Homework 2

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PHY 539: Introduction to String Theory

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1 Problem 1

First we find the mode expansion of the space-time Lorentz generator $J^{\mu\nu}$ of a bosonic open string. By definition we have

$$J^{\mu\nu} = \int_0^\pi J_0^{\mu\nu} d\sigma = \frac{1}{\pi l_s^2}\int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) d\sigma. \tag{1}$$

Plugging in the mode expansion of X field and its derivative:

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos(m\sigma) \\ \dot{X}^\mu(\tau,\sigma) &= l_s^2 p^\mu + l_s \sum_{m\neq 0} \alpha_m^\mu e^{-im\tau}\cos(m\sigma), \end{aligned} \tag{2}$$

we obtain(omitting terms that vanish after integration):

$$\begin{aligned} J^{\mu\nu} =& \frac{1}{\pi l_s^2}\int_0^\pi d\sigma (l_s^2 x^\mu p^\nu + l_s^4 p^\mu p^\nu \tau + i l_s^2 \cos(m\sigma)^2 \sum_{m=1}^\infty \frac{1}{m}(\alpha_m^\mu \alpha_{-m}^\nu - \alpha_{-m}^\mu \alpha_m^\nu + \alpha_m^\mu \alpha_m^\nu e^{-2im\tau} - \alpha_{-m}^\mu \alpha_{-m}^\nu e^{2im\tau}) \\ & - l_s^2 x^\nu p^\mu - l_s^4 p^\nu p^\mu \tau - i l_s^2 \cos(m\sigma)^2 \sum_{m=1}^\infty \frac{1}{m}(\alpha_m^\nu \alpha_{-m}^\mu - \alpha_{-m}^\nu \alpha_m^\mu + \alpha_m^\nu \alpha_m^\mu e^{-2im\tau} - \alpha_{-m}^\nu \alpha_{-m}^\mu e^{2im\tau})) \\ =& \frac{1}{\pi l_s^2}[\pi l_s^2 (x^\mu p^\nu - x^\nu p^\mu) + i l_s^2 \frac{\pi}{2} \sum_{m=1}^\infty \frac{1}{m}(2\alpha_m^\mu \alpha_{-m}^\nu - 2\alpha_m^\nu \alpha_{-m}^\mu)] \\ =& x^\mu p^\nu - x^\nu p^\mu + i \sum_{m=1}^\infty \frac{1}{m}(\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu). \end{aligned} \tag{3}$$

Applying the canonical commutation relation:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [x^\mu, x^\nu] = [p^\mu, p^\nu] = 0, \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{4}$$

we read that $[p^\mu, p^\nu] = 0$ immediately. And it is not hard to see that

$$[p^\mu, J^{\nu\sigma}] = [p^\mu, x^\nu p^\sigma - x^\sigma p^\nu] = i\eta^{\sigma\mu} p^\nu - i\eta^{\nu\mu} p^\sigma. \tag{5}$$

Lastly, $[J^{\mu\nu}, J^{\sigma\lambda}]$ can be separated into two terms:

$$[J^{\mu\nu}, J^{\sigma\lambda}] = [x^\mu p^\nu - x^\nu p^\mu, x^\sigma p^\lambda - x^\lambda p^\sigma] \\ - [\sum_{m=1}^{\infty} \frac{1}{m}(\alpha^\mu_m \alpha^\nu_{-m} - \alpha^\nu_m \alpha^\mu_{-m}), \sum_{m=1}^{\infty} \frac{1}{m}(\alpha^\sigma_m \alpha^\lambda_{-m} - \alpha^\lambda_m \alpha^\sigma_{-m})] \tag{6}$$

First we have $[x^\mu p^\nu, x^\sigma p^\lambda] = i\eta^{\mu\lambda} x^\sigma p^\nu - i\eta^{\sigma\nu} x^\mu p^\lambda$. Applying this we get:

$$[x^\mu p^\nu - x^\nu p^\mu, x^\sigma p^\lambda - x^\sigma p^\lambda] = i\eta^{\mu\lambda} x^\sigma p^\nu - i\eta^{\sigma\nu} x^\mu p^\lambda - i\eta^{\mu\sigma} x^\lambda p^\nu + i\eta^{\lambda\nu} x^\mu p^\sigma \\ - i\eta^{\nu\lambda} x^\sigma p^\mu + i\eta^{\sigma\mu} x^\nu p^\lambda + i\eta^{\nu\sigma} x^\lambda p^\mu - i\eta^{\lambda\mu} x^\nu p^\sigma. \tag{7}$$

Second, after applying $[\alpha^\mu_m \alpha^\nu_{-m}, \alpha^\sigma_m \alpha^\lambda_{-m}] = m\eta^{\mu\lambda} a^\sigma_m a^\nu_{-m} - m\eta^{\nu\sigma} a^\mu_m a^\lambda_{-m}$, we have

$$[\sum_{m=1}^{\infty} \frac{1}{m}(\alpha^\mu_m \alpha^\nu_{-m} - \alpha^\nu_m \alpha^\mu_{-m}), \sum_{m=1}^{\infty} \frac{1}{m}(\alpha^\sigma_m \alpha^\lambda_{-m} - \alpha^\lambda_m \alpha^\sigma_{-m})] = \sum_{m=1}^{\infty} \frac{1}{m}(\eta^{\mu\lambda} a^\sigma_m a^\nu_{-m} - \eta^{\nu\sigma} a^\mu_m a^\lambda_{-m} \\ - \eta^{\mu\sigma} a^\lambda_m a^\nu_{-m} + \eta^{\nu\lambda} a^\mu_m a^\sigma_{-m} - \eta^{\nu\lambda} a^\sigma_m a^\mu_{-m} + \eta^{\mu\sigma} a^\nu_m a^\lambda_{-m} + \eta^{\nu\sigma} a^\lambda_m a^\mu_{-m} - \eta^{\mu\lambda} a^\nu_m a^\sigma_{-m}). \tag{8}$$

It is not hard to see from now that matching equation 7 and 8, one can obtain the desired commutation relation for Lorentz generators:

$$[J^{\mu\nu}, J^{\sigma\lambda}] = -i\eta^{\nu\sigma} J^{\mu\lambda} + i\eta^{\mu\sigma} J^{\nu\lambda} + i\eta^{\nu\lambda} J^{\mu\sigma} - i\eta^{\mu\lambda} J^{\nu\sigma}. \tag{9}$$

2 Problem 2

a) First massive level of $D = 26$ open string is the combination of two kinds of excitations: $\alpha^\mu_{-1}\alpha^\nu_{-1} |0; k\rangle$ and $\alpha_{-2} |0; k\rangle$. A general state is $(s_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + v_\mu \alpha^\mu_{-2}) |0; k\rangle$. Physical state conditions are the constraints that $L_m |\phi\rangle = 0$ for $m > 0$ and $(L_0 - 1) |\phi\rangle = 0$. The second one, also called mass shell condition, gives $\alpha' M^2 = 1$ or $-\alpha' k^\mu k_\mu = 1$. The first one need to be examined more carefully. It is not hard to see that only $L_1$and L_2 conditions are not manifestly zero:

$$\begin{aligned} L_1 \alpha^\mu_{-2} |0; k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\lambda_{1-n} \alpha^\sigma_n \eta_{\lambda\sigma} \alpha^\mu_{-2} |0; k\rangle = \frac{1}{2}(\alpha^\lambda_2 \alpha^\sigma_{-1} \eta_{\lambda\sigma} \alpha^\mu_{-2} + \alpha^\lambda_{-1} \alpha^\sigma_2 \eta_{\lambda\sigma} \alpha^\mu_{-2}) |0; k\rangle \\ &= (\alpha^\sigma_{-1} \eta_{\lambda\sigma} \eta^{\lambda\mu} + \alpha^\lambda_{-1} \eta_{\lambda\sigma} \eta^{\sigma\mu}) |0; k\rangle = 2\alpha^\mu_{-1} |0; k\rangle, \\ L_1 \alpha^\mu_{-1} \alpha^\nu_{-1} |0; k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\lambda_{1-n} \alpha^\sigma_n \eta_{\lambda\sigma} \alpha^\mu_{-1} \alpha^\nu_{-1} |0; k\rangle = \frac{1}{2}(\alpha^\lambda_1 \alpha^\sigma_0 \eta_{\lambda\sigma} \alpha^\mu_{-1} \alpha^\nu_{-1} + \alpha^\lambda_0 \alpha^\sigma_1 \eta_{\lambda\sigma} \alpha^\mu_{-1} \alpha^\nu_{-1}) |0; k\rangle \\ &= \frac{1}{2}[\frac{1}{2} l_s k^\sigma \eta_{\lambda\sigma} (\eta^{\lambda\mu} \alpha^\nu_{-1} + \eta^{\lambda\nu} \alpha^\mu_{-1}) + \frac{1}{2} l_s k^\lambda \eta_{\lambda\sigma} (\eta^{\sigma\mu} \alpha^\nu_{-1} + \eta^{\sigma\nu} \alpha^\mu_{-1})] |0; k\rangle \\ &= \frac{1}{2} l_s (k^\mu \alpha^\nu_{-1} + k^\nu \alpha^\mu_{-1}) |0; k\rangle. \end{aligned} \tag{10}$$

Thus $L_1 |\phi\rangle$ means $\frac{1}{2} l_s s_{\mu\nu} k^\mu + v_\nu = 0$.

Similarly, when $m = 2$, $L_2 |0; k\rangle = 0$ gives: $s^\mu_\mu + l_s v_\sigma k^\sigma = 0$. We have $\frac{D^2+D}{2} + D - D - 1 = 350$ degrees of freedom right now. We will reduce them after considering gauge invariance.

b) Null state meets physical conditions but is orthogonal to every physical state including itself. These are states with zero norm: $\langle\phi|\phi\rangle = 0$: (real coefficients?)

$$\begin{aligned}
\langle 0;k|\,(s_{\delta\gamma}\alpha_1^\gamma\alpha_1^\delta + v_\lambda\alpha_2^\lambda)(s_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu + v_\sigma\alpha_{-2}^\sigma)\,|0;k\rangle &= 0 \\
\langle 0;k|\,(s_{\delta\gamma}s_{\mu\nu}\alpha_1^\gamma\alpha_1^\delta\alpha_{-1}^\mu\alpha_{-1}^\nu + v_\lambda v_\sigma\alpha_2^\lambda\alpha_{-2}^\sigma)\,|0;k\rangle &= 0 \\
\langle 0;k|\,[s_{\delta\gamma}s_{\mu\nu}(\eta^{\gamma\mu}\eta^{\delta\nu} + \eta^{\gamma\nu}\eta^{\mu\delta}) + 2v_\lambda v_\sigma\eta^{\lambda\sigma}]\,|0;k\rangle &= 0 \\
\langle 0;k|\,[s_{\delta\gamma}s_{\mu\nu}(\eta^{\gamma\mu}\eta^{\delta\nu} + \eta^{\gamma\nu}\eta^{\mu\delta}) + 2v_\lambda v_\sigma\eta^{\lambda\sigma}]\,|0;k\rangle &= 0.
\end{aligned} \tag{11}$$

This tells us that if coefficients have relation $s_\delta^\mu s_\mu^\delta + v_\lambda v^\lambda = 0$, the state is null. And $\langle\phi|\psi\rangle = 0$ for an arbitrary physical state $|\psi\rangle$ tells us that $s_\delta^\mu s'^\delta_\mu + v_\lambda v'^\lambda = 0$.

This seems hopeless. One should instead use the construction of null states basis and one can find $D+1-1 = D$ independent basis($L_{-1}c_\mu\alpha_{-1}^\mu\,|0;k\rangle$ and $(L_{-2} + \frac{3}{2}L_{-1}^2)\,|0;k\rangle$ with $c_\mu k^\mu = 0$). A gauge transformation is a shift like $|\phi\rangle + A_\mu\,|null\rangle^\mu$. This will not change any measurement $\langle\phi|\,\mathcal{O}\,|\phi\rangle$.

c) Thus the total dof is $350 - 26 = 324$. The rest is to show that the states transfer under $SO(25)$. ?

3 Problem 3

A general state is $t_{\beta\theta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + s_{\mu\nu}\alpha_{-2}^\mu\alpha_{-1}^\nu + v_\sigma\alpha_{-3}^\sigma\,|0;k\rangle$. $s_{\mu\nu}$ is not by definition symmetric any more. Mass-shell condition gives $-\alpha' k^\mu k_\mu = 2$. $L_m\,|\phi\rangle = 0$ gives non-trivial conditions only for $m = 1, 2,$ and 3.

$$\begin{aligned}
L_1\,|\phi\rangle = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{1-n}^\gamma\alpha_n^\delta\eta_{\gamma\delta}(t_{\beta\theta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + s_{\mu\nu}\alpha_{-2}^\mu\alpha_{-1}^\nu + v_\sigma\alpha_{-3}^\sigma\,|0;k\rangle) &= 0 \\
\frac{1}{2}[2\alpha_0^\gamma\alpha_1^\delta\eta_{\gamma\delta}t_{\beta\theta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + (2\alpha_0^\gamma\alpha_1^\delta\eta_{\gamma\delta} + \alpha_{-1}^\gamma\alpha_2^\delta\eta_{\gamma\delta})s_{\mu\nu}\alpha_{-2}^\mu\alpha_{-1}^\nu + \alpha_{-2}^\gamma\alpha_3^\delta\eta_{\gamma\delta}v_\sigma\alpha_{-3}^\sigma]\,|0;k\rangle &= 0 \\
\frac{1}{2}[2\alpha_0^\gamma\eta_{\gamma\delta}t_{\beta\theta\xi}(\eta^{\delta\beta}\alpha_{-1}^\theta\alpha_{-1}^\xi + \eta^{\theta\delta}\alpha_{-1}^\beta\alpha_{-1}^\xi + \eta^{\delta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta) + s_{\mu\nu}(2\alpha_0^\gamma\alpha_{-2}^\mu\eta^{\delta\nu}\eta_{\gamma\delta} + 2\alpha_{-1}^\gamma\alpha_{-1}^\nu\eta^{\delta\mu}\eta_{\gamma\delta}) & \\
+3\alpha_{-2}^\gamma\eta_{\gamma\delta}v_\sigma\eta^{\delta\sigma}]\,|0;k\rangle &= 0 \\
\frac{1}{2}[2t_{\beta\theta\xi}(\alpha_0^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + \alpha_0^\theta\alpha_{-1}^\beta\alpha_{-1}^\xi + \alpha_0^\xi\alpha_{-1}^\beta\alpha_{-1}^\theta) + s_{\mu\nu}(2\alpha_0^\nu\alpha_{-2}^\mu + 2\alpha_{-1}^\mu\alpha_{-1}^\nu) + 3v_\sigma\alpha_{-2}^\sigma]\,|0;k\rangle &= 0
\end{aligned} \tag{12}$$

The constraints that come from L_1 are $l_s s_{\mu\nu}k^\nu + 3v_\mu = 0$ and $3l_s t_{\beta\theta\xi}k^\beta + 2s_{\theta\xi} = 0$.

$$\begin{aligned}
L_2\,|\phi\rangle = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{2-n}^\gamma\alpha_n^\delta\eta_{\gamma\delta}(t_{\beta\theta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + s_{\mu\nu}\alpha_{-2}^\mu\alpha_{-1}^\nu + v_\sigma\alpha_{-3}^\sigma\,|0;k\rangle) &= 0 \\
\frac{1}{2}[\alpha_1^\gamma\alpha_1^\delta\eta_{\gamma\delta}t_{\beta\theta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + 2\alpha_0^\gamma\alpha_2^\delta\eta_{\gamma\delta}s_{\mu\nu}\alpha_{-2}^\mu\alpha_{-1}^\nu + \alpha_{-1}^\gamma\alpha_3^\delta\eta_{\gamma\delta}v_\sigma\alpha_{-3}^\sigma]\,|0;k\rangle &= 0 \\
\frac{1}{2}[6t_{\beta\theta\xi}\eta^{\theta\xi}\alpha_{-1}^\beta + 2l_s k^\mu s_{\mu\nu}\alpha_{-1}^\nu + 3v_\sigma\alpha_{-1}^\sigma]\,|0;k\rangle &= 0
\end{aligned} \tag{13}$$

The constraint that comes from L_2 is $6t_{\beta\theta\xi}\eta^{\theta\xi} + 2l_s k^\mu s_{\mu\beta} + 3v_\beta = 0$.

$$L_3 \left|\phi\right\rangle = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha^\gamma_{3-n}\alpha^\delta_n \eta_{\gamma\delta}(t_{\beta\theta\xi}\alpha^\beta_{-1}\alpha^\theta_{-1}\alpha^\xi_{-1} + s_{\mu\nu}\alpha^\mu_{-2}\alpha^\nu_{-1} + v_\sigma\alpha^\sigma_{-3}\left|0;k\right\rangle) = 0 \tag{14}$$

$$\frac{1}{2}[2\alpha^\gamma_2\alpha^\delta_1\eta_{\gamma\delta}s_{\mu\nu}\alpha^\mu_{-2}\alpha^\nu_{-1} + 2\alpha^\gamma_0\alpha^\delta_3\eta_{\gamma\delta}v_\sigma\alpha^\sigma_{-3}]\left|0;k\right\rangle = \frac{1}{2}[4s^\mu_\mu + 3l_s k^\sigma v_\sigma]\left|0;k\right\rangle = 0$$

The constraint that comes from L_3 is $4s^\mu_\mu + 3l_s k^\sigma v_\sigma = 0$.

Now we have $\frac{D(D-1)(D-2)}{6} + 3D(D-1) + D + \frac{D^2+D}{2} + D - D - \frac{D^2+D}{2} - D - 1 = 4549$ degrees of freedom. Gauge invariance will reduce it further.