**UNIVERSITY OF CALIFORNIA AT BERKELEY**
**Department of Mechanical Engineering**
**ME C231A / EE C220B**
**Experiential Advanced Control Design I**

## Midterm Exam　　　　　　　　November 5, 2019

**Your Name and Student ID:**

Please answer all questions. Make sure to review the Exam Instructions before completing the exam.

| Exercise | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Grade: | | | | | |

## Exercise 1 Controller Properties (10pts)

1. Consider the following controllers which compute $u_k$ for some time $k = 0, \ldots, N-1$. Check their properties in the corresponding boxes:
(Note that the symbol $k$ always denotes time, $x_k$ is the system state at time $k$, $f(\cdot)$ is a generic function of the variable $\cdot$)

| | Open loop | Closed loop | Time invariant | Time varying | Linear | Non-linear |
|---|---|---|---|---|---|---|
| 1. $u_k = 0$ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 2. $u_k = kx_0$ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 3. $u_k = f(k)x_0$ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 4. $u_k = sin(x_0)k^2$ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 5. $u_k = -x_k^2 + x_k$ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 6. $u_0 = f(x_0), u_1 = f(x_1), u_{N-1} = f(x_{N-1})$ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

## Exercise 2 NLPs, KKT Conditions (20pts)

The following problem will be solved by hand. Consider the optimization problem

$$\begin{aligned} \min \quad & \frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-1)^2 \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1 \\ & (x_1-1)^2 + (x_2-1)^2 \geq 1 \end{aligned}$$

1. Sketch the feasible set and some level sets of the objective function.
   **Answer**

2. Does the optimization problem have a unique solution?
   **Answer**

3. Write the KKT conditions
   **Answer**

4. The point $x_1^* = 0.5$, $x_2^* = 1 + \sqrt{0.75}$ is a solution to the problem above. Solve the KKT equations above to find the corresponding Lagrange multiplies. Hint: there are 3 Lagrange multipliers and it is very easy to find them by checking which constraint is active at $(x_1^*, x_2^*)$.
   **Answer**

## Exercise 3 Constrained Finite-Time Optimal Control (45pts)

*Deliverables list. Hand-written answers. One matlab code (answers to a) and Two plots with 2 figures each (answers to a,d)*

Consider the following simple finite-time optimal control problem (to be solved with YALMIP)

$$\begin{aligned}
\min_{x_0,\dots,x_N,u_0,\dots,u_{N-1}} \quad & p_\theta(x_N) + \sum_{k=0}^{k=N-1} q_k(x_k,u_k) \\
& x_{k+1} = x_k + 0.5u_k && \forall k = \{0,\dots,N-1\} \\
& -1 \le x_k \le 1 && \forall k = \{0,\dots,N\} \\
& -1 \le u_k \le 0 && \forall k = \{0,\dots,N-1\} \\
& x_0 = 1 \\
& |x_N| \le 0.5
\end{aligned} \tag{1}$$

where $N = 3$, $p_\theta(x_N) = \theta x_N^2$ is the terminal cost function parameterised by $\theta$ and $q_k(x,u) = R_k u^2$ is the time-varying stage cost with $R_0 = 1,\ R_1 = 0.5,\ R_2 = 2$.

(a) Set $\theta = 0.1$, compute the *optimal* solution $U^*(x_0)$ where $U^* = [u_0^*,\dots,u_2^*]$ and report below. Also, Submit your MATLAB code and the following plots in one figure (plot file should be in pdf or jpg – we will call this the "**optimal state/input plot**").

- Plot the *optimal* state trajectories as a function of time. This means $x$-axis: 0,1,2,3 and $y$-axis: $x_0^*,\dots,x_3^*$
- Plot the *optimal* solution $U^*((x_0)$ as a function of time. This means $x$-axis: 0,1,2 and $y$-axis: $u_0^*,\dots,u_2^*$

**Answer:**

(b) Is the above problem convex?
**Answer:**

(c) Can you solve this problem by forcing Yalmip to use a Linear Program solver?

```
options = sdpsettings('solver', 'linprog');
optimize(constraints, objective, options);
```

Explain the reason.
**Answer:**

(d) Can you re-tune the cost in order to have the state at the end of the horizon $x_3^*$ closer to the origin? If so, provide the tuning (i.e., write the new tuning in the space below) and the "optimal state/input plot" for the closed loop system when the new tuning is used.
**Answer:**

(e) Can you explain why with low values of $\theta (< 0.1)$, $x_3^*$ is always 0.5?
**Answer:**

**Exercise 4** **Constrained Finite-Time Optimal Control via Dynamic Programming (25pts)** *Deliverables list. Handwritten answer only is ok. If matlab is used, provide matlab code as well. There is only one question in this problem*

We try to solve the same problem in (1) via Dynamic Programming. You can solve this problem **by hand**. You are welcome to use matlab, if you do so, please upload your code, and still report your handwritten answers on this paper. We have solved it by hand in 10 minutes.

We assume that the inputs to the system in the finite-time optimal control problem (1) can be only $u_{min} = -1$ or $u_{max} = 0$ (instead of any value between -1 and 0). So each node of the graph in figure 1 is obtained by propagating the system state according to the model

$$x_{k+1} = x_k + 0.5u_k$$

where $u_k = u_{max}$ or $u_k = u_{min}$.

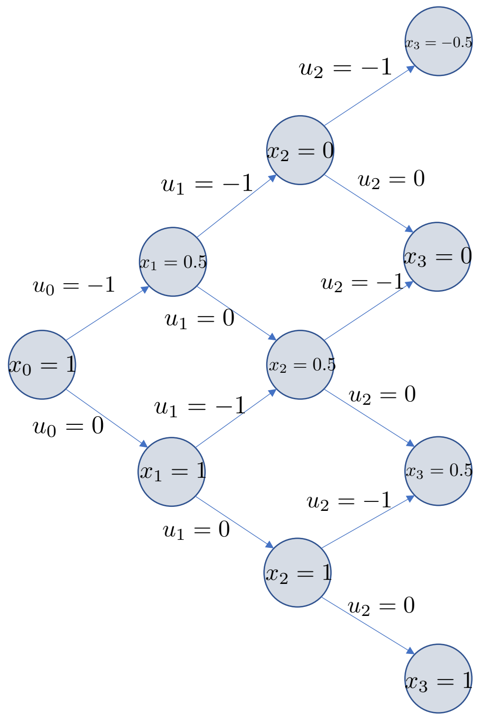


Figure 1: Graph describing state transition from time $k = 0$ to $k = 3$ by applying inputs $u_{min} = -1$ or $u_{max} = 0$

**QUESTION**: Solve the optimization problem (1) using Dynamic Programming, for the dynamics described by the graph in figure 1. Recall that $N = 3$, $p_\theta(x_N) = \theta x_N^2$, use $\theta = 5$, $q_k(x, u) = R_k u^2$ with $R_0 = 1,\ R_1 = 0.5,\ R_2 = 2,\ |x_N| \leq 0.5$.

(NOTE: make sure you use exactly the same cost and constraints as in problem (1)). In your solution, report

- $u_k^*(x_k)$ for each state of the graph for $k = 0, 1, 2$

- Cost-to-go $J_{k\to N}^*(x_k)$ for each state of the graph for $k = 0, 1, 2, 3$

Note that your optimal controller and cost-to-go will be defined only at discrete points shown in the graph. Your handwritten (or typed) solution at time $k = 1$ should look something like this:

$$u_1^*(x_1) \quad = \quad \begin{cases} \text{some number or NaN} & \text{if} \quad x_1 = 0.5 \\ \text{some number or NaN} & \text{if} \quad x_1 = 1 \end{cases} \tag{2}$$

$$J_{1\to 3}^*(x_1) \quad = \quad \begin{cases} \text{some number or infinty} & \text{if} \quad x_1 = 0.5 \\ \text{some number or infinty} & \text{if} \quad x_1 = 1 \end{cases} \tag{3}$$

We provide just time $k = 1$ as an example, you need to report $J_{k\to N}^*(x_k)$ for $k = 0, 1, 2, 3$ and $u_k^*(x_k)$ for $k = 0, 1, 2$.

**Answer:**

**Additional Space:**