ON 3N POINTS IN A PLANE

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In this note I prove the following theorem:

THEOREM 1. *Given* $3N$ *points in a plane, we can divide them into* N *triads such that, when we form a triangle with the points of each triad, the* N *triangles will all have a common point.*

The proof depends on three lemmas; the first two are well known, and are quoted without proof; the third I believe to be new, and leads directly to the proof of the theorem.

E^n denotes the solid unit ball in n-dimensional space R^n; Σ^{n-1} is the surface of E^n, and S^{n-1} too denotes the surface of a sphere.

LEMMA 1. (Fixed point theorem.) *If a continuous vector field be defined for all points of* E^n, *with the vectors pointing inwards at all points of* Σ^{n-1}, *then there is a point inside* E^n *at which the field vanishes.*

LEMMA 2. (Caratheodory.) *Given a set* A *in Euclidean* m*-space and any point* P *in the convex cover of* A, *we can find* $(m+1)$ *points* $a_0, \ldots, a_m$ *belonging to* A *which span a (closed) simplex containing* P $(a_0, \ldots, a_m$ *need not be distinct).*

LEMMA 3. *Let a mass-distribution in* E^n *be defined by an integrable density function* $\rho(x)$. (Thus $\rho(x) = 0$ for $x \notin E^n$.) *Then we can find a point* r *inside* E^n *so that every closed half-space with* r *on its boundary will contain at least* $1/(n+1)$ *of the total mass.*

Proof. Suppose the lemma untrue, so that for every point q of E^n there is at least one half-space which contains less than $1/(n+1)$ of the mass. We will obtain a contradiction. We first note that, given a point q, there is a one-one correspondence between the unit vectors from q and the half-spaces with q on their boundary; in fact, to the vector j will correspond the half-space given by $j\cdot(x-q) \geqslant 0$. Fixing our point q, we give a vector j the weight

$$w(j,q) = \left(\lambda - \frac{1}{n+1}\right)$$

if the corresponding half-space contains a proportion λ of the mass, where

$$\lambda < \frac{1}{n+1};$$

otherwise, we give it weight zero. Then $w(j,q)$ is continuous as a function of j and q, being an integral. We can regard the unit vectors j as points of a unit sphere S^{n-1}; take the integral

$$\int w(j,q)\, j\, dS^{n-1}$$

over the whole of S^{n-1}. This integral gives a vector $k(q)$; and $k(q)$ is a continuous function of q.

We have thus defined a continuous vector field $k(q)$, whose vectors evidently point inwards at all points of Σ^{n-1}. By Lemma 1, there is a point, r say, where the field vanishes. Consider the vectors from r as points of a unit sphere S^{n-1}. Denote by A the set of points j of S^{n-1} which were given a non-zero weight $w(j, r)$. By the hypothesis we desire to contradict, A is non-empty, and by the definition of the weighting function, A is open in S^{n-1}.

We assert that the centre of S^{n-1} lies in $H(A)$, the convex cover of A. For if not, there is a closed half-space (and hence an open half-space, since A is open) which contains A and has the centre of S^{n-1} on its boundary; that is, the set A is contained in a single hemisphere of S. But in this case, the vector integral

$$\int_{j\in A} w(j,r)j\,dS = k(r)$$

cannot possibly vanish, and we have a contradiction.

Applying Lemma 2, it follows that we can choose $(n+1)$ points $j_0, \ldots, j_n$ of A which span a simplex containing the centre of S^{n-1}. Every vector accordingly makes an angle less than or equal to $\frac{1}{2}\pi$ with at least one of $j_0, j_1, \ldots, j_n$; it follows that the $(n+1)$ closed half-spaces

$$j_i\cdot(x-r) \geqslant 0 \quad (i = 0, \ldots, n)$$

together account for the whole of space. But by our construction, they each contain less than $1/(n+1)$ of the total mass. We have thus obtained the contradiction we need.

Corollary 3·1. *The lemma remains true even if the mass distribution is no longer specified by a density function, e.g. we may allow concentrated point masses.*

In fact, such a distribution D will be the limit of a sequence of distributions corresponding to density functions; call such a sequence of density functions $\{\rho_n\}$. Then for each ρ_n there is a point r_n with the property described in the lemma; $\{r_n\}$ is a bounded infinite sequence of points, and so has at least one limit point, r say. We now see easily that any closed half-space with its boundary passing through r contains at least $1/(n+1)$ of D.

Corollary 3·2. *Let Y be a finite set consisting of M points in n-space, and suppose that $M > (n+1)R$. Then there is a point common to all the closed half-spaces which contain at least $(M-R)$ points of Y.*

Indeed, considering the set Y as a distribution of M equal point-masses, we see that all closed half-spaces containing $(M-R)$ points of Y must contain the point r constructed in Lemma 3 and its first corollary; for, if one did not, there would be another half-space opposite to it which had r on its boundary but contained no more than R points.

Proof of Theorem 1. We are given $3N$ points in a plane. By the corollary above, there is a point P common to all half-planes which contain $(2N+1)$ of the given points. We can set up a coordinate system with origin at P, so that the $3N$ given points have polar coordinates $(r_1, \theta_1), (r_2, \theta_2), \ldots, (r_{3N}, \theta_{3N})$. Number off the $3N$ points in order of increasing θ, and call them $b_1, \ldots, b_{3N}$. Now we assign the points to N triads, putting b_i

and b_j into the same triad if and only if $i \equiv j \pmod N$. In order to prove the theorem, it will be enough to show that the triangle $b_i b_{i+N} b_{i+2N}$ contains P for all $i = 1, \ldots, N$. This is easy to prove; in fact, if it were false, we would have $\theta_{j+N} - \theta_j > \pi$ for some j (suffices taken modulo $3N$); but there are at most $(N-1)$ points with θ between θ_{j+N} and θ_j, and so there is a half-plane containing $(2N+1)$ of the given points, but not P. This is a contradiction.

We conclude with a few remarks and references.

(1) By a slight modification in the above argument, we can prove the apparently stronger result:

THEOREM 1*. *Given a plane set Y consisting of $(3N-2)$ points, we can split it into N subsets $Y_1, \ldots, Y_N$ so that there is a point common to all the convex covers $H(Y_i)$.*

(2) Theorem 1* is very similar to the results of Rado (7). Given a dimension n and an integer N we may define $f(n, N)$ as the smallest integer with the following property: *Let Y be any finite n-dimensional set consisting of M points, where $M \geqslant f(n, N)$. Then we can split Y into N subsets $Y_1, \ldots, Y_N$ so that there is a point common to all the convex covers $H(Y_i)$.* It is easy to see that

$$f(n, N) \geqslant (n+1)N - n,$$

and in fact
$$f(1, N) = 2N - 1. \tag{1}$$

Rado's principal result is
$$f(n, N) \leqslant 2f(n-1, N) - n, \tag{2}$$

and he combines this with (1) to obtain his Theorem 1, which asserts

$$f(n, N) \leqslant 2^n(N-2) + n + 2; \tag{3}$$

in particular, when $N = 2$ this is $f(n, 2) = n + 2$, which is substantially Helly's theorem. The two-dimensional case of Rado's theorem is $f(2, N) \leqslant 4N - 4$; our Theorem 1* improves this to the best possible result

$$f(2, N) = 3N - 2,$$

and of course we can do rather better than (3) by combining this with (2). This improvement is hardly a valuable one—but it would be nice to know whether

$$f(n, N) = (n+1)N - n$$

is true in general.

Lemma 3 leads by a very simple argument to an estimate for $f(n, N)$ which is substantially stronger than Rado's when n and N are large. Let Y be an n-dimensional set consisting of $[n(n+1)(N-1)+1]$ points; then by Corollary 3·1 there is a point p such that every closed half-space containing p contains at least $[(N-1)n+1]$ points of Y. p is in the convex cover of Y, so by Lemma 2 there is a subset Y_1 of Y consisting of at most $(n+1)$ points such that $H(Y_1)$ is a simplex containing p, and every half-space containing p contains at least $[(N-1)n+1-n]$ points of $Y - Y_1$. Repeating the argument $(N-1)$ times, there are $(N-1)$ disjoint subsets $Y_1, \ldots, Y_{N-1}$ of Y such that p is common to $H(Y_1), \ldots, H(Y_{N-1})$, and every half-space containing p contains a point of $Y - \bigcup_{r=1}^{N-1} Y_r = Y_N$, say; thus p is in $H(Y_N)$. It follows that

$$f(n, N) \leqslant Nn(n+1) - n^2 - n + 1.$$

(3) Lemma 3 may be found more interesting than Theorem 1; so far as I know, it is new. The plane case of the lemma was proved in 1945 by Neumann (5); his proof, though elementary, is long, and does not extend to higher dimensions. Lemma 3 is clearly best possible, as we see when we consider the distribution consisting of $(n+1)$ point-masses.

(4) The method of Lemma 3 may be applied to several other problems which involve estimating

$$\min_x \max_j [g(x,j)],$$

where x runs through the points of E^n, j runs through the unit vectors corresponding to the points of an S^{n-1}, and $g(x,j)$ is a function of some class $\mathscr{G}$. For instance, Neumann (6) considered the plane case of another problem of this type. Let K be a closed convex body with interior points. Let p be a point inside K; then to any vector j there corresponds a directed chord of K through p. Suppose that p divides this chord so that the ratio of the segment in the positive direction from p to the whole chord is $\lambda: 1$. Then λ is a continuous function of p and j for p in the interior of K. Let $r(p)$ be the maximum of λ for all vectors j; obviously $r(p)$ is a continuous function, $\frac{1}{2} \leqslant r(p) \leqslant 1$, and by compactness $r(p)$ attains its minimum r^*. By a straightforward application of the ideas of Lemma 3, we can prove

Lemma 4. *Suppose that $r(q) = r^*$. Then there are $(n+1)$ vectors $j_0, \ldots, j_n$, possibly not all distinct, for all of which*

$$\lambda(q,j) = r(q) = r^*,$$

and which (viewed as points of S^{n-1}) span a simplex containing the centre of S.

From this lemma, we deduce

Corollary 4·1.

$$r^* \leqslant \frac{n}{n+1}, \tag{4}$$

with equality if and only if K is a simplex.

In fact, suppose that the chord through q parallel to j_i meets the boundary of K in s_i and t_i, so that qs_i is in the direction of j_i. Then q is in the simplex $s_0 s_1 \ldots s_n$. Drop duplicates, so that q lies in a simplex $s_0 s_1 \ldots s_m$, where $s_0, \ldots, s_m$ are all distinct, and $m \leqslant n$. Let the barycentric coordinates of q with respect to this simplex be $(\alpha_0, \alpha_1, \ldots, \alpha_m)$, and let the line $s_i q$ meet the opposite face of the simplex in the point u_i. Then

$$s_i q / s_i u_i = 1 - \alpha_i.$$

Also, since K is convex,

$$s_i u_i \leqslant s_i t_i \quad \text{for} \quad i = 0, \ldots, m; \tag{5}$$

and so

$$r^* = \frac{s_i q}{s_i t_i} \leqslant \frac{s_i q}{s_i u_i} = 1 - \alpha_i. \tag{6}$$

Summing (6) over $i = 0, \ldots, m$, we see that

$$(m+1) r^* \leqslant m + 1 - \Sigma \alpha_i = m,$$

so that

$$r^* \leqslant \frac{m}{m+1} \leqslant \frac{n}{n+1}.$$

This proves (4). Further, equality can be necessary in (4) only if $m = n$ and there is equality in (5) and (6) for every i. This implies that $\alpha_i = 1/(n+1)$ and $t_i = u_i$ for each i. Thus K is just the simplex $s_0 s_1 \ldots s_n$, and q is its barycentre.

Neumann (6) proved this corollary in the plane case. The n-dimensional inequality (4) was proved very simply by Süss (8), and independently by Hammer (3); both of them showed that, if g is the centroid of K,

$$r(g) \leqslant \frac{n}{n+1},$$

and from this (4) follows *a fortiori*; neither author gave the conditions for equality in (4). Another simple proof of (4) is suggested by Eggleston (2) as an exercise.

We may also apply Lemma 4 to sets of constant width; in this way we obtain yet another proof of Jung's theorem, that a set of diameter D is contained in a sphere of radius R, with $R \leqslant [n/\{2n+2\}]^{\frac{1}{2}} D$; see Jung (4). One very simple proof of this theorem has been given by Blumenthal and Wahlin (1), and an even more elegant one is contained in (2).

In conclusion, I would like to thank Professor Eggleston for his entertaining lectures, which led me to perpetrate this note.

REFERENCES

(1) Blumenthal, L. M. and Wahlin, G. E. On the smallest sphere enclosing a bounded n-dimensional set. *Bull. Amer. Math. Soc.* (2) 47 (1941), 771–7.

(2) Eggleston, H. G. *Convexity* (Cambridge, 1958).

(3) Hammer, P. C. The centroid of a convex body. *Proc. Amer. Math. Soc.* 2 (1951), 522–5.

(4) Jung, H. W. E. Uber die kleinst Kugel, die eine raumliche Figur einschliesst. *J. reine angew. Math.* 123 (1901), 241–57.

(5) Neumann, B. H. On some affine invariants of closed convex regions. *J. Lond. Math. Soc.* 14 (1939), 262–72.

(6) Neumann, B. H. On an invariant of plane regions and mass distributions. *J. Lond. Math. Soc.* 20 (1945), 226–37.

(7) Rado, R. Theorems on the intersection of convex sets of points. *J. Lond. Math. Soc.* 27 (1952), 320–8.

(8) Süss, W. Uber eine Affininvariante von Eibereichen. *Archiv. Math.* 1 (1948), 127–8.

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