Calculating Vladimirov derivatives

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1 Derivative of the χ character

Calculating Vladimirov derivative of $\chi(k_n y)$ where $k_n = p^{-n}$:

$$\begin{aligned}
D_y^s \chi(k_n y) &= \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-n}y)}{|x-y|_p^{1+s}} \\
&= \frac{1}{\Gamma_p(-s)} \left(\int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}(u+y))}{|u|_p^{1+s}} - \int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}y)}{|u|_p^{1+s}} \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(\int\limits_{\mathbb{Q}_p} du \frac{\chi(p^{-n}u)}{|u|_p^{1+s}} - \int\limits_{\mathbb{Q}_p} du \frac{1}{|u|_p^{1+s}} \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(-p^{(n-1)(s+1)-n} + \sum_{n}^{\infty} p^{ls}(1-\frac{1}{p}) - \sum_{-\infty}^{\infty} p^{ls}(1-\frac{1}{p}) \right) \\
&= \frac{1}{\Gamma_p(-s)} \chi(p^{-n}y) \left(p^{ns} \Gamma_p(-s) \right) = p^{ns} \chi(k_n y)
\end{aligned} \tag{1}$$

2 Check shift invariance of a general p-adic integral

Define a general integral over p-adic number field as:

$$\int\limits_{\mathbb{Q}_p} dx f(|x|_p) = \sum_{l=-\infty}^{\infty} \int\limits_{p^l \mathbb{U}_p} dx f(p^{-l}) = \sum_{l=-\infty}^{\infty} f(p^{-l})(1-\frac{1}{p})p^{-l} \tag{2}$$

A shift of the integral variable gives:

$$\int\limits_{\mathbb{Q}_p} dx f(|x|_p) = \int\limits_{\mathbb{Q}_p} d(u-y) f(|u-y|_p) \quad \text{where} \quad u = x + y \tag{3}$$

Suppose that $d(u-y)=du$ when y is constant. Then we can proceed by splitting $\mathbb{Q}_p$:

$$\int\limits_{\mathbb{Q}_p} d(u)f(|u-y|_p) = \sum_{l=-\infty}^{\infty} \int\limits_{p^l\mathbb{U}_p} du f(|u-y|_p) \tag{4}$$

Note that we cannot write $|u-y|_p$ as p^{-l} in each block any more. Suppose that $y = p^m(a_0+a_1p+a_2p^2+...)$, we have:

$$\begin{aligned}\sum_{l=-\infty}^{\infty} \int\limits_{p^l\mathbb{U}_p} du f(|u-y|_p) &= \sum_{l=-\infty}^{m-1} \int\limits_{p^l\mathbb{U}_p} du f(p^{-l}) + \sum_{l=m+1}^{\infty} \int\limits_{p^l\mathbb{U}_p} du f(p^{-m}) + (l=m) \text{ term} \\ &= \sum_{l=-\infty}^{m-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + f(p^{-m})p^{-(m+1)} + (l=m) \text{ term}\end{aligned} \tag{5}$$

This "contact term" can be written as:

$$\begin{aligned}\int\limits_{p^m\mathbb{U}_p} du f(|u-y|_p) &= \sum_{b_0=1,\neq a_0}^{p-1} \int\limits_{p^{m+1}\mathbb{U}_p} du f(p^{-m}) + (a_0=b_0) \text{ term} \\ &= f(p^{-m})(p-2)(1-\frac{1}{p})p^{-(m+1)} + \sum_{b_1=0,\neq a_1}^{p-1} \int\limits_{p^{m+2}\mathbb{U}_p} du f(p^{-(m+1)}) + (a_1=b_1) \text{ term} \\ &= f(p^{-m})(p-2)(1-\frac{1}{p})p^{-(m+1)} + \sum_{l=m+1}^{\infty} f(p^{-l})(1-\frac{1}{p})^2 p^{-(l+1)}\end{aligned} \tag{6}$$

In the end, all these terms add up to the integral after shift:

$$f(p^{-m})p^{-(m+1)}[(p-2)(1-\frac{1}{p})+1] + \sum_{l=-\infty}^{m-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + \sum_{l=m+1}^{\infty} f(p^{-l})(1-\frac{1}{p})^2 p^{-l} \tag{7}$$

This is not the integral before shift. What happened?????????
If we shift $z=p^n(c_0+c_1p+c_2p^2+...)$, the integral will be:

$$f(p^{-n})p^{-(n+1)}[(p-2)(1-\frac{1}{p})+1] + \sum_{l=-\infty}^{n-1} f(p^{-l})(1-\frac{1}{p})p^{-l} + \sum_{l=n+1}^{\infty} f(p^{-l})(1-\frac{1}{p})^2 p^{-l} \tag{8}$$

Are these two compatible? No, at least not for arbitrary f.
However, we should only consider a subset of function f, which has convergent integral on $\mathbb{Q}_p$. Using a test function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Z}_p \\ 0 & \text{otherwise} \end{cases} \tag{9}$$