Physics 539 Problem Set 3 Solutions*

November 5, 2018

Problem 1

(a)

The standard definition of the cross-ratio is in fact

$$(z_1, z_2, z_3, z_4) \equiv \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{1}$$

By direct calculation, (z_1, z_2, z_3, z_4) is invariant under an $SL(2, \mathbb{C})$ transformation

$$z_i \mapsto \frac{az_i + b}{cz_i + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1 \tag{2}$$

(in fact, it suffices to require that $ad - bc \neq 0$).

(b)

One can show that $z \mapsto (z, z_2, z_3, z_4)$ is the *unique* linear fractional transformation that maps z_2, z_3, z_4 to $1, 0, \infty$, in that order. Hence

$$z \mapsto (z, z_2, z_1, z_3) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \tag{3}$$

maps z_1, z_2, z_3 into $0, 1, \infty$, respectively.

Problem 2

Here, we consider the bc CFT with

$$h_b = \lambda, \quad h_c = 1 - \lambda, \tag{4}$$

and λ a positive integer.

(a)

Let us write the stress tensor as

$$T(z) = (1 - \lambda) : (\partial b)c : - \lambda : b(\partial c) : . \tag{5}$$

Being careful with signs from anticommutation and using

$$b(z)c(w) \sim c(z)b(w) \sim \frac{1}{z - w}, \tag{6}$$

*Please send comments or questions to Yale Fan, yalefan@gmail.com.

we compute the OPEs

$$
\begin{aligned}
:\partial b(z)c(z)::\partial b(w)c(w): &= -\frac{1}{(z-w)^4}+\frac{2}{(z-w)^2}:\partial b(w)c(w): \\
&\quad +\frac{1}{z-w}:\partial(\partial b(w)c(w)):+\cdots, \qquad (7)\\
:b(z)\partial c(z)::b(w)\partial c(w): &= -\frac{1}{(z-w)^4}-\frac{2}{(z-w)^2}:b(w)\partial c(w): \\
&\quad -\frac{1}{z-w}:\partial(b(w)\partial c(w)):+\cdots, \qquad (8)\\
:\partial b(z)c(z)::b(w)\partial c(w): &= -\frac{2}{(z-w)^4}+\frac{2}{(z-w)^3}:b(w)c(w): \\
&\quad +\frac{2}{(z-w)^2}:b(w)\partial c(w):+\frac{1}{z-w}:\partial(b(w)\partial c(w)):+\cdots, \qquad (9)\\
:b(z)\partial c(z)::\partial b(w)c(w): &= -\frac{2}{(z-w)^4}-\frac{2}{(z-w)^3}:b(w)c(w): \\
&\quad -\frac{2}{(z-w)^2}:\partial b(w)c(w):-\frac{1}{z-w}:\partial(\partial b(w)c(w)):+\cdots, \qquad (10)
\end{aligned}
$$

from which we deduce that

$$
T(z)T(w)=\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\cdots \qquad (11)
$$

where $c=-3(2\lambda-1)^2+1$.

(b)

Using the mode expansions

$$
b(z)=\sum_{k=-\infty}^{\infty}\frac{b_k}{z^{k+\lambda}}, \quad c(z)=\sum_{\ell=-\infty}^{\infty}\frac{c_\ell}{z^{\ell+1-\lambda}}, \qquad (12)
$$

the Virasoro generators are given by

$$
\begin{aligned}
L_m &= \frac{1}{2\pi i}\oint dz\, z^{m+1}T(z) \qquad (13)\\
&= \frac{1}{2\pi i}\oint dz\, z^{m+1}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\frac{\lambda(\ell+1-\lambda)-(1-\lambda)(k+\lambda)}{z^{k+\ell+2}}:b_k c_\ell: \qquad (14)\\
&= \sum_{n=-\infty}^{\infty}(m(\lambda-1)+n):b_{m-n}c_n: \qquad (15)\\
&= \sum_{n=-\infty}^{\infty}(m(\lambda-1)+n)\,{}^{\circ}_{\circ}\,b_{m-n}c_n\,{}^{\circ}_{\circ}+\delta_{m,0}a^g(\lambda). \qquad (16)
\end{aligned}
$$

Here, we distinguish between the default conformal normal ordering (:) and creation-annihilation normal ordering (${}^{\circ}_{\circ}$). The latter incurs a normal-ordering constant $a^g(\lambda)$ due to

$$
\{b_m, c_n\}=\delta_{m+n,0}. \qquad (17)
$$

It must be chosen so that the L_m obey the Virasoro algebra with the appropriate central charge. In fact, it suffices to demand that the global part of the algebra be realized correctly on the ground state(s). Recall that the doubly degenerate ground states $|\uparrow\rangle$ and $|\downarrow\rangle$ satisfy

$$c_0|\downarrow\rangle = |\uparrow\rangle, \quad b_0|\uparrow\rangle = |\downarrow\rangle, \quad c_0|\uparrow\rangle = b_0|\downarrow\rangle = 0; \tag{18}$$

they are annihilated by all $b_{n>0}$ and $c_{n>0}$, and in particular by all $L_{m>0}$. In BRST quantization ($\lambda = 2$ for the bosonic string), $|\downarrow\rangle$ is the physical ground state, so it is convenient to group b_0 with the lowering operators and c_0 with the raising operators. We compute that

$$\begin{aligned}
[L_1, L_{-1}]|\downarrow\rangle &= L_1 L_{-1}|\downarrow\rangle && (19)\\
&= L_1(1-\lambda)b_{-1}c_0|\downarrow\rangle && (20)\\
&= \lambda(1-\lambda)b_0 c_1 b_{-1} c_0|\downarrow\rangle && (21)\\
&= \lambda(1-\lambda)|\downarrow\rangle. && (22)
\end{aligned}$$

On the other hand, $2L_0|\downarrow\rangle = 2a^g(\lambda)|\downarrow\rangle$. Hence requiring that $[L_1, L_{-1}] = 2L_0$ fixes

$$a^g(\lambda) = \frac{\lambda(1-\lambda)}{2}.^{1} \tag{23}$$

(c)

We first compute the OPEs

$$\begin{aligned}
T(z)b(w) &= \frac{\lambda b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \cdots, && (24)\\
T(z)c(w) &= \frac{(1-\lambda)c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \cdots. && (25)
\end{aligned}$$

The standard contour integration method then gives

$$\begin{aligned}
[L_m, b_n] &= \frac{1}{2\pi i}\oint dw\, \mathrm{Res}_{z\to w}\, z^{m+1} w^{n+\lambda-1} T(z)b(w) && (26)\\
&= \frac{1}{2\pi i}\oint dw\, \mathrm{Res}_{z\to w}\, z^{m+1} w^{n+\lambda-1}\left[\frac{\lambda b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}\right] && (27)\\
&= \frac{1}{2\pi i}\oint dw\, \left[(m+1)\lambda w^{m+n+\lambda-1} b(w) + w^{m+n+\lambda}\partial b(w)\right] && (28)\\
&= (m+1)\lambda b_{m+n} - (m+n+\lambda)b_{m+n} = (m(\lambda-1)-n)b_{m+n} && (29)
\end{aligned}$$

and, similarly,

$$[L_m, c_n] = [L_m, b_n]|_{\lambda\to 1-\lambda, b\to c} = -(m\lambda+n)c_{m+n}. \tag{30}$$

Alternatively, we have directly from the mode expansion that

$$\begin{aligned}
[L_m, b_n] &= [\textstyle\sum_k (m(\lambda-1)+k) : b_{m-k}c_k :, b_n] && (31)\\
&= (m(\lambda-1)-n)[b_{m+n}c_{-n}, b_n] && (32)
\end{aligned}$$

[1] The difference between the two normal-ordering prescriptions can equivalently be written as

$$T(z) = (1-\lambda) : (\partial b)c : - \lambda : b(\partial c) : = (1-\lambda) \,\text{\textdegree\!\textdegree}\, (\partial b)c \,\text{\textdegree\!\textdegree}\, - \lambda \,\text{\textdegree\!\textdegree}\, b(\partial c) \,\text{\textdegree\!\textdegree}\, + \frac{\lambda(1-\lambda)}{2z^2},$$

which follows from the more basic relation

$$: b(z)c(w) : - \,\text{\textdegree\!\textdegree}\, b(z)c(w) \,\text{\textdegree\!\textdegree}\, = \frac{(z/w)^{1-\lambda}-1}{z-w}.$$

$$= (m(\lambda-1)-n)b_{m+n}\{b_n, c_{-n}\} \tag{33}$$
$$= (m(\lambda-1)-n)b_{m+n} \tag{34}$$

and

$$[L_m, c_n] = [\textstyle\sum_k (m(\lambda-1)+k) : b_{m-k}c_k :, c_n] \tag{35}$$
$$= (m\lambda+n)[b_{-n}c_{m+n}, c_n] \tag{36}$$
$$= -(m\lambda+n)c_{m+n}\{b_{-n}, c_n\} \tag{37}$$
$$= -(m\lambda+n)c_{m+n}. \tag{38}$$

Normal ordering can only produce a constant shift, which does not affect the commutator.

(d)

By the usual rules of the state-operator correspondence, well-definedness of $\lim_{z\to 0} b(z)|1\rangle$ and $\lim_{z\to 0} c(z)|1\rangle$ requires that

$$b_m|1\rangle = 0 \text{ for } m \geq 1-\lambda, \tag{39}$$
$$c_m|1\rangle = 0 \text{ for } m \geq \lambda, \tag{40}$$

where $|1\rangle$ is the conformal vacuum (the state corresponding to the identity operator). Hence b_m for $m \leq -\lambda$ and c_m for $m \leq \lambda - 1$ are interpreted as raising operators.

(e)

In the following, we work with the conformal vacuum rather than the "zero-mode vacua." Since $b_{1-n}|1\rangle = 0$ for $n \leq \lambda$ and $c_n|1\rangle = 0$ for $n \geq \lambda$, we have that

$$L_1|1\rangle = \sum_{n=-\infty}^{\infty} (\lambda - 1 + n)b_{1-n}c_n|1\rangle = 0. \tag{41}$$

Since $b_{-1-n}|1\rangle = 0$ for $n \leq \lambda - 2$ and $c_n|1\rangle = 0$ for $n \geq \lambda$, we have that

$$L_{-1}|1\rangle = \sum_{n=-\infty}^{\infty} (1 - \lambda + n)b_{-1-n}c_n|1\rangle = 0 \cdot b_{-\lambda}c_{\lambda-1}|1\rangle = 0. \tag{42}$$

Using conformal normal ordering, namely

$$L_0 = \sum_{n=-\infty}^{\infty} n : b_{-n}c_n : = \sum_{n=\lambda}^{\infty} nb_{-n}c_n - \sum_{n=-\infty}^{\lambda-1} nc_nb_{-n} \tag{43}$$

(here, all raising operators are to the left of lowering operators), we likewise find that $L_0|1\rangle = 0$. Hence $L_{0,\pm 1}$ annihilate the conformal vacuum, as expected.

(f)

Given the ranges of their indices, all of the raising operators $b_{m\leq -\lambda}$ and $c_{m\leq \lambda-1}$ anticommute with each other. Given the commutation relations

$$[L_0, b_n] = -nb_n, \quad [L_0, c_n] = -nc_n, \tag{44}$$

a generic state and its L_0-eigenvalue are

$$b_{m_1}\cdots b_{m_k}c_{n_1}\cdots c_{n_\ell}|1\rangle, \quad -(m_1 + \cdots + m_k + n_1 + \cdots + n_\ell). \tag{45}$$

Since λ is assumed to be a positive integer, the states of lowest L_0-eigenvalue are

$$c_0 c_1 \cdots c_{\lambda-1}|1\rangle, \quad c_1 \cdots c_{\lambda-1}|1\rangle, \tag{46}$$

and this lowest eigenvalue is $\lambda(1-\lambda)/2$ (which we recognize as the creation-annihilation normal-ordering constant from before). In other words, we have the identifications

$$|{\downarrow}\rangle = c_1 \cdots c_{\lambda-1}|1\rangle \Longleftrightarrow b_{1-\lambda} \cdots b_{-1}|{\downarrow}\rangle = |1\rangle, \quad c_0|{\downarrow}\rangle = |{\uparrow}\rangle. \tag{47}$$

When $\lambda = 1$, the states of lowest (i.e., vanishing) L_0-eigenvalue are simply $c_0|1\rangle$ and $|1\rangle$. When $\lambda = 2$, we have the familiar representation $|1\rangle = b_{-1}|{\downarrow}\rangle$.

Problem 3

Here, we consider the $c = 1$ CFT of a single free boson, where $[\alpha_m, \alpha_n] = m\delta_{m+n,0}$ ($\alpha_0 = \ell_s p$).

(a)

Using

$$L_{-1} = \alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1 + \cdots, \quad L_{-2} = \frac{1}{2}\alpha_{-1}^2 + \alpha_{-2}\alpha_0 + \cdots, \tag{48}$$

the level-1 descendant can be written as

$$\begin{aligned} |\chi\rangle &= (L_{-2} + aL_{-1}^2)|0;k\rangle & (49) \\ &= \left[(1+a)\ell_s k\alpha_{-2} + \left(\frac{1}{2} + a(\ell_s k)^2\right)\alpha_{-1}^2\right]|0;k\rangle. & (50) \end{aligned}$$

Vanishing ($|\chi\rangle = 0$) requires that

$$a = -1, \quad \ell_s k = \pm\frac{1}{\sqrt{2}}. \tag{51}$$

(b)

Using

$$L_1 = \alpha_0\alpha_1 + \alpha_{-1}\alpha_2 + \cdots, \quad L_2 = \frac{1}{2}\alpha_1^2 + \alpha_0\alpha_2 + \cdots, \tag{52}$$

consider the putative highest-weight state(s)

$$|\psi\rangle = (\alpha_{-2} + b\alpha_{-1}^2)|0;k\rangle. \tag{53}$$

The condition $L_1|\psi\rangle = 0$ gives

$$b\ell_s k = -1. \tag{54}$$

The condition $L_2|\psi\rangle = 0$ gives

$$2\ell_s k + b = 0. \tag{55}$$

Hence we require that

$$\ell_s k = \pm\frac{1}{\sqrt{2}}, \quad b = \mp\sqrt{2}. \tag{56}$$

Such states have positive norm:

$$\langle\psi|\psi\rangle = 2(1 + b^2) = 6. \tag{57}$$

Problem 4

For a highest-weight (actually lowest-weight) state $|\phi\rangle$ of weight h in a CFT of central charge c, the Gram matrix of descendants at level 3 is

$$M_3(c,h) = \begin{pmatrix} \langle\phi|L_3L_{-3}|\phi\rangle & \langle\phi|L_1L_2L_{-3}|\phi\rangle & \langle\phi|L_1^3L_{-3}|\phi\rangle \\ \langle\phi|L_3L_{-2}L_{-1}|\phi\rangle & \langle\phi|L_1L_2L_{-2}L_{-1}|\phi\rangle & \langle\phi|L_1^3L_{-2}L_{-1}|\phi\rangle \\ \langle\phi|L_3L_{-1}^3|\phi\rangle & \langle\phi|L_1L_2L_{-1}^3|\phi\rangle & \langle\phi|L_1^3L_{-1}^3|\phi\rangle \end{pmatrix} \tag{58}$$

$$= \begin{pmatrix} 2(3h+c) & 10h & 24h \\ 10h & h(8(h+1)+c) & 12h(3h+1) \\ 24h & 12h(3h+1) & 24h(h+1)(2h+1) \end{pmatrix}. \tag{59}$$

The corresponding Kac determinant is

$$\det M_3(c,h) = 2304(h-h_{1,1})^2(h-h_{2,1})(h-h_{1,2})(h-h_{3,1})(h-h_{1,3}) \tag{60}$$

where $h_{1,1}=0$ and

$$h_{2,1}, h_{1,2} = \frac{5-c\pm\sqrt{(c-1)(c-25)}}{16}, \quad h_{3,1}, h_{1,3} = \frac{7-c\pm\sqrt{(c-1)(c-25)}}{6}. \tag{61}$$

Since

$$\det M_1(c,h) = M_1(c,h) = 2h \tag{62}$$

and

$$M_2(c,h) = \begin{pmatrix} 8h^2+4h & 6h \\ 6h & 4h+c/2 \end{pmatrix} \implies \det M_2(c,h) = 32h(h-h_{2,1})(h-h_{1,2}), \tag{63}$$

we see that $h=h_{1,1}$ corresponds to a null state at level 1 while $h=h_{2,1},h_{1,2}$ correspond to null states at level 2. Hence only $h=h_{3,1},h_{1,3}$ yield null states at level 3. This can also be seen by solving explicitly for the null states at level 3.

Problem 5

(a)

For Virasoro primaries A_i of weights $(h_i,\tilde{h}_i)$, we have the conformal Ward identity

$$\langle T(z)A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)\rangle = \sum_{p=1}^n \left[\frac{h_p}{(z-z_p)^2} + \frac{1}{z-z_p}\frac{\partial}{\partial z_p}\right]\langle A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)\rangle. \tag{64}$$

The fact that $T(z)\sim z^{-4}$ as $z\to\infty$ implies that the terms of order z^{-1}, z^{-2}, and z^{-3} in the RHS must vanish. At large z, the RHS becomes

$$\sum_{p=1}^n \left[h_p\left(\frac{1}{z^2}+\frac{2z_p}{z^3}\right) + \left(\frac{1}{z}+\frac{z_p}{z^2}+\frac{z_p^2}{z^3}\right)\frac{\partial}{\partial z_p} + O(z^{-4})\right]\langle A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)\rangle, \tag{65}$$

so we obtain the constraints

$$\sum_{p=1}^n \frac{\partial}{\partial z_p}\langle A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)\rangle = 0, \tag{66}$$

$$\sum_{p=1}^n \left(h_p + z_p\frac{\partial}{\partial z_p}\right)\langle A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)\rangle = 0, \tag{67}$$

$$\sum_{p=1}^{n}\left(2h_p z_p + z_p^2 \frac{\partial}{\partial z_p}\right)\langle A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)\rangle = 0. \tag{68}$$

These constraints are equivalent to

$$\langle 0|[L_*, A_1(z_1,\bar{z}_1)\cdots A_n(z_n,\bar{z}_n)]|0\rangle = 0 \tag{69}$$

where $* = -1, 0, 1$, respectively, i.e., to global conformal ($SL(2,\mathbb{C})$-) covariance of the correlation function. They correspond individually to invariance under translations, covariance under dilatations (including rotations), and covariance under special conformal transformations.

(b)

Let us refer to the above constraints, along with the corresponding ones following from the Ward identity for $\bar{T}(\bar{z})$, as the L_{-1}, L_0, and L_1 constraints, respectively.

Consider the two-point function

$$f(z_1, z_2, \bar{z}_1, \bar{z}_2) \equiv \langle A_1(z_1,\bar{z}_1) A_2(z_2,\bar{z}_2)\rangle. \tag{70}$$

The L_{-1} constraints give

$$f(z_1, z_2, \bar{z}_1, \bar{z}_2) = f(z_1 - z_2, \bar{z}_1 - \bar{z}_2). \tag{71}$$

The L_0 constraints then give

$$f(z_1 - z_2, \bar{z}_1 - \bar{z}_2) = \frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}(\bar{z}_1 - \bar{z}_2)^{\tilde{h}_1+\tilde{h}_2}} \tag{72}$$

for some constant of integration C_{12}. Finally, the L_1 constraints show that

$$h_1 = h_2, \quad \tilde{h}_1 = \tilde{h}_2 \tag{73}$$

must hold for the two-point function to be nonvanishing, so that

$$f(z_1 - z_2, \bar{z}_1 - \bar{z}_2) = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\tilde{h}}} \tag{74}$$

with $h_1 = h_2 = h$ and $\tilde{h}_1 = \tilde{h}_2 = \tilde{h}$.