Physics 539 Problem Set 2 Solutions*

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Problem 1 (Becker-Becker-Schwarz, Problem 2.6)

As explained in BBS, Exercise 2.10, the open-string mode expansion

$$X^\mu(\tau,\sigma) = x^\mu + \ell_s^2 p^\mu \tau + i\ell_s \sum_{m\neq 0} \frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos(m\sigma) \tag{1}$$

(with Neumann boundary conditions) implies that

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{m=1}^\infty \frac{1}{m}(\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu). \tag{2}$$

Note that when quantizing, there are no normal-ordering ambiguities in this expression because $\mu \neq \nu$. For completeness, recall also that in covariant quantization,

$$[X^\mu(\tau,\sigma), T\dot{X}^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \implies [x^\mu, p^\nu] = i\eta^{\mu\nu},\ [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{3}$$

by the Poisson resummation formula $\sum_{k=-\infty}^\infty \delta(t-k\pi) = \frac{1}{\pi}\sum_{n=-\infty}^\infty e^{2int}$. One then computes, for example, that

$$2[x^{[\mu}p^{\nu]}, x^{[\rho}p^{\sigma]}] = i(\eta^{\mu\rho}x^{[\nu}p^{\sigma]} - \eta^{\mu\sigma}x^{[\nu}p^{\rho]} - \eta^{\nu\rho}x^{[\mu}p^{\sigma]} + \eta^{\nu\sigma}x^{[\mu}p^{\rho]}), \tag{4}$$

$$2[\alpha_{-m}^{[\mu}\alpha_m^{\nu]}, \alpha_{-m}^{[\rho}\alpha_m^{\sigma]}] = -m(\eta^{\mu\rho}\alpha_{-m}^{[\nu}\alpha_m^{\sigma]} - \eta^{\mu\sigma}\alpha_{-m}^{[\nu}\alpha_m^{\rho]} - \eta^{\nu\rho}\alpha_{-m}^{[\mu}\alpha_m^{\sigma]} + \eta^{\nu\sigma}\alpha_{-m}^{[\mu}\alpha_m^{\rho]}) \tag{5}$$

$$\implies [J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\mu\rho}). \tag{6}$$

Hence Poincaré symmetry is preserved quantum-mechanically in covariant quantization.

Problem 2

(a)

An arbitrary state at the first massive level of the open string can be written as

$$|\phi\rangle = (s_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + v_\mu \alpha_{-2}^\mu)|0;k\rangle. \tag{7}$$

The physical state conditions are

$$(L_0 - 1)|\phi\rangle = L_m|\phi\rangle = 0, \quad m \geq 1 \tag{8}$$

(in our case, it suffices to consider $m = 1, 2$).

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In light of $L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \frac{1}{2}\alpha_0^2 + N$ and $\alpha_0^\mu = \ell_s p^\mu$, the L_0 constraint simply gives the mass-shell condition

$$k^2 = -\frac{1}{\alpha'} \tag{9}$$

($\alpha' = \ell_s^2/2$). Since $L_1 = \alpha_0 \cdot \alpha_1 + \alpha_{-1} \cdot \alpha_2 + \cdots$, the L_1 constraint gives

$$\ell_s k^\mu s_{\mu\nu} + v_\nu = 0. \tag{10}$$

Since $L_2 = \frac{1}{2}\alpha_1^2 + \alpha_0 \cdot \alpha_2 + \cdots$, the L_2 constraint gives

$$s_\mu{}^\mu + 2\ell_s k^\mu v_\mu = 0. \tag{11}$$

We have used the commutation relations (3).

(b)

In old covariant quantization, gauge transformations arise from shifts of physical states by null states ($\mathcal{H}_{\text{OCQ}} = \mathcal{H}_{\text{phys}}/\mathcal{H}_{\text{null}}$). A null state is both physical and spurious. The general form of a spurious state is

$$|\psi_{\text{gen}}\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle, \quad (L_0 - 1 + n)|\chi_n\rangle = 0, \tag{12}$$

or WLOG (by the Virasoro algebra), a linear combination of the form

$$|\psi_{\text{gen}}\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle. \tag{13}$$

At level $N = 2$, a general spurious state takes the form

$$|\psi\rangle = (L_{-1}\epsilon_\mu \alpha_{-1}^\mu + \xi L_{-2})|0; k\rangle \tag{14}$$

where ϵ_μ is a vector and ξ is a scalar. Keeping only the relevant terms in $L_{-1} = \alpha_{-1} \cdot \alpha_0 + \alpha_{-2} \cdot \alpha_1 + \cdots$ and $L_{-2} = \frac{1}{2}\alpha_{-1}^2 + \alpha_{-2} \cdot \alpha_0 + \cdots$, this simplifies to

$$|\psi\rangle = \left[\left(\frac{1}{2}\xi\eta_{\mu\nu} + \ell_s \epsilon_{(\mu}k_{\nu)}\right)\alpha_{-1}^\mu \alpha_{-1}^\nu + (\epsilon_\mu + \ell_s \xi k_\mu)\alpha_{-2}^\mu\right]|0; k\rangle. \tag{15}$$

For the state $|\psi\rangle$ to be null, it must satisfy the physical state conditions (9), (10), (11), which require (after some simplification) that

$$\ell_s^2 k^2 = -2, \quad 3\xi + \ell_s(\epsilon \cdot k) = 0, \quad (D - 26)\xi = 0. \tag{16}$$

Since $D = 26$ by assumption, the middle condition in (16) fixes ξ in terms of ϵ_μ, so that shifts of the form $|\phi\rangle \to |\phi\rangle + |\psi\rangle$ give rise to gauge transformations

$$\begin{aligned} s_{\mu\nu} &\to s_{\mu\nu} + \ell_s \epsilon_{(\mu}k_{\nu)} - \frac{1}{6}\ell_s(\epsilon \cdot k)\eta_{\mu\nu}, \\ v_\mu &\to v_\mu + \epsilon_\mu - \frac{1}{3}\ell_s^2(\epsilon \cdot k)k_\mu. \end{aligned} \tag{17}$$

If instead $D \neq 26$, we would have found that $\xi = \epsilon \cdot k = 0$, as appropriate for a massive vector.

More physically, the null states at this level comprise both a massive vector and a massive scalar ($(D-1)+1 = D$) in $D = 26$. This can be seen by considering separately the components of ϵ_μ orthogonal to k_μ and parallel to k_μ in (17). If $\epsilon \cdot k = 0$, then (17) becomes

$$s_{\mu\nu} \to s_{\mu\nu} + \ell_s \epsilon_{(\mu}k_{\nu)}, \quad v_\mu \to v_\mu + \epsilon_\mu. \tag{18}$$

If $\epsilon_\mu = c\ell_s k_\mu$ for some dimensionless c, then (keeping in mind that $\ell_s^2 k^2 = -2$) (17) becomes

$$s_{\mu\nu} \to s_{\mu\nu} + c\left(\ell_s^2 k_\mu k_\nu + \frac{1}{3}\eta_{\mu\nu}\right), \quad v_\mu \to v_\mu + \frac{5}{3}c\ell_s k_\mu. \tag{19}$$

The transformations (18) correspond to shifts by null states of the form

$$L_{-1}\epsilon_\mu \alpha_{-1}^\mu |0;k\rangle = (\ell_s \epsilon_{(\mu}k_{\nu)}\alpha_{-1}^\mu \alpha_{-1}^\nu + \epsilon_\mu \alpha_{-2}^\mu)|0;k\rangle \tag{20}$$

where both (9) and $\epsilon \cdot k = 0$ must hold for (10) and (11) to be satisfied. The transformations (19) correspond to shifts by the null state

$$\left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|0;k\rangle = \left[\left(\frac{1}{2}\eta_{\mu\nu} + \frac{3}{2}\ell_s^2 k_\mu k_\nu\right)\alpha_{-1}^\mu \alpha_{-1}^\nu + \frac{5}{2}\ell_s k_\mu \alpha_{-2}^\mu\right]|0;k\rangle \tag{21}$$

(see BBS, Section 2.4), with $c = 3/2$.

Above, we assumed that null states are orthogonal to all physical states and then imposed that they be physical. In principle, one could proceed in the opposite order by assuming that null states are physical and then imposing that they be orthogonal to all physical states:

$$\langle 0;k|(\tilde{s}_{\rho\sigma}\alpha_1^\rho \alpha_1^\sigma + \tilde{v}_\rho \alpha_2^\rho)(s_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + v_\mu \alpha_{-2}^\mu)|0;k\rangle = 0 \Longleftrightarrow \tilde{s}_{\mu\nu}s^{\mu\nu} + \tilde{v}_\mu v^\mu = 0. \tag{22}$$

This seems more difficult.

(c)

An arbitrary state (7) has

$$\binom{D+1}{2} + D \tag{23}$$

components. We obtain $D+1$ constraints from (10) and (11), so physical states have

$$\binom{D+1}{2} - 1 \tag{24}$$

components. We obtain D further constraints from modding out by null states, giving

$$\binom{D+1}{2} - D - 1 = \binom{D}{2} - 1 = 324 \tag{25}$$

states at this level. This is the **324** (traceless symmetric rank-two tensor) of $SO(25)$.

Problem 3

A general open string state at the second massive level ($N = 3$) is

$$|\phi\rangle = (w_{\mu\nu\rho}\alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\rho + b_{\mu\nu}\alpha_{-2}^\mu \alpha_{-1}^\nu + v_\mu \alpha_{-3}^\mu)|0;k\rangle \tag{26}$$

where the symmetric tensor $w_{\mu\nu\rho}$ has $\binom{D+2}{3}$ components, $b_{\mu\nu}$ has D^2 components, and v_μ has D components. The nontrivial physical state conditions are

$$(L_0 - 1)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle = 0 \tag{27}$$

(note that $L_3|\phi\rangle = 0$ follows automatically, since $L_3 = [L_2, L_1]$). In addition to $\ell_s^2 k^2 = -4$, they yield the independent constraints

$$3\ell_s k^\rho w_{\rho\mu\nu} + 2b_{(\mu\nu)} = 0, \tag{28}$$

$$\ell_s k^\nu b_{\mu\nu} + 3v_\mu = 0, \tag{29}$$

$$3\eta^{\nu\rho} w_{\nu\rho\mu} + 2\ell_s k^\nu b_{\nu\mu} + 3v_\mu = 0, \tag{30}$$

of which there are $\binom{D+1}{2} + 2D$. We now take a shortcut relative to our treatment of the first massive level and simply write down the tensor and vector null states separately. Using $L_{-1} = \alpha_{-1}\cdot\alpha_0 + \alpha_{-2}\cdot\alpha_1 + \alpha_{-3}\cdot\alpha_2 + \cdots$, the tensor null states take the form

$$|\psi_{\text{tensor}}\rangle = L_{-1}(s_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + u_\mu\alpha^\mu_{-2})|0;k\rangle \tag{31}$$

$$= (\ell_s s_{(\mu\nu}k_{\rho)}\alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_{-1} + (\ell_s u_\mu k_\nu + 2s_{\mu\nu})\alpha^\mu_{-2}\alpha^\nu_{-1} + 2u_\mu\alpha^\mu_{-3})|0;k\rangle, \tag{32}$$

subject to the constraints (10) and (11) with u_μ in the place of v_μ. There are $\binom{D+1}{2} - 1$ such states. They generate the gauge transformations

$$w_{\mu\nu\rho} \to w_{\mu\nu\rho} + \ell_s s_{(\mu\nu}k_{\rho)}, \quad b_{\mu\nu} \to b_{\mu\nu} + \ell_s u_\mu k_\nu + 2s_{\mu\nu}, \quad v_\mu \to v_\mu + 2u_\mu. \tag{33}$$

Using $L_{-2} = \frac{1}{2}\alpha^2_{-1} + \alpha_{-2}\cdot\alpha_0 + \alpha_{-3}\cdot\alpha_1 + \cdots$, the vector null states take the form

$$|\psi_{\text{vector}}\rangle = \left(L_{-2} + \frac{3}{2}L^2_{-1}\right) a_\mu\alpha^\mu_{-1}|0;k\rangle \tag{34}$$

$$= (V_{\mu\nu\rho}\alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_{-1} + V_{\mu\nu}\alpha^\mu_{-2}\alpha^\nu_{-1} + V_\mu\alpha^\mu_{-3})|0;k\rangle \tag{35}$$

subject to $a\cdot k = 0$, where we have set

$$V_{\mu\nu\rho} \equiv \frac{1}{2}\eta_{(\mu\nu}a_{\rho)} + \frac{3}{2}\ell_s^2 a_{(\mu}k_\nu k_{\rho)}, \quad V_{\mu\nu} \equiv 2\ell_s a_{(\mu}k_{\nu)} + \frac{7}{2}\ell_s a_\mu k_\nu, \quad V_\mu \equiv 4a_\mu. \tag{36}$$

There are $D-1$ such states. Their corresponding gauge transformations are easily read off from (36). Seeing as there are $\binom{D+1}{2} + D - 2$ null states in all, there are

$$\binom{D+2}{3} + D^2 + D - \left[\binom{D+1}{2} + 2D\right] - \left[\binom{D+1}{2} + D - 2\right] \tag{37}$$

states at this level, which can be regrouped as

$$\left[\binom{D+1}{3} - (D-1)\right] + \binom{D-1}{2} = 3200. \tag{38}$$

They comprise the **2900** $\oplus$ **300** (traceless symmetric rank-three tensor and antisymmetric rank-two tensor) of $SO(25)$. To go beyond dimension-counting and see this decomposition directly, it is helpful to go to the rest frame.

Problem 4

(a)

We take a heuristic approach. In light-cone gauge, the Lorentz generators can be grouped into $J^{ij}, J^{i+}, J^{i-}, J^{+-}$. From (2), we would expect that

$$J^{i-} \sim x^i p^- - x^- p^i - i\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^i_{-m}\alpha^-_m - \alpha^-_{-m}\alpha^i_m). \tag{39}$$

However, this expression suffers from normal-ordering ambiguities at the quantum level.

As x^i and p^- do not commute, we first symmetrize this expression to ensure Hermiticity:

$$J^{i-} = \underbrace{\frac{1}{2}(x^i p^- + p^- x^i) - x^- p^i}_{\ell^{i-}} \underbrace{- i \sum_{m=1}^{\infty} \frac{1}{m}(\alpha^i_{-m}\alpha^-_m - \alpha^-_{-m}\alpha^i_m)}_{E^{i-}} . \tag{40}$$

We then make ordering ambiguities explicit by writing the $-$ oscillators in terms of transverse Virasoro generators:

$$\ell_s^2 p^- = \frac{1}{p^+}(L_0 - a), \quad \alpha^-_m = \frac{1}{\ell_s p^+}(L_m - a\delta_{m,0}), \quad L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{i=1}^{D-2} \alpha^i_{m-n}\alpha^i_n. \tag{41}$$

This gives

$$J^{i-} = \frac{1}{2\ell_s^2 p^+}(x^i(L_0 - a) + (L_0 - a)x^i) - x^- p^i - \frac{i}{\ell_s p^+}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^i_{-m}L_m - L_{-m}\alpha^i_m) \tag{42}$$

for the open string in light-cone gauge.

(b)

We are instructed to use the representation $x^- = -i\frac{\partial}{\partial p^+}$, and in particular

$$[x^-, p^+] = -i, \tag{43}$$

to compute the commutator $[J^{i-}, J^{j-}]$. Some of the ingredients in this calculation are

$$[x^i, p^j] = i\eta^{ij}, \quad [\alpha^i_m, \alpha^j_n] = m\eta^{ij}\delta_{m+n,0}, \quad [L_m, x^i] = -i\ell_s\alpha^i_m, \quad [L_m, \alpha^j_n] = -n\alpha^j_{m+n}, \tag{44}$$

and the fact that the transverse Virasoro generators obey

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D-2}{12}m(m^2-1)\delta_{m+n,0} \tag{45}$$

with central charge $D-2$. We find, for example, that $[\ell^{i-}, \ell^{j-}] = 0$ with $J^{i-} \equiv \ell^{i-} + E^{i-}$ as in (40). I will not record the details here;[1] the end result is

$$[J^{i-}, J^{j-}] = -\frac{2}{\ell_s^2(p^+)^2}\sum_{m=1}^{\infty}\left[m\left(1 - \frac{D-2}{24}\right) + \frac{1}{m}\left(\frac{D-2}{24} - a\right)\right](\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m), \tag{46}$$

from which we read off $D = 26$ and $a = 1$. We thus conclude (with some effort) that Poincaré symmetry is also preserved in light-cone quantization.

[1] A slick approach is presented in Section 2.3 of GSW. For the barehanded approach, see Appendix B of the original paper by Goddard, Goldstone, Rebbi, and Thorne, Nucl. Phys. B56 (1973). I have also been informed that the explicit calculation can be found in the string theory lecture notes of Gleb Arutyunov (Section 4.2.1).