HW2

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Problem 1

Solutions:

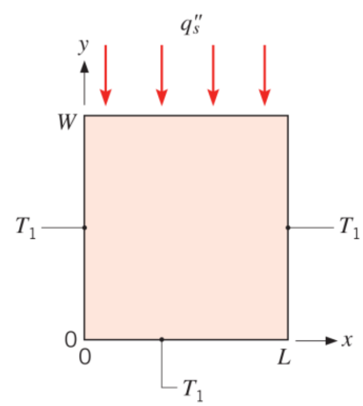


Figure 1: two-dimensional rectangular plate

(a)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\begin{cases} T(0, y) = T_1 = 30°C \\ T(x, 0) = T_1 = 30°C \\ T(L, y) = T_1 = 30°C \\ T(x, W) = q_s'' = -\left. q_y'' \right|_{y=W} = -\left(-\left. k\frac{\partial T}{\partial y} \right|_{y=W} \right) = \left. k\frac{\partial T}{\partial y} \right|_{y=W} \end{cases}$$

Define

$$\theta(x, y) = T(x, y) - T_1$$

So the boundary conditions can be turned to

$$\begin{cases} \theta(0,y) = 0 \\ \theta(x,0) = 0 \\ \theta(L,y) = 0 \\ \theta(x,W) = q_s'' = -\left. q_y'' \right|_{y=W} = -\left(-\left. k\frac{\partial(T-T_1)}{\partial y} \right|_{y=W} \right) = \left. k\frac{\partial T}{\partial y} \right|_{y=W} \end{cases}$$

Use SOV to solve this problem

$$\theta(x,y) = X(x)Y(y)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

Define

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \pm\lambda^2$$

So we can get

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0$$
$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0$$

$$\begin{cases} X(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x) \\ Y(y) = C_3\cosh(\lambda y) + C_4\sinh(\lambda y) \end{cases}$$

So

$$\theta(x,y) = [C_1\cos(\lambda x) + C_2\sin(\lambda x)][C_3\cosh(\lambda y) + C_4\sinh(\lambda y)]$$

Use boundary conditions to solve the coefficient

$$\begin{cases} \theta(0,y) = C_1[C_3\cosh(\lambda y) + C_4\sinh(\lambda y)] = 0 \\ C_1 = 0 \end{cases}$$

$$\begin{cases} \theta(x,0) = [C_1\cos(\lambda x) + C_2\sin(\lambda x)]C_3 = 0 \\ C_3 = 0 \end{cases}$$

$$\begin{cases} \theta(L,y) = [C_2\sin(\lambda L)][C_4\sinh(\lambda y)] = 0 \\ \lambda_n = \frac{n\pi}{L} \quad n = 1,2,3,... \end{cases}$$

So

$$\begin{aligned} \theta(x,y) &= C_2C_4\sin(\lambda_n x)\sinh(\lambda_n y) \\ &= C_n\sin(\lambda_n x)\sinh(\lambda_n y) \end{aligned}$$

General solution

$$\theta(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right)$$

Use the forth boundary condition to solve C_n

$$\left(\frac{\partial\theta}{\partial y}\right)\bigg|_{y=W} = \frac{q_s''}{k}$$

$$\theta(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

$$\frac{\partial\theta}{\partial y}\bigg|_{y=W} = \sum_{n=1}^{\infty} C_n\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi W}{L}\right) = \frac{q_s''}{k}$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\left[\sum_{n=1}^{\infty} C_n\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi W}{L}\right)\right]dx = \int_0^L \frac{q_s''}{k}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow C_n \frac{n\pi}{L} \cosh\left(\frac{n\pi W}{L}\right) \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{q_s''}{k} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

$$C_n = \frac{q_s''}{k} \cdot \frac{L}{n\pi} \cdot \frac{1}{\cosh\left(\frac{n\pi W}{L}\right)} \cdot \frac{\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$$

$$= \frac{q_s''}{k} \cdot \frac{L}{n\pi} \cdot \frac{1}{\cosh\left(\frac{n\pi W}{L}\right)} \cdot \frac{2}{L} \cdot \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

So

$$\begin{aligned}
C_n &= \frac{q_s''}{k} \cdot \frac{1}{\cosh\left(\frac{n\pi W}{L}\right)} \cdot \frac{2}{n\pi} \cdot \left[\frac{-\cos\left(\frac{n\pi x}{L}\right)}{n\pi/L}\right]_0^L \\
&= \frac{2q_s''}{k\cosh\left(\frac{n\pi W}{L}\right)} \cdot \frac{1}{n\pi} \cdot [1 - \cos(n\pi)] \cdot \frac{L}{n\pi} \\
&= \frac{q_s''}{k} \cdot \frac{1}{\cosh\left(\frac{n\pi W}{L}\right)} \cdot \frac{2L}{(n\pi)^2} \cdot [1 - \cos(n\pi)] \\
&= \frac{q_s''}{k} \cdot \frac{1}{\cosh\left(\frac{n\pi W}{L}\right)} \cdot \frac{2L}{(n\pi)^2} \cdot \left[1 + (-1)^{n+1}\right] \\
&= \frac{2q_s''L}{k(n\pi)^2\cosh\left(\frac{n\pi W}{L}\right)} \cdot \left[1 + (-1)^{n+1}\right]
\end{aligned}$$

So the final solution is

$$\theta(x,y) = \sum_{n=1}^{\infty} \frac{2q_s''L}{k(n\pi)^2\cosh\left(\frac{n\pi W}{L}\right)} \cdot \left[1 + (-1)^{n+1}\right] \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

$$T(x,y) = T_1 + \sum_{n=1}^{\infty} \frac{2q_s''L}{k(n\pi)^2\cosh\left(\frac{n\pi W}{L}\right)} \cdot \left[1 + (-1)^{n+1}\right] \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

Substitute known conditions into the equation, the analytical solution for the steady-state temperature distribution is

$$T(x,y) = 303.15 + \sum_{n=1}^{\infty} \frac{400}{(n\pi)^2\cosh\left(\frac{n\pi}{L}\right)} \cdot \left[1 + (-1)^{n+1}\right] \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

(b)

Using AI-assisted programming tools, develop and verify a computational model to obtain and visualize the steady-state temperature field.

We use the finite difference method(FDM)

The $L \times W$ plate is discretized into a grid of $N_x \times N_y$ nodes. The coordinates of a node (i, j) are (x_i, y_j), where $x_i = i \cdot \Delta x$ and $y_j = j \cdot \Delta y$. The grid spacings are $\Delta x = L/(N_x - 1)$ and $\Delta y = W/(N_y - 1)$. The temperature at a node (i, j) is denoted as $T_{i,j}$.

For any internal node, the governing Laplace equation ($\nabla^2 T = 0$) is approximated using a central difference scheme. Assuming a uniform grid where $\Delta x = \Delta y = h$, the equation becomes:

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Rearranging for iterative solution, the temperature at any internal node is the average of its four neighbors:

$$T_{i,j} = \frac{1}{4}(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1})$$

The boundary conditions specified in the problem are applied to the nodes on the edges of the grid:

- **Left boundary ($i = 0$):** $T_{0,j} = T_1$
- **Right boundary ($i = N_x - 1$):** $T_{N_x-1,j} = T_1$
- **Bottom boundary ($j = 0$):** $T_{i,0} = T_1$
- **Top boundary (flux, $j = N_y - 1$):** At this boundary, we must apply the heat flux condition $\frac{\partial T}{\partial y}\Big|_{y=W} = \frac{q_s''}{k}$

To handle the flux boundary at $j = N_y - 1$ (where $y = W$), we introduce a "ghost node" T_{i,N_y} outside the domain. Applying a central difference approximation for the derivative at the boundary:

$$\frac{T_{i,N_y} - T_{i,N_y-2}}{2\Delta y} \approx \frac{\partial T}{\partial y}\Big|_{y=W} = \frac{q_s''}{k}$$

We can solve for the ghost node temperature:

$$T_{i,N_y} = T_{i,N_y-2} + \frac{2\Delta y q_s''}{k}$$

Now, we apply the standard FDM equation (from Step 2) at the top boundary node T_{i,N_y-1}:

$$T_{i,N_y-1} = \frac{1}{4}(T_{i+1,N_y-1} + T_{i-1,N_y-1} + T_{i,N_y} + T_{i,N_y-2})$$

Substitute the expression for the ghost node T_{i,N_y}:

$$T_{i,N_y-1} = \frac{1}{4}\left(T_{i+1,N_y-1} + T_{i-1,N_y-1} + \left(T_{i,N_y-2} + \frac{2\Delta y q_s''}{k}\right) + T_{i,N_y-2}\right)$$

This simplifies to the iterative equation for the top boundary (for $0 < i < N_x - 1$):

$$T_{i,N_y-1} = \frac{1}{4}\left(T_{i+1,N_y-1} + T_{i-1,N_y-1} + 2T_{i,N_y-2} + \frac{2\Delta y q_s''}{k}\right)$$

The FDM equations for all nodes form a large system of linear algebraic equations, $A\vec{T} = \vec{B}$.

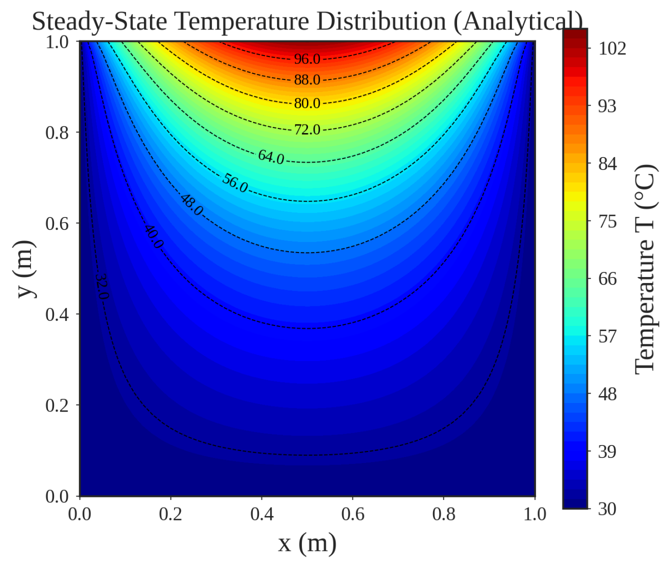
This system can be solved iteratively (e.g., using the Gauss-Seidel method) by repeatedly applying the equations until the temperature field converges (i.e., changes between iterations are below a small tolerance).

Validation: The computed FDM solution $T_{i,j}$ can be verified by comparing it at several points against the analytical solution $T(x_i, y_j)$ derived in part (a).

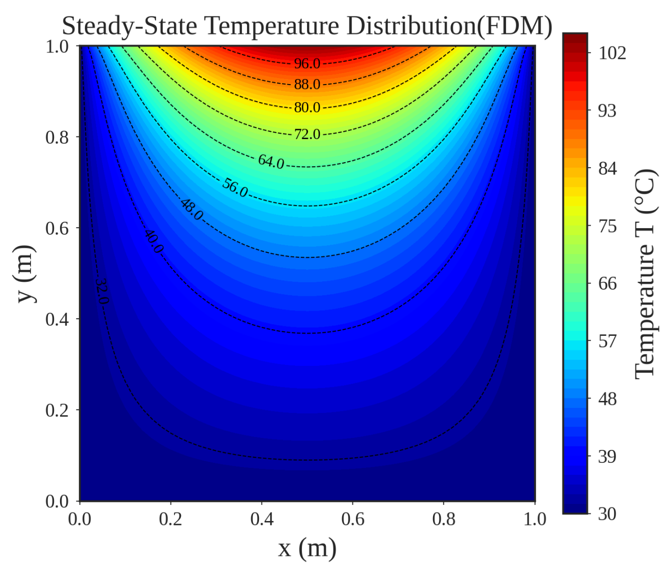
Visualization: The resulting $N_y \times N_x$ temperature matrix T is then visualized, typically using a contour plot or a heatmap, to show the steady-state temperature field.

Use Python to plot the 2D temperatu distribution by FDM

Also to verify with the analytical solution in (a), use Python to plot the 2D temperature analytical distribution



(a) Analytical Solution



(b) Numerical Solution (FDM)

Figure 2: Comparison of steady-state temperature distributions for Problem 1.

The numerical model was validated against the analytical solution derived in part (a). The iterative FDM solution converged after 4674 iterations (9.8814 seconds). A comparison at the center point of the plate ($x = 0.5, y = 0.5$) shows excellent agreement:

- FDM calculated result: $T(0.5, 0.5) = 46.0082$ °C
- Analytical solution result: $T(0.5, 0.5) = 46.0122$ °C
- **Absolute error: 0.003970 °C**

The low absolute error confirms the high accuracy of the FDM computational model.

(c)

Based on the computed solution, determine the heat transfer rate per unit thickness through the lower surface.($0 \leq x \leq L, y = 0$)

Define Heat rate through the lower surface:

$$q_y''\Big|_{out,y=0} = q_y''\Big|_{y=0} = -k\frac{\partial T}{\partial y}\Big|_{y=0}$$

And the total heat rate is the integral of the flux along the x axis

$$q'_{lower,out} = -\int_0^L k\frac{\partial T}{\partial y}\Big|_{y=0} dx$$

Based on the FDM solution in Problem1 (b)

$$\frac{\partial T}{\partial y}\Big|_{y=0,x=x_i} \approx \frac{T_{i,1} - T_{i,0}}{\Delta y}$$

So

$$q'_{lower,out} = -\sum_{i=1}^{N_x-1}\left(\frac{T_{i,1} - T_{i,0}}{\Delta y}\right)\Delta x$$

Use python to calculate the the heat transfer rate per unit thickness through the lower surface

$$q'_{lower,out} = -139.9277W/m$$

Problem 2

Solutions:

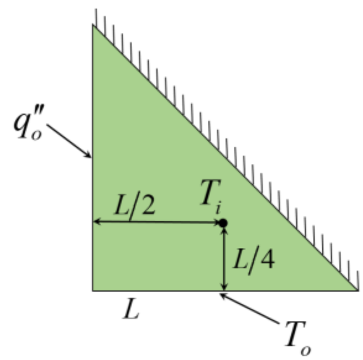


Figure 3: two-dimensional right angle isosceles triangle

We can rotate this triangle clockwise by 90° and combine it to form a $L \times L$ square with adiabatic diagonals.

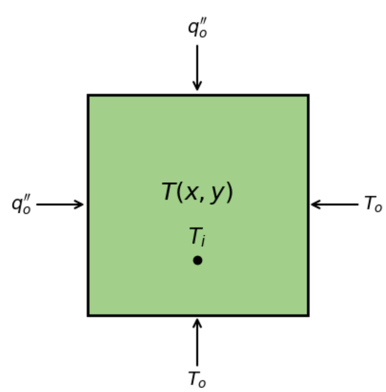


Figure 4: a $L \times L$ square with adiabatic diagonals

If we still define $\theta = T - T_o$, then the square will not have three homogeneous conditions

So we redefine

$$\theta = \frac{T - T_o}{q_s'' L/k}$$

$$\alpha = x/L$$
$$\beta = y/L$$

Then, the equation becomes:

$$\frac{\partial^2 \theta}{\partial \alpha^2} + \frac{\partial^2 \theta}{\partial \beta^2} = 0$$

So the boundary conditions are:

$$\theta(1, \beta) = 0$$
$$\theta(0, \alpha) = 0$$
$$\left.\frac{\partial \theta}{\partial \alpha}\right|_{\alpha=0} = -1$$
$$\left.\frac{\partial \theta}{\partial \beta}\right|_{\beta=0} = 1$$

And then separate the square to two squares

$$\theta = \theta_1 + \theta_2$$

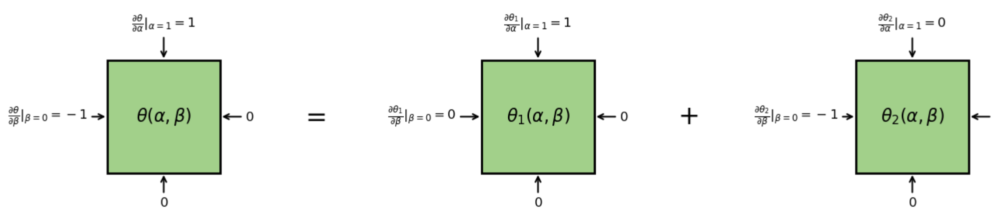


Figure 5: two separate squares θ_1 and θ_2

So

$$\theta(\alpha, \beta) = [C_1 \cos(\lambda\alpha) + C_2 \sin(\lambda\alpha)][C_3 \cosh(\lambda\beta) + C_4 \sinh(\lambda\beta)]$$

Boundary conditions:

Left:

$$\frac{\partial \theta_1}{\partial \alpha}|_{\alpha=0} = (C_2 \lambda)(C_3 \cosh\ \lambda\beta + C_4 \sinh\ \lambda\beta) = 0 \quad C_2 = 0$$

Bottom:

$$\theta_1(\alpha, 0) = C_1 \cos\ (\lambda\alpha) C_3 = 0 \quad C_3 = 0$$

Right:

$$\theta_1(1, \beta) = C_n \cos\ \lambda\ \sinh\ \lambda\beta = 0$$

$$\lambda_n = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \ldots$$

General solution:

$$\theta_1(\alpha, \beta) = \sum_{n=1}^{\infty} C_n \cos\ \lambda_n \alpha\ \sinh\ \lambda_n \beta$$

Apply the last boundary condition:

$$\frac{\partial \theta_1}{\partial \beta}|_{\beta=1} = \sum_{n=1}^{\infty} C_n \lambda_n \cos\ \lambda_n \alpha\ \cosh\ \lambda_n = 1$$

$$C_n \lambda_n \cosh\ \lambda_n \int_0^1 \cos^2 \lambda_n \alpha d\ \alpha = \int_0^1 \cos\ \lambda_n \alpha d\ \alpha$$

$$C_n = \frac{\int_0^1 \cos\ \lambda_n \alpha d\alpha}{\lambda_n \cosh\ \lambda_n \int_0^1 \cos^2 \lambda_n \alpha d\alpha} = \frac{(-1)^{n+1}/\lambda_n}{\frac{1}{2}\lambda_n \cosh\ \lambda_n} = \frac{2(-1)^{n+1}}{\lambda_n^2 \cosh\ \lambda_n}$$

$$\theta_1(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\lambda_n^2 \cosh\ \lambda_n} \cos\ \lambda_n \alpha\ \sinh\ \lambda_n \beta$$

Boundary conditions:

Bottom:

$$\theta_2(\alpha, 0) = (C_1 \cosh \lambda\alpha + C_2 \sinh \lambda\alpha) C_3 = 0 \quad C_3 = 0$$

Top:

$$\frac{\partial \theta_2}{\partial \beta}|_{\beta=1} = (C_1 \cosh \lambda\alpha + C_2 \sinh \lambda\alpha)(C_4 \lambda \cos \lambda) = 0 \quad \cos \lambda = 0$$

$$\lambda_n = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \dots$$

Right:

$$\theta_2(1, \beta) = (C_1 \cosh \lambda_n + C_2 \sinh \lambda_n)(C_4 \sin \lambda_n \beta) = 0$$

$$C_1 \cosh \lambda_n + C_2 \sinh \lambda_n = 0$$
$$C_1 = -C_2 \frac{\sinh \lambda_n}{\cosh \lambda_n}$$

$$\begin{aligned} \theta_2(\alpha, \beta) &= C_n \left[\sinh \lambda_n \alpha \cosh \lambda_n - \cosh \lambda_n \alpha \sinh \lambda_n\right] \sin \lambda_n \beta \\ &= C_n \sinh(\lambda_n(\alpha - 1)) \sin \lambda_n \beta \\ &= -C_n \sinh(\lambda_n(1 - \alpha)) \sin \lambda_n \beta \end{aligned}$$

General solution:

$$\theta_2(\alpha, \beta) = \sum_{n=1}^{\infty} C_n \sinh(\lambda_n(1 - \alpha)) \sin(\lambda_n \beta)$$

Apply the last boundary condition:

$$\frac{\partial \theta_2}{\partial \alpha}|_{\alpha=0} = \sum_{n=1}^{\infty} C_n \left[-\lambda_n \cosh(\lambda_n(1-\alpha))\right]|_{\alpha=0} \sin(\lambda_n \beta) = -1$$

$$\sum_{n=1}^{\infty} -C_n \lambda_n \cosh(\lambda_n) \sin(\lambda_n \beta) = -1$$

$$\sum_{n=1}^{\infty} C_n \lambda_n \cosh(\lambda_n) \sin(\lambda_n \beta) = 1$$

$$C_n \lambda_n \cosh(\lambda_n) \int_0^1 \sin^2(\lambda_n \beta) d\beta = \int_0^1 1 \cdot \sin(\lambda_n \beta) d\beta$$

$$C_n = \frac{\int_0^1 \sin(\lambda_n \beta) d\beta}{\lambda_n \cosh(\lambda_n) \int_0^1 \sin^2(\lambda_n \beta) d\beta} = \frac{1/\lambda_n}{\lambda_n \cosh(\lambda_n) \cdot (1/2)} = \frac{2}{\lambda_n^2 \cosh(\lambda_n)}$$

Therefore:

$$\theta_2(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\lambda_n^2 \cosh\ \lambda_n} \cos\ \lambda_n(1-\beta) \sinh\ \lambda_n(1-\alpha)$$

So we can calculate T as

$$\begin{aligned} T &= \theta \frac{q_s'' L}{k} + T_o \\ &= (\theta_1 + \theta) \frac{q_s'' L}{k} + T_o \end{aligned}$$

Substitute the parameters T_i with $(L/2,\ L/4)$

Use Python to plot this distribution, and calculate T_i

$$\begin{aligned} T &= T(L/2, L/4) = T(0.5, 0.25) \\ &= 71.9°C \end{aligned}$$

Problem 3

Solutions:

First of all, we calculate the Bi number to determine if this problem can be solved with The lumped capacitance method:

$$Bi = \frac{hr_c}{k} = \frac{1200 \times 0.025}{0.627} = 47.85 \gg 0.1$$

So we need to use the 1D transient conduction equation of the sphere:

$$\theta(\gamma, \tau) = \sum_{n=1}^{\infty} C_n \frac{1}{\lambda_n \gamma} \sin(\lambda_n \gamma) \exp(-\lambda_n^2 \tau)$$

Where:

$$\theta = \frac{T - T_\infty}{T_i - T_\infty}$$

$$\gamma = r/r_o$$

$$\tau = Fo = \frac{\alpha t}{r_0^2}$$

$$C_n = \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)}$$

$$1 - \lambda_n \cot \lambda_n = Bi$$

Use Pyhton to calculating the temperature distributions and evolutions with time, and plot the temperature distributions for the egg after 2, 4, 6, 8, 10 and 15 minutes, Here is the figure:

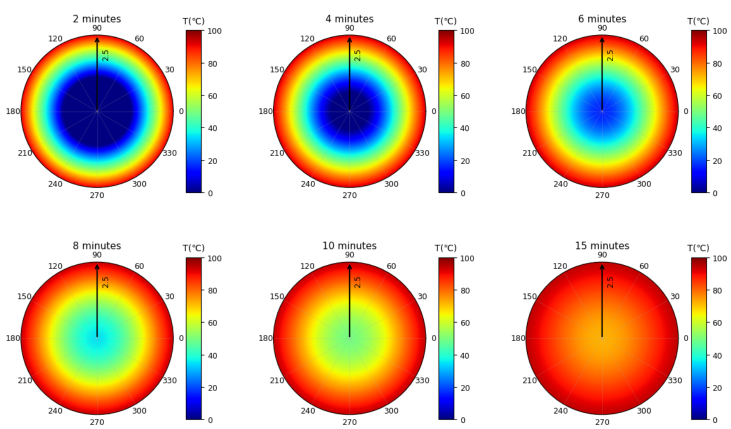


Figure 6: the boiling egg temperature distributions and evolutions with time

We can also draw a animation to see the whole boiling process of this egg

It is saved as animation.gif

All the original codes can be found at:

Advanced-Heat-Transfer-HW2 original code