Homework 1

Due Date: December 21, 2020 - 8pm

Problem 1

Identify which of the following sets are convex, and provide a brief justification or proof for each.

1. A set of the form $\{x \in \mathcal{R}^n | \alpha \leq a^T x \leq \beta\}$.
2. A set of the form $\{x \in \mathcal{R}^n | \alpha_i \leq x_i \leq \beta_i\}$.
3. The set of points closer to a given point than a given set, i.e., $\{x | \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S\}$, where $S \subseteq \mathcal{R}^n$.
4. The set of points closer to a set than another, $\{x | dist(x, S) \leq dist(x, T)\}$, where $S, T \subseteq \mathcal{R}^n$.
5. A polyhedron $\{x \in \mathcal{R}^n | Ax \leq b\}$ for some $A \in \mathcal{R}^{m \times n}, b \in \mathcal{R}^m$.
6. The intersection $\cap_i S_i$, where $S_i \subset \mathcal{R}^n, i \in I$ is a collection of convex sets.

Problem 2

Specify whether the function is strongly convex, strictly convex, convex, or non-convex, and give a brief justification for each.

1. $f(x) = x^4$.
2. $f(x_i) = \sum_i x_i log(x_i)$, for $x > 0$.
3. $f(x) = log(1 + e^x)$.
4. $f(x, y) = |x| + |y| + 2x - 2$.
5. $f(x, y) = x^2 - xy + 2y^2 + 3$.
6. $f(x, y) = 2xy + 3x^2 + y^2$.
7. $f(x, y) = e^{(x^2 - xy + y^2 + 2)} + \frac{1}{\sqrt{x + 2y}}$, for $x, y > 0$.
8. $f(x) = \frac{\|Ax - b\|^2}{1 - x^T x}$, on set $\{x : \|x\|_2 < 1\}$, where $\{x \in \mathcal{R}^n$.

Problem 3

(a) Show that the following statements are equivalent.

- $\nabla f(x)$ is Lipschitz with constant L.
- $\nabla^2 f(x) \preceq LI$, for all x.
- $f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|_2^2$, for all x, y.

(b) Assume a strongly convex function $f(x)$ ($x \in \mathcal{R}^n$) and let $x(k+1) = x(k) - t\nabla f(x(k))$ (i.e. the standard gradient descent step). Let further $g(x) = x - t\nabla f(x)$ be such that $\|g(y) - g(x)\|_2 \leq \beta \|y - x\|_2$, for some $\beta < 1$, and any x, y. Let finally x^* denote the global minimum of $f(x)$. Show that

$$\|x(k) - x^*\|_2 \leq \beta^k \|x(0) - x^*\|_2$$

Hint: the goal is to start with $\|x(k) - x^*\|_2$, the distance from the optimal of the current point $x(k)$, and step-by-step compare it with the distance from the optimal at earlier steps $\|x(k-1) - x^*\|_2$, $\|x(k-2) - x^*\|_2$, etc.

Problem 4

For each of these functions, answer the following questions:

1. $f(x, y) = xlog(x) + ylog(y)$, for $x, y > 1$.
2. $f(x, y) = |x| + |y| + 2x - 2y + 3$.
3. $f(x, y) = x^2 - xy + 2y^2$.

(a): which of the methods we learned (gradient descent, subgradient method, proximal gradient method) are applicable for each function.

(b): for the applicable methods, what is the convergence rate when an approximation error of ϵ is needed: $O(1/\epsilon)$?$O(1/\epsilon^2)$?$O(log(1/\epsilon))$?

(c): pick the fastest method (among the three) for each function, and show the basic update step (i.e., the gradient step, the subgradient step, or the proximal gradient step, depending on the method). Explain also how to pick the step sizes to ensure convergence.