Sheet #10 – Solution

Due:

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1 Kalman Filter (10 pt)

In the lecture we discussed the following linear dynamical system, consisting of latent variables $\mathbf{z}$ and observable variables $\mathbf{x}$, following the relationship

$$\mathbf{z}_1 = \boldsymbol{\mu}_0 + \mathbf{u}, \qquad \text{where} \quad \mathbf{u} \sim \mathcal{N}(\mathbf{u}|0, \mathbf{P}_0) \tag{1}$$

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{w}_t, \qquad \text{where} \quad \mathbf{w}_t \sim \mathcal{N}(\mathbf{w}_t|0, \boldsymbol{\Gamma}) \tag{2}$$

$$\mathbf{x}_t = \mathbf{C}\mathbf{z}_t + \mathbf{v}_t, \qquad \text{where} \quad \mathbf{v}_t \sim \mathcal{N}(\mathbf{v}_t|0, \boldsymbol{\Sigma}). \tag{3}$$

We assume $\boldsymbol{\mu}_0, \mathbf{P}_0, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}, \mathbf{A}$, and $\mathbf{C}$ to be known.[1] Slightly reformulating the results from the lecture we get that $p(\mathbf{z}_t|\mathbf{x}_{1:t}) \sim \mathcal{N}(\mathbf{z}_t|\boldsymbol{\mu}_t, \mathbf{V}_t)$ with

$$\boldsymbol{\mu}_t = \mathbf{A}\boldsymbol{\mu}_{t-1} + \mathbf{K}_t(\mathbf{x}_t - \mathbf{C}\mathbf{A}\boldsymbol{\mu}_{t-1}) \tag{4}$$

$$\mathbf{V}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{C})\mathbf{P}_{t-1}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix of suitable dimensionality. $\mathbf{P}_{t-1}$ and $\mathbf{K}_t$ are given as

$$\mathbf{P}_{t-1} = \mathbf{A}\mathbf{V}_{t-1}\mathbf{A}^T + \boldsymbol{\Gamma} \tag{6}$$

$$\mathbf{K}_t = \mathbf{P}_{t-1}\mathbf{C}^T(\mathbf{C}\mathbf{P}_{t-1}\mathbf{C}^T + \boldsymbol{\Sigma})^{-1}. \tag{7}$$

i) Consider the situation where the observation noise goes to zero for $\mathbf{x}_t$, i.e. we now have a time dependent $\boldsymbol{\Sigma}_t$, with $\boldsymbol{\Sigma}_j = \boldsymbol{\Sigma}$ for $j = 1, ..., t-1$ and $\boldsymbol{\Sigma}_t = \mathbf{0}$. Assuming $\mathbf{C} = \mathbf{I}$, what do we get for $p(\mathbf{z}_t|\mathbf{x}_1, ..., \mathbf{x}_t)$? Does that agree with your intuition?

Solution: Intuitively we would expect to end up with a deterministic solution, i.e. a delta-distribution on the observation independent of any prior uncertainties. This is actually the case. We have that

$$\mathbf{K}_t = \mathbf{P}_{t-1}\mathbf{C}^T(\mathbf{C}\mathbf{P}_{t-1}\mathbf{C}^T + \boldsymbol{\Sigma})^{-1} = \mathbf{P}_{t-1}\mathbf{C}^T(\mathbf{C}\mathbf{P}_{t-1}\mathbf{C}^T)^{-1} = \mathbf{C}^{-1}. \tag{8}$$

The variance term then becomes $\mathbf{V}_t = \mathbf{0}$ and for the mean we get that

$$\boldsymbol{\mu}_t = \mathbf{A}\boldsymbol{\mu}_{t-1} + \mathbf{K}_t(\mathbf{x}_t - \mathbf{C}\mathbf{A}\boldsymbol{\mu}_{t-1}) = \mathbf{A}\boldsymbol{\mu}_{t-1} + \mathbf{C}^{-1}\mathbf{x}_t - \mathbf{A}\boldsymbol{\mu}_{t-1} = \mathbf{C}^{-1}\mathbf{x}_t = \mathbf{x}_t, \tag{9}$$

given our assumption of $\mathbf{C} = \mathbf{I}$.

ii) Implement the Kalman filter updates and plot both the observations $\mathbf{x}$ as well as the estimated means $\boldsymbol{\mu}_t$ you get for `kalman.npy`. We assume the following system. $\mathbf{z} \in \mathbb{R}^4$ with two spatial dimensions and the other two giving the velocities in each of the directions. We assume

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{10}$$

i.e. we observe only the locations. The initial location and velocity we assume to be given as $\boldsymbol{\mu}_0 = [8, 10, 1, 0]$ and the noise covariances are given as $\mathbf{P}_0 = \mathbf{I}_4$, $\Gamma = 0.001 \cdot \mathbf{I}_4$, and $\boldsymbol{\Sigma} = \mathbf{I}_2$, where the subscript indicates the dimensionality of the corresponding identity matrix.

[1] We haven't discussed the case where we want to learn those in the lecture. If you want to know how, have a look e.g. at *Bishop* 13.3.2.

Solution: See `kalman.ipynb`.

2 Ilastik + Multicut for Instance Segmentation (20 pt)

In the lecture, we discussed how we get from raw images to segmented cells. The goal of this exercise is to walk through the whole pipeline from raw images to the final segmentation using the ilastik software.[2]

i) Download and setup Ilastik, as well as the *Helmstaedter* data set, a 3d data set of the mouse retina.[3]

ii) In order to segment the individual cells in the second part of the pipeline we first need a good boundary predictor. Following along with the *Pixel Classification* tutorial on the website,[4] interactively label some cells and cell boundaries to train such a classifier.

iii) Finally, use the multicut workflow,[5] to end up with a final cell segmentation.

For each step document your results via screenshots.

Solution: See `screen01...`,...`screen05...` for an example.

3 Integer Linear Program (10 pt)

The image partition problem as discussed in the lecture comes down to

$$\arg\min_{\mathbf{z}} \underbrace{\mathbf{w}^T\mathbf{z}}_{=:\mathcal{L}(\mathbf{z})} \quad \text{s.t.} \quad \mathbf{z} \text{ gives a valid clustering and } \mathbf{z} \in \{0,1\}^{|E|}. \tag{11}$$

Assuming we are given the graph in Figure 1, with weights $\mathbf{w} = (w_{ab}, w_{ac}, w_{bd}, w_{cd})$ and edge cut indicators $\mathbf{z} = (z_{ab}, z_{ac}, z_{bd}, z_{cd})$, give the loss $\mathcal{L}(\mathbf{z})$ for all combinations of $\mathbf{z}$ and the final result subject to the validity constraints. The weight vector is given as $\mathbf{w} = (0.1, 0.6, 0.9, -1.2)$.

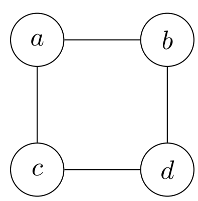

Figure 1: The graph.

Solution: Considering only the integrality constrains, we need to consider 16 cases and get that the optimal solution is $\mathbf{z} = (0, 0, 0, 1)$ with a value of -1.2. However that does not fulfill the constraints, c and d should both be in separate clusters as well as in the same. The optimal solution is then given via $\mathbf{z} = (1, 0, 0, 1)$ with -1.1. See Table 1 for all cases.

[2] www.ilastik.org
[3] See https://www.ilastik.org/download.html for both.
[4] https://www.ilastik.org/documentation/pixelclassification/pixelclassification
[5] Follow this tutorial https://www.ilastik.org/documentation/multicut/multicut

Table 1: Solution for the ILP. The valid column specifies whether we have a solution fulfilling the constraints.

valid?	z_{ab}	z_{ac}	z_{bd}	z_{cd}	$\mathcal{L}$
✓	0	0	0	0	0.0
✗	0	0	0	1	-1.2
✗	0	0	1	0	0.9
✗	0	1	0	0	0.6
✗	1	0	0	0	0.1
✓	0	0	1	1	-0.3
✓	0	1	0	1	-0.6
✓	0	1	1	0	1.5
✓	1	0	0	1	-1.1
✓	1	0	1	0	1.0
✓	1	1	0	0	0.7
✓	0	1	1	1	0.3
✓	1	0	1	1	-0.2
✓	1	1	0	1	-0.5
✓	1	1	1	0	1.6
✓	1	1	1	1	0.4