Homework 3

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PHY 539: Introduction to String Theory

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1 Problem 1

a) It is not hard to show that under $z \to \frac{az+b}{cz+d}$, the cross ratio becomes

$$\frac{\left(\frac{az_1+b}{cz_1+d}-\frac{az_2+b}{cz_2+d}\right)\left(\frac{az_3+b}{cz_3+d}-\frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d}-\frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d}-\frac{az_4+b}{cz_4+d}\right)}=\frac{(z_1-z_2)(z_3-z_4)(bc-ad)^2}{(z_1-z_3)(z_2-z_4)(bc-ad)^2}=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \tag{1}$$

b) We set the transformation to be $z \to \frac{az+b}{cz+d}$. Solving the set of equations

$$az_1+b=0 \wedge az_2+b=cz_2+d \wedge cz_3+d=0 \wedge ad-bc=1, \tag{2}$$

we have

$$\begin{aligned} a &\to -\frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to \frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \\ c &\to \frac{z_2-z_1}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_1-z_2)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}} \end{aligned} \tag{3}$$

or

$$\begin{aligned} a &\to \frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to -\frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \\ c &\to \frac{z_1-z_2}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_2-z_1)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}. \end{aligned} \tag{4}$$

2 Problem 2

We refer to the b c system as a $2D$ field theory with a Grassmann action

$$S=\frac{1}{2\pi}\int d^2z\ b\bar{\partial}c \tag{5}$$

a)

$$
\begin{aligned}
T(z)T(w) &= (:(\partial_z b)c(z): - \lambda\partial_z :bc(z):)(:(\partial_w b)c(w): - \lambda\partial_w :bc(w):) \\
&=:(\partial_z b)c(z)::(\partial_w b)c(w): - \lambda\partial_z :bc(z)::(\partial_w b)c(w): \\
&\quad - \lambda :(\partial_z b)c(z):\partial_w :bc(w): + \lambda^2\partial_z :bc(z):\partial_w :bc(w):
\end{aligned} \tag{6}
$$

The full contraction(quartic) part is

$$
\lambda^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \omega}\frac{1}{(z-\omega)(z-\omega)} - \lambda\frac{\partial}{\partial \omega}\frac{\frac{\partial}{\partial z}\frac{1}{z-\omega}}{z-\omega} - \lambda\frac{\partial}{\partial z}\frac{\frac{\partial}{\partial \omega}\frac{1}{z-\omega}}{z-\omega} + \frac{\partial}{\partial z}\frac{1}{z-\omega}\frac{\partial}{\partial \omega}\frac{1}{z-\omega} = \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} \tag{7}
$$

The quadratic part is

$$
123 \tag{8}
$$