Homework 1

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PHY 539: Introduction to String Theory

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1 Problem 1

Adopting the static gauge, we have:

$$-detg_{\alpha\beta} = 1 - \partial_0 X^i \partial_0 X_i + \partial_1 X^i \partial_1 X_i + \partial_2 X^i \partial_2 X_i \tag{1}$$ $\boxed{1}$

Define $g'_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X_i$, then

$$-detg_{\alpha\beta} = 1 + g'_{\alpha\beta}\eta^{\alpha\beta} \tag{2}$$ $\boxed{2}$

Consider derivatives of $X^i(\sigma^0, \sigma^1, \sigma^2)$ as small quantities and expand action to the fourth order of them:

$$S = -\mathcal{T} \int d^3\sigma (1 + \frac{1}{2} g'_{\alpha\beta}\eta^{\alpha\beta} - \frac{1}{8}(g'_{\gamma\delta}\eta^{\gamma\delta})^2) \tag{3}$$ $\boxed{3}$

Contract the stress-energy tensor with $\eta^{\alpha\beta}$, we have $T_{\alpha\beta}\eta^{\alpha\beta} = -\frac{1}{2} g'_{\alpha\beta}\eta^{\alpha\beta}$. So the fourth order term is:

$$-\frac{1}{8}(g'_{\alpha\beta}\eta^{\alpha\beta})^2 = -\frac{1}{2}(T^\alpha_\beta)^2 \tag{4}$$ $\boxed{4}$

2 Problem 2

a) This is the same as two straight open strings that their endpoints coincide. Adopting the open string solution:

$$X^0 = A\tau, X^1 = A\cos\tau\cos\sigma, X^2 = A\sin\tau\cos\sigma \tag{5}$$ $\boxed{5}$

but let σ runs from 0 to 2π.

b) At any endpoint, for example $\sigma = 0$, we have:

$$\sqrt{(\frac{dX^1}{dX^0})^2 + (\frac{dX^2}{dX^0})^2} = \sqrt{(-\sin\tau)^2 + (\cos\tau)^2} = 1 \tag{6} \boxed{6}$$

So it moves at the speed of light.

c) Energy can be obtained by:

$$E = P^0 = \mathcal{T}\int_0^{2\pi} d\sigma \frac{dX^0}{d\tau} = 2\pi A\mathcal{T} \tag{7} \boxed{7}$$

While angular momentum is:

$$J = \mathcal{T}\int_0^{2\pi} d\sigma (X^1 \frac{dX^2}{d\tau} - X^2 \frac{dX^1}{d\tau}) = \pi A^2 \mathcal{T} \tag{8} \boxed{8}$$

So we have $J = \frac{1}{4\pi T}E^2$. The slope is half of the open string case.

d) One rotating period is 2π. So we have:

$$2\pi\hbar n = \mathcal{T}\int_0^{2\pi} d\sigma \int_0^{2\pi} d\tau (-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu) = 4\pi^2 A^2 \mathcal{T} \tag{9} \boxed{9}$$

So the angular momentum is $J = \pi A^2 \mathcal{T} = \frac{1}{2}\hbar n$.

3 Problem 3

Let σ be from 0 to 2π, a pulsating solution is:

$$X^0 = A\tau, X^1 = A\cos\tau\cos\sigma, X^2 = A\cos\tau\sin\sigma \tag{10} \boxed{10}$$

Using the semi-classical quantization condition, we have:

$$2\pi\hbar n = \mathcal{T}\int_0^{2\pi} d\sigma \int_0^{2\pi} d\tau (-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu) = 4\pi^2 A^2 \mathcal{T} \tag{11} \boxed{11}$$