

$$y_1 = -\frac{l_1}{2} \cos \theta_1 \quad x_2 = l_1 \sin \theta_1 + \frac{l_2}{2} \sin \theta_2 \quad y_2 = -l_1 \cos \theta_1 - \frac{l_2}{2} \cos \theta_2$$

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = \frac{l_1^2}{4} \dot{\theta}_1^2 \quad v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + \frac{l_2^2}{4} \dot{\theta}_2^2 + l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$T = \frac{1}{2} \left(\frac{m_1 l_1^2}{4} + I_1 + m_2 l_1^2 \right) \dot{\theta}_1^2 + \frac{1}{2} \left(\frac{m_2 l_2^2}{4} + I_2 \right) \dot{\theta}_2^2 + \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$V = m_1 g y_1 + m_2 g y_2 \quad V = -\frac{1}{2} m_1 g l_1 \cos \theta_1 - m_2 g (l_1 \cos \theta_1 + \frac{l_2}{2} \cos \theta_2)$$

$$L = T - V$$

$$\frac{\partial L}{\partial \theta_1} = \left(\frac{m_1 l_1^2}{4} + I_1 + m_2 l_1^2 \right) \dot{\theta}_1 + \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = \left(\frac{m_1 l_1^2}{4} + I_1 + m_2 l_1^2 \right) \ddot{\theta}_1 + \frac{1}{2} m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \left(\frac{1}{2} m_1 + m_2 \right) g l_1 \sin \theta_1$$

$$\tau_1 = \left(\frac{m_1 l_1^2}{4} + m_2 l_1^2 + I_1 \right) \ddot{\theta}_1 + \frac{1}{2} m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + \left(\frac{1}{2} m_1 + m_2 \right) g l_1 \sin \theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \left(\frac{m_2 l_2^2}{4} + I_2 \right) \dot{\theta}_2 + \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = \left(\frac{m_2 l_2^2}{4} + I_2 \right) \ddot{\theta}_2 + \frac{1}{2} m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{1}{2} m_2 g l_2 \sin \theta_2$$

$$\tau_2 = \left(\frac{m_2 l_2^2}{4} + I_2 \right) \ddot{\theta}_2 + \frac{1}{2} m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \frac{1}{2} m_2 g l_2 \sin \theta_2$$

$$M(\theta) \ddot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + G(\theta) = \tau$$

$$M(\theta) = \begin{bmatrix} \frac{1}{4} m_1 l_1^2 + m_2 l_1^2 + I_1 & \frac{1}{2} m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2} m_2 l_1 l_2 \cos(\theta_1 - \theta_2) & \frac{1}{4} m_2 l_2^2 + I_2 \end{bmatrix}$$

$$C(\theta, \dot{\theta}) = \begin{bmatrix} 0 & \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ -\frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) & 0 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} \left(\frac{1}{2} m_1 + m_2 \right) g l_1 \sin \theta_1 \\ \frac{1}{2} m_2 g l_2 \sin \theta_2 \end{bmatrix}$$

$$\ddot{\theta} = M(\theta)^{-1} [\tau - C(\theta, \dot{\theta}) \dot{\theta} - G(\theta)]$$

