China-UK Low Carbon College, Shanghai Jiao Tong University

PE6201:Advanced Heat Transfer **Project of Fall 2025**

Due date: 01/05/2025, Thursday

Transient 2-D Heat Conduction with Convective and Radiative Boundary Conditions: Numerical Methods and Physics-Informed Neural Networks

Introduction

Transient heat conduction with surface convection and radiation is commonly encountered in thermal engineering systems such as furnace walls, heat shields, battery casings, and electronic cooling structures. In this project, you will simulate the temperature evolution inside a thin plate subject to convective and radiative heat losses at its surfaces. You will first solve the governing partial differential equation using a conventional numerical method, and then apply Physics-Informed Neural Networks (PINNs) to solve the same physical problem. Finally, you will modify the boundary conditions to study how they affect the dimensionality and complexity of the heat transfer process.

Project Description

This project considers transient heat conduction within a thin rectangular metal plate that exchanges heat with the surrounding environment. The plate thickness in the z-direction is much smaller than its in-plane dimensions (width W and height H), i.e. $t_z \ll W, H$. Because conduction across the thickness occurs much faster than within the plane, temperature gradients in the z direction are negligible. Therefore, the temperature can be assumed uniform through the thickness, and the temperature field is modeled as $T(x, y, t)$, reducing the problem to two-dimensional transient conduction in the x–y plane.

Although conduction occurs primarily within the plane, heat is simultaneously lost from both large surfaces of the plate by convection and thermal radiation to the ambient environment at temperature T_a. The surface heat loss per unit area is

$$q'' = h(T - T_a) + \varepsilon\sigma(T^4 - T_a^4),$$

and because both surfaces exchange heat, the total heat loss per unit base area is $2q''$. When distributed uniformly across the thickness, this becomes an equivalent volumetric heat sink $2q''/t_z$, which affects the transient temperature evolution within the plate.

Under these assumptions, the dimensional governing equation is

$$\rho C_p t_z \frac{\partial T}{\partial t} = \kappa t_z \nabla^2 T - 2h(T - T_a) - 2\varepsilon\sigma(T^4 - T_a^4),$$

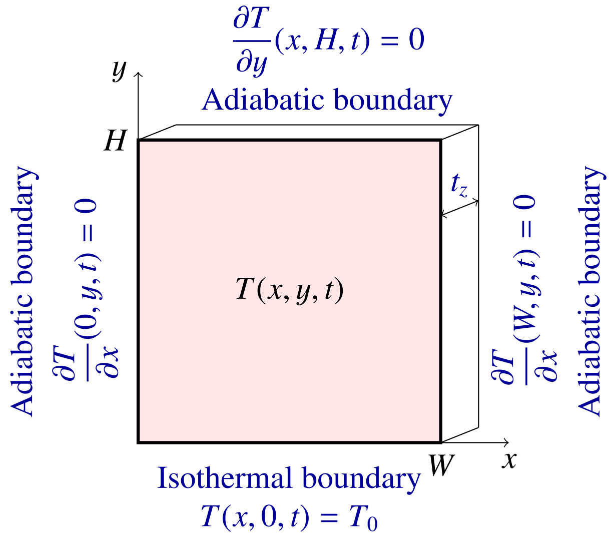

Figure 1: Schematic of the physical domain and boundary conditions for the thin-plate transient heat conduction problem. Heat is lost from the two large faces normal to the z-direction by combined convection and radiation to the ambient at temperature T_a, represented by the surface heat flux $q'' = h(T - T_a) + \varepsilon\sigma(T^4 - T_a^4)$.

where the left-hand side represents transient heat storage per unit base area, the first term on the right-hand side represents in-plane conduction, and the last two terms represent convective and radiative heat losses.

To generalize the problem and facilitate analysis, we introduce the non-dimensional temperature

$$\theta = \frac{T - T_a}{T_0 - T_a},$$

where $\theta = 1$ corresponds to the heated boundary and $\theta = 0$ to ambient conditions. The non-dimensional spatial and temporal variables are defined as

$$X = \frac{x}{W}, \qquad Y = \frac{y}{H}, \qquad \tau = \frac{\alpha t}{L^2}, \qquad \alpha = \frac{\kappa}{\rho C_p},$$

and the Biot number and radiation parameter are expressed as

$$\mathrm{Bi} = \frac{2hW^2}{\kappa t_z}, \qquad \mathrm{Ra} = \frac{2\varepsilon\sigma W^2 (T_0 - T_a)^3}{\kappa t_z}.$$

For the material and operating conditions adopted in this project (copper plate with $\kappa = 400\ \mathrm{W/(m \cdot K)}$, $t_z = 0.01$ m, $h = 1\ \mathrm{W/(m^2 \cdot K)}$, $\varepsilon = 0.5$, $\sigma = 5.670373 \times 10^{-8}\ \mathrm{W/(m^2 \cdot K^4)}$, $T_0 = 1000$ K, $T_a = 300$ K, and $W = L = 1$ m), these expressions give the numerical values

$$\mathrm{Bi} = 0.5, \qquad \mathrm{Ra} \approx 4.86.$$

The non-dimensional governing equation then becomes

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} - \mathrm{Bi}\,\theta - \mathrm{Ra}\left[(\theta + \theta_a)^4 - \theta_a^4\right], \qquad \theta_a = \frac{T_a}{T_0 - T_a}.$$

Non-dimensional boundary conditions (baseline case):

$$\theta(X,0,\tau) = 1, \qquad \frac{\partial\theta}{\partial Y}(X,1,\tau) = 0, \qquad \frac{\partial\theta}{\partial X}(0,Y,\tau) = 0, \qquad \frac{\partial\theta}{\partial X}(1,Y,\tau) = 0.$$

Initial condition:

$$\theta(X,Y,0) = 0.$$

Tasks

1. Baseline Numerical Solution

Solve the governing equation using a conventional numerical approach based on the **Method of Lines (MOL)**, in which the 2-D spatial domain is discretized (e.g. using finite differences), while time remains continuous and is integrated using an ODE solver such as `solve_ivp` in Python. This part of the project requires students to independently study and implement the MOL framework.

You are expected to:

- Discretize the spatial domain in the x–y plane and construct the resulting system of ordinary differential equations,
- Integrate the system in time using a stable ODE solver (such as `solve_ivp` in Python) until $\tau = 0.6$,
- Generate the 2-D temperature field $T(x, y, t)$ at three prescribed time instants:

$$\tau = 0.10, \quad \tau = 0.30, \quad \tau = 0.60,$$

- Plot 1-D temperature profiles along the centerlines at the prescribed time instants

$$T(x = 0.5W,\ y, \tau_i) \quad \text{and} \quad T(x,\ y = 0.5H, \tau_i),$$

  where

$$\tau_i = 0.10,\ 0.30,\ 0.60.$$

  for each of the above time instants,

- Plot the temperature time history at the plate center

$$T(x = 0.5W,\ y = 0.5H, \tau), \qquad 0 \le \tau \le 0.60,$$

This part of the project requires you to **independently study the Method of Lines** and apply it to build a working numerical solver for the thin-plate transient heat conduction problem.

2. PINN Solution

In this part, you will develop a **Physics-Informed Neural Network (PINN)** to solve the same governing equation and boundary/initial conditions as in the baseline numerical solution. The goal of this section is to explore how machine-learning-based PDE solvers behave compared to traditional numerical methods.

You are expected to:

- Design and implement a fully-connected neural network (MLP) that takes (x, y, t) as inputs and outputs the temperature field $T(x, y, t)$,
- Formulate the PINN loss function by combining PDE residual loss, boundary-condition loss, and initial-condition loss. Explicitly write out your loss components and discuss their roles,
- Select appropriate sampling strategies for collocation points in the spatial-temporal domain (e.g., random sampling, Latin hypercube sampling, structured grid sampling),
- Explain choices of network depth, width, activation function, optimizer, and learning rate,
- Train the PINN model and report training performance (loss history, convergence behavior),
- Evaluate the trained network on the same spatial and temporal sampling points used in the baseline numerical method for fair comparison:

$$\tau = 0.10, \quad 0.30, \quad 0.60; \qquad (x = 0.5W,\ y = 0.5H).$$

- Plot the 2-D temperature fields $T(x, y, t)$ at the prescribed time instants and generate 1-D centerline profiles using the same required formats as in Part 1,
- Compute and report quantitative error metrics such as L_2 error and relative error between the PINN prediction and the numerical solution,
- Compare training and inference time with the numerical baseline and discuss efficiency,
- Reflect on the advantages and limitations of PINNs for the current problem and discuss conditions under which PINNs may outperform traditional numerical solvers.

3. Boundary Condition Variation

Modify the bottom boundary condition to introduce non-uniform heating and re-solve the problem to generate a temperature field with stronger two–dimensional characteristics. For example, apply a spatially varying temperature distribution at the bottom boundary:

$$\theta(X, 0, \tau) = 1 + 0.5\sin(\pi X),$$

where the sinusoidal variation produces a non-uniform thermal input along the plate width. After modifying the boundary condition, repeat the same plotting requirements from Part 1 and Part 2.

Deliverables

- A full report including problem description, modeling assumptions, numerical implementation, PINN design, results, and discussion.
- Figures comparing temperature fields, time histories, and computational performance.
- Source codes (baseline numerical + PINN solver).
- Presentation slides.

Schedule

- **Phase-1 (10 points):** Due 12/09 by 5:00 PM. Submit a completed baseline numerical model (Method of Lines solution) including 2-D temperature fields, centerline profiles, and center-point time history for the baseline case. A short project timeline must also be included.
- **Phase-2 (10 points):** Due 12/23 by 5:00 PM. Submit implementation progress on the PINN model, including a working code framework (network structure, loss formulation, sampling strategy, and training script). Preliminary PINN results are encouraged and will receive additional credit.
- **Phase-3 (60 points):** Due 01/05 by 5:00 PM. Submit the full final report, complete numerical and PINN results, comparisons, boundary-condition modification, and final figures. Upload complete code and presentation slides.
- **Presentation (20 points):** 01/06/2025 at 12:55 PM. A 5+2 minute presentation summarizing methods, results, and conclusions.

Generative AI Usage Policy: You may use generative AI tools for coding, explanation, and report writing. Clearly indicate the extent of AI assistance in your report.