Homework 3

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PHY 539: Introduction to String Theory

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1 Problem 1

a) It is not hard to show that under $z \to \frac{az+b}{cz+d}$, the cross ratio becomes

$$\frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right)\left(\frac{az_3+b}{cz_3+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}\right)} = \frac{(z_1-z_2)(z_3-z_4)(bc-ad)^2}{(z_1-z_3)(z_2-z_4)(bc-ad)^2} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \tag{1}$$

b) We set the transformation to be $z \to \frac{az+b}{cz+d}$. Solving the set of equations

$$az_1 + b = 0 \wedge az_2 + b = cz_2 + d \wedge cz_3 + d = 0 \wedge ad - bc = 1, \tag{2}$$

we have

$$a \to -\frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to \frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \tag{3}$$
$$c \to \frac{z_2-z_1}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_1-z_2)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}$$

or

$$a \to \frac{\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, b \to -\frac{z_1\sqrt{z_3-z_2}}{\sqrt{(z_1-z_2)(z_1-z_3)}}, \tag{4}$$
$$c \to \frac{z_1-z_2}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}, d \to \frac{(z_2-z_1)z_3}{\sqrt{(z_1-z_2)(z_1-z_3)}\sqrt{z_3-z_2}}.$$

2 Problem 2

We refer to the b c system as a $2D$ field theory with a Grassmann action

$$S = \frac{1}{2\pi}\int d^2z\ b\bar{\partial}c \tag{5}$$

The contraction of b and c field is

$$b(z)c(\omega) - :b(z)c(\omega): = \frac{1}{z-\omega}. \tag{6}$$

a)

$$\begin{aligned} T(z)T(w) &= (:(\partial_z b)c(z):-\lambda\partial_z:bc(z):)\,(:(\partial_w b)c(w):-\lambda\partial_w:bc(w):) \\ &=:(\partial_z b)c(z)::(\partial_w b)c(w):-\lambda\partial_z:bc(z)::(\partial_w b)c(w): \\ &\quad -\lambda:(\partial_z b)c(z):\partial_w:bc(w):+\lambda^2\partial_z:bc(z):\partial_w:bc(w): \end{aligned} \tag{7}$$

The full contraction(quartic) part is

$$\lambda^2\frac{\partial}{\partial z}\frac{\partial}{\partial\omega}\frac{1}{(z-\omega)(z-\omega)}-\lambda\frac{\partial}{\partial\omega}\frac{\frac{\partial}{\partial z}\frac{1}{z-\omega}}{z-\omega}-\lambda\frac{\partial}{\partial z}\frac{\frac{\partial}{\partial\omega}\frac{1}{z-\omega}}{z-\omega}+\frac{\partial}{\partial z}\frac{1}{z-\omega}\frac{\partial}{\partial\omega}\frac{1}{z-\omega}=\frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4} \tag{8}$$

The other terms are(omitting normal ordering symbols for simplicity)

$$\begin{aligned} c(\omega)\frac{\partial b(z)}{\partial z}\frac{\partial}{\partial\omega}\frac{1}{z-\omega}+c(z)\frac{\partial b(\omega)}{\partial\omega}\frac{\partial}{\partial z}\frac{1}{z-\omega} &= -\frac{2\left(c(\omega)b'(\omega)\right)}{(z-\omega)^2}+\frac{-c(\omega)b''(\omega)-b'(\omega)c'(\omega)}{z-\omega} \\ &\quad +O\left((z-\omega)^0\right) \\ -\lambda\frac{\partial}{\partial z}\left(b(z)c(\omega)\frac{\partial}{\partial\omega}\frac{1}{z-\omega}+\frac{c(z)\frac{\partial b(\omega)}{\partial\omega}}{z-\omega}\right) &= \frac{2\lambda b(\omega)c(\omega)}{(z-\omega)^3}+\frac{4\lambda c(\omega)b'(\omega)}{(z-\omega)^2} \\ &\quad +\frac{\lambda\left(2c(\omega)b''(\omega)+2b'(\omega)c'(\omega)\right)}{z-\omega}+O\left((z-\omega)^0\right) \\ -\lambda\frac{\partial}{\partial\omega}\left(b(\omega)c(z)\frac{\partial}{\partial z}\frac{1}{z-\omega}+\frac{c(\omega)\frac{\partial b(z)}{\partial z}}{z-\omega}\right) &= -\frac{2(\lambda b(\omega)c(\omega))}{(z-\omega)^3}+\frac{\lambda\left(2c(\omega)b'(\omega)-2b(\omega)c'(\omega)\right)}{(z-\omega)^2} \\ &\quad +\frac{\lambda\left(c(\omega)b''(\omega)-b(\omega)c''(\omega)\right)}{z-\omega}+O\left((z-\omega)^0\right) \\ \lambda^2\frac{\partial}{\partial z}\frac{\partial}{\partial\omega}\left(\frac{b(\omega)c(z)}{z-\omega}+\frac{b(z)c(\omega)}{z-\omega}\right) &= -\frac{4\left(\lambda^2c(\omega)b'(\omega)\right)}{(z-\omega)^2}+\frac{\lambda^2\left(-2c(\omega)b''(\omega)-2b'(\omega)c'(\omega)\right)}{z-\omega} \\ &\quad +O\left((z-\omega)^0\right) \end{aligned} \tag{9}$$

In conclusion, we have

$$\begin{aligned} T(z)T(\omega) \sim \frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4}-\frac{2\left((\lambda-1)(2\lambda-1)c(\omega)b'(\omega)+\lambda b(\omega)c'(\omega)\right)}{(z-\omega)^2} \\ \frac{((3-2\lambda)\lambda-1)c(\omega)b''(\omega)+(-2(\lambda-1)\lambda-1)b'(\omega)c'(\omega)-\lambda b(\omega)c''(\omega)}{z-\omega} \end{aligned} \tag{10}$$

???Shouldn't $T(z)T(\omega)\sim\frac{-6(\lambda-1)\lambda-1}{(z-\omega)^4}+\frac{2T(\omega)}{(z-\omega)^2}+\frac{\partial T(\omega)}{z-\omega}$? I only noticed this contradiction in the last minute but could not make it right.

b)

$$T(z)=\sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}\,,\qquad L_m=\frac{1}{2\pi i}\oint dz\ z^{m+1}T(z) \tag{11}$$

We can just expand $T(z)$ in terms of b and c modes and pick up the power $-(m+2)$ terms for $m \neq 0$.

$$T(z) =: (\partial_z b)c(z) : -\lambda\partial_z : bc(z) := \sum_l \sum_k : \frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} : \tag{12}$$
$$- : \lambda(\frac{-(k+\lambda) : b_k}{z^{k+\lambda+1}} \frac{c_l :}{z^{l+1-\lambda}} + \frac{: b_k}{z^{k+\lambda}} \frac{-(l+1-\lambda)c_l :}{z^{l+2-\lambda}}) :$$
$$(\text{setting } l = m-k) = \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (-(k+\lambda) + \lambda(k+\lambda) + \lambda(m-k+1-\lambda))$$
$$= \sum_m \sum_k \frac{: b_k c_{m-k} :}{z^{m+2}} (\lambda m - k)$$

It is easy to read off L_m from above as $L_m = \sum_k (\lambda m - k) : b_k c_{m-k} :$. For $m = 0$, an appropriate normal ordering constant should be included: $L_0 = \sum_k k(c_{-k}b_k + b_{-k}c_k) - k$.

c)

$$[L_m, b_n] = [\sum_k (\lambda m - k) : b_k c_{m-k} :, b_n] \tag{13}$$
$$= ((\lambda-1)m - n)[: b_{m+n}c_{-n} :, b_n] = ((\lambda-1)m-n)b_{m+n}$$
$$[L_m, c_n] = [\sum_k (\lambda m - k) : b_k c_{m-k} :, c_n] = (\lambda m + n)[: b_{-n}c_{m+n} :, c_n] = (\lambda m + n)c_{m+n} \tag{14}$$

d) Because $c(z)\,|1\rangle$ has no pole at $z \to 0$, $c_m\,|1\rangle$ has to be zero for $m > \lambda - 1$. Because $b(z)\,|1\rangle$ has no pole at $z \to 0$, $b_m\,|1\rangle$ has to be zero for $m > -\lambda$.

e)

$$L_0\,|1\rangle = (\sum_{k>0} k(c_{-k}b_k + b_{-k}c_k) + a^g)\,|1\rangle \tag{15}$$
$$= (\sum_1^{[-\lambda]} k c_{-k}b_k + \sum_1^{[\lambda-1]} k b_{-k}c_k + a^g)\,|1\rangle$$
$$= (\{\frac{1+[-\lambda]}{2}[-\lambda], \text{ if } \lambda \leq -1; \text{ 0 otherwise } \}$$
$$+ \{\frac{1+[\lambda-1]}{2}[\lambda-1], \text{ if } \lambda \geq 2; \text{ 0 otherwise } \} + a^g)\,|1\rangle$$

For L_1, exchanging b and c operators does not give extra constant term. So the only non zero terms are in the region $2-\lambda \leq k \leq -\lambda, k \in \mathbb{Z}$ which is an empty set. Thus,

$$L_1\,|1\rangle = (\sum_k (\lambda - k) : b_k c_{1-k} :))\,|1\rangle = 0 \tag{16}$$

For L_{-1} the non zero term is $k = -\lambda$. If $\lambda \notin \mathbb{Z}$, it is empty; if $\lambda \in \mathbb{Z}$, the only term left is still zero. So $L_{-1}\,|1\rangle = 0$.

f) Remember that $[L_0, b_n] = -nb_n$ and $[L_0, c_n] = nc_n$...not finished

3 Problem 3

$L_{-1}|0;0\rangle = 0$ then this $|0;0\rangle$ is the identity state($h = 0$). For $|0;k\rangle$ the weight is $\frac{k^2}{2}$.

a)

$$\begin{aligned} \langle k;0|\,(L_2 + aL_1^2)(L_{-2} + aL_{-1}^2)\,|0;k\rangle &= 0 \\ \langle k;0|\,L_2L_{-2} + aL_2L_{-1}^2 + aL_1^2L_{-2} + a^2L_1^2L_{-1}^2\,|0;k\rangle &= 0 \\ \langle k;0|\,4h + \frac{c}{2} + 6ah + 6ah + a^2 4h(2h+1)\,|0;k\rangle &= 0 \\ 8a^2h^2 + \left(4a^2 + 12a + 4\right)h + \frac{c}{2} &= 0 \end{aligned} \tag{17}$$

Then we have

$$\begin{aligned} \frac{k^2}{2} &= -\frac{a(a+3) + \sqrt{(a+1)^2(a(a+4)+1)} + 1}{4a^2} \\ \text{or } \frac{k^2}{2} &= \frac{-a(a+3) + \sqrt{(a+1)^2(a(a+4)+1)} - 1}{4a^2} \end{aligned} \tag{18}$$

b) The norm of the state is $(2 + 2b^2)\langle k;0|0;k\rangle$, which is positive for real coefficient b. For its weight, we have

$$L_0(\alpha_{-2} + b\alpha_{-1}^2)\,|0;k\rangle = (2+h)\alpha_{-2} + (2b + hb)\alpha_{-1}^2\,|0;k\rangle \tag{19}$$

So the weight is $2 + h = 2 + \frac{k^2}{2}$. For $L_1|\phi\rangle = 0$, $b = \frac{-1}{k}$.

4 Problem 4

The matrix is

$$\begin{pmatrix} L_3L_{-3} & L_3L_{-2}L_{-1} & L_3L_{-1}^3 \\ L_1L_2L_{-3} & L_1L_2L_{-2}L_{-1} & L_1L_2L_{-1}^3 \\ L_1^3L_{-3} & L_1^3L_{-2}L_{-1} & L_1^3L_{-1}^3 \end{pmatrix} \tag{20}$$

Evaluating this with the highest-weight state, we have $\langle\phi|\,M\,|\phi\rangle =$

$$\begin{pmatrix} 2c + 6h & 10h & 16h \\ 10h & 8h^2 + ch + 8h & 36h^2 + 12h \\ 16h & 36h^2 + 12h & 4h(6h+6)(2h+1) \end{pmatrix} \tag{21}$$

If the determinant is zero, then there is combinations of the basis that give zero, or there exists null state.

$$\begin{aligned} \det\, M = 16h^2(3c^2(h+1)(2h+1) &+ c(h(3h(22h-5)-19)+6) \\ &+ 2h(h(3h(24h-71)+53)-10)) \end{aligned} \tag{22}$$

Difficult to solve?

5 Problem 5

a) Using the conformal Ward identity

$$\delta_\epsilon \langle A \rangle = -\frac{1}{2\pi i} \oint dz\, \epsilon(z) \langle T(z) A \rangle = 0 \tag{23}$$

When the infinitesimal transformations are $\epsilon \sim O(1) + O(z) + O(z^2)$. These means the correlation function is invariant under all the conformal transformations.

b)

$$\langle A_1(z_1, \bar{z}_1) A_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}, \text{ if } h_1 = h_2 = h \text{ and } \bar{h}_1 = \bar{h}_2 = \bar{h} \tag{24}$$

If the conformal dimensions are different then the correlation function vanishes.