i) $y_n \sim N(\beta^T x_n, \sigma^2)$.

$$L(\beta^T x, \sigma^2 | x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - \beta x_i)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot e^{-\frac{\sum_{i=1}^{n}(y_i - \beta x_i)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot e^{-\frac{(Y-\beta^T X)^T (Y-\beta^T X)}{2\sigma^2}}$$

get natural logarithms in both sides. we get

$$\ln(L(\beta^T x, \sigma^2 | x)) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{(Y-\beta^T X)^T (Y-\beta X)}{2\sigma^2}$$

get partial differentiate of the above equation from β.

$$\frac{\partial \ln(L(\beta^T x, \sigma^2 | X))}{\partial \beta} = 0 \Rightarrow \frac{1}{2\sigma^2}(0 - 2X^T Y + 2X^T \beta X) = 0.$$

$$\beta = \frac{X^T Y}{X^T X}.$$

$$\beta = (X^T X)^{-1} \cdot X^T Y.$$

ii). $\sigma^2 = \arg\max_{\sigma^2} \sum_{i=1}^{N} \log N(y_n | \beta^T x, \sigma^2)$.

solve this is again to get differentiation of the equation and set it to zero.

$$\frac{\partial}{\partial \sigma^2}\left[-\frac{N}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(\beta^T x - y)^T(\beta^T x - y)\right] = 0.$$

$$\sigma^2 = \frac{1}{N}(\beta^T x - y)^T(\beta^T x - y)$$