ways of regulating the integral are equivalent:

- Including  $(-\chi(p^{-n}y))$  in the numerator;
- “Analytic continuation” of  $\Gamma_p(-s) = \int_{\mathbb{Q}_p} \frac{dx}{|x|_p} \chi(x) |x|_p^{-s}$  to the region  $-s > 0$  by  $\Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(1-s)}$ .

It is really like defining all the quantities modulo  $\sum_{-\infty}^\infty p^{ls}$ .

## 2 Is V derivative equipped with a Leibniz rule and/or a product rule?

Let's find the answer.

Sadly, there is no general ordinary product rule. But a reduced version comes out.  
What about a Leibniz rule?

### 3 Check shift invariance of a general $p$ -adic integral

#### 3.1 Integrand depending only on the $p$ -adic norm

Define a general integral over  $p$ -adic number field as:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} dx f(p^{-l}) = \sum_{l=-\infty}^{\infty} f(p^{-l}) (1 - \frac{1}{p}) p^{-l} \quad (2)$$

A shift of the integral variable gives:

$$\int_{\mathbb{Q}_p} dx f(|x|_p) = \int_{\mathbb{Q}_p} d(u - y) f(|u - y|_p) \quad \text{where} \quad u = x + y \quad (3)$$

Suppose that  $d(u - y) = du$  when  $y$  is constant. Then we can proceed by splitting  $\mathbb{Q}_p$ :

$$\int_{\mathbb{Q}_p} d(u) f(|u - y|_p) = \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u - y|_p) \quad (4)$$

Note that we cannot write  $|u - y|_p$  as  $p^{-l}$  in each block any more. Suppose that  $y = p^m(a_0 + a_1 p + a_2 p^2 + \dots)$ , we have:

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \int_{p^l \mathbb{U}_p} du f(|u - y|_p) &= \sum_{l=-\infty}^{m-1} \int_{p^l \mathbb{U}_p} du f(p^{-l}) + \sum_{l=m+1}^{\infty} \int_{p^l \mathbb{U}_p} du f(p^{-m}) + (l = m) \text{ term} \\ &= \sum_{l=-\infty}^{m-1} f(p^{-l}) (1 - \frac{1}{p}) p^{-l} + f(p^{-m}) p^{-(m+1)} + (l = m) \text{ term} \end{aligned} \quad (5)$$

This “contact term” can be written as:

$$\begin{aligned} \int_{p^m \mathbb{U}_p} du f(|u - y|_p) &= \sum_{b_0=1, \neq a_0}^{p-1} \int_{p^{m+1} \mathbb{Z}_p} du f(p^{-m}) + (a_0 = b_0) \text{ term} \\ &= f(p^{-m}) (p - 2) p^{-(m+1)} + \sum_{b_1=0, \neq a_1}^{p-1} \int_{p^{m+2} \mathbb{Z}_p} du f(p^{-(m+1)}) + (a_1 = b_1) \text{ term} \\ &= f(p^{-m}) (p - 2) p^{-(m+1)} + \sum_{l=m+1}^{\infty} f(p^{-l}) (1 - \frac{1}{p}) p^{-l} \end{aligned} \quad (6)$$

In the end, all these terms add up to the integral after shift:

$$f(p^{-m})p^{-(m+1)}(p-1) + \sum_{l=-\infty}^{m-1} f(p^{-l})(1 - \frac{1}{p})p^{-l} + \sum_{l=m+1}^{\infty} f(p^{-l})(1 - \frac{1}{p})p^{-l} \quad (7)$$

This is exactly  $\sum_{l=-\infty}^{\infty} f(p^{-l})(1 - \frac{1}{p})p^{-l}$ , the integral before shifting.

### 3.2 Integrand depending also on $p$ -adic digits

## 4 Useful Fourier(and maybe Mellin) transforms

A most common one is the Fourier transform of  $|k|_p^s$  or a Mellin transform of  $\chi(kx)$ :

$$\int_{\mathbb{Q}_p} dk \chi(kx) |k|_p^s = \Gamma_p(s+1) \frac{1}{|x|_p^{s+1}}, \quad (8)$$

recalling that the definition of  $\Gamma_p$  is:

$$\Gamma_p(s) = \int_{\mathbb{Q}_p} \frac{du}{|u|} \chi(u) |u|^s. \quad (9)$$

A slight variation is:

$$\int_{\mathbb{Q}_p} dk \chi(kx) |k|_p^s \left( \frac{k}{p} \right)_L = up^{s+\frac{1}{2}} \frac{\left( \frac{x}{p} \right)_L}{|x|_p^{s+1}}, \quad (10)$$

where  $\left( \frac{k}{p} \right)_L$  is the Legendre symbol,  $u$  equals 1 for  $p = 1 \bmod 4$  while  $i$  for  $p = 3 \bmod 4$ . A useful Gauss sum here is

$$\sum_{\alpha \in \mathbb{F}_p} \left( \frac{k}{p} \right)_L \chi \left( -\frac{k_0 \alpha}{p} \right) = \frac{\sqrt{p}}{u} \left( \frac{k_0}{p} \right)_L. \quad (11)$$

We want to know about the Fourier transform of the characteristic function itself:

$$\int_{\mathbb{Q}_p} dk \chi(kx) \chi(ky) = \frac{1}{|x+y|} \Gamma_p(1). \quad (12)$$

### 4.1 A simple example of Adelic product relation

We need to know the Mellin transform of  $e^{-\pi x^2}$  in  $\mathbb{R}$  and  $\gamma_0(x)$  the indicator(or characteristic) function of  $\mathbb{Z}_p$  in their respective fields:

$$\int_{\mathbb{R}} \frac{dx}{|x|} |x|^s e^{-\pi x^2} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \equiv \zeta_{\infty}(s), \quad \int_{\mathbb{Q}_p} \frac{dx}{|x|_p} |x|_p^s \gamma_0(x) = \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{-s}}\right) \equiv \left(1 - \frac{1}{p}\right) \zeta_p(s) \quad (13)$$

and the Mellin transform of respective character functions are definitions of the Gelfan-Graev gamma functions:

$$\int_{\mathbb{R}} dx |x|^s e^{2\pi i x} \equiv \Gamma_{\infty}(s), \quad \int_{\mathbb{Q}_p} \frac{dx}{|x|_p} |x|_p^s \chi(x) \equiv \Gamma_p(s). \quad (14)$$

Interestingly, we have

$$\Gamma_{\infty}(s) = \frac{\zeta_{\infty}(s)}{\zeta_{\infty}(1-s)}, \quad \Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(1-s)}. \quad (15)$$

And remember that the definition of completed Riemann Zeta function  $\xi(s)$  is an Adelic Mellin transform of the indicator functions( $e^{-\pi x^2}$  is an analogue in Reals?) over Haar measures, which cancels the  $(1 - \frac{1}{p})$  factor in the integral over  $\mathbb{Q}_p$ . This is

$$\xi(s) = \zeta_{\infty}(s) \prod_p \zeta_p(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s). \quad (16)$$

This completed Riemann Zeta function solves the functional equation

$$\xi(s) = \xi(1-s). \quad (17)$$

Then

$$\Gamma_{\infty}(s) \prod_p \Gamma_p(s) = \frac{\zeta_{\infty}(s)}{\zeta_{\infty}(1-s)} \prod_p \frac{\zeta_p(s)}{\zeta_p(1-s)} = \frac{\xi(s)}{\xi(1-s)} = 1. \quad (18)$$

Note that indicator functions( $e^{-\pi x^2}$  is an analogue in Reals?) are their own Fourier transforms.

## 5 Gauss integral

$$\int_{\mathbb{Q}_p} \chi(\alpha x^2 + \beta x) dx = \lambda_p(\alpha) |2\alpha|_p^{-\frac{1}{2}} \chi(\frac{\beta^2}{4\alpha}), \quad \alpha \neq 0, \quad (19)$$

where  $\lambda_p(\alpha)$  is an arithmetic complex-valued function [1].

## References

- [1] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-adic Analysis and Mathematical Physics*. Series on Soviet and East European mathematics. World Scientific, 1994.  
[https://books.google.com/books?id=bp7Ui\\_jt1fMC](https://books.google.com/books?id=bp7Ui_jt1fMC).