Homework 3

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PHY 539: Introduction to String Theory

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1 Problem 1

a) It is not hard to show that under $z \to \frac{az+b}{cz+d}$, the cross ratio becomes

$$\frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right)\left(\frac{az_3+b}{cz_3+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}\right)} = \frac{(z_1-z_2)(z_3-z_4)(bc-ad)^2}{(z_1-z_3)(z_2-z_4)(bc-ad)^2} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \quad (1)$$

b) We set the transformation to be $z \to \frac{az+b}{cz+d}$. Solving the set of equations

$$az_1 + b = 0 \wedge az_2 + b = cz_2 + d \wedge cz_3 + d = 0 \wedge ad - bc = 1, \quad (2)$$

we have

$$a \to -\frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to \frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \quad (3)$$
$$c \to \frac{z_2-z_1}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_1-z_2)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}$$

or

$$a \to \frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to -\frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \quad (4)$$
$$c \to \frac{z_1-z_2}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_2-z_1)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}.$$

2 Problem 2

We refer to the b c system as a $2D$ field theory with a Grassmann action

$$S = \frac{1}{2\pi}\int d^2z\ b\bar{\partial}c \quad (5)$$

The contraction of b and c field is

$$b(z)c(\omega) - :b(z)c(\omega):= \frac{1}{z-\omega}. \quad (6)$$

a)

$$\begin{aligned}
T(z)T(w) &= (:(\partial_z b)c(z): -\lambda\partial_z :bc(z):)\,(:(\partial_w b)c(w): -\lambda\partial_w :bc(w):)\\
&=: (\partial_z b)c(z) :: (\partial_w b)c(w): -\lambda\partial_z :bc(z)::(\partial_w b)c(w):\\
&\quad -\lambda :(\partial_z b)c(z): \partial_w :bc(w): +\lambda^2\partial_z :bc(z):\partial_w :bc(w):
\end{aligned} \tag{7}$$

The full contraction(quartic) part is

$$\lambda^2\frac{\partial}{\partial z}\frac{\partial}{\partial\omega}\frac{1}{(z-\omega)(z-\omega)} - \lambda\frac{\partial}{\partial\omega}\frac{\frac{\partial}{\partial z}\frac{1}{z-\omega}}{z-\omega} - \lambda\frac{\partial}{\partial z}\frac{\frac{\partial}{\partial\omega}\frac{1}{z-\omega}}{z-\omega} + \frac{\partial}{\partial z}\frac{1}{z-\omega}\frac{\partial}{\partial\omega}\frac{1}{z-\omega} = \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} \tag{8}$$

The other terms are(omitting normal ordering symbols for simplicity)

$$\begin{aligned}
c(\omega)\frac{\partial b(z)}{\partial z}\frac{\partial}{\partial\omega}\frac{1}{z-\omega} + c(z)\frac{\partial b(\omega)}{\partial\omega}\frac{\partial}{\partial z}\frac{1}{z-\omega} &= -\frac{2\left(c(\omega)b'(\omega)\right)}{(z-\omega)^2} + \frac{-c(\omega)b''(\omega)-b'(\omega)c'(\omega)}{z-\omega}\\
&\quad +O\left((z-\omega)^0\right)\\
-\lambda\frac{\partial}{\partial z}\left(b(z)c(\omega)\frac{\partial}{\partial\omega}\frac{1}{z-\omega} + \frac{c(z)\frac{\partial b(\omega)}{\partial\omega}}{z-\omega}\right) &= \frac{2\lambda b(\omega)c(\omega)}{(z-\omega)^3} + \frac{4\lambda c(\omega)b'(\omega)}{(z-\omega)^2}\\
&\quad +\frac{\lambda\left(2c(\omega)b''(\omega)+2b'(\omega)c'(\omega)\right)}{z-\omega} + O\left((z-\omega)^0\right)\\
-\lambda\frac{\partial}{\partial\omega}\left(b(\omega)c(z)\frac{\partial}{\partial z}\frac{1}{z-\omega} + \frac{c(\omega)\frac{\partial b(z)}{\partial z}}{z-\omega}\right) &= -\frac{2(\lambda b(\omega)c(\omega))}{(z-\omega)^3} + \frac{\lambda\left(2c(\omega)b'(\omega)-2b(\omega)c'(\omega)\right)}{(z-\omega)^2}\\
&\quad +\frac{\lambda\left(c(\omega)b''(\omega)-b(\omega)c''(\omega)\right)}{z-\omega} + O\left((z-\omega)^0\right)\\
\lambda^2\frac{\partial}{\partial z}\frac{\partial}{\partial\omega}\left(\frac{b(\omega)c(z)}{z-\omega} + \frac{b(z)c(\omega)}{z-\omega}\right) &= -\frac{4\left(\lambda^2 c(\omega)b'(\omega)\right)}{(z-\omega)^2} + \frac{\lambda^2\left(-2c(\omega)b''(\omega)-2b'(\omega)c'(\omega)\right)}{z-\omega}\\
&\quad +O\left((z-\omega)^0\right)
\end{aligned} \tag{9}$$

In conclusion, we have

$$\begin{aligned}
T(z)T(\omega) \sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} - \frac{2\left((\lambda-1)(2\lambda-1)c(\omega)b'(\omega)+\lambda b(\omega)c'(\omega)\right)}{(z-\omega)^2}\\
\frac{((3-2\lambda)\lambda-1)c(\omega)b''(\omega)+(-2(\lambda-1)\lambda-1)b'(\omega)c'(\omega)-\lambda b(\omega)c''(\omega)}{z-\omega}
\end{aligned} \tag{10}$$

???Shouldn't $T(z)T(\omega) \sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}$? I only noticed this contradiction in the last minute but could not make it right.

b)

$$T(z) = \sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}\,, \qquad L_m = \frac{1}{2\pi i}\oint dz\; z^{m+1}T(z) \tag{11}$$

We can just expand $T(z)$ in terms of b and c modes and pick up the power $-(m+2)$ terms for $m \neq 0$.

$$
\begin{aligned}
T(z) =: (\partial_z b)c(z) : -\lambda\partial_z : bc(z) :&= \sum_l \sum_k : \frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} : \\
&- : \lambda\Big(\frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} + \frac{: b_k}{z^{k+\lambda}} \frac{-(l+1-\lambda)c_l :}{z^{l+2-\lambda}}\Big) : \\
(\text{setting } l = m-k) &= \sum_m \sum_k \frac{: b_k c_{m-k}:}{z^{m+2}} (-(k+\lambda) + \lambda(k+\lambda) + \lambda(m-k+1-\lambda)) \\
&= \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (\lambda m - k)
\end{aligned}
\tag{12}
$$

It is easy to read off L_m from above as $L_m = \sum_k (\lambda m - k) : b_k c_{m-k} :$. For $m = 0$, an appropriate normal ordering constant should be included: $L_0 = \sum_k -k : b_k c_{-k} : + \frac{1}{2}\lambda(1-\lambda)$