IDX G10 AP Calculus
Study Guide Issue 1
By 10(10) Emma Li, Edited by 10(3) Eric Wang

2.1 Rate of Change and Limits

For any $\varepsilon > 0$, there exists a $\delta > 0$, such that whenever $0 < |x - c| < \delta$,

$$|f(x) - L| < \varepsilon$$

Then we say *f* has a limit *L* as *x* approaches *c*. for notation, we write

$$\lim_{x \to c} f(x) = L$$

Properties of Limits

If *L, M, c, k* are real numbers, and $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$

- $\lim_{x \to c} k = k$
- $\lim_{x \to c} x = c$

- $\lim_{x\to c}\left(f(x)+g(x)\right)=L+M$
- $\lim_{x\to c}\left(f(x)-g(x)\right)=L-M$
- $\lim_{x\to c}\left(k\bullet f(x)\right)=k\bullet L$
- $\lim_{x\to c}\left(f(x)\bullet g(x)\right)=L\bullet M$
- If $M\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M}$
- If r and s are integers and $s\neq 0$, $L^{\frac{r}{s}}$ is a real number, then $\lim_{x\to c}\left(f(x)\right)^{\frac{r}{s}}=L^{\frac{r}{s}}$

Left-handed and right-handed limits

$\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$

- $\lim_{x\to 0}\frac{\sin x}{x}=1$

Sandwich Theorem (Squeeze Theorem)

If $g(x)\le f(x)\le h(x)$ for all $x\neq c$ in the neighborhood of *c*, and

$$\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L$$

Then

$$\lim_{x\to c} f(x)=L$$

Average and Instantaneous Speed

Average speed: $\frac{\Delta y}{\Delta t}$

Instantaneous speed: $\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}$

2.2 Limits Involving Infinity

Finite Limits as $x \to \pm\infty$

Horizontal Asymptote:

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \ or \ \lim_{x\to-\infty} f(x) = b$$

Note: Rational functions have at most one horizontal asymptote, while other functions have at most two horizontal asymptotes.

Vertical Asymptote:

The line $y = a$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^-} f(x) = \infty \ or - \infty$$

Or

$$\lim_{x\to a^+} f(x) = \infty \ or - \infty$$

Note: all properties that apply to limits apply to limits involving infinity, including the sandwich theorem.

End Behavior Model

The function *g* is

(a) a right end behavior model for $f \Leftrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

(b) a left end behavior model for $f \Leftrightarrow \lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$

If one function provides both a left and right end behavior model, it is simply called end behavior model.

2.3 Continuity

Continuity at a Point

Interior point:

A function $y = f(x)$ is continuous at an interior point *c* of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

Endpoint:

A function $y = f(x)$ is continuous at left endpoint *a* of its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

A function $y = f(x)$ is continuous at right endpoint *b* of its domain if

$$\lim_{x \to b^-} f(x) = f(b)$$

Types of Discontinuity

Removable discontinuity: $f(x) \neq \lim_{x \to c} f(x)$

Jump discontinuity: $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x) \Longrightarrow \lim_{x \to c} f(x)\ DNE$

Infinite discontinuity: $\lim_{x \to c^-} f(x) = \pm\infty$ or $\lim_{x \to c^+} f(x) = \pm\infty \Longrightarrow \lim_{x \to c} f(x)\ DNE$

Oscillating discontinuity

Continuous functions

Functions are continuous on an interval iff it is continuous at every point of its domain

Properties of Continuous Functions

If the functions *f* and *g* are continuous at $x = c$, then the following combinations are continuous at $x = c$,

1. sums: $f + g$
2. differences: $f - g$
3. products: $f \bullet g$
4. constant multiples: $k \bullet f$, for any number *k*
5. quotients: $\frac{f}{g}$, provided $g(c) \neq 0$

If the function *f* is continuous at *c* and *g* is continuous at $f(c)$, then the composite $g \circ f$ is continuous at *c*.

Intermediate Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$

Corollary 1: Extreme value theorem (EVT)

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ must have its maximum and minimum, and hence takes on every value between its maximum and minimum values that occur on $[a, b]$.

Corollary 2: Zero's Theorem

If a function $y = f(x)$ that is continuous on a closed interval $[a, b]$, and if

$$\left(f(a) \bullet f(b)\right) < 0$$

Then there exists $c \in (a, b)$ such that $f(c) = 0$

2.4 The Rate of Change of Tangent Lines

Difference Quotient

The difference quotient for *f* at *a* is given by the expression

$$\frac{f(a + h) - f(a)}{h}$$

Slope of a Curve

The slope of a curve $y = f(x)$ at a point $P\left(a, f(a)\right)$ is the number

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Provided the limit exists.

- The tangent to the curve at *P* is the line through *P* with this slope.

Normal to a Curve

Normal line: the line perpendicular to the tangent at a point

Rates of Change

The average rate of change of a function over an interval is

$$\frac{amount\ of\ change\ in\ function\ value}{length\ of\ the\ interval} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The instantaneous rate of change of a function at a point $(x_1, f(x_1))$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

3.1 Derivative of a Function

Definition of Derivatives

Derivative of a point:

The derivative of the function *f* at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Provided the limit exists.

Derivative as a function with respect to *x*:

The derivative of the function *f* with respect to *x* is the function *f* whose value at *x* is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Provided the limit exists.

Alternate definition: the derivative of the function *f* at $x = a$ is limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Provided the limit exists.

Notations

y'; $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f(x)$, $\dot{f}(x)$

One-sided Derivatives

Right-handed derivative: $f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

Left-handed derivative: $f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$

If $f(x) > 0$, the slope of the tangent line is positive, so *f* is increasing.

If $f(x) < 0$, the slope of the tangent line is negative, so *f* is decreasing.

3.2 Differentiability

Types of Nonexistence of $f'(a)$

If $f'(a)$ exists, then

1. $f(a)$ is defined
2. $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$

Nonexistence:

1. Corner: $f_+'(a)$ and $f'_-(a)$ exist but differ
2. Vertical tangent: $f'_+(a) = \infty$, $f'_-(a) = \infty$ or $f'_+(a) = -\infty$, $f'_-(a) = -\infty$
3. Cusp: $f'_+(a) = \infty$, $f'_-(a) = -\infty$ or $f'_+(a) = -\infty$, $f'_-(a) = \infty$
4. Discontinuity

Derivatives on a Calculator

Symmetric difference quotient (numerical derivative of *f*):

$$NDER(f(x), a) = \frac{f(a+h) - f(a-h)}{2h}$$

Where default $h = 0.001$, but still can be changed

$$\text{Continuity} \not\Rightarrow \text{Differentiability}$$

$$\text{Differentiability} \Rightarrow \text{Continuity}$$

Note: IVT also applies for derivatives

If *a* and *b* are any two points in an interval on which *f* is differentiable, then *f'* takes on every value between $f'(a)$ and $f'(b)$

3.3 Rules for Differentiation

Rule 1 Derivative of a Constant

If *c* is a constant, then

$$\frac{d}{dx}(c) = 0$$

Rule 2 Power Rules for Positive Integer Powers of *x*

If *n* is a positive integer, then

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Rule 3 The Constant Multiple Rule

If *u* is a differentiable function of *x* and *c* is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Rule 4 The Sum and Difference Rule

If *u* and *v* are differentiable functions of *x*, then their sum and difference are differentiable at every point where *u* and *v* are differentiable. At each points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Rule 5 the Products Rule

The product of two differentiable function *u* and *v* is differentiable, and

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Rule 6 The Quotient Rule

At a point where $v \neq 0$, the quotient u/v of two differentiable functions is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Rule 7 Power Rules for Negative Integer Powers of *x*

If *n* is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Second and Higher Order Derivatives

Second derivative: derivative of *y'*

$$f''(x) = y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Third derivative:

$$f'''(x) = y''' = \frac{dy''}{dx} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The *n*-th derivative:

$$f^{(n)}(x) = y^{(n)} = \frac{dy^{(n-1)}}{dx} = \frac{d^ny}{dx^n}$$

3.5 Derivative of Trigonometric Functions

Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivatives of Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \bullet \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \bullet \cot x$$

These can be derived by differentiation of quotient rule.

3.6 Chain Rule

If *f* is differentiable at the point $u = g(x)$, and *g* is differentiable at *x*, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at *x*, and

$$(f \circ g)'(x) = f'(g(x)) \bullet g'(x)$$

Alternatively:

If $y = f(u)$ and $u = g(x)$, the

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

Where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

3.7 Implicit Differentiation

Implicit Defined Function

An equation that defines several functions implicitly (hidden inside the equation), without giving explicit formulas. A obvious example would be the function $y = e^{x+y} + 2$.

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to *x* (or another destinated variable)
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation
3. Factor out $\frac{dy}{dx}$
4. Solve for $\frac{dy}{dx}$

Power Rule for Rational Powers

If *n* is any rational number, then

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

If $n < 1$, then the derivative does not exist at $x = 0$ due to the existence of a infinite discontinuity at the point x=0