Calculating Vladimirov derivatives

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1 Derivative of the χ character

It seems to me that there are two ways to do the calculation: one is to split $\mathbb{Q}_p$ into different domains($\xi\mathbb{U}_p$) and do the integral of χ; the other is to decompose the χ character function into piece-wise constant functions γ_n and apply the V-derivatives on them.

a) Splitting $\mathbb{Q}_p$

$$D_y^s \chi(k_n y) = \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|x - p^{n-m}y_0|_p^{1+s}} \tag{1}$$

Where $k_n = p^{-n}$ and $y = p^{n-m}y_0$ with $m \geq 1$(other wise χ will be a constant function and the derivative is zero).
Useful integrals are:

$$\int\limits_{\xi\mathbb{U}_p} dx\chi(x) = |\xi|_p'(\gamma_0(\xi) - \frac{1}{p}\gamma_0(p\xi)) = \begin{cases} 1 \\ 2 \\ 3 \end{cases} \tag{2}$$

Consider the integral in each $x \in p^l\mathbb{U}_p$ block($x = p^l x_0$ where $x_0 \in \mathbb{U}_p$) and we have 3 parts:
First, for $l < n - m$, we have $|x - y|_p = |x|_p = p^{-l}$ and the integral becomes:

$$\begin{aligned}
&\frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int\limits_{p^l\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{p^{-l(1+s)}} \\
&= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int\limits_{p^{l-n}\mathbb{U}_p} dz |p^n|_p \frac{\chi(z) - \chi(p^{-m}y_0)}{p^{-l(1+s)}} \\
&= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} p^{-n} \frac{0 - \chi(p^{-m}y_0)}{p^{-l(1+s)}} (1 - \frac{1}{p}) |p^{l+n}|_p \\
&= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} -\chi(p^{-m}y_0)(1 - \frac{1}{p}) p^{ls} \\
&= \frac{-\chi(p^{-m}y_0)}{\Gamma_p(-s)} \frac{p-1}{p^s - 1} p^{s(n-m)-1}.
\end{aligned} \tag{3}$$

The second part, $l = n - m$, is more complicated. The derivative is:

$$\begin{aligned}
&\frac{1}{\Gamma_p(-s)} \int\limits_{p^{n-m}\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}(x_0 - y_0)|_p^{1+s}} \\
&= \frac{1}{\Gamma_p(-s)} \int\limits_{p^{n-m}\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}(x_0 - y_0)|_p^{1+s}}
\end{aligned} \tag{4}$$

The last term, $l > n - m$, is:

$$\begin{aligned}
&\frac{1}{\Gamma_p(-s)} \sum_{l=n-m+1}^{\infty} \int\limits_{p^l\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|p^{n-m}y_0|_p^{1+s}} \\
&= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \sum_{l=n-m+1}^{\infty} \int\limits_{p^l\mathbb{U}_p} dx (\chi(p^{-n}x) - \chi(p^{-m}y_0)) \\
&= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \Big(\sum_{l=n-m+1}^{\infty} \int\limits_{p^l\mathbb{U}_p} dx \chi(p^{-n}x) - \sum_{l=n-m+1}^{\infty} p^{-l}(1 - \frac{1}{p}) \chi(p^{-m}y_0) \Big) \\
&= \frac{1}{\Gamma_p(-s)} p^{(n-m)(1+s)} \Big(\sum_{l=n-m+1}^{\infty} \int\limits_{p^l\mathbb{U}_p} dx \chi(p^{-n}x) - p^{-1+m-n} \chi(p^{-m}y_0) \Big)
\end{aligned} \tag{5}$$

While this integral $\sum_{l=n-m+1}^{\infty} \int\limits_{p^l\mathbb{U}_p} dx \chi(p^{-n}x)$ should be considered in two cases:

- When $m = 1$, this is $\sum_{l=n}^{\infty} p^l(1 - \frac{1}{p}) = p^{-n}$;
- When $m \geq 2$, this is $-p^{-n} + \sum_{l=n}^{\infty} p^l(1 - \frac{1}{p}) = 0$.

So in this region, the integral becomes:

$$\begin{cases} \frac{1}{\Gamma_p(-s)} p^{(n-m)s-1}(p^{1-m} - \chi(p^{-m}y_0)), & m = 1 \\ \frac{-1}{\Gamma_p(-s)} p^{(n-m)s-1} \chi(p^{-m}y_0), & m \geq 2 \end{cases} \tag{6}$$