Homework 2

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PHY 539: Introduction to String Theory

October 10, 2018

1 Problem 1

First we find the mode expansion of the space-time Lorentz generator $J^{\mu\nu}$ of a bosonic open string. By definition we have

$$J^{\mu\nu} = \int_0^\pi J_0^{\mu\nu} d\sigma = \frac{1}{\pi l_s^2} \int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) d\sigma. \tag{1}$$

Plugging in the mode expansion of X field and its derivative:

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos(m\sigma) \\ \dot{X}^\mu(\tau,\sigma) &= l_s^2 p^\mu + l_s \sum_{m\neq 0} \alpha_m^\mu e^{-im\tau} \cos(m\sigma), \end{aligned} \tag{2}$$

we obtain(omitting terms that vanish after integration):

$$\begin{aligned} J^{\mu\nu} =& \frac{1}{\pi l_s^2} \int_0^\pi d\sigma (l_s^2 x^\mu p^\nu + l_s^4 p^\mu p^\nu \tau + i l_s^2 \cos(m\sigma)^2 \sum_{m=1}^\infty \frac{1}{m} (\alpha_m^\mu \alpha_{-m}^\nu - \alpha_{-m}^\mu \alpha_m^\nu + \alpha_m^\mu \alpha_m^\nu e^{-2im\tau} - \alpha_{-m}^\mu \alpha_{-m}^\nu e^{2im\tau}) \\ & - l_s^2 x^\nu p^\mu - l_s^4 p^\nu p^\mu \tau - i l_s^2 \cos(m\sigma)^2 \sum_{m=1}^\infty \frac{1}{m} (\alpha_m^\nu \alpha_{-m}^\mu - \alpha_{-m}^\nu \alpha_m^\mu + \alpha_m^\nu \alpha_m^\mu e^{-2im\tau} - \alpha_{-m}^\nu \alpha_{-m}^\mu e^{2im\tau})) \\ =& \frac{1}{\pi l_s^2} [\pi l_s^2 (x^\mu p^\nu - x^\nu p^\mu) + i l_s^2 \frac{\pi}{2} \sum_{m=1}^\infty \frac{1}{m} (2\alpha_m^\mu \alpha_{-m}^\nu - 2\alpha_m^\nu \alpha_{-m}^\mu)] \\ =& x^\mu p^\nu - x^\nu p^\mu + i \sum_{m=1}^\infty \frac{1}{m} (\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu). \end{aligned} \tag{3}$$

Applying the canonical commutation relation:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [x^\mu, x^\nu] = [p^\mu, p^\nu] = 0, \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{4}$$

we read that $[p^\mu, p^\nu] = 0$ immediately. And it is not hard to see that

$$[p^\mu, J^{\nu\sigma}] = [p^\mu, x^\nu p^\sigma - x^\sigma p^\nu] = i\eta^{\sigma\mu} p^\nu - i\eta^{\nu\mu} p^\sigma. \tag{5}$$

Lastly, $[J^{\mu\nu}, J^{\sigma\lambda}]$ can be separated into two terms:

$$\begin{aligned}[J^{\mu\nu}, J^{\sigma\lambda}] &= [x^\mu p^\nu - x^\nu p^\mu, x^\sigma p^\lambda - x^\lambda p^\sigma] \\ &- [\sum_{m=1}^{\infty} \frac{1}{m}(\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu), \sum_{m=1}^{\infty} \frac{1}{m}(\alpha_m^\sigma \alpha_{-m}^\lambda - \alpha_m^\lambda \alpha_{-m}^\sigma)]\end{aligned} \tag{6}$$

First we have $[x^\mu p^\nu, x^\sigma p^\lambda] = i\eta^{\mu\lambda} x^\sigma p^\nu - i\eta^{\sigma\nu} x^\mu p^\lambda$. Applying this we get:

$$\begin{aligned}[x^\mu p^\nu - x^\nu p^\mu, x^\sigma p^\lambda - x^\sigma p^\lambda] = i\eta^{\mu\lambda} x^\sigma p^\nu - i\eta^{\sigma\nu} x^\mu p^\lambda - i\eta^{\mu\sigma} x^\lambda p^\nu + i\eta^{\lambda\nu} x^\mu p^\sigma \\ - i\eta^{\nu\lambda} x^\sigma p^\mu + i\eta^{\sigma\mu} x^\nu p^\lambda + i\eta^{\nu\sigma} x^\lambda p^\mu - i\eta^{\lambda\mu} x^\nu p^\sigma.\end{aligned} \tag{7}$$

Second, after applying $[\alpha_m^\mu \alpha_{-m}^\nu, \alpha_m^\sigma \alpha_{-m}^\lambda] = m\eta^{\mu\lambda} a_m^\sigma a_{-m}^\nu - m\eta^{\nu\sigma} a_m^\mu a_{-m}^\lambda$, we have

$$\begin{aligned}[\sum_{m=1}^{\infty} \frac{1}{m}(\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu), \sum_{m=1}^{\infty} \frac{1}{m}(\alpha_m^\sigma \alpha_{-m}^\lambda - \alpha_m^\lambda \alpha_{-m}^\sigma)] = \sum_{m=1}^{\infty} \frac{1}{m}(\eta^{\mu\lambda} a_m^\sigma a_{-m}^\nu - \eta^{\nu\sigma} a_m^\mu a_{-m}^\lambda \\ - \eta^{\mu\sigma} a_m^\lambda a_{-m}^\nu + \eta^{\nu\lambda} a_m^\mu a_{-m}^\sigma - \eta^{\nu\lambda} a_m^\sigma a_{-m}^\mu + \eta^{\mu\sigma} a_m^\nu a_{-m}^\lambda + \eta^{\nu\sigma} a_m^\lambda a_{-m}^\mu - \eta^{\mu\lambda} a_m^\nu a_{-m}^\sigma).\end{aligned} \tag{8}$$

It is not hard to see from now that matching equation 7 and 8, one can obtain the desired commutation relation for Lorentz generators:

$$[J^{\mu\nu}, J^{\sigma\lambda}] = -i\eta^{\nu\sigma} J^{\mu\lambda} + i\eta^{\mu\sigma} J^{\nu\lambda} + i\eta^{\nu\lambda} J^{\mu\sigma} - i\eta^{\mu\lambda} J^{\nu\sigma}. \tag{9}$$

2 Problem 2

a) First massive level of $D = 26$ open string is the combination of two kinds of excitations: $\alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle$ and $\alpha_{-2} |0; k\rangle$. A general state is $(s_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + v_\mu \alpha_{-2}^\mu) |0; k\rangle$. Physical state conditions are the constraints that $L_m |\phi\rangle = 0$ for $m > 0$ and $(L_0 - 1) |\phi\rangle = 0$. The second one, also called mass shell condition, gives $\alpha' M^2 = 1$ or $-\alpha' k^\mu k_\mu = 1$. The first one need to be examined more carefully. See what we have in hand:

$$\begin{aligned}L_m \alpha_{-2}^\mu |0; k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^\lambda \alpha_n^\sigma \eta_{\lambda\sigma} \alpha_{-2}^\mu |0; k\rangle = \frac{1}{2} \alpha_{m-2}^\lambda \alpha_2^\sigma \eta_{\lambda\sigma} \alpha_{-2}^\mu |0; k\rangle = \alpha_{m-2}^\mu |0; k\rangle \\ L_m \alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^\lambda \alpha_n^\sigma \eta_{\lambda\sigma} \alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle = (\frac{1}{2} \alpha_{m-1}^\mu \alpha_{-1}^\nu + \frac{1}{2} \alpha_{m-1}^\nu \alpha_{-1}^\mu) |0; k\rangle\end{aligned} \tag{10}$$

where we define sign^μ to be a vector $(-1, 1, 1, ..., 1)$. It is not hard to see that when $m > 2$, all these terms are manifestly zero. When $m = 1$, $L_1 |0; k\rangle = 0$ gives:

$$\begin{aligned}[s_{\mu\nu}(\frac{1}{2} \alpha_0^\mu \alpha_{-1}^\nu + \frac{1}{2} \alpha_0^\nu \alpha_{-1}^\mu) + v_\sigma \alpha_{-1}^\sigma)] |0; k\rangle = 0 \\ [s_{\mu\nu}(\frac{1}{4} l_s k^\mu \alpha_{-1}^\nu + \frac{1}{4} l_s k^\nu \alpha_{-1}^\mu) + v_\sigma \alpha_{-1}^\sigma)] |0; k\rangle = 0\end{aligned} \tag{11}$$