Since the likelihood is Multinomial distribution.

$$P(x|\theta, n) = \text{Multinomial}(\theta, n)$$

$$= \frac{n!}{x_1! x_2! \cdots x_k!} \prod_{i=1}^{k} \theta_i^{x}$$

And Prior is Dirchlet Distribution

$$P(\theta|\alpha) = \text{Dirchlet}(\alpha)$$

$$= \frac{1}{B(\alpha)} \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

By using Bayes' therom, $P(\theta|X, n, \alpha) = \dfrac{P(x|\theta, n)\, P(\theta|\alpha)}{P(x)}$

$$P(\theta|X, n, \alpha) \propto \prod_{i=1}^{k} \theta_i^{x_i} \cdot \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

$$P(\theta|X, n, \alpha) \propto \prod_{i=1}^{k} \theta^{x_i + \alpha_i - 1}$$

We can tell $P(\theta|X, n, \alpha) \propto \prod_{i=1}^{k} \theta_i^{(x_i + \alpha_i) - 1}$ is dirchlet distribution

The posterior distribution is proportional to

$$P(\theta|x, n, \alpha) \propto P(x|\theta, n) \cdot P(\theta|\alpha)$$

The posterior is Dirichlet distribution since the conjugacy of the Dirchlet prior with Multinomial likelihood.

For the parameterization of Posterior;

$$\alpha_{posterior} = \alpha + x$$

So the posterior distribution is $P(\theta|x, n, \alpha) = \text{Dirchlet}(\alpha + x)$