

Calculating Vladimirov derivatives

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1 Derivative of the χ character

It seems to me that there are two ways to do the calculation: one is to split \mathbb{Q}_p into different domains($\xi\mathbb{U}_p$) and do the integral of χ ; the other is to decompose the χ character function into piece-wise constant functions γ_n and apply the V-derivatives on them.

a) Splitting \mathbb{Q}_p

$$D_y^s \chi(k_n y) = \frac{1}{\Gamma_p(-s)} \int_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|x - p^{n-m}y_0|_p^{1+s}} \quad (1)$$

Where $k_n = p^{-n}$ and $y = p^{n-m}y_0$ with $m \geq 1$ (other wise χ will be a constant function and the derivative is zero).

Useful integrals are:

$$\int_{\xi\mathbb{U}_p} dx \chi(x) = |\xi|_p (\gamma_0(\xi) - \frac{1}{p} \gamma_0(p\xi)) = \begin{cases} |\xi|_p (1 - \frac{1}{p}), & v_p(\xi) \geq 0 \\ -1, & v_p(\xi) = -1 \\ 0, & v_p(\xi) \leq -1 \end{cases} \quad (2)$$

Consider the integral in each $x \in p^l\mathbb{U}_p$ block($x = p^l x_0$ where $x_0 \in \mathbb{U}_p$) and we have 3 parts: First, for $l < n - m$, we have $|x - y|_p = |x|_p = p^{-l}$ and the integral becomes:

$$\begin{aligned} & \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int_{p^l\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{p^{-l(1+s)}} \\ &= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int_{p^{l-n}\mathbb{U}_p} dz |p^n|_p \frac{\chi(z) - \chi(p^{-m}y_0)}{p^{-l(1+s)}} \\ &= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} p^{-n} \frac{0 - \chi(p^{-m}y_0)}{p^{-l(1+s)}} (1 - \frac{1}{p}) |p^{l+n}|_p \\ &= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} -\chi(p^{-m}y_0) (1 - \frac{1}{p}) p^{ls} \\ &= \frac{-\chi(p^{-m}y_0)}{\Gamma_p(-s)} \frac{p-1}{p^s-1} p^{s(n-m)-1}. \end{aligned} \quad (3)$$

The second part, $l = n - m$, is more complicated. Note that $\mathbb{U}_p = \{a_0\} + p\mathbb{Z}_p$, where a_0 is the first digit of numbers in \mathbb{U}_p . So we can do integrals in this way as well:

$$\int_{\mathbb{U}_p} dy = |\{a_0\}| \times \int_{p\mathbb{Z}_p} dx = (p-1)|p|_p = 1 - \frac{1}{p}. \quad (4)$$

In this sense we can do the splitting infinitely as $p^{n-m}(\{a_0\} + \{a_1\} + \{a_2\} + \{a_3\} + \dots + \{a_{n-1}\} + p^n\mathbb{Z}_p)$, and the integral is equal to $(p-1)p^{n-1}|p^n|_p = 1 - \frac{1}{p}$ correctly.

Thus the integral region $p^{n-m}\mathbb{U}_p$ can be split into $p^{n-m}(\{a_0\} + \{a_1\} + \{a_2\} + \{a_3\} + \dots + \{a_{n-1}\} + p^n\mathbb{Z}_p)$. For a given $y_0 \in \mathbb{U}_p$ and all $x_0 \in \mathbb{U}_p$, there is 1 case where they have the same first digit (where $|x_0 - y_0|_p$ is determined by following digits) and $p-2$ cases where they have different first digits (where $|x_0 - y_0|_p = 1$); there is 1 case where they have the same second digit (where $|x_0 - y_0|_p$ is determined by following digits) and $p-1$ cases where they have different second digits (where $|x_0 - y_0|_p = p^{-1}$); there is 1 case where they have the same third digit (where $|x_0 - y_0|_p$ is determined by following digits) and $p-1$ cases where they have different third digits (where $|x_0 - y_0|_p = p^{-2}$); ... Continue this splitting till $n \rightarrow \infty$ and we should obtain the summation form of the integral.

Note that here different digits are weighted by $\chi(p^{-n}x) = \chi(p^{-m}x_0)$ and the summation is trickier. Consider for example for a **particular** first digit of x_0 , a_0 . Without any pre-factor and constant term, the integral is:

$$\int_{p\mathbb{Z}_p} dx_0 \frac{\chi(p^{-m}(a_0 + a_1p + \dots))}{|x_0 - y_0|_p^{1+s}} = \int_{\mathbb{Z}_p} dx_1 \chi(p^{-m}a_0) \chi(p^{-m+1}x_1) \quad (5)$$

The derivative is:

$$\begin{aligned} & \frac{1}{\Gamma_p(-s)} \int_{p^{n-m}\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}(x_0 - y_0)|_p^{1+s}} \\ &= \frac{1}{\Gamma_p(-s)} \int_{p^{n-m}\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}(x_0 - y_0)|_p^{1+s}} \end{aligned} \quad (6)$$

The last term, $l > n - m$, is:

$$\begin{aligned}
& \frac{1}{\Gamma_p(-s)} \sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}y_0|_p^{1+s}} \\
&= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx (\chi(p^{-n}x) - \chi(p^{-m}y_0)) \\
&= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \left(\sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \chi(p^{-n}x) - \sum_{l=n-m+1}^{\infty} p^{-l} \left(1 - \frac{1}{p}\right) \chi(p^{-m}y_0) \right) \\
&= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \left(\sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \chi(p^{-n}x) - p^{-1+m-n} \chi(p^{-m}y_0) \right)
\end{aligned} \tag{7}$$

While this integral $\sum_{l=n-m+1}^{\infty} \int_{p^l \mathbb{U}_p} dx \chi(p^{-n}x)$ should be considered in two cases:

- When $m = 1$, this is $\sum_{l=n}^{\infty} p^l \left(1 - \frac{1}{p}\right) = p^{-n}$;
- When $m \geq 2$, this is $-p^{-n} + \sum_{l=n}^{\infty} p^l \left(1 - \frac{1}{p}\right) = 0$.

So in this region, the integral becomes:

$$\begin{cases} \frac{1}{\Gamma_p(-s)} p^{(n-m)s-1} (p^{1-m} - \chi(p^{-m}y_0)), & m = 1 \\ \frac{-1}{\Gamma_p(-s)} p^{(n-m)s-1} \chi(p^{-m}y_0), & m \geq 2 \end{cases} \tag{8}$$