

Homework 3

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PHY 539: Introduction to String Theory

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1 Problem 1

a) It is not hard to show that under $z \rightarrow \frac{az+b}{cz+d}$, the cross ratio becomes

$$\frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right) \left(\frac{az_3+b}{cz_3+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}\right) \left(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}\right)} = \frac{(z_1 - z_2)(z_3 - z_4)(bc - ad)^2}{(z_1 - z_3)(z_2 - z_4)(bc - ad)^2} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \quad (1)$$

b) We set the transformation to be $z \rightarrow \frac{az+b}{cz+d}$. Solving the set of equations

$$az_1 + b = 0 \wedge az_2 + b = cz_2 + d \wedge cz_3 + d = 0 \wedge ad - bc = 1, \quad (2)$$

we have

$$\begin{aligned} a &\rightarrow -\frac{\sqrt{z_3 - z_2}}{\sqrt{(z_1 - z_2)(z_1 - z_3)}}, b \rightarrow \frac{z_1 \sqrt{z_3 - z_2}}{\sqrt{(z_1 - z_2)(z_1 - z_3)}}, \\ c &\rightarrow \frac{z_2 - z_1}{\sqrt{(z_1 - z_2)(z_1 - z_3)}\sqrt{z_3 - z_2}}, d \rightarrow \frac{(z_1 - z_2)z_3}{\sqrt{(z_1 - z_2)(z_1 - z_3)}\sqrt{z_3 - z_2}} \end{aligned} \quad (3)$$

or

$$\begin{aligned} a &\rightarrow \frac{\sqrt{z_3 - z_2}}{\sqrt{(z_1 - z_2)(z_1 - z_3)}}, b \rightarrow -\frac{z_1 \sqrt{z_3 - z_2}}{\sqrt{(z_1 - z_2)(z_1 - z_3)}}, \\ c &\rightarrow \frac{z_1 - z_2}{\sqrt{(z_1 - z_2)(z_1 - z_3)}\sqrt{z_3 - z_2}}, d \rightarrow \frac{(z_2 - z_1)z_3}{\sqrt{(z_1 - z_2)(z_1 - z_3)}\sqrt{z_3 - z_2}}. \end{aligned} \quad (4)$$

2 Problem 2

We refer to the b c system as a $2D$ field theory with a Grassmann action

$$S = \frac{1}{2\pi} \int d^2 z \, b \bar{\partial} c \quad (5)$$

The contraction of b and c field is

$$b(z)c(\omega) - :b(z)c(\omega): = \frac{1}{z - \omega}. \quad (6)$$

a)

$$\begin{aligned}
T(z)T(w) &= (: (\partial_z b)c(z) : -\lambda \partial_z : bc(z) :) (: (\partial_w b)c(w) : -\lambda \partial_w : bc(w) :) \\
&=: (\partial_z b)c(z) :: (\partial_w b)c(w) : -\lambda \partial_z : bc(z) :: (\partial_w b)c(w) : \\
&\quad -\lambda : (\partial_z b)c(z) : \partial_w : bc(w) : +\lambda^2 \partial_z : bc(z) : \partial_w : bc(w) :
\end{aligned} \tag{7}$$

The full contraction(quartic) part is

$$\lambda^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \omega} \frac{1}{(z-\omega)(z-\omega)} - \lambda \frac{\partial}{\partial \omega} \frac{\frac{\partial}{\partial z} \frac{1}{z-\omega}}{z-\omega} - \lambda \frac{\partial}{\partial z} \frac{\frac{\partial}{\partial \omega} \frac{1}{z-\omega}}{z-\omega} + \frac{\partial}{\partial z} \frac{1}{z-\omega} \frac{\partial}{\partial \omega} \frac{1}{z-\omega} = \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} \tag{8}$$

The other terms are(omitting normal ordering symbols for simplicity)

$$\begin{aligned}
c(\omega) \frac{\partial b(z)}{\partial z} \frac{\partial}{\partial \omega} \frac{1}{z-\omega} + c(z) \frac{\partial b(\omega)}{\partial \omega} \frac{\partial}{\partial z} \frac{1}{z-\omega} &= -\frac{2(c(\omega)b'(\omega))}{(z-\omega)^2} + \frac{-c(\omega)b''(\omega) - b'(\omega)c'(\omega)}{z-\omega} \\
&\quad + O((z-\omega)^0) \\
-\lambda \frac{\partial}{\partial z} \left(b(z)c(\omega) \frac{\partial}{\partial \omega} \frac{1}{z-\omega} + \frac{c(z)\frac{\partial b(\omega)}{\partial \omega}}{z-\omega} \right) &= \frac{2\lambda b(\omega)c(\omega)}{(z-\omega)^3} + \frac{4\lambda c(\omega)b'(\omega)}{(z-\omega)^2} \\
&\quad + \frac{\lambda(2c(\omega)b''(\omega) + 2b'(\omega)c'(\omega))}{z-\omega} + O((z-\omega)^0) \\
-\lambda \frac{\partial}{\partial \omega} \left(b(\omega)c(z) \frac{\partial}{\partial z} \frac{1}{z-\omega} + \frac{c(\omega)\frac{\partial b(z)}{\partial z}}{z-\omega} \right) &= -\frac{2(\lambda b(\omega)c(\omega))}{(z-\omega)^3} + \frac{\lambda(2c(\omega)b'(\omega) - 2b(\omega)c'(\omega))}{(z-\omega)^2} \\
&\quad + \frac{\lambda(c(\omega)b''(\omega) - b(\omega)c''(\omega))}{z-\omega} + O((z-\omega)^0) \\
\lambda^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \omega} \left(\frac{b(\omega)c(z)}{z-\omega} + \frac{b(z)c(\omega)}{z-\omega} \right) &= -\frac{4(\lambda^2 c(\omega)b'(\omega))}{(z-\omega)^2} + \frac{\lambda^2(-2c(\omega)b''(\omega) - 2b'(\omega)c'(\omega))}{z-\omega} \\
&\quad + O((z-\omega)^0)
\end{aligned} \tag{9}$$

In conclusion, we have

$$\begin{aligned}
T(z)T(w) &\sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} - \frac{2((\lambda-1)(2\lambda-1)c(\omega)b'(\omega) + \lambda b(\omega)c'(\omega))}{(z-\omega)^2} \\
&\quad \frac{((3-2\lambda)\lambda-1)c(\omega)b''(\omega) + (-2(\lambda-1)\lambda-1)b'(\omega)c'(\omega) - \lambda b(\omega)c''(\omega))}{z-\omega}
\end{aligned} \tag{10}$$

???Shouldn't $T(z)T(w) \sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}$? I only noticed this contradiction in the last minute but could not make it right.

b)

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \quad L_m = \frac{1}{2\pi i} \oint dz z^{m+1} T(z) \tag{11}$$

We can just expand $T(z)$ in terms of b and c modes and pick up the power $-(m+2)$ terms for $m \neq 0$.

$$\begin{aligned}
T(z) &= :(\partial_z b)c(z) : - \lambda \partial_z : bc(z) : = \sum_l \sum_k : \frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} : \\
&\quad - : \lambda \left(\frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} + \frac{: b_k}{z^{k+\lambda}} \frac{-(l+1-\lambda)c_l :}{z^{l+2-\lambda}} \right) : \\
&\quad (\text{setting } l = m - k) = \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (-(k+\lambda) + \lambda(k+\lambda) + \lambda(m-k+1-\lambda)) \\
&\quad = \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (\lambda m - k)
\end{aligned} \tag{12}$$

It is easy to read off L_m from above as $L_m = \sum_k (\lambda m - k) : b_k c_{m-k} :$. For $m = 0$, an appropriate normal ordering constant should be included: $L_0 = \sum_k k(c_{-k} b_k + b_{-k} c_k) - k$.

c)

$$[L_m, b_n] = \left[\sum_k (\lambda m - k) : b_k c_{m-k} :, b_n \right] \tag{13}$$

$$= ((\lambda - 1)m - n) : b_{m+n} c_{-n} :, b_n = ((\lambda - 1)m - n) b_{m+n}$$

$$[L_m, c_n] = \left[\sum_k (\lambda m - k) : b_k c_{m-k} :, c_n \right] = (\lambda m + n) : b_{-n} c_{m+n} :, c_n = (\lambda m + n) c_{m+n} \tag{14}$$

d) Because $c(z) |1\rangle$ has no pole at $z \rightarrow 0$, $c_m |1\rangle$ has to be zero for $m > \lambda - 1$. Because $b(z) |1\rangle$ has no pole at $z \rightarrow 0$, $b_m |1\rangle$ has to be zero for $m > -\lambda$.

e)

$$\begin{aligned}
L_0 |1\rangle &= \left(\sum_{k>0} k(c_{-k} b_k + b_{-k} c_k) + a^g \right) |1\rangle \\
&= \left(\sum_1^{[-\lambda]} k c_{-k} b_k + \sum_1^{[\lambda-1]} k b_{-k} c_k + a^g \right) |1\rangle \\
&= \left(\left\{ \frac{1+[-\lambda]}{2} [-\lambda], \text{ if } \lambda \leq -1; 0 \text{ otherwise} \right\} \right. \\
&\quad \left. + \left\{ \frac{1+[\lambda-1]}{2} [\lambda-1], \text{ if } \lambda \geq 2; 0 \text{ otherwise} \right\} + a^g \right) |1\rangle
\end{aligned} \tag{15}$$

For L_1 , exchanging b and c operators does not give extra constant term. So the only non zero terms are in the region $2 - \lambda \leq k \leq -\lambda, k \in \mathbb{Z}$ which is an empty set. Thus,

$$L_1 |1\rangle = \left(\sum_k (\lambda - k) : b_k c_{1-k} : \right) |1\rangle = 0 \tag{16}$$

For L_{-1} the non zero term is $k = -\lambda$. If $\lambda \notin \mathbb{Z}$, it is empty; if $\lambda \in \mathbb{Z}$, the only term left is still zero. So $L_{-1} |1\rangle = 0$.

f) Remember that $[L_0, b_n] = -nb_n$ and $[L_0, c_n] = nc_n$