IDX G10 AP Precalculus
Study Guide Issue S1 M1
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2.1 Polynomials

Polynomials is a function of a form:

- $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + {}_2x^2 + a_1x + a_0$ (n is a nonnegative integer and a are real numbers)
- Polynomial Function of degree n
- **Terms:** ax... ; **Coefficients:** a...
- **Leading term:** the term containing the highest power of x : a_nx^n ; **Leading Coefficient:** a_n

Degree	Name	Example (please try to write one by yourself)
0	constant	
1	linear	
2	quadratic	
3	cubic	
4	quartic	
5	quintic	

- **zeros of a function:** the roots of the equation f(x)=0

2.2 Long Divison; the Remainder and Factor Theorems

Long Division: dividing $f(x) = 6x^3 - 2x^2 + 5x + 1$ by $x - 1$ yields

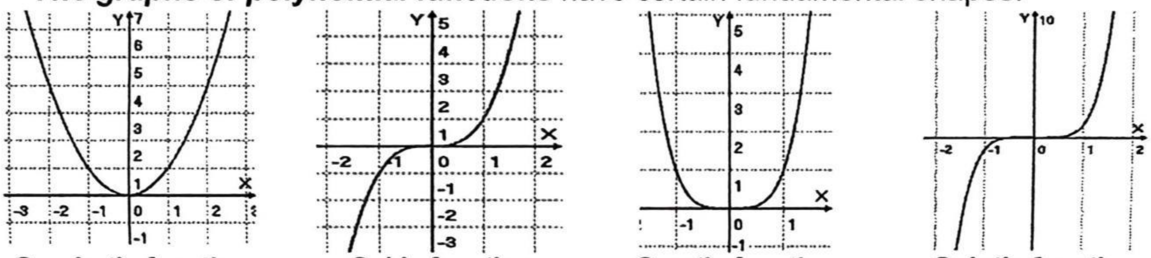
Thus, $6x^3 - 2x^2 + 5x + 1 = (x-1)\left(6x^2 + 4x + 9\right) + 10$.

$$\begin{array}{rl} & \underline{6x^2 + 4x + 9} \quad \leftarrow \textit{Quotient} \\ \textit{Divisor} \rightarrow x-1 \Big) & 6x^3 - 2x^2 + 5x + 1 \leftarrow \textit{Dividend} \\ & \underline{6x^3 - 6x^2} \\ & \quad 4x^2 + 5x \\ & \quad \underline{4x^2 - 4x} \\ & \qquad 9x + 1 \\ & \qquad \underline{9x - 9} \\ & \qquad\quad 10 \leftarrow \textit{Remainder} \end{array}$$

- **Remainder Theorem:** When polynomial P(x) is divided by x-a, the remainder is P(a)
    - P(x)=(x-a) • Q(x) + R ==> P(a) = R
- **Factor Theorem:** A number a is a solution of the polynomial equation f(x) = 0 if and only if x-a is a factor of f(x)

2.3 Graphing Polynomial Functions

The graphs of polynomial functions have certain fundamental shapes.



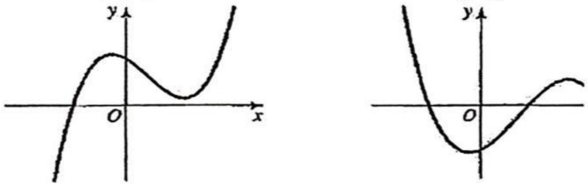
Quadratic function $y = x^2$

Cubic function $y = x^3$

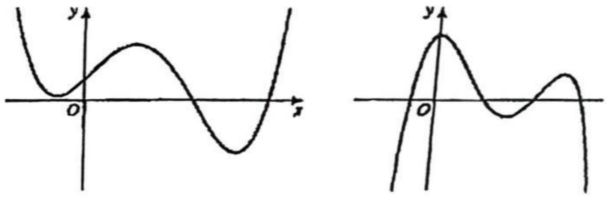
Quartic function $y = x^4$

Quintic function $y = x^5$

Graph of $f(x)=ax^3+bx^2+cx+d$



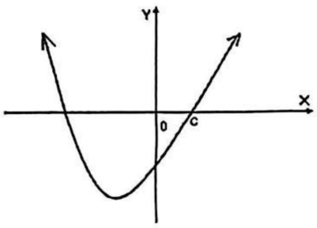
Graph of $f(x)=ax^4+bx^3+cx^2+dx+e$



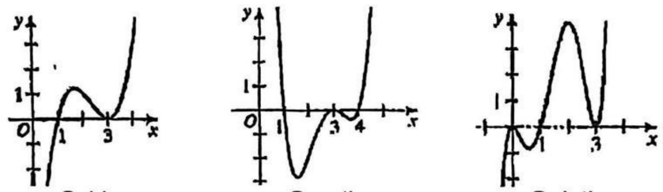
➢ ***Effect of a Squared or Cubed Factor***

- A polynomial function will have a **zero** that is a real number at each point at which it *crosses* the x-axis.

★ If a polynomial $P(x)$ has a factor such as $(x-c)$ **only once**, the graph cross the x-axis at $x=c$ straight.

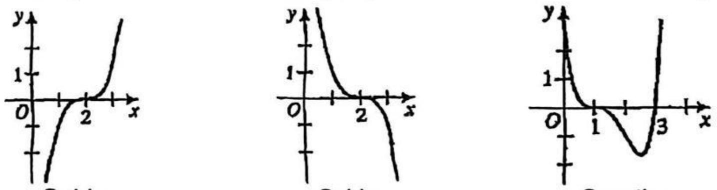


★ If a polynomial $P(x)$ has a factor such as $(x-c)^2$, then $x=c$ is a **double root** of $P(x)=0$ and the graph is tangent to the x-axis at $x=c$.



Cubic $y=(x-1)(x-3)^2$ | Quartic $y=(x-1)(x-3)^2(x-4)$ | Quintic $y=x^2(x-1)(x-3)^2$

★ If a polynomial $P(x)$ has a factor such as $(x-c)^3$, then $x=c$ is a **triple root** of $P(x)=0$ and the graph flattens out around $(c,0)$ and cross the x-axis at the point.



Cubic $y=(x-2)^3$ | Cubic $y=-(x-2)^3$ | Quartic $y=(x-1)^3(x-3)$

2.4 Finding Maximums and Minimums of Polynomial

- **Quadratic Functions**: the **Maximum or Minimum value** of the quadratic function y=ax^2 + bx +c occurs at x=-b/2a
- **Cubic functions**
  - Point of inflection: **Points of inflection** of a polynomial function occur at input vales where the **rate of change** of the function changes from **increasing** to **decreasing** or from **decreasing** to **increasing**. This occur where the graph of a polynomial function changes from **concave down** to **concave up** or from **concave up** to **concave down**
- **Local maximum and local minimum**

- When the graph of a cubic function has a "**peak**" and a "**valley**", the function has a **local maximum** at the **highest point** of the **peak** and a **local minimum** at the **lowest point** of the **valley**
- the **highest point** among the **local maximum** is the **global maximum**; the **lowest point** among the **local minimum** is the **global minimum**

- **Local Extrema Theorem:** A polynomial function of degree n has at most n-1 relative maxima/minima
- **Extreme Value Theorem:** If a polynomial function f is on a closed interval [a,b], then f has both a maximum and a minimum value on [a,b]
- **Intermediate Value Theorem:** If P(x) is a polynomial function such that a<b and P(a) is not equal to P(b), then P(x) takes on every value between P(a) and P(b) in the interval [a,b]
- **The Location Principle:** if P(a) · P(b) < 0, then P(x) has at least one zero between a and b.

2.6 Solving Polynomial Equations by Factoring

- **Rational Root Theorem**

➢ **Rational Root Theorem**

Use factoring to solve the general polynomial equation,

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$$

$$(qx - p)(\qquad\qquad\qquad) = 0$$

If $qx - p$ is a factor (so that $x = \frac{p}{q}$ is a root), then q must divide a_n, and p must divide a_0.

Let $P(x)$ be a polynomial of degree n with integral coefficients and a nonzero constant terms:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0, \text{ where } a_0 \neq 0$$

If one of the roots of the equation $P(x) = 0$ is $x = \frac{p}{q}$ where p and q are nonzero integers with no common factor other than 1, then p must be a factor of the **constant term** a_0 and q must be a factor of the **leading coefficient** a_n.

- **Other ways**
  - Cross multiplication
  - use one variable to represent one complicated element
    - E.g. t=2x^2
  - $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
  - $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

2.7 General results for polynomial equations

- **The Fundamental Theorem of Algebra:** In the **complex number system** consisting of all real and imaginary numbers, if P(x) is a polynomial function of degree n (n<0) with **complex coefficients**, then the equation P(x)=0 has **exactly n roots** (provided a double root is counted as 2 roots. a triple root is counted as 3 roots, and so on)
- **Complex Conjugates Theorem:** If P(x) is a polynomial with **real coefficients**, and **a+bi** is an imaginary root of the equation P(x)=0, the **a-bi** is also a root
- **Theorem 3**: Suppose P(x) is a polynomial with **rational coefficients**, and a and b are rational numbers such that √b is irrational, if **a+√b** is an imaginary root of the equation P(x)=0, then **a-√b** is also a root
- **Theorem 4**: If P(x) is a polynomial of **odd degree with real coefficients,** then the equation P(x)=0 has **at least one real root**
- **Vieta's Formula:** For the equation $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0=0$, with $a\neq0$: the sum of the roots is $-a_{n-1}/a_n$; the product of the roots is a_0/a_n if n is even, and $-a_0/a_n$ if n is odd.

3.1 Linear Inequalities; Absolute Value

Quick Review

1. You can add the same number to (or subtract the same number from) both sides of an inequality.

$$a<b \quad \Rightarrow \quad a+c<b+c$$

2. You can multiply (or divide) both sides of an inequality by the same **positive** number.

$$a<b,\ c>0 \quad \Rightarrow \quad ac<bc$$

3. You can multiply (or divide) both sides of an inequality by the same **negative** number if you *reverse the inequality sign.* $a<b,\ c<0 \quad \Rightarrow \quad ac>bc$

4. Another definition of absolute value of $x\in$ is: $|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$

➢ *Notations*

- $1 < x < 5$ is equivalent to $(1,\ 5)$
- $0 \le x \le 4$ is equivalent to $[0,\ 4]$
- $10 \le x < 40$ is equivalent to $[10,\ 40)$
- $x > 1$ is equivalent to $(1,\ \infty)$
- $x \le 9$ is equivalent to $(-\infty,\ 9]$.

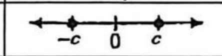
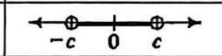
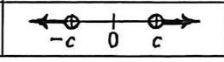
➢ *Graphing*

- A solid dot (•) → an endpoint is included in the graph;
- An open dot (o) → the endpoint is not included in the graph
- **Combined inequality** → its graph is the intersection of the graphs

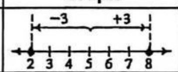
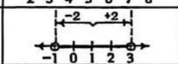
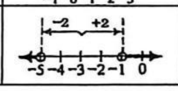
Wrong combined inequalities: for example, $6 < x < 1$, *it is the* ***empty set***. And we use $\varnothing$ to describe.

➢ *Absolute Value*

- The absolute value of a number x, denoted $|x|$, is interpreted geometrically the ***distance*** from x to zero in either direction on the number line.

Sentences Involving $\lvert x\rvert$ $c \ge 0$			
Sentence	**Meaning**	**Graph**	**Solution**
$\lvert x\rvert = c$	The distance from x to 0 is *exactly* c units.		$x = c$ or $x = -c$
$\lvert x\rvert < c$	The distance from x to 0 is *less than* c units.		$-c < x < c$
$\lvert x\rvert > c$	The distance from x to 0 is *greater than* c units.		$x < -c$ or $x > c$

- $|x-k|$ is interpreted the distance from x to k in either direction on the number line

Sentences Involving $\lvert x-k\rvert$			
Sentence	**Meaning**	**Graph**	**Solution**
$\lvert x-5\rvert = 3$	The distance from x to 5 is 3 units.		$x = 2$ or $x = 8$
$\lvert x-1\rvert < 2$	The distance from x to 1 is *less than* 2 units.		$-1 < x < 3$
$\lvert x+3\rvert > 2$ ($\lvert x-(-3)\rvert > 2$)	The distance from x to -3 is *greater than* 2 units.		$x < -5$ or $x > -1$

Sentence	Equivalent Sentence	*For example*
$\lvert ax+b\rvert = c$	$ax+b = \pm c$	$\lvert 2x-3\rvert = 4 \triangleright 2x-3 = \pm 4$
$\lvert ax+b\rvert < c$	$-c < ax+b < c$	$\lvert 2x-3\rvert < 4 \triangleright -4 < 2x-3 < 4$
$\lvert ax+b\rvert > c$	$ax+b < -c$ or $ax+b > c$	$\lvert 2x-3\rvert > 4 \triangleright 2x-3 < -4$ OR $2x-3 > 4$