Physics 539 Fall 2018

Problem Set 3

Due on Nov. 5

1. a) Show that the cross-ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

is invariant under $SL(2, \mathbf{C})$.

b) Find an $SL(2, \mathbf{C})$ transformation which maps points z_1, z_2, z_3 into 0, 1, and ∞ respectively.

2. Consider the anti-commuting b c system where $h_b = \lambda$, $h_c = 1 - \lambda$, and λ is a positive integer. The stress-energy tensor is

$$T(z) =: (\partial b)c : -\lambda \partial(: bc :)$$

a) Find the OPE of $T(z)$ with $T(w)$.

b) Using the mode expansions

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}} , \qquad c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}} ,$$

express the Virasoro operators L_m in terms of b_k and c_l.

c) Calculate the commutators $[L_m, b_n]$ and $[L_m, c_n]$.

d) For what range of m do operators c_m annihilate the state $|1\rangle$ corresponding to the identity operator? For what range of m do operators b_m annihilate it?

e) Find $L_n|1\rangle$ for $n = -1, 0, 1$.

f) Write down the states in the Hilbert space which have the lowest eigenvalue of L_0? What is the lowest eigenvalue?

3. Consider the $c = 1$ conformal field theory of a free field $X(z, \bar{z})$. The descendant state $L_{-1}|0; k\rangle$ vanishes for $k = 0$. In turn, a new highest weight state $\alpha_{-1}|0; 0\rangle$ appears.

a) Find the values of k and a where descendants of the form

$$(L_{-2} + aL_{-1}^2)|0; k\rangle$$

vanish.

b) Show that at these values of k there exist highest weight, positive norm states of the form

$$(\alpha_{-2} + b\alpha_{-1}^2)|0;k\rangle$$

4. Consider a highest-weight state $|\phi\rangle$ of weight h in a CFT with central charge c. Calculate the 3×3 matrix of inner products for descendants at level 3, i.e.

$$L_{-3}|\phi\rangle\ , \quad L_{-2}L_{-1}|\phi\rangle\ , \quad L_{-1}^3|\phi\rangle$$

Find the weights h for which there is a null state at level 3.

5. The Ward identity for insertion of $T(z)$ into a correlation function of Virasoro primary fields A_i of conformal weights $(h_i, \tilde{h}_i)$ is

$$\langle T(z)A_1(z_1,\bar{z}_1)\ldots A_n(z_n,\bar{z}_n)\rangle = \sum_{p=1}^{n}\left[\frac{h_p}{(z-z_p)^2} + \frac{1}{z-z_p}\frac{\partial}{\partial z_p}\right]\langle A_1(z_1,\bar{z}_1)\ldots A_n(z_n,\bar{z}_n)\rangle$$

a) Derive the three constraints obeyed by $\langle A_1(z_1,\bar{z}_1)\ldots A_n(z_n,\bar{z}_n)\rangle$ that follow from the fact that $T(z) \sim z^{-4}$ at large z. What is the meaning of these constraints?

b) Using these constraints, and the corresponding constraints stemming from the Ward identity for insertion of $\tilde{T}(\bar{z})$, determine the coordinate dependence of the 2-point function $\langle A_1(z_1,\bar{z}_1)A_2(z_2,\bar{z}_2)\rangle$. Under what circumstances is it non-vanishing?