Homework 3

Ziming Ji
PHY 539: Introduction to String Theory

November 5, 2018

1 Problem 1

a) It is not hard to show that under $z \to \frac{az+b}{cz+d}$, the cross ratio becomes

$$\frac{\left(\frac{az_1+b}{cz_1+d}-\frac{az_2+b}{cz_2+d}\right)\left(\frac{az_3+b}{cz_3+d}-\frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d}-\frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d}-\frac{az_4+b}{cz_4+d}\right)}=\frac{(z_1-z_2)(z_3-z_4)(bc-ad)^2}{(z_1-z_3)(z_2-z_4)(bc-ad)^2}=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \tag{1}$$

b) We set the transformation to be $z \to \frac{az+b}{cz+d}$. Solving the set of equations

$$az_1+b=0 \wedge az_2+b=cz_2+d \wedge cz_3+d=0 \wedge ad-bc=1, \tag{2}$$

we have

$$a \to -\frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to \frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \tag{3}$$
$$c \to \frac{z_2-z_1}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_1-z_2)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}$$

or

$$a \to \frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to -\frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \tag{4}$$
$$c \to \frac{z_1-z_2}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_2-z_1)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}.$$

2 Problem 2

We refer to the b c system as a $2D$ field theory with a Grassmann action

$$S=\frac{1}{2\pi}\int d^2z\; b\bar{\partial}c \tag{5}$$

The contraction of b and c field is

$$b(z)c(\omega)-:b(z)c(\omega):=\frac{1}{z-\omega}. \tag{6}$$

a)

$$
\begin{aligned}
T(z)T(w) &= (:(\partial_z b)c(z): - \lambda\partial_z :bc(z):)(:(\partial_w b)c(w): - \lambda\partial_w :bc(w):) \\
&=: (\partial_z b)c(z) :: (\partial_w b)c(w): - \lambda\partial_z :bc(z)::(\partial_w b)c(w): \\
&\quad - \lambda :(\partial_z b)c(z): \partial_w :bc(w): + \lambda^2 \partial_z :bc(z): \partial_w :bc(w):
\end{aligned} \tag{7}
$$

The full contraction(quartic) part is

$$
\lambda^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \omega}\frac{1}{(z-\omega)(z-\omega)} - \lambda \frac{\partial}{\partial \omega}\frac{\frac{\partial}{\partial z}\frac{1}{z-\omega}}{z-\omega} - \lambda \frac{\partial}{\partial z}\frac{\frac{\partial}{\partial \omega}\frac{1}{z-\omega}}{z-\omega} + \frac{\partial}{\partial z}\frac{1}{z-\omega}\frac{\partial}{\partial \omega}\frac{1}{z-\omega} = \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} \tag{8}
$$

The other terms are(omitting normal ordering symbols for simplicity)

$$
\begin{aligned}
c(\omega)\frac{\partial b(z)}{\partial z}\frac{\partial}{\partial \omega}\frac{1}{z-\omega} + c(z)\frac{\partial b(\omega)}{\partial \omega}\frac{\partial}{\partial z}\frac{1}{z-\omega} &= -\frac{2\left(c(\omega)b'(\omega)\right)}{(z-\omega)^2} + \frac{-c(\omega)b''(\omega) - b'(\omega)c'(\omega)}{z-\omega} \\
&\quad + O\left((z-\omega)^0\right) \\
-\lambda \frac{\partial}{\partial z}\left(b(z)c(\omega)\frac{\partial}{\partial \omega}\frac{1}{z-\omega} + \frac{c(z)\frac{\partial b(\omega)}{\partial \omega}}{z-\omega}\right) &= \frac{2\lambda b(\omega)c(\omega)}{(z-\omega)^3} + \frac{4\lambda c(\omega)b'(\omega)}{(z-\omega)^2} \\
&\quad + \frac{\lambda\left(2c(\omega)b''(\omega) + 2b'(\omega)c'(\omega)\right)}{z-\omega} + O\left((z-\omega)^0\right) \\
-\lambda \frac{\partial}{\partial \omega}\left(b(\omega)c(z)\frac{\partial}{\partial z}\frac{1}{z-\omega} + \frac{c(\omega)\frac{\partial b(z)}{\partial z}}{z-\omega}\right) &= -\frac{2(\lambda b(\omega)c(\omega))}{(z-\omega)^3} + \frac{\lambda\left(2c(\omega)b'(\omega) - 2b(\omega)c'(\omega)\right)}{(z-\omega)^2} \\
&\quad + \frac{\lambda\left(c(\omega)b''(\omega) - b(\omega)c''(\omega)\right)}{z-\omega} + O\left((z-\omega)^0\right) \\
\lambda^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \omega}\left(\frac{b(\omega)c(z)}{z-\omega} + \frac{b(z)c(\omega)}{z-\omega}\right) &= -\frac{4\left(\lambda^2 c(\omega)b'(\omega)\right)}{(z-\omega)^2} + \frac{\lambda^2\left(-2c(\omega)b''(\omega) - 2b'(\omega)c'(\omega)\right)}{z-\omega} \\
&\quad + O\left((z-\omega)^0\right)
\end{aligned} \tag{9}
$$

In conclusion, we have

$$
\begin{aligned}
T(z)T(\omega) \sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} - \frac{2\left((\lambda-1)(2\lambda-1)c(\omega)b'(\omega) + \lambda b(\omega)c'(\omega)\right)}{(z-\omega)^2} \\
\frac{((3-2\lambda)\lambda-1)c(\omega)b''(\omega) + (-2(\lambda-1)\lambda-1)b'(\omega)c'(\omega) - \lambda b(\omega)c''(\omega)}{z-\omega}
\end{aligned} \tag{10}
$$

???Shouldn't $T(z)T(\omega) \sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}$? I only noticed this contradiction in the last minute but could not make it right.

b)

$$
T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}\,, \qquad L_m = \frac{1}{2\pi i}\oint dz\; z^{m+1} T(z) \tag{11}
$$

We can just expand $T(z)$ in terms of b and c modes and pick up the power $-(m+2)$ terms for $m \neq 0$.

$$T(z) =: (\partial_z b)c(z) : -\lambda\partial_z : bc(z) := \sum_l \sum_k : \frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} : \tag{12}$$

$$- : \lambda(\frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} + \frac{: b_k}{z^{k+\lambda}} \frac{-(l+1-\lambda)c_l :}{z^{l+2-\lambda}}) :$$

$$(\text{setting } l = m-k) = \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (-(k+\lambda) + \lambda(k+\lambda) + \lambda(m-k+1-\lambda))$$

$$= \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (\lambda m - k)$$

It is easy to read off L_m from above as $L_m = \sum_k (\lambda m - k) : b_k c_{m-k} :$. For $m = 0$, an appropriate normal ordering constant should be included: $L_0 = \sum_k k(c_{-k}b_k + b_{-k}c_k) - k$.

c)

$$[L_m, b_n] = [\sum_k (\lambda m - k) : b_k c_{m-k} :, b_n] \tag{13}$$

$$= ((\lambda - 1)m - n)[: b_{m+n}c_{-n} :, b_n] = ((\lambda - 1)m - n)b_{m+n}$$

$$[L_m, c_n] = [\sum_k (\lambda m - k) : b_k c_{m-k} :, c_n] = (\lambda m + n)[: b_{-n}c_{m+n} :, c_n] = (\lambda m + n)c_{m+n} \tag{14}$$

d) Because $c(z)\left|1\right\rangle$ has no pole at $z \to 0$, $c_m\left|1\right\rangle$ has to be zero for $m > \lambda - 1$. Because $b(z)\left|1\right\rangle$ has no pole at $z \to 0$, $b_m\left|1\right\rangle$ has to be zero for $m > -\lambda$.

e)

$$L_0\left|1\right\rangle = (\sum_{k>0} k(c_{-k}b_k + b_{-k}c_k) + a^g)\left|1\right\rangle \tag{15}$$

$$= (\sum_1^{[-\lambda]} kc_{-k}b_k + \sum_1^{[\lambda-1]} kb_{-k}c_k + a^g)\left|1\right\rangle$$

$$= (\{\frac{1+[-\lambda]}{2}[-\lambda], \text{ if } \lambda \leq -1; \text{ 0 otherwise } \}$$

$$+ \{\frac{1+[\lambda-1]}{2}[\lambda-1], \text{ if } \lambda \geq 2; \text{ 0 otherwise } \} + a^g)\left|1\right\rangle$$

For L_1, exchanging b and c operators does not give extra constant term. So the only non zero terms are in the region $2 - \lambda \leq k \leq -\lambda, k \in \mathbb{Z}$ which is an empty set. Thus,

$$L_1\left|1\right\rangle = (\sum_k (\lambda - k) : b_k c_{1-k} :))\left|1\right\rangle = 0 \tag{16}$$

For L_{-1} the non zero term is $k = -\lambda$. If $\lambda \notin \mathbb{Z}$, it is empty; if $\lambda \in \mathbb{Z}$, the only term left is still zero. So $L_{-1}\left|1\right\rangle = 0$.

f) Remember that $[L_0, b_n] = -nb_n$ and $[L_0, c_n] = nc_n$

3 Problem 3

$L_{-1}|0;0\rangle = 0$ then this $|0;0\rangle$ is the identity state($h=0$). For $|0;k\rangle$ the weight is $\frac{k^2}{2}$.

a)

$$\begin{aligned} \langle k;0|\,(L_2 + aL_1^2)(L_{-2} + aL_{-1}^2)\,|0;k\rangle &= 0 \\ \langle k;0|\,L_2L_{-2} + aL_2L_{-1}^2 + aL_1^2L_{-2} + a^2L_1^2L_{-1}^2\,|0;k\rangle &= 0 \end{aligned} \tag{17}$$