Physics 539 Problem Set 1 Solutions*

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Problem 1

As usual, we use "mostly plus" signature for $\eta_{\mu\nu}$. Working directly in terms of the components of the stress tensor, we have in static gauge that

$$\begin{aligned}
\det g_{\alpha\beta} &= \begin{vmatrix} -1+\partial_0 X^i \partial_0 X_i & \partial_0 X^i \partial_1 X_i & \partial_0 X^i \partial_2 X_i \\ \partial_1 X^i \partial_0 X_i & 1+\partial_1 X^i \partial_1 X_i & \partial_1 X^i \partial_2 X_i \\ \partial_2 X^i \partial_0 X_i & \partial_2 X^i \partial_1 X_i & 1+\partial_2 X^i \partial_2 X_i \end{vmatrix} \\
&= \begin{vmatrix} -1+T_{11}+T_{22} & T_{01} & T_{02} \\ T_{01} & 1+T_{00}-T_{22} & T_{12} \\ T_{02} & T_{12} & 1+T_{00}-T_{11} \end{vmatrix} \\
&= -1+2(-T_{00}+T_{11}+T_{22})+3(T_{00}T_{11}+T_{00}T_{22}-T_{11}T_{22}) \\
&\quad -T_{00}^2-T_{11}^2-T_{22}^2-T_{01}^2-T_{02}^2+T_{12}^2+\cdots \\
&= -1+2T_\alpha{}^\alpha+\tfrac{1}{2}T_{\alpha\beta}T^{\alpha\beta}-\tfrac{3}{2}(T_\alpha{}^\alpha)^2+\cdots,
\end{aligned} \tag{1}$$

where we have kept only terms up to fourth order in derivatives (or second order in $T_{\alpha\beta}$). This form makes manifest that the expansion of the action to arbitrary order can be written in terms of $T_{\alpha\beta}$. Taylor expanding then gives

$$S = -\mathcal{T}\int d^3\sigma\sqrt{-\det g_{\alpha\beta}} = -\mathcal{T}\int d^3\sigma\left(1-T_\alpha{}^\alpha-\frac{1}{4}T_{\alpha\beta}T^{\alpha\beta}+\frac{1}{4}(T_\alpha{}^\alpha)^2+\cdots\right). \tag{2}$$

The first term evaluates to $-\mathcal{T}L_1L_2\int d\tau$, where L_1 and L_2 are the respective lengths of the σ^1 and σ^2 intervals. Hence $\mathcal{T}$ has the interpretation as the surface tension of the membrane.

Problem 2

(a)

We follow the conventions of Becker-Becker-Schwarz (in particular, $\sigma \sim \sigma + \pi$). Let us make the ansatz

$$\begin{aligned}
X^0 &= \#\tau, \\
X^1 &= R\cos(a\tau)\cos(2\sigma), \\
X^2 &= R\sin(a\tau)\cos(2\sigma),
\end{aligned} \tag{3}$$

and $X^{i>2}=0$, where R is interpreted as the radius of the folded string. In conformal gauge, the equation of motion

$$\partial_+\partial_- X^\mu = 0 \tag{4}$$

*Please send comments or questions to Yale Fan, yalefan@gmail.com.

fixes $a = \pm 2$ (the sign determines the direction in which the string spins). The constraints

$$\partial_+ X^\mu \partial_+ X_\mu = \partial_- X^\mu \partial_- X_\mu = 0 \tag{5}$$

fix the coefficient # to be $2R$ (again, up to a sign that we choose to be +).

(b)

The folds occur at $\sigma = 0, \pi/2$. We compute that

$$\dot{X}^\mu = 2R(1, -\sin(2\tau)\cos(2\sigma), \cos(2\tau)\cos(2\sigma), \vec{0}) \implies \dot{X}^\mu \dot{X}_\mu|_{\sigma=0,\pi/2} = 0. \tag{6}$$

Hence $\dot{X}^\mu$ is lightlike at $\sigma = 0, \pi/2$, which implies that the folds move at the speed of light.

(c)

The conserved worldsheet currents corresponding to center-of-mass momentum and angular momentum in spacetime are

$$P^\mu_\alpha = T\partial_\alpha X^\mu, \tag{7}$$
$$J^{\mu\nu}_\alpha = T(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu) \tag{8}$$

(these are the Noether currents for Poincaré symmetry on the worldsheet). The corresponding conserved charges are

$$P^\mu = \int_0^\pi d\sigma\, P^\mu_0 = T\int_0^\pi d\sigma\, \dot{X}^\mu, \tag{9}$$
$$J^{\mu\nu} = \int_0^\pi d\sigma\, J^{\mu\nu}_0 = T\int_0^\pi d\sigma\, (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu). \tag{10}$$

For our solution, we compute that

$$(P^0, P^1, P^2) = (2\pi RT, 0, 0), \tag{11}$$
$$(J^{01}, J^{02}, J^{12}) = (0, 0, \pi R^2 T). \tag{12}$$

Hence the relation between energy and angular momentum is

$$J^{12} = \frac{\alpha'}{2}(P^0)^2, \tag{13}$$

with $T = (2\pi\alpha')^{-1}$. For an open string with Neumann boundary conditions, we would find that

$$J^{12} = \alpha'(P^0)^2 \tag{14}$$

(see Becker-Becker-Schwarz, Problem 2.1). Either way, we obtain a Regge trajectory.

(d)

In conformal gauge, the Polyakov action simplifies to

$$S = -\frac{T}{2}\int d^2\sigma\, \partial^\alpha X^\mu \partial_\alpha X_\mu = \frac{T}{2}\int d^2\sigma\, (\dot{X}^2 - X'^2). \tag{15}$$

For our solution, $\partial^\alpha X^\mu \partial_\alpha X_\mu = 8R^2 \sin^2(2\sigma)$, and computing the action for one period gives

$$S = -4R^2 T \int_0^{\pi/2} d\tau \int_0^\pi d\sigma\, \sin^2(2\sigma) = -\pi^2 R^2 T = -\pi J^{12}. \tag{16}$$

Here, it is important to note that one period corresponds to τ going from 0 to $\pi/2$, since the two ends of the string are physically indistinguishable. Imposing the Bohr-Sommerfeld quantization condition then yields

$$J^{12} = 2n\hbar, \tag{17}$$

so the spin is quantized to be an *even* integer.

As a check of this result, recall that in the quantum theory of the closed bosonic string, the mass spectrum takes the form $\frac{1}{4}\alpha' M^2 \sim$ integer, up to a normal-ordering shift.[1] Combining this fact with (13) gives precisely $J \sim 2(\text{integer})$, as we found. By contrast, the open bosonic string has $\alpha' M^2 \sim$ integer and therefore, by (14), $J \sim$ integer.

Problem 3

We now make the ansatz (in conformal gauge)

$$\begin{aligned} X^0 &= 2R\tau, \\ X^1 &= R\cos(2\tau)\cos(2\sigma), \\ X^2 &= R\cos(2\tau)\sin(2\sigma), \end{aligned} \tag{18}$$

$X^{i>2} = 0$. One can check that with the coefficients above, the equation of motion (4) and the constraints (5) are satisfied. For this solution, we compute that $P^0 = 2\pi RT$ while the angular momentum vanishes, as expected. The action computed over one period (again, $\tau \in [0, \pi/2)$) is

$$S = -4R^2T\int_0^\pi d\sigma \int_0^{\pi/2} d\tau\ \cos^2(2\tau) = -\pi^2R^2T = -\frac{1}{4T}(P^0)^2, \tag{19}$$

so the Bohr-Sommerfeld quantization condition yields

$$(P^0)^2 = \frac{4n\hbar}{\alpha'} \tag{20}$$

(again, with $T = (2\pi\alpha')^{-1}$).

Problem 4

Let us first solve the problem with arbitrary momentum and winding, and then comment on the intended solution.

To solve for the exact spectrum of transverse oscillations of the wrapped string, write

$$X^\mu(\sigma,\tau) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \tag{21}$$

with $\sigma^\pm = \tau \pm \sigma$. The mode expansions of the right- and left-movers are

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \ell_s\alpha_0^\mu\sigma^- + \frac{i\ell_s}{2}\sum_{n\neq0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}, \tag{22}$$

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \ell_s\tilde{\alpha}_0^\mu\sigma^+ + \frac{i\ell_s}{2}\sum_{n\neq0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}. \tag{23}$$

The fact that the string wraps the X^1 circle w times requires the boundary condition

$$X^1(\sigma+\pi,\tau) = X^1(\sigma,\tau) + wL \tag{24}$$

[1] Which is, in fact, an integer for the critical string.

(this is consistent because $X^1 \sim X^1 + L$), which implies that

$$\alpha_0^1 - \tilde{\alpha}_0^1 = -\frac{wL}{\pi \ell_s}. \tag{25}$$

On the other hand, the coefficient of τ in $X^\mu(\sigma,\tau)$ is $\ell_s^2 p^\mu$ where p^μ is the center-of-mass momentum of the string: hence

$$\alpha_0^\mu + \tilde{\alpha}_0^\mu = \ell_s p^\mu. \tag{26}$$

Combining (25) and (26) gives

$$\alpha_0^1 = \frac{1}{2}\left(\ell_s p^1 - \frac{wL}{\pi\ell_s}\right), \quad \tilde{\alpha}_0^1 = \frac{1}{2}\left(\ell_s p^1 + \frac{wL}{\pi\ell_s}\right), \tag{27}$$

while $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{\ell_s}{2} p^\mu$ for $\mu \neq 1$. Now recall that in terms of the number operators for right- and left-moving excitations, we have

$$L_0 = \frac{1}{2}\alpha_0^2 + N, \quad \tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^2 + \tilde{N}. \tag{28}$$

At the quantum level, the mass-shell condition is that physical states satisfy

$$(L_0 - a)|\phi\rangle = (\tilde{L}_0 - a)|\phi\rangle = 0 \tag{29}$$

where $a = \frac{D-2}{24}$; moreover, the momentum p^1 is quantized as

$$p^1 = \frac{2\pi k}{L}, \quad k \in \mathbb{Z}. \tag{30}$$

Finally, we are actually interested in the effective energy spectrum from the point of view of an observer in the $D-1$ noncompact dimensions, so that we may write

$$M^2 = (p^0)^2 - \sum_{i=2}^{D-1} (p^i)^2 = -p^2 + (p^1)^2. \tag{31}$$

Combining (27)–(31) and setting $\alpha' = \ell_s^2/2$ (hence $T = (\pi\ell_s^2)^{-1}$) shows that

$$M^2 = \left(\frac{2\pi k}{L}\right)^2 + \left(\frac{wL}{2\pi\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2a), \tag{32}$$

where the level-matching condition $L_0 = \tilde{L}_0$ gives $N - \tilde{N} = wk$. The momentum and winding contributions are exchanged by T-duality.

Let us now examine the degeneracies of the first three energy levels in the large-L limit. In this limit, the k-dependent term in the formula (32) is negligible compared to the $N + \tilde{N}$ term. Indeed, at large L, we have

$$M = \underbrace{\frac{wL}{2\pi\alpha'}}_{wTL} + \frac{2\pi}{wL}(N + \tilde{N} - 2a) + O(L^{-3}), \tag{33}$$

which has no explicit dependence on k up to $O(L^{-1})$. Since $N, \tilde{N} \geq 0$, we have (for $w = 1$) that:

- The ground state has $N + \tilde{N} = 0$ ($k = 0$) and is nondegenerate.
- The first excited states have $N + \tilde{N} = 1$, so that $(N, \tilde{N}) = (1,0)$ or $(0,1)$ and $k = 1, -1$. They are $2(D-2)$-fold degenerate and arise from excitations of the transverse oscillator modes α_{-1}^i and $\tilde{\alpha}_{-1}^i$ ($i = 1, \ldots, D-2$).

- The second excited states have $N + \tilde{N} = 2$. From the large-L formula (33), one would conclude that $(N, \tilde{N}) = (2, 0)$ or $(1, 1)$ or $(0, 2)$ and $k = 2, 0, -2$. For $k = \pm 2$, there are

$$(D-2) + \binom{D-1}{2}$$

states coming from excitations of $\{\alpha^i_{-2}, \alpha^i_{-1}\alpha^j_{-1}\}$ and $\{\tilde{\alpha}^i_{-2}, \tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}\}$. However, the exact formula (32) reveals that these states are actually slightly higher than the $k = 0$ states. Hence the second excited states have only $k = 0$ and are $(D-2)^2$-fold degenerate, coming from excitations of $\{\alpha^i_{-1}, \tilde{\alpha}^i_{-1}\}$.

Strictly speaking, the above makes sense only for $a = 1$ and $D = 26$. We have been implicitly working in light-cone gauge, where the oscillator modes of X^+ are eliminated and those of X^- are fixed by those of X^i via the Virasoro constraints, leaving only transverse oscillators. These transverse modes can be used to construct physical states.

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Having said all this, one could make several objections to the above solution. For example, the meaning of "transverse" in light-cone gauge ($i = 1, \ldots, D-2$) differs from the physical meaning of "transverse" for this string configuration ($i = 2, \ldots, D-1$). So why is light-cone gauge a valid way to get the spectrum? Moreover, why can we treat the oscillators in the compact direction ($i = 1$) in the same way as those in the noncompact directions?

We will largely ignore these objections, but here are some half-hearted justifications. First, it is in fact possible to adapt light-cone gauge to single out X^0, X^1, despite the different boundary conditions in these directions. Second, the winding boundary condition only affects α^1_0, $\tilde{\alpha}^1_0$ and not the α^1_{-n}, $\tilde{\alpha}^1_{-n}$ (which we use to construct states).

The intended solution (below) is to assume that $p^1 = 0$, which simplifies the level-matching condition and changes the counting of degeneracies, as well as $w = 1$.

(a)

To get the approximate spectrum, we work perturbatively in static gauge ($X^0 = \tau$ and $X^1 = \sigma$, the latter because $w = 1$) and *assume* that all states created by transverse oscillators (in the physical sense) are physical.

By expanding the Nambu-Goto action to leading order, we obtain the Hamiltonian

$$H = TL + \frac{T}{2}\int_0^L d\sigma\,((\dot{X}^i)^2 + (X'^i)^2) \tag{34}$$

for transverse excitations ($i = 2, \ldots, D-1$). Setting the center-of-mass momentum p^i to zero, we obtain the spectrum simply from $E = p^0$. Substituting the relevant mode expansions gives

$$H = TL + \frac{2\pi}{L}(N + \tilde{N} - 2a), \tag{35}$$

where $p^1 = 0$ implies the level-matching condition $N = \tilde{N}$ and the a-dependent shift comes from normal ordering. At levels $N + \tilde{N} = 0, 1, 2$, we obtain degeneracies of

$$1, \quad (D-2)^2, \quad \left[(D-2) + \binom{D-1}{2}\right]^2, \tag{36}$$

respectively (think "closed string = (open string)^2").

(b)

The exact formula, via the mass-shell condition, is

$$E = \sqrt{(TL)^2 + 8\pi T(N-a)} = TL + \frac{4\pi}{L}(N-a) + O(L^{-3}) \tag{37}$$

where $N = \tilde{N}$.