Knee Joint — Compact Mathematical Derivation (Matrix form)

$$q = \begin{bmatrix} \theta_2 \\ \phi \end{bmatrix}, \qquad L = 0.407\frac{BH}{160},\ m = 2.76\frac{BW}{60},\ I = \tfrac{1}{3}mL^2,\ g = 9.8$$

$$r_{CM} = \frac{L}{2}\begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix}, \qquad \dot r_{CM} = \frac{L}{2}\dot\theta_2 \begin{bmatrix} -\sin\theta_2 \\ \cos\theta_2 \end{bmatrix}$$

$$\|\dot r_{CM}\|^2 = \left(\frac{L}{2}\right)^2 \dot\theta_2^2$$

$$T = \tfrac{1}{2}I\dot\theta_2^2 + \tfrac{1}{2}m\|\dot r_{CM}\|^2 + T_{\phi,\mathrm{eff}} = \tfrac{1}{2}\Big(I + \frac{mL^2}{4}\Big)\dot\theta_2^2 + \tfrac{1}{2}\Big(\frac{mL^2}{4}\Big)\sin^2\theta_2\,\dot\phi^2$$

$$V = mg\frac{L}{2}(1 - \cos\theta_2), \qquad \partial_{\theta_2}V = mg\frac{L}{2}\sin\theta_2$$

$$\frac{\partial T}{\partial\dot\theta_2} = \Big(I + \frac{mL^2}{4}\Big)\dot\theta_2, \quad \frac{d}{dt}\Big(\frac{\partial T}{\partial\dot\theta_2}\Big) = \Big(I + \frac{mL^2}{4}\Big)\ddot\theta_2$$

$$\frac{\partial T}{\partial\theta_2} = \frac{mL^2}{4}\sin\theta_2\cos\theta_2\,\dot\phi^2$$

Euler–Lagrange ($Q_{\theta_2} = \tau_{\mathrm{rad}}$):

$$\Big(I + \frac{mL^2}{4}\Big)\ddot\theta_2 - \frac{mL^2}{4}\sin\theta_2\cos\theta_2\,\dot\phi^2 + mg\frac{L}{2}\sin\theta_2 = \tau_{\mathrm{rad}}$$

$$\boxed{\ddot\theta_2 = \frac{\tau_{\mathrm{rad}} + \frac{mL^2}{4}\dot\phi^2\sin\theta_2\cos\theta_2 - mg\frac{L}{2}\sin\theta_2}{I + \frac{mL^2}{4}}}$$

$$\frac{\partial T}{\partial\dot\phi} = \Big(\frac{mL^2}{4}\Big)\sin^2\theta_2\,\dot\phi$$

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial\dot\phi}\Big) = \Big(\frac{mL^2}{4}\Big)\big(\sin^2\theta_2\,\ddot\phi + 2\sin\theta_2\cos\theta_2\,\dot\theta_2\dot\phi\big) = 0$$

$$\boxed{\ddot\phi = -2\dot\theta_2\dot\phi\cot\theta_2}$$

$$M(q) = \begin{bmatrix} I + \dfrac{mL^2}{4} & 0 \\ 0 & \dfrac{mL^2}{4}\sin^2\theta_2 \end{bmatrix}, \quad C(q,\dot q) = \begin{bmatrix} -\dfrac{mL^2}{4}\sin\theta_2\cos\theta_2\,\dot\phi^2 \\ 2\,\dfrac{mL^2}{4}\sin\theta_2\cos\theta_2\,\dot\theta_2\dot\phi \end{bmatrix},$$

$$G(q) = \begin{bmatrix} mg\frac{L}{2}\sin\theta_2 \\ 0 \end{bmatrix}, \qquad \tau = \begin{bmatrix} \tau_{\mathrm{rad}} \\ 0 \end{bmatrix}$$

$$\boxed{M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau}$$

$$BH = 172\ (\mathrm{cm}), \quad BW = 70\ (\mathrm{kg})$$

$$L = 0.407\frac{172}{160} = 0.437525\ \mathrm{m}$$

$$m = 2.76\frac{70}{60} = 3.22\ \mathrm{kg}$$

$$I = \tfrac{1}{3}mL^2 \approx 0.2054661882\ \mathrm{kg\,m^2}$$

$$\frac{mL^2}{4} \approx 0.1540996411, \quad D := I + \frac{mL^2}{4} \approx 0.3595658293$$

$$mg\frac{L}{2} \approx 6.90326945\ \mathrm{N\,m}$$

$$\boxed{\ddot{\theta}_2 = \frac{\tau_{\mathrm{rad}} + 0.15409964\,\dot{\phi}^2\sin\theta_2\cos\theta_2 - 6.90326945\,\sin\theta_2}{0.35956583}} \quad , \quad \boxed{\ddot{\phi} = -2\dot{\theta}_2\dot{\phi}\cot\theta_2}$$