

# Homework 2

Ziming Ji

PHY 539: Introduction to String Theory

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## 1 Problem 1

First we find the mode expansion of the space-time Lorentz generator  $J^{\mu\nu}$  of a bosonic open string. By definition we have

$$J^{\mu\nu} = \int_0^\pi J_0^{\mu\nu} d\sigma = \frac{1}{\pi l_s^2} \int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) d\sigma. \quad (1)$$

Plugging in the mode expansion of  $X$  field and its derivative:

$$\begin{aligned} X^\mu(\tau, \sigma) &= x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos(m\sigma) \\ \dot{X}^\mu(\tau, \sigma) &= l_s^2 p^\mu + l_s \sum_{m \neq 0} \alpha_m^\mu e^{-im\tau} \cos(m\sigma), \end{aligned} \quad (2)$$

we obtain(omitting terms that vanish after integration):

$$\begin{aligned} J^{\mu\nu} &= \frac{1}{\pi l_s^2} \int_0^\pi d\sigma (l_s^2 x^\mu p^\nu + l_s^4 p^\mu p^\nu \tau + i l_s^2 \cos(m\sigma)^2 \sum_{m=1}^\infty \frac{1}{m} (\alpha_m^\mu \alpha_{-m}^\nu - \alpha_{-m}^\mu \alpha_m^\nu + \alpha_m^\mu \alpha_m^\nu e^{-2im\tau} - \alpha_{-m}^\mu \alpha_{-m}^\nu e^{2im\tau}) \\ &\quad - l_s^2 x^\nu p^\mu - l_s^4 p^\nu p^\mu \tau - i l_s^2 \cos(m\sigma)^2 \sum_{m=1}^\infty \frac{1}{m} (\alpha_m^\nu \alpha_{-m}^\mu - \alpha_{-m}^\nu \alpha_m^\mu + \alpha_m^\nu \alpha_m^\mu e^{-2im\tau} - \alpha_{-m}^\nu \alpha_{-m}^\mu e^{2im\tau})) \\ &= \frac{1}{\pi l_s^2} [\pi l_s^2 (x^\mu p^\nu - x^\nu p^\mu) + i l_s^2 \frac{\pi}{2} \sum_{m=1}^\infty \frac{1}{m} (2\alpha_m^\mu \alpha_{-m}^\nu - 2\alpha_m^\nu \alpha_{-m}^\mu)] \\ &= x^\mu p^\nu - x^\nu p^\mu + i \sum_{m=1}^\infty \frac{1}{m} (\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu). \end{aligned} \quad (3)$$

Applying the canonical commutation relation:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [x^\mu, x^\nu] = [p^\mu, p^\nu] = 0, \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m+n,0}, \quad (4)$$

we read that  $[p^\mu, p^\nu] = 0$  immediately. And it is not hard to see that

$$[p^\mu, J^{\nu\sigma}] = [p^\mu, x^\nu p^\sigma - x^\sigma p^\nu] = i\eta^{\sigma\mu} p^\nu - i\eta^{\nu\mu} p^\sigma. \quad (5)$$

Lastly,  $[J^{\mu\nu}, J^{\sigma\lambda}]$  can be separated into two terms:

$$\begin{aligned} [J^{\mu\nu}, J^{\sigma\lambda}] &= [x^\mu p^\nu - x^\nu p^\mu, x^\sigma p^\lambda - x^\lambda p^\sigma] \\ &- \left[ \sum_{m=1}^{\infty} \frac{1}{m} (\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu), \sum_{m=1}^{\infty} \frac{1}{m} (\alpha_m^\sigma \alpha_{-m}^\lambda - \alpha_m^\lambda \alpha_{-m}^\sigma) \right] \end{aligned} \quad (6)$$

First we have  $[x^\mu p^\nu, x^\sigma p^\lambda] = i\eta^{\mu\lambda} x^\sigma p^\nu - i\eta^{\sigma\nu} x^\mu p^\lambda$ . Applying this we get:

$$\begin{aligned} [x^\mu p^\nu - x^\nu p^\mu, x^\sigma p^\lambda - x^\lambda p^\sigma] &= i\eta^{\mu\lambda} x^\sigma p^\nu - i\eta^{\sigma\nu} x^\mu p^\lambda - i\eta^{\mu\sigma} x^\lambda p^\nu + i\eta^{\lambda\nu} x^\mu p^\sigma \\ &- i\eta^{\nu\lambda} x^\sigma p^\mu + i\eta^{\sigma\mu} x^\nu p^\lambda + i\eta^{\nu\sigma} x^\lambda p^\mu - i\eta^{\lambda\mu} x^\nu p^\sigma. \end{aligned} \quad (7)$$

Second, after applying  $[\alpha_m^\mu \alpha_{-m}^\nu, \alpha_m^\sigma \alpha_{-m}^\lambda] = m\eta^{\mu\lambda} a_m^\sigma a_{-m}^\nu - m\eta^{\nu\sigma} a_m^\mu a_{-m}^\lambda$ , we have

$$\begin{aligned} \left[ \sum_{m=1}^{\infty} \frac{1}{m} (\alpha_m^\mu \alpha_{-m}^\nu - \alpha_m^\nu \alpha_{-m}^\mu), \sum_{m=1}^{\infty} \frac{1}{m} (\alpha_m^\sigma \alpha_{-m}^\lambda - \alpha_m^\lambda \alpha_{-m}^\sigma) \right] &= \sum_{m=1}^{\infty} \frac{1}{m} (\eta^{\mu\lambda} a_m^\sigma a_{-m}^\nu - \eta^{\nu\sigma} a_m^\mu a_{-m}^\lambda \\ &- \eta^{\mu\sigma} a_m^\lambda a_{-m}^\nu + \eta^{\nu\lambda} a_m^\mu a_{-m}^\sigma - \eta^{\nu\lambda} a_m^\sigma a_{-m}^\mu + \eta^{\mu\sigma} a_m^\nu a_{-m}^\lambda + \eta^{\nu\sigma} a_m^\lambda a_{-m}^\mu - \eta^{\mu\lambda} a_m^\nu a_{-m}^\sigma). \end{aligned} \quad (8)$$

It is not hard to see from now that matching equation 7 and 8, one can obtain the desired commutation relation for Lorentz generators:

$$[J^{\mu\nu}, J^{\sigma\lambda}] = -i\eta^{\nu\sigma} J^{\mu\lambda} + i\eta^{\mu\sigma} J^{\nu\lambda} + i\eta^{\nu\lambda} J^{\mu\sigma} - i\eta^{\mu\lambda} J^{\nu\sigma}. \quad (9)$$

## 2 Problem 2

a) First massive level of  $D = 26$  open string is the combination of two kinds of excitations:  $\alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle$  and  $\alpha_{-2} |0; k\rangle$ . A general state is  $(s_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + v_\mu \alpha_{-2}^\mu) |0; k\rangle$ . Physical state conditions are the constraints that  $L_m |\phi\rangle = 0$  for  $m > 0$  and  $(L_0 - 1) |\phi\rangle = 0$ . The second one, also called mass shell condition, gives  $\alpha' M^2 = 1$  or  $-\alpha' k^\mu k_\mu = 1$ . The first one need to be examined more carefully. It is not hard to see that only  $L_1$  and  $L_2$  conditions are not manifestly zero:

$$\begin{aligned} L_1 \alpha_{-2}^\mu |0; k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{1-n}^\lambda \alpha_n^\sigma \eta_{\lambda\sigma} \alpha_{-2}^\mu |0; k\rangle = \frac{1}{2} (\alpha_2^\lambda \alpha_{-1}^\sigma \eta_{\lambda\sigma} \alpha_{-2}^\mu + \alpha_{-1}^\lambda \alpha_2^\sigma \eta_{\lambda\sigma} \alpha_{-2}^\mu) |0; k\rangle \\ &= (\alpha_{-1}^\sigma \eta_{\lambda\sigma} \eta^{\lambda\mu} + \alpha_{-1}^\lambda \eta_{\lambda\sigma} \eta^{\sigma\mu}) |0; k\rangle = 2\alpha_{-1}^\mu |0; k\rangle, \\ L_1 \alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{1-n}^\lambda \alpha_n^\sigma \eta_{\lambda\sigma} \alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle = \frac{1}{2} (\alpha_1^\lambda \alpha_0^\sigma \eta_{\lambda\sigma} \alpha_{-1}^\mu \alpha_{-1}^\nu + \alpha_0^\lambda \alpha_1^\sigma \eta_{\lambda\sigma} \alpha_{-1}^\mu \alpha_{-1}^\nu) |0; k\rangle \\ &= \frac{1}{2} \left[ \frac{1}{2} l_s k^\sigma \eta_{\lambda\sigma} (\eta^{\lambda\mu} \alpha_{-1}^\nu + \eta^{\lambda\nu} \alpha_{-1}^\mu) + \frac{1}{2} l_s k^\lambda \eta_{\lambda\sigma} (\eta^{\sigma\mu} \alpha_{-1}^\nu + \eta^{\sigma\nu} \alpha_{-1}^\mu) \right] |0; k\rangle \\ &= \frac{1}{2} l_s (k^\mu \alpha_{-1}^\nu + k^\nu \alpha_{-1}^\mu) |0; k\rangle. \end{aligned} \quad (10)$$

Thus  $L_1 |\phi\rangle$  means  $\frac{1}{2} l_s s_{\mu\nu} k^\mu + v_\nu = 0$ .

Similarly, when  $m = 2$ ,  $L_2 |0; k\rangle = 0$  gives:  $s_\mu^\mu + l_s v_\sigma k^\sigma = 0$ . We have  $\frac{D^2+D}{2} + D - D - 1 = 350$  degrees of freedom right now. We will reduce them after considering gauge invariance.

b) Null state meets physical conditions but is orthogonal to every physical state including itself. These are states with zero norm:  $\langle\phi|\phi\rangle = 0$ : (real coefficients?)

$$\begin{aligned}
\langle 0; k | (s_{\delta\gamma}\alpha_1^\gamma\alpha_1^\delta + v_\lambda\alpha_2^\lambda)(s_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu + v_\sigma\alpha_{-2}^\sigma) | 0; k \rangle &= 0 \\
\langle 0; k | (s_{\delta\gamma}s_{\mu\nu}\alpha_1^\gamma\alpha_1^\delta\alpha_{-1}^\mu\alpha_{-1}^\nu + v_\lambda v_\sigma\alpha_2^\lambda\alpha_{-2}^\sigma) | 0; k \rangle &= 0 \\
\langle 0; k | [s_{\delta\gamma}s_{\mu\nu}(\eta^{\gamma\mu}\eta^{\delta\nu} + \eta^{\gamma\nu}\eta^{\mu\delta}) + 2v_\lambda v_\sigma\eta^{\lambda\sigma}] | 0; k \rangle &= 0 \\
\langle 0; k | [s_{\delta\gamma}s_{\mu\nu}(\eta^{\gamma\mu}\eta^{\delta\nu} + \eta^{\gamma\nu}\eta^{\mu\delta}) + 2v_\lambda v_\sigma\eta^{\lambda\sigma}] | 0; k \rangle &= 0.
\end{aligned} \tag{11}$$

This tells us that if coefficients have relation  $s_\delta^\mu s_\mu^\delta + v_\lambda v^\lambda = 0$ , the state is null. And  $\langle\phi|\psi\rangle = 0$  for an arbitrary physical state  $|\psi\rangle$  tells us that  $s_\delta^\mu s_\mu^\delta + v_\lambda v^\lambda = 0$ .

This is actually hopeless. One should instead use the construction of null states basis and one can find  $D + 1 - 1 = D$  independent basis. Thus the total dof is  $350 - 26 = 24$ .

### 3 Problem 3

A general state is  $t_{\beta\theta\xi}\alpha_{-1}^\beta\alpha_{-1}^\theta\alpha_{-1}^\xi + s_{\mu\nu}\alpha_{-2}^\mu\alpha_{-1}^\nu + v_\sigma\alpha_{-3}^\sigma | 0; k \rangle$ .  $s_{\mu\nu}$  is not by definition symmetric any more. Mass-shell condition gives  $-\alpha' k^\mu k_\mu = 2$ .  $L_m |\phi\rangle = 0$  gives non-trivial conditions only for  $m = 1, 2$ , and 3

$$\begin{aligned}
\frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^\gamma \alpha_n^\delta \eta_{\gamma\delta} (t_{\beta\theta\xi} \alpha_{-1}^\beta \alpha_{-1}^\theta \alpha_{-1}^\xi + s_{\mu\nu} \alpha_{-2}^\mu \alpha_{-1}^\nu + v_\sigma \alpha_{-3}^\sigma | 0; k \rangle) &= 0 \\
\frac{1}{2} [\alpha_{m-1}^\gamma \alpha_1^\delta \eta_{\gamma\delta} t_{\beta\theta\xi} \alpha_{-1}^\beta \alpha_{-1}^\theta \alpha_{-1}^\xi + (\alpha_{m-1}^\gamma \alpha_1^\delta \eta_{\gamma\delta} + \alpha_{m-2}^\gamma \alpha_2^\delta \eta_{\gamma\delta}) s_{\mu\nu} \alpha_{-2}^\mu \alpha_{-1}^\nu + \alpha_{m-3}^\gamma \alpha_3^\delta \eta_{\gamma\delta} v_\sigma \alpha_{-3}^\sigma] | 0; k \rangle &= 0 \\
\frac{1}{2} [\alpha_{m-1}^\gamma \eta_{\gamma\delta} t_{\beta\theta\xi} (\eta^{\delta\beta} \alpha_{-1}^\theta \alpha_{-1}^\xi + \eta^{\theta\delta} \alpha_{-1}^\beta \alpha_{-1}^\xi + \eta^{\delta\xi} \alpha_{-1}^\beta \alpha_{-1}^\theta) + s_{\mu\nu} (\alpha_{m-1}^\gamma \alpha_{-2}^\mu \eta^{\delta\nu} \eta_{\gamma\delta} + 2\alpha_{m-2}^\gamma \alpha_{-1}^\nu \eta^{\delta\mu} \eta_{\gamma\delta}) & \\
+ 3\alpha_{m-3}^\gamma \eta_{\gamma\delta} v_\sigma \eta^{\delta\sigma}] | 0; k \rangle &= 0
\end{aligned} \tag{12}$$