Calculating Vladimirov derivatives

Ziming Ji

October 1, 2018

1 Derivative of the χ character

It seems to me that there are two ways to do the calculation: one is to split $\mathbb{Q}_p$ into different domains($\xi\mathbb{U}_p$) and do the integral of χ; the other is to decompose the χ character function into piece-wise constant functions γ_n and apply the V-derivatives on them.

a) Splitting $\mathbb{Q}_p$

$$D^s_y \chi(k_n y) = \frac{1}{\Gamma_p(-s)} \int\limits_{\mathbb{Q}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{|x - p^{n-m}y_0|_p^{1+s}} \tag{1}$$

Where $k_n = p^{-n}$ and $y = p^{n-m}y_0$ with $m \geq 1$(other wise χ will be a constant function and the derivative is zero).
Useful integrals are:

$$\int\limits_{\xi\mathbb{U}_p} dx \chi(x) = |\xi|_p(\gamma_0(\xi) - \frac{1}{p}\gamma_0(p\xi)) = \begin{cases} |\xi|_p(1-\frac{1}{p}), & v_p(\xi) \geq 0 \\ -1, & v_p(\xi) = -1 \\ 0, & v_p(\xi) \leq -1 \end{cases} \tag{2}$$

Consider the integral in each $x \in p^l\mathbb{U}_p$ block($x = p^l x_0$ where $x_0 \in \mathbb{U}_p$) and we have 3 parts:
First, for $l < n-m$, we have $|x-y|_p = |x|_p = p^{-l}$ and the integral becomes:

$$\begin{aligned}
&\frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int\limits_{p^l\mathbb{U}_p} dx \frac{\chi(p^{-n}x) - \chi(p^{-m}y_0)}{p^{-l(1+s)}} \\
&= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} \int\limits_{p^{l-n}\mathbb{U}_p} dz |p^n|_p \frac{\chi(z) - \chi(p^{-m}y_0)}{p^{-l(1+s)}} \\
&= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} p^{-n} \frac{0 - \chi(p^{-m}y_0)}{p^{-l(1+s)}} (1 - \frac{1}{p})|p^{l+n}|_p \\
&= \frac{1}{\Gamma_p(-s)} \sum_{l=-\infty}^{n-m-1} -\chi(p^{-m}y_0)(1-\frac{1}{p})p^{ls} \\
&= \frac{-\chi(p^{-m}y_0)}{\Gamma_p(-s)} \frac{p-1}{p^s-1} p^{s(n-m)-1}.
\end{aligned} \tag{3}$$

The second part, $l = n - m$, is more complicated. Note that $\mathbb{U}_p = \{a_0\} + p\mathbb{Z}_p$, where a_0 is the first digit of numbers in $\mathbb{U}_p$. So we can do integrals in this way as well:

$$\int\limits_{\mathbb{U}_p} dy = |\{a_0\}| \times \int\limits_{p\mathbb{Z}_p} dx = (p-1)|p|_p = 1 - \frac{1}{p}. \tag{4}$$

In this sense we can do the splitting infinitely as $p^{n-m}(\{a_0\} + \{a_1\} + \{a_2\} + \{a_3\} + ... + \{a_{n-1}\} + p^n\mathbb{Z}_p)$, and the integral is equal to$(p-1)p^{n-1}|p^n|_p = 1 - \frac{1}{p}$ correctly.

Thus the integral region $p^{n-m}\mathbb{U}_p$ can be split into $p^{n-m}(\{a_0\} + \{a_1\} + \{a_2\} + \{a_3\} + ... + \{a_{n-1}\} + p^n\mathbb{Z}_p)$. For a given $y_0 \in \mathbb{U}_p$ and all $x_0 \in \mathbb{U}_p$, there is 1 case where they have the same first digit(where $|x_0 - y_0|_p$ is determined by following digits) and $p-2$ cases where they have different first digits(where $|x_0 - y_0|_p = 1$) ; there is 1 case where they have the same second digit(where $|x_0 - y_0|_p$ is determined by following digits) and $p-1$ cases where they have different second digits(where $|x_0 - y_0|_p = p^{-1}$); there is 1 case where they have the same third digit(where $|x_0 - y_0|_p$ is determined by following digits) and $p-1$ cases where they have different third digits(where $|x_0 - y_0|_p = p^{-2}$);...Continue this spiting till $n \to \infty$ and we should obtain the summation form of the integral.

Note that here different digits are weighted by $\chi(p^{-n}x) = \chi(p^{-m}x_0)$ and the summation is trickier. Consider for example for a **particular** first digit of x_0, $a_0 \neq b_0$, which is the first digit of y_0. Without any pre-factor and constant term, the integral is:

$$\begin{aligned}
&\int\limits_{p\mathbb{Z}_p} dx_0 \frac{\chi(p^{-m}(a_0 + a_1 p + ...))}{|x_0 - y_0|_p^{1+s}} \\
&= \int\limits_{\mathbb{Z}_p} dx_1 \chi(p^{-m}a_0)\chi(p^{-m+1}x_1) \\
&= S_0 \chi(p^{-m}a_0).
\end{aligned} \tag{5}$$

We define S_0 to be the remaining integral of $\chi(p^{-m+1}x_1)$. Then we do the summation over all $a_0 \neq b_0$:

$$\begin{aligned}
&\sum_{a_0=1,\neq b_0}^{p-1} S_0 \chi(p^{-m}a_0) \\
&= S_0\left(\frac{e^{2\pi i p^{1-m}-1}}{e^{2\pi i p^{-m}} - 1} - 1 - \chi(p^{-m}b_0)\right) \\
&= S_0(\Sigma_p(m) - \chi(p^{-m}b_0) - 1).
\end{aligned} \tag{6}$$

This discussion is valid for all digits, except that for not the first digits we have summations that start from $a_i = 0$ and the result is $S_i(\Sigma_p(m) - \chi(p^{-m}b_i))$ for $i \geq 1$.

The derivative is:

$$
\begin{aligned}
&\frac{1}{\Gamma_p(-s)}\int\limits_{p^{n-m}\mathbb{U}_p} dx\frac{\chi(p^{-n}x)-\chi(p^{-m}y_0)}{|p^{n-m}(x_0-y_0)|_p^{1+s}}\\
&=\frac{1}{\Gamma_p(-s)}\int\limits_{p^{n-m}\mathbb{U}_p} dx\frac{\chi(p^{-n}x)-\chi(p^{-m}y_0)}{|p^{n-m}(x_0-y_0)|_p^{1+s}}
\end{aligned} \tag{7}
$$

The last term, $l>n-m$, is:

$$
\begin{aligned}
&\frac{1}{\Gamma_p(-s)}\sum_{l=n-m+1}^{\infty}\int\limits_{p^l\mathbb{U}_p} dx\frac{\chi(p^{-n}x)-\chi(p^{-m}y_0)}{|p^{n-m}y_0|_p^{1+s}}\\
&=\frac{1}{\Gamma_p(-s)}p^{(n-m)(1+s)}\sum_{l=n-m+1}^{\infty}\int\limits_{p^l\mathbb{U}_p} dx(\chi(p^{-n}x)-\chi(p^{-m}y_0))\\
&=\frac{1}{\Gamma_p(-s)}p^{(n-m)(1+s)}(\sum_{l=n-m+1}^{\infty}\int\limits_{p^l\mathbb{U}_p} dx\chi(p^{-n}x)-\sum_{l=n-m+1}^{\infty}p^{-l}(1-\frac{1}{p})\chi(p^{-m}y_0))\\
&=\frac{1}{\Gamma_p(-s)}p^{(n-m)(1+s)}(\sum_{l=n-m+1}^{\infty}\int\limits_{p^l\mathbb{U}_p} dx\chi(p^{-n}x)-p^{-1+m-n}\chi(p^{-m}y_0))
\end{aligned} \tag{8}
$$

While this integral $\sum_{l=n-m+1}^{\infty}\int\limits_{p^l\mathbb{U}_p} dx\chi(p^{-n}x)$ should be considered in two cases:

- When $m=1$, this is $\sum_{l=n}^{\infty}p^l(1-\frac{1}{p})=p^{-n}$;
- When $m\geq 2$, this is $-p^{-n}+\sum_{l=n}^{\infty}p^l(1-\frac{1}{p})=0$.

So in this region, the integral becomes:

$$
\begin{cases}
\frac{1}{\Gamma_p(-s)}p^{(n-m)s-1}(p^{1-m}-\chi(p^{-m}y_0)), & m=1\\
\frac{-1}{\Gamma_p(-s)}p^{(n-m)s-1}\chi(p^{-m}y_0), & m\geq 2
\end{cases} \tag{9}
$$