# Homework 3

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## 1 Problem 1

**a)** It is not hard to show that under $z \to \frac{az+b}{cz+d}$, the cross ratio becomes

$$\frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right)\left(\frac{az_3+b}{cz_3+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}\right)} = \frac{(z_1-z_2)(z_3-z_4)(bc-ad)^2}{(z_1-z_3)(z_2-z_4)(bc-ad)^2} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \quad (1)$$

**b)** We set the transformation to be $z \to \frac{az+b}{cz+d}$. Solving the set of equations

$$az_1 + b = 0 \wedge az_2 + b = cz_2 + d \wedge cz_3 + d = 0 \wedge ad - bc = 1, \quad (2)$$

we have

$$a \to -\frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to \frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \quad (3)$$

$$c \to \frac{z_2-z_1}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_1-z_2)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}$$

or

$$a \to \frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to -\frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \quad (4)$$

$$c \to \frac{z_1-z_2}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_2-z_1)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}.$$

## 2 Problem 2

We refer to the $b$ $c$ system as a $2D$ field theory with a Grassmann action

$$S = \frac{1}{2\pi}\int d^2z\ b\bar{\partial}c \quad (5)$$

The contraction of $b$ and $c$ field is

$$(b(z)c(\omega) - :b(z)c(\omega):)(-1)^{\#} = \frac{1}{z-\omega}. \quad (6)$$

**a)**

$$\begin{aligned} T(z)T(w) &= (:(\partial_z b)c(z):-\lambda\partial_z:bc(z):)\,(:(\partial_w b)c(w):-\lambda\partial_w:bc(w):) \\ &=:(\partial_z b)c(z)::(\partial_w b)c(w):-\lambda\partial_z:bc(z)::(\partial_w b)c(w): \\ &\quad -\lambda:(\partial_z b)c(z):\partial_w:bc(w):+\lambda^2\partial_z:bc(z):\partial_w:bc(w): \end{aligned} \tag{7}$$

The full contraction(quartic) part is

$$\lambda^2\frac{\partial}{\partial z}\frac{\partial}{\partial\omega}\frac{1}{(z-\omega)(z-\omega)}-\lambda\frac{\partial}{\partial\omega}\frac{\frac{\partial}{\partial z}\frac{1}{z-\omega}}{z-\omega}-\lambda\frac{\partial}{\partial z}\frac{\frac{\partial}{\partial\omega}\frac{1}{z-\omega}}{z-\omega}+\frac{\partial}{\partial z}\frac{1}{z-\omega}\frac{\partial}{\partial\omega}\frac{1}{z-\omega}=\frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} \tag{8}$$

The other terms are(omitting normal ordering symbols for simplicity)

$$c(\omega)\frac{\partial b(z)}{\partial z}\frac{\partial}{\partial\omega}\frac{1}{z-\omega}+c(z)\frac{\partial b(\omega)}{\partial\omega}\frac{\partial}{\partial z}\frac{1}{z-\omega}=\frac{c(\omega)b''(\omega)-b'(\omega)c'(\omega)}{z-\omega}+O\left((z-\omega)^0\right) \tag{9}$$

$$-\lambda\frac{\partial}{\partial z}\left(b(z)c(\omega)\frac{\partial}{\partial\omega}\frac{1}{z-\omega}+\frac{c(z)\frac{\partial b(\omega)}{\partial\omega}}{z-\omega}\right)=\frac{2\lambda b(\omega)c(\omega)}{(z-\omega)^3}+\frac{2\lambda c(\omega)b'(\omega)}{(z-\omega)^2}+O\left((z-\omega)^0\right)$$

$$\begin{aligned} -\lambda\frac{\partial}{\partial\omega}\left(b(\omega)c(z)\frac{\partial}{\partial z}\frac{1}{z-\omega}+\frac{c(\omega)\frac{\partial b(z)}{\partial z}}{z-\omega}\right)&=\frac{2\lambda b(\omega)c(\omega)}{(z-\omega)^3}+\frac{2\lambda b(\omega)c'(\omega)}{(z-\omega)^2} \\ &\quad+\frac{\lambda\left(b(\omega)c''(\omega)-c(\omega)b''(\omega)\right)}{z-\omega}+O\left((z-\omega)^0\right) \end{aligned}$$

$$\begin{aligned} \lambda^2\frac{\partial}{\partial z}\frac{\partial}{\partial\omega}\left(\frac{b(\omega)c(z)}{z-\omega}+\frac{b(z)c(\omega)}{z-\omega}\right)&=-\frac{4\left(\lambda^2 b(\omega)c(\omega)\right)}{(z-\omega)^3}+\frac{\lambda^2\left(-2c(\omega)b'(\omega)-2b(\omega)c'(\omega)\right)}{(z-\omega)^2} \\ &\quad+O\left((z-\omega)^0\right) \end{aligned}$$

In conclusion, we have

$$\begin{aligned} T(z)T(\omega)\sim{}&\frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4}-\frac{4((\lambda-1)\lambda b(\omega)c(\omega))}{(z-\omega)^3}-\frac{2\left((\lambda-1)\lambda\left(c(\omega)b'(\omega)+b(\omega)c'(\omega)\right)\right)}{(z-\omega)^2} \\ &+\frac{-(\lambda-1)c(\omega)b''(\omega)-b'(\omega)c'(\omega)+\lambda b(\omega)c''(\omega)}{z-\omega} \end{aligned} \tag{10}$$

**b)**

$$T(z)=\sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}\ ,\qquad L_m=\frac{1}{2\pi i}\oint dz\ z^{m+1}T(z) \tag{11}$$

We can just expand $T(z)$ in terms of $b$ and $c$ modes and pick up the power $-(m+2)$ terms.

$$\begin{aligned} T(z)=:(\partial_z b)c(z):-\lambda\partial_z:bc(z):=\sum_l\sum_k:\frac{-(k+\lambda)b_k}{z^{k+\lambda+1}}\frac{c_l}{z^{l+1-\lambda}}: \\ -:\lambda(\frac{-(k+\lambda)b_k}{z^{k+\lambda+1}}\frac{c_l}{z^{l+1-\lambda}}+\frac{b_k}{z^{k+\lambda}}\frac{-(l+1-\lambda)c_l}{z^{l+2-\lambda}}): \end{aligned} \tag{12}$$