Homework 1

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PHY 539: Introduction to String Theory

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1   Problem 1

Adopting the static gauge, we have:

$$-det g_{\alpha\beta} = 1 - \partial_0 X^i \partial_0 X_i + \partial_1 X^i \partial_1 X_i + \partial_2 X^i \partial_2 X_i. \tag{1} \boxed{\texttt{Det}}$$

Define $g'_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X_i$, then

$$-det g_{\alpha\beta} = 1 + g'_{\alpha\beta}\eta^{\alpha\beta}. \tag{2} \boxed{\texttt{DetShort}}$$

Consider derivatives of $X^i(\sigma^0, \sigma^1, \sigma^2)$ as small quantities and expand action to the fourth order of them:

$$S = -\mathcal{T} \int d^3\sigma (1 + \frac{1}{2} g'_{\alpha\beta}\eta^{\alpha\beta} - \frac{1}{8}(g'_{\gamma\delta}\eta^{\gamma\delta})^2 + \mathcal{O}(\partial_i X^j)^6). \tag{3} \boxed{\texttt{Expansion}}$$

Contract the stress-energy tensor with $\eta^{\alpha\beta}$, we have $T_{\alpha\beta}\eta^{\alpha\beta} = -\frac{1}{2}g'_{\alpha\beta}\eta^{\alpha\beta}$. So the fourth order term is:

$$-\frac{1}{8}(g'_{\alpha\beta}\eta^{\alpha\beta})^2 = -\frac{1}{2}(T_\beta^\alpha)^2. \tag{4} \boxed{\texttt{Fourth}}$$

2   Problem 2

a)   This is the same as two straight open strings that their endpoints coincide. Adopting the open string solution:

$$X^0 = A\tau, X^1 = A\cos\tau\cos\sigma, X^2 = A\sin\tau\cos\sigma, \tag{5} \boxed{\texttt{FoldSol}}$$

but let σ runs from 0 to 2π.

b) At any endpoint, for example $\sigma = 0$, we have:

$$\sqrt{(\frac{dX^1}{dX^0})^2 + (\frac{dX^2}{dX^0})^2} = \sqrt{(-\sin\tau)^2 + (\cos\tau)^2} = 1. \tag{6} \boxed{\texttt{FoldSpeed}}$$

So it moves at the speed of light.

c) Energy can be obtained by:

$$E = P^0 = \mathcal{T} \int_0^{2\pi} d\sigma \frac{dX^0}{d\tau} = 2\pi A\mathcal{T}. \tag{7} \boxed{\texttt{FoldEnergy}}$$

While angular momentum is:

$$J = \mathcal{T} \int_0^{2\pi} d\sigma (X^1 \frac{dX^2}{d\tau} - X^2 \frac{dX^1}{d\tau}) = \pi A^2 \mathcal{T}. \tag{8} \boxed{\texttt{FoldAngular}}$$

So we have $J = \frac{1}{4\pi T} E^2$. The slope is half of the open string case.

d) One rotating period is 2π. So we have:

$$2\pi\hbar n = |-\frac{\mathcal{T}}{2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\tau (-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu)| = 2\pi^2 A^2 \mathcal{T}. \tag{9} \boxed{\texttt{SemiQuanti}}$$

So the angular momentum is $J = \pi A^2 \mathcal{T} = n\hbar$.

3 Problem 3

Let σ be from 0 to 2π, a pulsating solution is:

$$X^0 = R\tau, X^1 = R\cos\tau\cos\sigma, X^2 = R\cos\tau\sin\sigma. \tag{10} \boxed{\texttt{PulsatingS}}$$

Energy of this string is:

$$E = P^0 = \mathcal{T} \int_0^{2\pi} d\sigma \frac{dX^0}{d\tau} = 2\pi R\mathcal{T}. \tag{11} \boxed{\texttt{PulsatingE}}$$

Using the semi-classical quantization condition, we have(one period is π):

$$2\pi\hbar n = |-\frac{\mathcal{T}}{2} \int_0^{2\pi} d\sigma \int_0^\pi d\tau (-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu)| = \pi^2 R^2 \mathcal{T}. \tag{12} \boxed{\texttt{SemiQuanti}}$$

Thus, $E = \sqrt{2\pi n\hbar\mathcal{T}}$.

4   Problem 4

a)   The perturbation process goes as follows: Using the static gauge $\tau = X^0, \sigma = X^1$ for Nambu-Goto action, and expand the squared root; By treating the derivatives as small quantities, we can approximate the original action by just the first two terms. In this manner, we obtain an approximate quadratic action but without any constraints:

$$
\begin{aligned}
L_{string} &= -\mathcal{T} \int_0^L d\sigma \sqrt{-det g_{\alpha\beta}} \\
&= -\mathcal{T} \int_0^L d\sigma \sqrt{1 - \partial_0 X^i \partial_0 X_i + \partial_1 X^i \partial_1 X_i + \mathcal{O}(\partial_\alpha X^i)^4} \qquad (13) \quad \boxed{\texttt{StaticNambu}} \\
&= -\mathcal{T}L - \frac{\mathcal{T}}{2} \int_0^L d\sigma (-\partial_0 X^i \partial_0 X_i + \partial_1 X^i \partial_1 X_i + \mathcal{O}(\partial_\alpha X^i)^4).
\end{aligned}
$$

. Then the Hamiltonian is:

$$
H = \mathcal{T}(\dot{X}^i)^2 - L_{string} = \mathcal{T}L + \frac{2\pi}{L} \sum_{-\infty}^{\infty} (a^i_{-n} a^i_n + \widetilde{a}^i_{-n} \widetilde{a}^i_n). \qquad (14) \quad \boxed{\texttt{Hamiltonian}}
$$

Now the remaining transverse modes are all of positive signature and can be quantized as usual harmonic oscillators. This is to say the spectrum is:

$$
E = \mathcal{T}L + \frac{2\pi}{L}(N + \widetilde{N}). \qquad (15) \quad \boxed{\texttt{Spectrum1}}
$$

First three eigenstates are $(N + \widetilde{N}) = 0$ (degeneracy 1), $(N + \widetilde{N}) = 1$ (degeneracy $D - 2 + D - 2 = 2D - 4$) and $(N + \widetilde{N}) = 2$ (degeneracy $2D - 4 + 2(D-2)^2/2 + (D-2)^2 = 2(D-2)(D-1)$).(The degeneracy is very different from the non-perturbative quantization? No level matching constraints.)

b)   (Assume that X^1 is a "pure winding" direction, so we do not consider p^1.)To derive the exact spectrum, we adopt the canonical quantization and consider all the constraints. We first derive the modes in X^1 direction. In this so-called "winding sector" $\alpha^1_0 = -\widetilde{\alpha}^1_0$, we have $X^1 = X^1_L + X^1_R = 2\widetilde{\alpha}^1_0 \sqrt{\frac{\alpha'}{2}} \sigma$ (no momentum in this direction). Imposing the boundary condition, we have $\sqrt{\frac{\alpha'}{2}} \widetilde{\alpha}^1_0 = \frac{L}{4\pi}$. Then the next convenient thing to do is to set

$X^0 = 2\alpha_0^0 \tau \sqrt{\frac{\alpha'}{2}}$ and we have $E = \sqrt{\frac{\alpha'}{2}} \widetilde{\alpha}_0^0$. At last we impose the constraints:

$$L_0 = \widetilde{L}_0 = a$$

$$-\frac{(\alpha_0^0)^2}{2} + \frac{(\alpha_0^1)^2}{2} + \sum_{n=2}^{D-1} \alpha_{-n}^i \alpha_n^i = a \qquad (16) \quad \boxed{\text{Constraint}}$$

$$-\frac{E^2}{4}\alpha' + \frac{L^2}{16\pi^2\alpha'} + N - a = 0.$$

The constant being $\alpha' = \frac{1}{2\pi\mathcal{T}}$, we finally have:

$$E = \sqrt{(\mathcal{T}L)^2 + 8\pi\mathcal{T}(N - a)}. \qquad (17) \quad \boxed{\text{ExactSpect}}$$

correction to a) Constraints from choosing the static gauge should be considered properly:

$$\frac{\partial\sqrt{-det g_{\alpha\beta}}}{\partial\tau} = \frac{1}{2\sqrt{-det g_{\alpha\beta}}}\frac{\partial(-det g_{\alpha\beta})}{\partial\tau} = 0 \tag{18}$$

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