1. Kinematics: Position and Velocity

$$x_1 = a_1\sin\theta_1$$
$$y_1 = -a_1\cos\theta_1$$
$$x_2 = l_1\sin\theta_1 + a_2\sin\theta_2$$
$$y_2 = -l_1\cos\theta_1 - a_2\cos\theta_2$$
$$x_3 = l_1\sin\theta_1 + l_2\sin\theta_2 + a_3\sin\theta_3$$
$$y_3 = -l_1\cos\theta_1 - l_2\cos\theta_2 - a_3\cos\theta_3$$

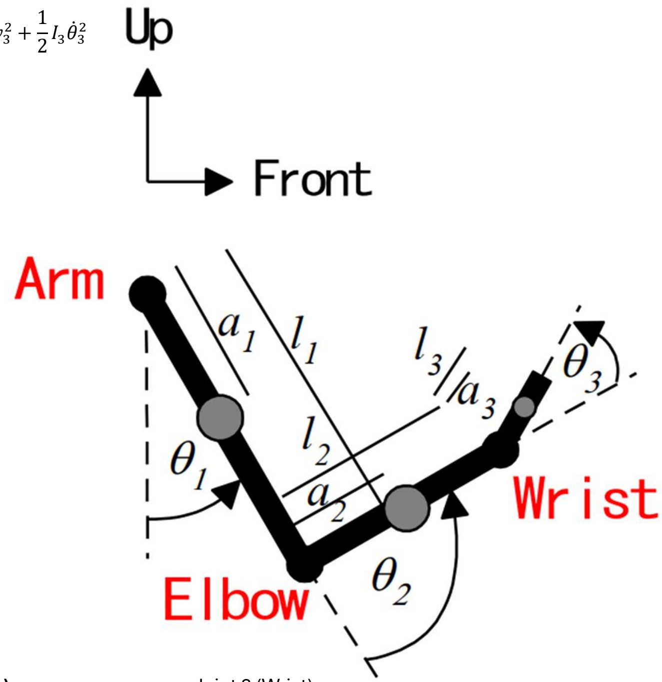
Squared Velocity magnitudes ($v_i^2 = \dot{x}_i^2 + \dot{y}_i^2$):

$$v_1^2 = a_1^2\dot{\theta}_1^2$$
$$v_2^2 = l_1^2\dot{\theta}_1^2 + a_2^2\dot{\theta}_2^2 + 2l_1a_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$
$$v_3^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + a_3^2\dot{\theta}_3^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + 2l_1a_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + 2l_2a_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)$$

2. Energy Systems

Total Kinetic Energy (E_k):

$$E_k = \frac{1}{2}m_1v_1^2 + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m_3v_3^2 + \frac{1}{2}I_3\dot{\theta}_3^2$$



Expanded:

$$\begin{aligned}E_k = &\frac{1}{2}(I_1 + m_1a_1^2 + m_2l_1^2 + m_3l_1^2)\dot{\theta}_1^2\\ &+\frac{1}{2}(I_2 + m_2a_2^2 + m_3l_2^2)\dot{\theta}_2^2\\ &+\frac{1}{2}(I_3 + m_3a_3^2)\dot{\theta}_3^2\\ &+(m_2l_1a_2 + m_3l_1l_2)\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\\ &+(m_3l_1a_3)\cos(\theta_1-\theta_3)\dot{\theta}_1\dot{\theta}_3\\ &+(m_3l_2a_3)\cos(\theta_2-\theta_3)\dot{\theta}_2\dot{\theta}_3\end{aligned}$$

Total Potential Energy (E_p):

$$E_p = m_1gy_1 + m_2gy_2 + m_3gy_3$$

Expanded:

$$\begin{aligned}E_p = &-(m_1a_1 + m_2l_1 + m_3l_1)g\cos\theta_1\\ &-(m_2a_2 + m_3l_2)g\cos\theta_2\\ &-(m_3a_3)g\cos\theta_3\end{aligned}$$

3. The Lagrangian

$$\mathcal{L} = E_k - E_p$$

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}K_{11}\dot{\theta}_1^2 + \frac{1}{2}K_{22}\dot{\theta}_2^2 + \frac{1}{2}K_{33}\dot{\theta}_3^2\\ &+K_{12}\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\\ &+K_{13}\cos(\theta_1-\theta_3)\dot{\theta}_1\dot{\theta}_3\\ &+K_{23}\cos(\theta_2-\theta_3)\dot{\theta}_2\dot{\theta}_3\\ &+G_{c1}\cos\theta_1 + G_{c2}\cos\theta_2 + G_{c3}\cos\theta_3\end{aligned}$$

4. Equations of Motion (Expanded Scalar Form)

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i}\right) - \frac{\partial\mathcal{L}}{\partial\theta_i} = \tau_i$$

Joint 1 (Shoulder):

$$\begin{aligned}\tau_1 = &(I_1 + m_1a_1^2 + (m_2+m_3)l_1^2)\ddot{\theta}_1\\ &+(l_1(m_2a_2 + m_3l_2)\cos(\theta_1-\theta_2))\ddot{\theta}_2\\ &+(m_3l_1a_3\cos(\theta_1-\theta_3))\ddot{\theta}_3\\ &+l_1(m_2a_2 + m_3l_2)\sin(\theta_1-\theta_2)\dot{\theta}_2^2\\ &+m_3l_1a_3\sin(\theta_1-\theta_3)\dot{\theta}_3^2\\ &+(m_1a_1 + (m_2+m_3)l_1)g\sin\theta_1\end{aligned}$$

Joint 2 (Elbow):

$$\begin{aligned}\tau_2 = &(l_1(m_2a_2 + m_3l_2)\cos(\theta_1-\theta_2))\ddot{\theta}_1\\ &+(I_2 + m_2a_2^2 + m_3l_2^2)\ddot{\theta}_2\\ &+(m_3l_2a_3\cos(\theta_2-\theta_3))\ddot{\theta}_3\\ &-l_1(m_2a_2 + m_3l_2)\sin(\theta_1-\theta_2)\dot{\theta}_1^2\\ &+m_3l_2a_3\sin(\theta_2-\theta_3)\dot{\theta}_3^2\\ &+(m_2a_2 + m_3l_2)g\sin\theta_2\end{aligned}$$

Joint 3 (Wrist):

$$\begin{aligned}\tau_3 = &(m_3l_1a_3\cos(\theta_1-\theta_3))\ddot{\theta}_1\\ &+(m_3l_2a_3\cos(\theta_2-\theta_3))\ddot{\theta}_2\\ &+(I_3 + m_3a_3^2)\ddot{\theta}_3\\ &-m_3l_1a_3\sin(\theta_1-\theta_3)\dot{\theta}_1^2\\ &-m_3l_2a_3\sin(\theta_2-\theta_3)\dot{\theta}_2^2\\ &+m_3a_3g\sin\theta_3\end{aligned}$$

5. Equations of Motion (Matrix Form)

$$\mathrm{M}(\theta)\ddot{\theta} + \mathrm{V}(\theta,\dot{\theta}) + \mathrm{G}(\theta) = \tau$$

Inertia Matrix (M):

$$\begin{bmatrix}K_{11} & K_{12}c_{12} & K_{13}c_{13}\\ K_{12}c_{12} & K_{22} & K_{23}c_{23}\\ K_{13}c_{13} & K_{23}c_{23} & K_{33}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{bmatrix}$$

Coriolis/Centrifugal (V), Gravity (G), and Torque (τ):

$$+\begin{bmatrix}K_{12}s_{12}\dot{\theta}_2^2 + K_{13}s_{13}\dot{\theta}_3^2\\ -K_{12}s_{12}\dot{\theta}_1^2 + K_{23}s_{23}\dot{\theta}_3^2\\ -K_{13}s_{13}\dot{\theta}_1^2 - K_{23}s_{23}\dot{\theta}_2^2\end{bmatrix} + \begin{bmatrix}G_{c1}\sin\theta_1\\ G_{c2}\sin\theta_2\\ G_{c3}\sin\theta_3\end{bmatrix} = \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}$$

Note: $c_{ij} = \cos(\theta_i - \theta_j)$ *and* $s_{ij} = \sin(\theta_i - \theta_j)$.