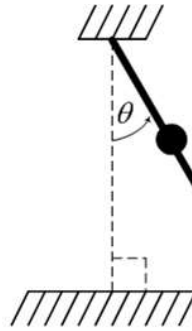
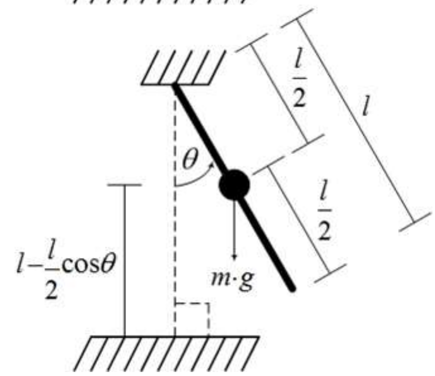


1. Kinematics of Center of Mass (COM)

$$r_{cm} = \begin{bmatrix} \frac{l}{2}\sin\theta\cos\phi \\ \frac{l}{2}\sin\theta\sin\phi \\ -\frac{l}{2}\cos\theta \end{bmatrix} \qquad v^2 = ||\ \dot{\mathbf{r}}_{cm}\ ||^2 = \frac{l^2}{4}(\dot{\theta}^2 + \dot{\phi}^2\sin\theta^2$$

2. Lagrangian Energy System

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(\frac{ml^2}{4} + I\right)\dot{\theta}^2 + \frac{ml^2}{8}\dot{\phi}^2\sin\theta^2$$



$$V = -mg\frac{l}{2}\cos\theta \qquad \mathcal{L} = T - V$$

3. Euler-Lagrange Equation for θ

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \tau_{net}$$

Derivatives:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \left(\frac{ml^2}{4} + I\right)\dot{\theta} \Rightarrow \frac{d}{dt}\cdots = \left(\frac{ml^2}{4} + I\right)\ddot{\theta} \qquad \frac{\partial\mathcal{L}}{\partial\theta} = \frac{ml^2}{8}\dot{\phi}^2(2\sin\ \theta\cos\ \theta) - \frac{mgl}{2}\sin\ \theta$$

4. Net Torque Model

$$\tau_{net} = \tau_{active} + \tau_{passive} \qquad \tau_{active} = F_{mus}\cdot d_{arm} \qquad \tau_{passive} = -c\dot{\theta} + k_1 e^{-k_2(\theta-\phi_1)} - k_3 e^{-k_4(\phi_2-\theta)}$$

5. Motion Equation (Acceleration)

$$\left(\frac{ml^2}{4} + I\right)\ddot{\theta} - \left(\frac{ml^2}{8}\dot{\phi}^2\sin(2\theta) - \frac{mgl}{2}\sin\ \theta\right) = \tau_{net}$$

$$\boxed{\ddot{\theta} = \frac{\tau_{net} + \frac{ml^2}{8}\dot{\phi}^2\sin(2\theta) - \frac{mgl}{2}\sin\ \theta}{\frac{ml^2}{4} + I}}$$

6. State-Space Matrix Form (Runge-Kutta Input)

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ f_1(\theta, \dot{\theta}, \phi, \dot{\phi}) \\ \dot{\phi} \\ f_2(\theta, \dot{\theta}, \phi, \dot{\phi}) \end{bmatrix}$$