Assignment 2

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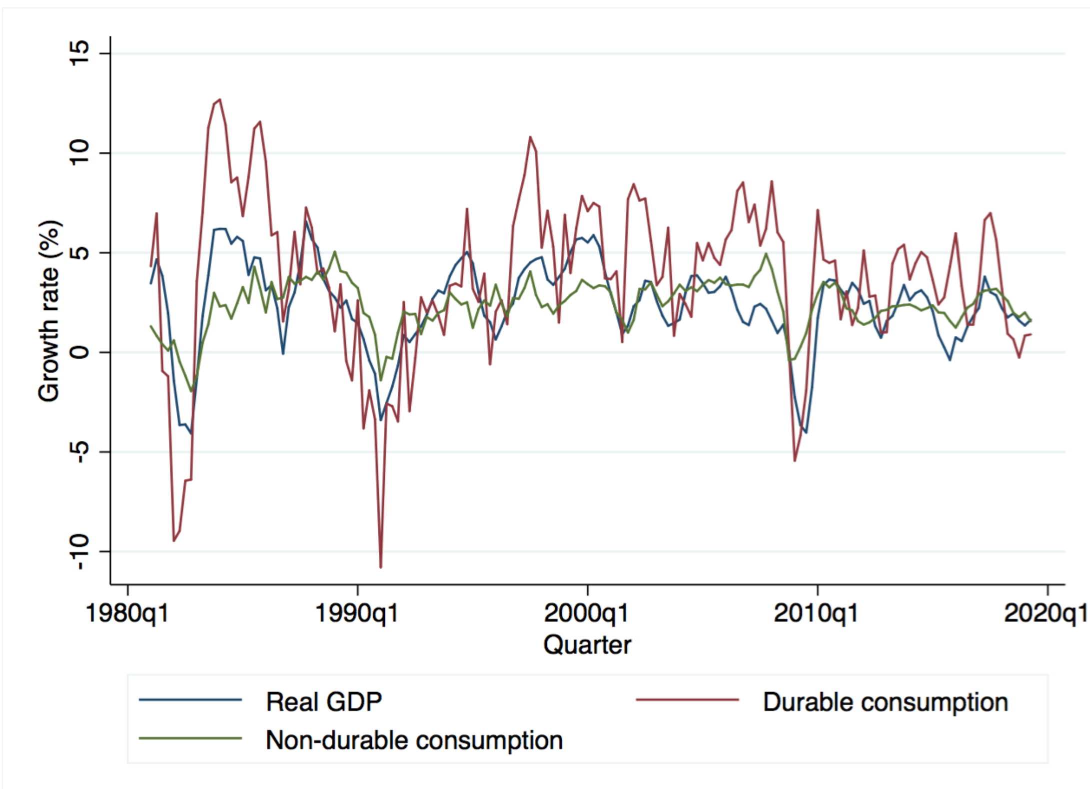
1

(a)

Figure 1: Y-O-Y Growth Rate of Real GDP and Consumption in Canada, 1981q1-2019q2



Notes: The quarterly data of the year-over-year growth rate from 1981q1 to 2019q2. All values are seasonally adjusted, in constant 2012 Canadian dollars. Source: Statistics Canada,Table 36-10-0104-01.

(b)

Table 1: Real GDP and Consumption ACF in Canada 1981Q1-2019Q2

	Correlation					
Lags	0	1	2	3	4	5
Real GDP	1.00	0.88*	0.66*	0.39*	0.14	-0.01
Durables Consumption	1.00	0.84*	0.67*	0.47*	0.25*	0.17
Non-Durables Consumption	1.00	0.76*	0.59*	0.40*	0.12	0.11

Notes: Auto-correlation table for year-over-year growth rates using seasonally adjusted real quarterly data from 1981Q1 to 2019Q2. * indicates significance at the 0.01 percent level. Source: Statistics Canada table 36-10-0104-01.

It is difficult to assess the smoothness of time series individually through inspection. Instead, a mathematical (but still arbitrary) approach can be taking by assessing the half-life of a series' auto-correlation. Smoothness can be relatively assessed by the count of lags before the auto-correlation drops below 0.5; the higher the count, the smoother the series.

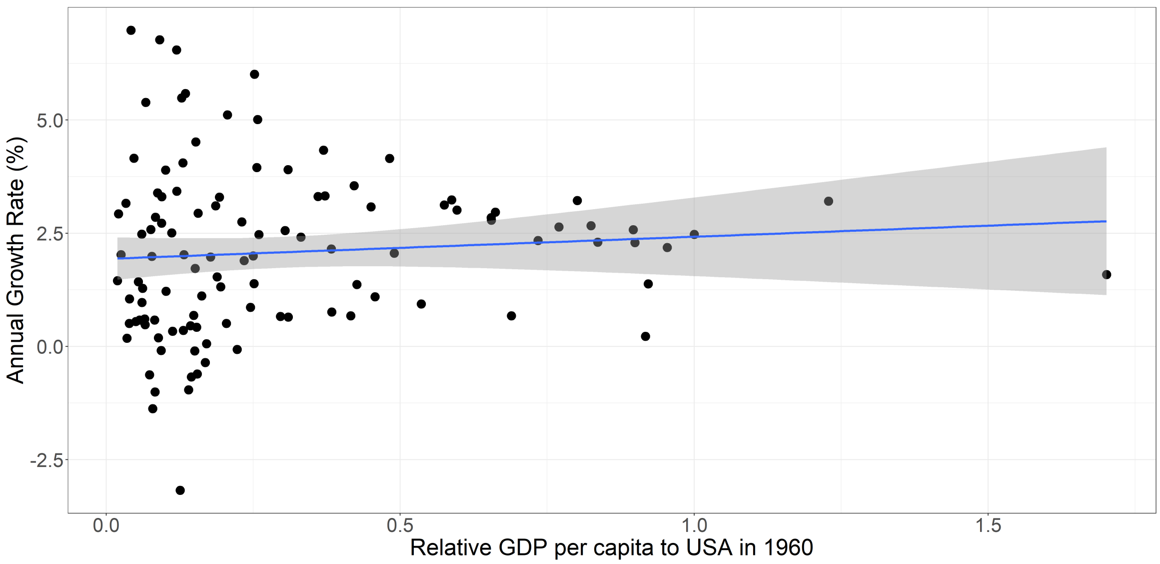
Referring to Table 1's auto-correlation functions of Real GDP, durable consumption and non-durable consumption, all three series exhibit the same half-life, reaching an auto-correlation below 0.5 at a lag of 3. However, the auto-correlation of durable consumption and real GDP at lag 2 are at 0.67 and 0.66 respectively, while the value of non-durable consumption is at 0.59. Then we may say that durable consumption is less smooth than real GDP and non-durable consumption.

Another observation can be made in the fourth lag: Non-Durable consumption correlation remains significant, while real GDP loses its correlation significance. Since correlation is declining slower on average in the non-durable consumption, it can be inferred that non-durable consumption is smoother than real GDP.

2

(a)

Figure 2: Comparison of Output Growth 1960-2000 and Output Level in 1960 across Countries

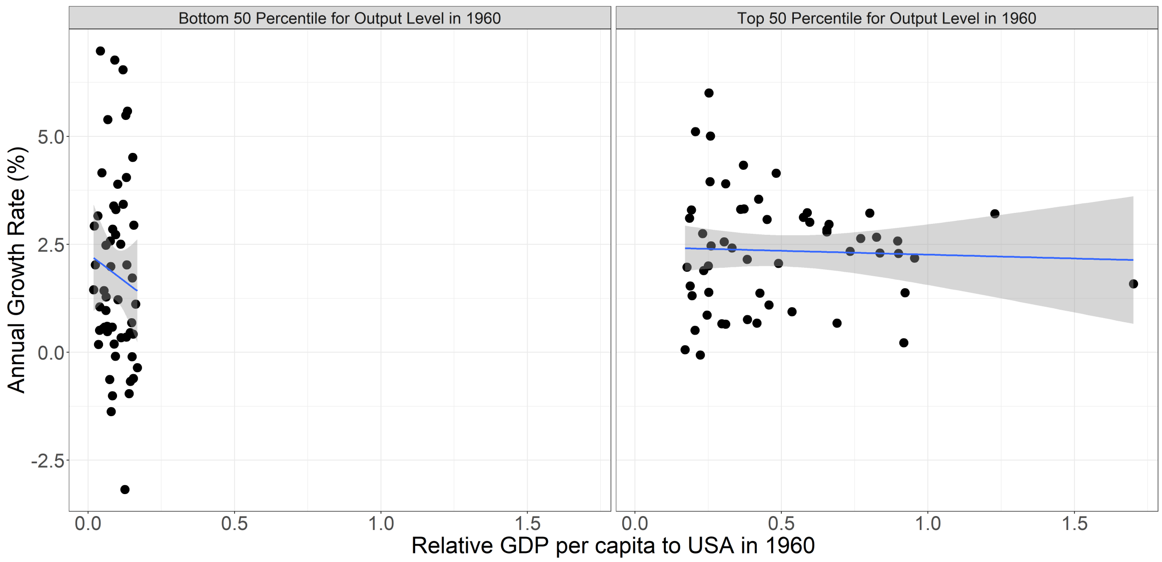


Source: Adapted from Penn World Table 9.1's real GDP using national account growth rates from 1960 - 2000.

There is no significant correlation as assessed by a Pearson's correlation test.

(b)

Figure 3: A Comparison of Output Growth 1960-2000 and Output Level in 1960 across Countries

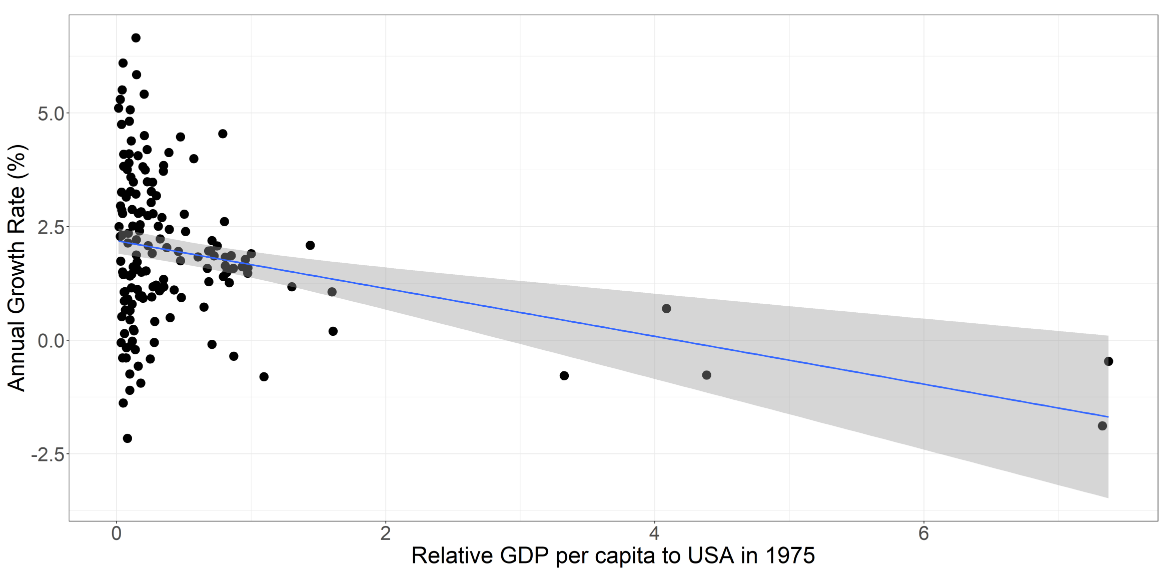


Source: Adapted from Penn World Table 9.1's real GDP using national account growth rates from 1960 - 2000. Data is separated according to top and bottom 50 output percentile relative to USA in 1960.

There is no significant correlation as assessed by a Pearson's correlation test on both groups.

(c)

Figure 4: A Comparison of Output Growth 1975-2015 and Output Level in 1975

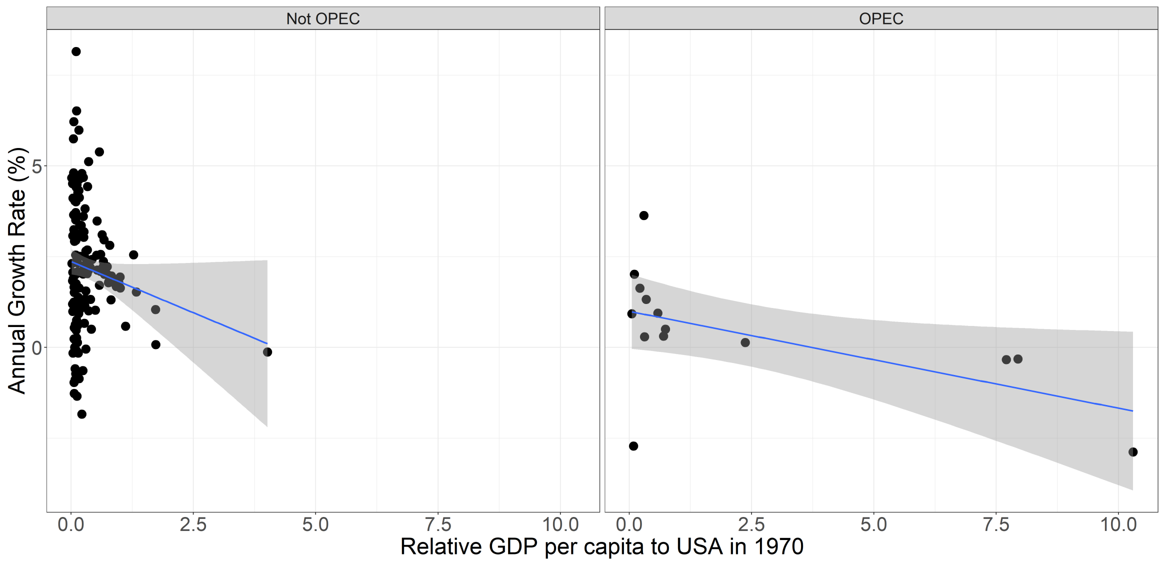


Source: Adapted from Penn World Table 9.1's real GDP using national account growth rates from 1975 - 2015.

A significant correlation of -0.31 can be assessed from a Pearson's correlation test at a 5% level of significance.

(d)

Figure 5: OPEC Countries: A Comparison of Output Growth 1970-2010 and Output Level in 1970



Source: Adapted from Penn World Table 9.1's real GDP using national account growth rates from 1970 - 2010. Countries are split according to The Organization of the Petroleum Exporting Countries (OPEC). The 15 members of OPEC are classified according to Jodidata, exempting the lack of data on Libya.

From the OPEC countries, A significant correlation of -0.56 can be assessed from a Pearson's correlation test at a 5% level of significance.

From the Non-OPEC countries, a significant correlation cannot be found at the 5% level of significance with a correlation of -0.15 and a p-value of 0.075 using a Pearson's correlation test.

3

Given:

$$Y_t = zK_t^{\alpha}N_t^{1-\alpha} \tag{3.1}$$

$s = \frac{S_t}{Y_t}$
$K_{t+1} = (1-\delta)K_t + I_t$

(a)

$$\begin{aligned}
Y_t &= zK_t^\alpha N_t^{1-\alpha} \\
\frac{Y_t}{N_t} &= \frac{zK_t^\alpha N_t^{1-\alpha}}{N_t} \\
y_t &= zK_t^\alpha N_t^{-\alpha} && y_t \text{ is the output per worker (per capital) and and } k_t = \frac{K_t}{N_t} \\
y_t &= zk_t^\alpha && (3.2)
\end{aligned}$$

(b)

Note that in the closed economy solow model without government.

$$Y_t = C_t + I_t \tag{3.3}$$

However, savings $S_t = Y_t - C_t$ so in this model, $S_t = I_t$ and $s = \frac{I_t}{Y_t}$.

Then

$$sY_t = I_t \tag{3.4}$$

subbing 3.1 to 3.4, we have

$$I_t = szK_t^\alpha N_t^{1-\alpha}$$

so we have

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + szK_t^\alpha N_t^{1-\alpha} \\
\frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}} &= \frac{(1-\delta)K_t}{N_t} + \frac{szK_t^\alpha N_t^{1-\alpha}}{N_t} \\
k_{t+1}(1+n) &= (1-\delta)k_t + szk_t^\alpha && \text{result from 3.2} \\
k_{t+1} &= \frac{(1-\delta)k_t + szk_t^\alpha}{1+n} && (3.5)
\end{aligned}$$

(c)

In steady state, variables are constant. Using equation 3.5, we have

$$
\begin{aligned}
k &= \frac{(1-\delta)k + szk^{\alpha}}{1+n} \\
k[(1+n)-(1-\delta)] &= szk^{\alpha} \\
k &= \frac{szk^{\alpha}}{n+\delta} \\
k^{1-\alpha} &= \frac{sz}{n+\delta} \\
k &= \left(\frac{sz}{n+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

From 3.3 we have $C_t = Y_t - I_t$, then immediately following 3.4 we have

$$
\begin{aligned}
C_t &= (1-s)Y_t \\
&= (1-s)zK_t^{\alpha}N_t^{1-\alpha}
\end{aligned}
$$

In steady state for consumption,

$$
\begin{aligned}
C &= (1-s)zK^{\alpha}N^{1-\alpha} && \text{divide by N} \\
c &= (1-s)zk^{\alpha} \\
&= (1-s)z\left(\frac{sz}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} && \text{from 3.6}
\end{aligned}
$$

From 3.2, we have i the steady state

$$
\begin{aligned}
y &= zk^{\alpha} && \text{Substitute 3.6} \\
&= z\left(\frac{sz}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} && (3.7)
\end{aligned}
$$

(d)

From equation 3.7, we can see that n has an inverse relationship to the steady state of y: an increase in n, results in a decrease for the steady state value of y. This behavior is shown in the derivative below, which has a negative value.

$$
\frac{\partial y}{\partial n} = \frac{\alpha}{(\alpha-1)s}\left(\frac{sz}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}+1} \qquad \alpha < 1
$$

(e)

From 3.5 steady state, $szk^{\alpha} = (n+\delta)k$

Maximize consumption in steady state:

$$\begin{aligned}
\max_k \{(1-s)zk^{\alpha}\} \\
\frac{\partial}{\partial k} &= \alpha z k^{\alpha-1} - (n+\delta) && \text{set to 0} \\
\frac{n+\delta}{\alpha z} &= k^{\alpha-1} \\
\left(\frac{n+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}} &= k*
\end{aligned}$$

$$\frac{\partial^2}{\partial k^2} = \alpha(\alpha-1)zk^{\alpha-2}$$

SOC is less or equal than 0 if $0 < \alpha < 1$

Therefore the golden rule of consumption is at $c* = (1-s)zk*$

(f)

From 3.5,

$$\begin{aligned}
k_{t+1} &= \frac{(1-\delta)k_t + szk_t^{\alpha}}{1+n} \\
\frac{k_{t+1}-k_t}{k_t} &= \frac{(1-\delta)k_t + szk_t^{\alpha}}{k_t(1+n)} - 1 \\
\frac{\Delta k_{t+1}}{k_t} &= \frac{1-\delta + szk_t^{\alpha-1}}{1+n} - 1 \\
&= \frac{s(1+\mu)^t k_t^{\alpha-1} - \delta - n}{1+n}
\end{aligned}$$

(g)

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
\Delta K_{t+1} &= I_t - \delta K_t \\
\frac{\Delta K_{t+1}}{K_t} &= \frac{sY_t}{K_t} - \delta \\
&= \frac{sy_t}{k_t} - \delta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta k_{t+1}}{k_t} &\approx \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta N_{t+1}}{N_t} \\
\frac{\Delta k_{t+1}}{k_t} + n &\approx \frac{sy_t}{k_t} - \delta \\
&= \frac{sy_t}{k_t} - \delta \\
\frac{\Delta k_{t+1}}{k_t} &= \frac{sy_t}{k_t} - (\delta + n) = \gamma
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta c_{t+1}}{c_t} &= \frac{(1-s)y_{t+1} - (1-s)y_t}{(1-s)y_t} \\
&= \frac{\Delta y_{t+1}}{y_t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta y_{t+1}}{y_t} &\approx \log y_{t+1} - \log y_t \\
&= (t+1)\log(1+\mu) + \alpha \log k_{t+1} - t\log(1+\mu) + \alpha \log k_t \\
&= \log(t+1) + \alpha(\log k_{t+1} - \log k_t) \\
&\approx \mu + \alpha\gamma
\end{aligned}
$$

If $\gamma = \frac{\mu}{1-\alpha}$ then we have a balanced growth path where

$$
\frac{\Delta y_{t+1}}{y_t} = \frac{\Delta c_{t+1}}{c_t} = \frac{\Delta k_{t+1}}{k_t}
$$

4

(a)

$$
\begin{aligned}
Y_t &= zK_t \\
\frac{Y_t}{N_t} &= \frac{zK_t}{N_t} \\
y_t &= zk_t
\end{aligned}
$$

(b)

By the same derivations as 3.4, $sY_t = I_t$ an $I_t = szK_t$

so we have

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + szK_t \\ \frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}} &= \frac{(1-\delta)K_t + zK_t}{N_t} \\ k_{t+1}(1+n) &= (1-\delta)k_t + szk_t \\ k_{t+1} &= \frac{(1-\delta)k_t + szk_t}{1+n} \end{aligned} \tag{4.1}$$

(c)

$$\begin{aligned} \frac{\Delta Y_{t+1}}{Y_t} &= \frac{zK_{t+1} - zK_t}{zK_t} \\ &= \frac{\Delta K_{t+1}}{K_t} \\ \frac{\Delta y_{t+1}}{y_t} &= \frac{\Delta k_{t+1}}{k_t} \end{aligned}$$

$$\begin{aligned} \frac{\Delta c_{t+1}}{c_t} &= \frac{(1-s)y_{t+1} - (1-s)y_t}{(1-s)y_t} \\ &= \frac{\Delta y_{t+1}}{y_t} \end{aligned}$$

Since we have $\frac{\Delta y_{t+1}}{y_t} = \frac{\Delta c_{t+1}}{c_t} = \frac{\Delta k_{t+1}}{k_t}$, the economy is in a balanced growth path.

(d)

From equation 4.1, we have

$$\begin{aligned} k_{t+1} &= \left(\frac{1-\delta+sz}{1+n}\right)k_t \\ \frac{k_{t+1}}{k_t} &= \frac{1-\delta+sz}{1+n} \end{aligned}$$

$$\begin{aligned}\frac{y_{t+1}}{y_t} &= \frac{zk_{t+1}}{zk_t} \\ &= \frac{k_{t+1}}{k_t} \\ &= \frac{1-\delta+sz}{1+n}\end{aligned}$$

$$\begin{aligned}G_{Ct} &= \frac{c_{t+1}}{c_t} \\ &= \frac{(1-s)y_{t+1}}{(1-s)y_t} \\ &= \frac{y_{t+1}}{y_t} \\ &= \frac{1-\delta+sz}{1+n}\end{aligned}$$

Now we can derive the 3 expressions of how consumption per worker grows for an increase in s, δ, or z

$$\frac{\partial G_{Ct}}{\partial s} = \frac{z}{1+n}$$

$$\frac{\partial G_{Ct}}{\partial z} = \frac{s}{1+n}$$

$$\frac{\partial G_{Ct}}{\partial \delta} = \frac{-1}{1+n}$$

5

(a)

$$\begin{aligned}Y_t &= C_t + I_t + G_t \\ Y_t - C_t - G_t &= I_t \\ Y_t - (1-s)(Y_t - G_t) - G_t &= I_t \\ Y_t - Y_t + G_t + sY_t - sG_t - G_t &= I_t \\ sY_t - sG_t &= I_t\end{aligned}$$

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + sz_tK_t^{\alpha}N_t^{1-\alpha} - sgN_t \\
\frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}} &= \frac{(1-\delta)K_t + sz_tK_t^{\alpha}N_t^{1-\alpha} - sgN_t}{N_t} \\
k_{t+1} &= \frac{(1-\delta)k_t + szk_t^{\alpha} - sg}{1+n}
\end{aligned}
$$

$$
\begin{aligned}
C_t &= (1-s)(Y_t - T_t) \\
C_t &= (1-s)(z_tK_t^{\alpha}N^{1-\alpha} - gN_t) \\
c_t &= (1-s)(z_tk_t^{\alpha} - g)
\end{aligned}
$$

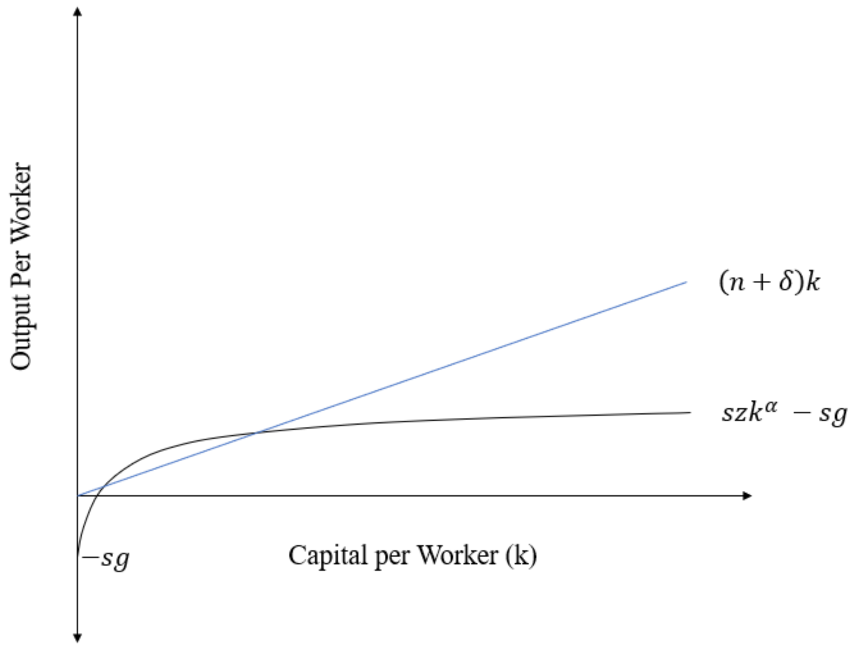
$$
\begin{aligned}
Y_t &= z_tK_t^{\alpha}N^{1-\alpha} \\
y_t &= z_tk_t^{\alpha}
\end{aligned}
$$

(b) and (c)

In steady state, we have

$$
\begin{aligned}
k &= \frac{(1-\delta)k - szk^{\alpha} - sg}{1+n} \\
k(n+\delta) &= szk^{\alpha} - sg
\end{aligned}
$$

Figure 6: Steady State for Solow Model with Government



Graph to model the steady state for a Cobb Douglas Solow Model with a government, $k(n+\delta) = szk^{\alpha} - sg$

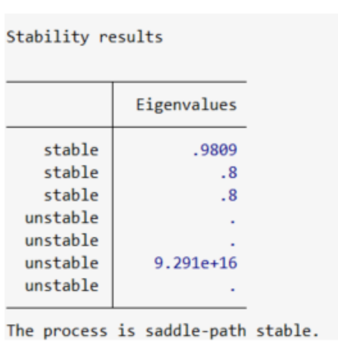
As depicted on the graph, we have in the steady state $k(n+\delta) = szk^{\alpha} - sg$. Note that $szk^{\alpha} - sg$ has a positive first order derivative and negative second order derivative, while $(n+\delta)k$ has a constant rate of change. Then it is possible for the right hand side to have a higher rate of change than $(n+\delta)$, enough for $szk^{\alpha} - sg > k(n+\delta)$, and eventually the rate of change will diminish so that $szk^{\alpha} - sg < k(n+\delta)$ causing the two equations to intersect twice at two different values of k.

(e)

Plugging in the values and z = 1 as an exogenous variable, we have two results: k = 0.0532 and k = 0.5798

(f)

Figure 7: Solow Model Stability Test



Stability results

	Eigenvalues
stable	.9809
stable	.8
stable	.8
unstable	.
unstable	.
unstable	9.291e+16
unstable	.

The process is saddle-path stable.

According to the stability results the process is saddle-path stable.

(g)

Figure 8: Solow Model Steady State Values

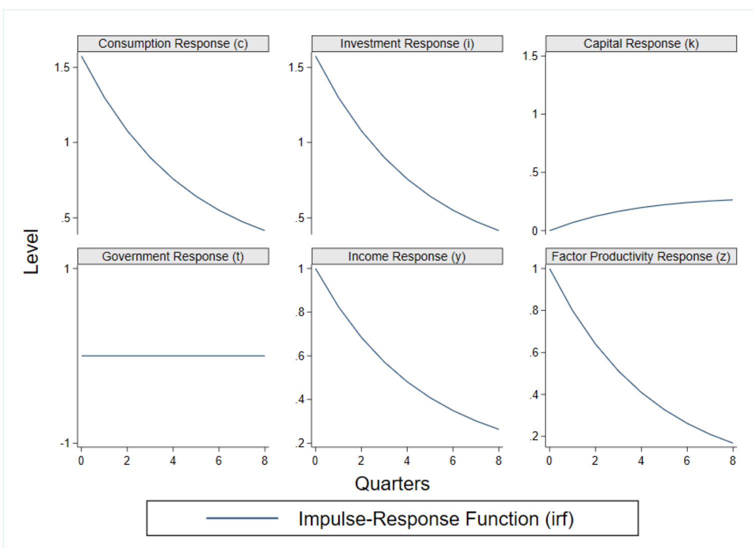
Location of model steady-state

	Coef.
k	0.58
z	1.00
e	1.00
c	0.50
t	0.30
y	0.82
i	0.03

Note: In our model, we used t as the variable name for $g_t = g * e_t$. After implementing the shocks, and the new variable t, the value of k remained the same in the previous steady state.

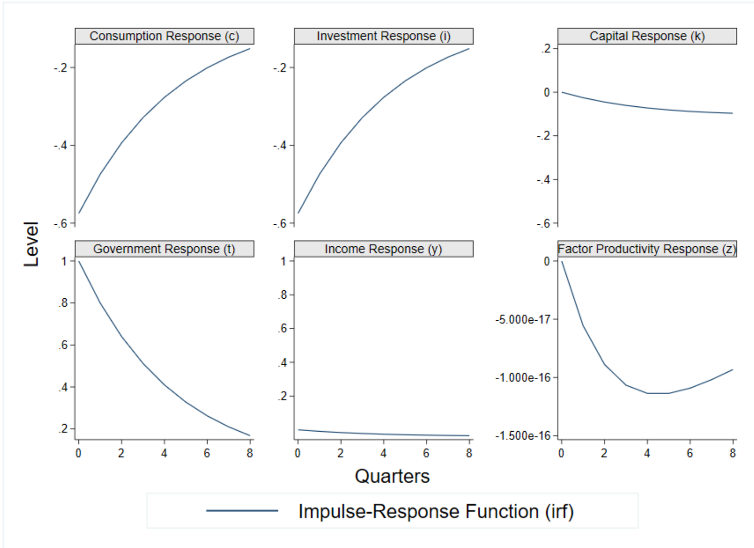
(h)

Figure 9: ϵ_t shock's Impulse-Response Function



IRFs modeled after a closed economy Solow Model with a government from a Cobb-Douglas Production Function and introducing shocks: $z_{t+1} = z_t^{\rho_z}\epsilon_t$, and $e_{t+1} = e_t^{\rho_g}$. ϵ_t is the impulse.

Figure 10: e_t shock's Impulse-Response Function



IRFs modeled after a closed economy Solow Model with a government from a Cobb-Douglas Production Function and introducing shocks: $z_{t+1} = z_t^{\rho_z} \epsilon_t$, and $e_{t+1} = e_t^{\rho_g}$. e_t is the impulse.

From ϵ_t impulse to e_t impulse:

Consumption and investment responses changed from negative slopes to positive (negative to positive shocks). Capital response changed from positive to negative slope (positive to negative shock). Government changed from constant to negative slope (no shock to negative shock). Income and factor productivity changed from declining to near-zero values (negative to near-zero shocks).

6

(a)

$$Y_t = C_t + I_t$$
$$I_t = sY_t$$
$$K_{t+1} = K_t + I_t$$
$$K_{t+1} - Kt = I_t = sY_t$$
$$\frac{K_{t+1} - K_t}{K_t} = \frac{sY_t}{K_t}$$

The growth rate of K_{t+1}:

$$\begin{aligned}
\frac{K_{t+1}-K_t}{K_t} &= \frac{sY_t}{K_t} \\
g_{K_t} &= \frac{s[(1-a_K)K_t]^{\alpha}[A_t(1-a_N)N_t]^{1-\alpha}}{K_t} \\
&= s(1-a_K)^{\alpha}{K_t}^{\alpha-1}[A_t(1-a_N)N_t]^{1-\alpha} \\
&= s(1-a_K)^{\alpha}(1-a_N)^{1-\alpha}(\frac{A_tN_t}{K_t})^{1-\alpha} \\
&= c_K[\frac{A_tN_t}{K_t}]^{1-\alpha}
\end{aligned}$$

where c_K=$s(1-a_K)^{\alpha}(1-a_N)^{1-\alpha}$

The growth rate of A_{t+1}:

$$\begin{aligned}
\frac{A_{t+1}-A_t}{A_t} &= \frac{A_{t+1}}{A_t}-1 \\
g_{A_t} &= \frac{B[a_KK_t]^{\beta}[a_NN_t]^{\gamma}}{A_t^{1-\theta}}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\Delta g_{K_t}}{g_{K_t}} &\approx ln(\frac{g_{K_{t+1}}}{g_{K_t}}) \\
g_{K_{t+1}} &= c_K[\frac{A_{t+1}N_{t+1}}{K_{t+1}}]^{1-\alpha} \\
g_{K_t} &= c_K[\frac{A_tN_t}{K_t}]^{1-\alpha} \\
\frac{g_{K_{t+1}}}{g_{K_t}} &= [\frac{A_{t+1}}{A_t}\frac{N_{t+1}}{N_t}\frac{K_t}{K_{t+1}}]^{1-\alpha} \\
\frac{g_{K_{t+1}}}{g_{K_t}} &= (\frac{A_{t+1}}{A_t})^{1-\alpha}(\frac{N_{t+1}}{N_t})^{1-\alpha}(\frac{K_{t+1}}{K_t})^{-(1-\alpha)} \\
ln(\frac{g_{K_{t+1}}}{g_{K_t}}) &= (1-\alpha)ln(\frac{A_{t+1}}{A_t})+(1-\alpha)ln(\frac{N_{t+1}}{N_t})-(1-\alpha)ln(\frac{K_{t+1}}{K_t}) \\
\frac{\Delta g_{K_t}}{g_{K_t}} &= (1-\alpha)[g_{A_t}+g_{N_t}-g_{K_t}] \\
\frac{\Delta g_{K_t}}{g_{K_t}} &= (1-\alpha)[g_{A_t}+n-g_{K_t}]
\end{aligned}$$

The net growth rate is equal to the logarithm of the gross growth rate,
then we get:

$$\frac{\Delta g_{A_t}}{g_{A_t}} \approx ln(\frac{g_{A_{t+1}}}{g_{A_t}})$$

$$g_{A_{t+1}} = c_A {K_{t+1}}^{\beta} N_{t+1}^{\gamma} A_{t+1}^{\theta-1} and g_{A_t} = c_A K_t^{\beta} N_t^{\gamma} A_t^{\theta-1}$$

$$\begin{aligned}
\frac{g_{A_{t+1}}}{g_{A_t}} &= (\frac{K_{t+1}}{K_t})^{\beta}(\frac{N_{t+1}}{N_t})^{\gamma}(\frac{A_{t+1}}{A_t})^{\theta-1} \\
ln(\frac{g_{A_{t+1}}}{g_{A_t}}) &= \beta ln(\frac{K_{t+1}}{K_t}) + \gamma ln(\frac{N_{t+1}}{N_t}) + (\theta-1)ln(\frac{A_{t+1}}{A_t}) \\
\frac{\Delta g_{A_t}}{g_{A_t}} &= \beta g_{K_t} + \gamma g_{N_t} + (\theta-1) g_{A_t} \\
\frac{\Delta g_{A_t}}{g_{A_t}} &= \beta g_{K_t} + \gamma n - (1-\theta) g_{A_t}
\end{aligned}$$

(c)

$$\begin{aligned}
K_{t+1} &= K_t + sY_t \\
\frac{K_{t+1}N_{t+1}}{N_{t+1}N_t} &= k_t + s\frac{Y_t}{N_t} \\
k_{t+1}(1+n) &= k_t + s\frac{Y_t}{N_t} \\
(k_{t+1}-k_t)(1+n) &= s\frac{Y_t}{N_t} \\
\left(\frac{k_{t+1}-k_t}{k_t}\right)(1+n) &= \frac{s\frac{Y_t}{N_t}}{k_t} \\
&= \frac{s[(1-a_K)k_t]^{\alpha}[A_t(1-a_N)N_t]^{1-\alpha}}{k_t} N_t^{\alpha-1} \\
&= s(1-a_K)^{\alpha} {k_t}^{\alpha-1}[A_t(1-a_N)]^{1-\alpha} \\
&= s(1-a_K)^{\alpha}(1-a_N)^{1-\alpha}\left(\frac{A_t}{k_t}\right)^{1-\alpha} \\
\frac{k_{t+1}-k_t}{k_t} &= \frac{c_K[\frac{A_t}{k_t}]^{1-\alpha}}{1+n}
\end{aligned}$$

where c_K=$s(1-a_K)^{\alpha}(1-a_N)^{1-\alpha}$

As savings (s) is in the numerator, a change in savings affects the growth rate according to its first order derivative $\frac{\partial g_k}{\partial s} = (1-a_K)^{\alpha}(1-a_N)^{1-\alpha}\left(\frac{A_t}{k_t}\right)^{1-\alpha}$. In the example of the United States where savings declined from 9% to 3%, growth rate would similarly decline by $(0.06)(1-a_K)^{\alpha}(1-a_N)^{1-\alpha}\left(\frac{A_t}{k_t}\right)^{1-\alpha}$.

7

(a)

$$
\begin{aligned}
y_1 &= c_t + s \\
c_{t+1} &= y_2 + s(1+r) \\
s &= y_1 - c_t \\
s &= \frac{c_{t+1} - y_2}{1+r} \\
\frac{c_{t+1} - y_2}{1+r} &= y_1 - c_t \\
\frac{c_{t+1}}{1+r} + c_t &= y_1 + \frac{y_2}{1+r} = we \\
we &= y_1 + \frac{y_2}{1+r} \text{is total household wealth}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\max_{c_t, c_{t+1}} U(c_t) + \beta U(c_{t+2}) \\
&\text{s.t. } \frac{c_{t+1}}{1+r} + c_t = y_1 + \frac{y_2}{1+r}
\end{aligned}
$$

$$
\begin{aligned}
L(c_t, c_{t+1}, \lambda) &= U(c_t) + \beta U(c_{t+2}) - \lambda[\frac{c_{t+1}}{1+r} + c_t - y_1 - \frac{y_2}{1+r}] \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\partial c_t} &= U'_{c_t} - \lambda = 0 \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\partial c_{t+1}} &= \beta U'_{c_{t+1}} - \frac{\lambda}{1+r} = 0 \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\lambda} &= \frac{c_{t+1}}{1+r} + c_t - y_1 - \frac{y_2}{1+r} = 0 \\
MRS_{c_t, c_{t+1}} &= \frac{U'_{c_t}}{U'_{c_{t+1}}} = \beta(1+r)
\end{aligned}
$$

(c)

$$\text{According to,} U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma < 1.$$
$$U'_{c_t} = c_t^{-\gamma}$$
$$U'_{c_{t+1}} = c_{t+1}^{-\gamma}$$
$$\frac{U'_{c_t}}{U'_{c_{t+1}}} = \beta(1+r) = \frac{c_t^{-\gamma}}{c_{t+1}^{-\gamma}}$$
$$c_{t+1} = c_t \sqrt[\gamma]{\beta(1+r)}$$

let's substitute it back to (b), we can get:

$$\frac{c_t \sqrt[\gamma]{\beta(1+r)}}{1+r} + c_t = y_1 + \frac{y_2}{1+r}$$
$$c_t \sqrt[\gamma]{\beta(1+r)} + (1+r)c_t = (1+r)y_1 + y_2$$
$$c_t = \frac{(1+r)y_1 + y_2}{\sqrt[\gamma]{\beta(1+r)} + (1+r)}$$
$$c_{t+1} = \frac{[(1+r)y_1 + y_2] \sqrt[\gamma]{\beta(1+r)}}{\sqrt[\gamma]{\beta(1+r)} + (1+r)}$$
$$s = y_1 - \frac{(1+r)y_1 + y_2}{\sqrt[\gamma]{\beta(1+r)} + (1+r)}$$

(d)

$$\frac{\partial c_t}{\partial y_2} = \frac{1}{\sqrt[\gamma]{\beta(1+r)} + (1+r)} > 0$$
$$\frac{\partial c_{t+1}}{\partial y_2} = \frac{\sqrt[\gamma]{\beta(1+r)}}{\sqrt[\gamma]{\beta(1+r)} + (1+r)} > 0$$
$$\frac{\partial s}{\partial y_2} = -\frac{1}{\sqrt[\gamma]{\beta(1+r)} + (1+r)} < 0$$

That means:

$$y_2 \uparrow \Rightarrow c_t \uparrow \text{and} c_{t+1} \uparrow$$
$$y_2 \uparrow \Rightarrow s \downarrow$$

When increasing second period income,it leads to increase in the first and second period consumption by decrease the first period saving

8

(a)

$$
\begin{aligned}
y_1 &= c_t + s \\
c_{t+1} &= s(1+r) \\
s &= y_1 - c_t \\
s &= \frac{c_{t+1}}{1+r} \\
\frac{c_{t+1}}{1+r} &= y_1 - c_t \\
\frac{c_{t+1}}{1+r} + c_t &= y_1 = we \\
\max_{c_t, c_{t+1}} U(c_t) + \beta U(c_{t+2}) &\text{subject to} \frac{c_{t+1}}{1+r} + c_t = y_1 \\
L(c_t, c_{t+1}, \lambda) &= U(c_t) + \beta U(c_{t+2}) - \lambda[\frac{c_{t+1}}{1+r} + c_t - y_1] \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\partial c_t} &= U'_{c_t} - \lambda = 0 \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\partial c_{t+1}} &= \beta U'_{c_{t+1}} - \frac{\lambda}{1+r} = 0 \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\lambda} &= \frac{c_{t+1}}{1+r} + c_t - y_1 = 0 \\
MRS_{c_t, c_{t+1}} &= \frac{U'_{c_t}}{U'_{c_{t+1}}} = \beta(1+r)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\text{According to}, U(c) &= ln(c), \\
U'_{c_t} &= c_t^{-1} \\
U'_{c_{t+1}} &= c_{t+1}^{-1} \\
\frac{U'_{c_t}}{U'_{c_{t+1}}} &= \beta(1+r) = \frac{c_t^{-1}}{c_{t+1}^{-1}} \\
c_{t+1} &= c_t\beta(1+r)
\end{aligned}
$$

let's substitute it back to (b), we can get:

$$
\begin{aligned}
\frac{c_t\beta(1+r)}{1+r} + c_t &= y_1 \\
c_t\beta + c_t &= y_1 \\
c_t &= \frac{y_1}{(1+\beta)} \\
c_{t+1} &= \frac{\beta y_1(1+r)}{(1+\beta)} \\
s &= y_1 - \frac{y_1}{(1+\beta)} \\
&= \frac{\beta y_1}{(1+\beta)}
\end{aligned}
$$

(c)

$$\begin{aligned}
y_1 - \tau &= c_t + s \\
c_{t+1} &= \tau(1+r) + s(1+r) \\
s &= y_1 - \tau - c_t \\
s &= \frac{c_{t+1}}{1+r} - \tau \\
\frac{c_{t+1}}{1+r} - \tau &= y_1 - \tau - c_t \\
\frac{c_{t+1}}{1+r} + c_t &= y_1 = we \\
\max_{c_t, c_{t+1}} U(c_t) + \beta U(c_{t+2}) &\text{subject to} \frac{c_{t+1}}{1+r} + c_t = y_1 \\
L(c_t, c_{t+1}, \lambda) &= U(c_t) + \beta U(c_{t+2}) - \lambda[\frac{c_{t+1}}{1+r} + c_t - y_1] \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\partial c_t} &= U'_{c_t} - \lambda = 0 \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\partial c_{t+1}} &= \beta U'_{c_{t+1}} - \frac{\lambda}{1+r} = 0 \\
\frac{\partial L(c_t, c_{t+1}, \lambda)}{\lambda} &= \frac{c_{t+1}}{1+r} + c_t - y_1 = 0 \\
MRS_{c_t, c_{t+1}} &= \frac{U'_{c_t}}{U'_{c_{t+1}}} = \beta(1+r)
\end{aligned}$$

(d)

$$
\begin{aligned}
\text{According to, } U(c) &= ln(c),\\
U'_{c_t} &= c_t^{-1}\\
U'_{c_{t+1}} &= c_{t+1}^{-1}\\
\frac{U'_{c_t}}{U'_{c_{t+1}}} &= \beta(1+r) = \frac{c_t^{-1}}{c_{t+1}^{-1}}\\
c_{t+1} &= c_t\beta(1+r)
\end{aligned}
$$

let's substitute it back to (c), we can get:

$$
\begin{aligned}
\frac{c_t\beta(1+r)}{1+r} + c_t &= y_1\\
c_t\beta + c_t &= y_1\\
c_t &= \frac{y_1}{(1+\beta)}\\
c_{t+1} &= \frac{\beta y_1(1+r)}{(1+\beta)}\\
s &= y_1 - \tau - \frac{y_1}{(1+\beta)}\\
&= \frac{\beta y_1}{(1+\beta)} - \tau
\end{aligned}
$$

9

Iso-elastic utility: $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$

Then $U'(c) = c^{-\gamma}$ and $U''(c) = -\gamma c^{-(1+\gamma)}$

$$
\begin{aligned}
R_r(c) &= -c\frac{U''(c)}{U'(c)}\\
&= -c\frac{-\gamma c^{-(1+\gamma)}}{c^{-\gamma}}\\
&= \gamma
\end{aligned}
$$

Therefore, iso-elastic utility is CCRA.

Log utility: $U(c) = \ln(c)$

Then $U'(c) = c^{-1}$ and $U''(c) = -c^{-2}$

$$
\begin{aligned}
R_r(c) &= -c\frac{-c^{-2}}{c^{-1}} \\
&= \frac{c^{-1}}{c^{-1}} \\
&= 1
\end{aligned}
$$

Therefore, log utility is CCRA.

Linear quadratic utility: $U(c) = c - \frac{a}{2}c^2, a > 0$

Then $U'(c) = 1 - ac$ and $U''(c) = -a$

$$
\begin{aligned}
R_r(c) &= -c\frac{-a}{1-ac} \\
&= \frac{ac}{1-ac}
\end{aligned}
$$

IRRA if 0<ac<1, DRRA if ac >1

Exponential utility: $U(c) = 1 - e^{ac}, \alpha > 0$

Then $U'(c) = \alpha e^{-\alpha c}$ and $U''(c) = -\alpha^2 e^{-\alpha c}$

$$
\begin{aligned}
R_r(c) &= -c\frac{-\alpha^2 e^{-\alpha c}}{\alpha e^{-\alpha e}} \\
&= ca
\end{aligned}
$$

Since c,a > 0, we have IRRA.

Power utility: $U(c) = \frac{c^{\rho}-1}{v}, v > 0$

Then $U'(c) = \frac{\rho c^{\rho-1}}{v}$ and $U''(c) = \frac{\rho(\rho-1)c^{\rho-2}}{v}$

$$
\begin{aligned}
R_r(c) &= -c\frac{\frac{\rho(\rho-1)c^{\rho-2}}{v}}{\frac{\rho c^{\rho-1}}{v}} \\
&= -(p-1)
\end{aligned}
$$

Therefore power utility is CCRA

Linear Utility: $U(c) = \mu c, \mu > 0$

Then $U'(c) = \mu$ and $U''(c) = 0$

$$
\begin{aligned}
R_r(c) &= -c\frac{0}{\mu} \\
&= 0
\end{aligned}
$$

Therefore linear utility is CCRA.

Appendix

```
library(dplyr)
library(ggplot2)

# Question 1----

*BY STATA
clear all
*Log file
capture log close
 log using "/Users/wuyi/Desktop/assignment2.log",replace

*Options
set linesize 158
set more off
set maxiter 100

*Queston 1

*Load Data
insheet using /Users/wuyi/Desktop/Q1data.csv

set more off
timer on 1

gen quarter=tq(1980q1)+_n-1
format %tq quarter

tsset quarter

*real GDP
gen gy=((y[_n]-y[_n-4])/y[_n-4])*100

*Durable consumption
gen gd=((d[_n]-d[_n-4])/d[_n-4])*100

*Non-durable consumption
gen gn=((n[_n]-n[_n-4])/n[_n-4])*100



sum gy gd gn


twoway line gy gd gn quarter, ytitle("Growth rate (%)") xtitle("Quarter") graphregion(color(white)) legend(label(1 Real GDP) label(2 Durable consumptio
graph export "yygrowth.png", replace width(4000)

*(b)
pwcorr gy L1.gy L2.gy L3.gy L4.gy L5.gy F1.gy F2.gy F3.gy F4.gy F5.gy, star(0.01)
pwcorr gd L1.gd L2.gd L3.gd L4.gd L5.gd F1.gd F2.gd F3.gd F4.gy F5.gd, star(0.01)
pwcorr gn L1.gn L2.gn L3.gn L4.gn L5.gn F1.gn F2.gn F3.gn F4.gn F5.gn, star(0.01)

# Loading Data Frame
setwd('EC640 macroeconomics/A2')
df <- read.csv("Q1Data.csv", header = TRUE)
df[,-1] <- as.data.frame(sapply(df[-1], as.numeric))
df[,1] <- as.Date(as.character(df[,1]), format = "%m/%d/%Y")
df <- df[-c(1:4),]

# png("Q1A.png", width = 465, height = 225, units='mm', res = 300)
# ggplot(data = df, aes(x = Date)) +
#    # geom_line(aes(y = Durables_Y2Y, color = 'dodgerblue1'), size = 1.1) +
#     geom_line(aes(y = Nondurables_Y2Y, color = I('grey2')), size = 1.1) +
#     geom_line(aes(y = GDP_Y2Y, color = 'red'), size = 1.1) +
#     theme_bw() +
#     scale_color_manual(labels = c('Durables', 'Nondurables', 'GDP'),
#                        values = c("dodgerblue1", 'grey2', 'red')) +
#     labs(color = 'Series') +
#     theme(legend.position="bottom",
#           legend.text = element_text(size=22),
#           axis.text = element_text(size=22),
#           axis.title = element_text(size=26),
#           plot.title=element_text(size = 26),
#           legend.title=element_text(size= 26),
#           # Change legend key size and key width
#           legend.key.size = unit(1.5, "cm"),
#           legend.key.width = unit(2.0,"cm")) +
#     labs(x = 'Year', y = 'Year-to-year Growth Rate')
# dev.off()

### Part B

q<-acf(df['GDP_Y2Y'])
abline(h = 0.5, col = 'red')
q

q <- acf(df['Nondurables_Y2Y'])
abline(h = 0.5, col = 'red')
q
```

```
q <- acf(df['Durables_Y2Y'])
abline(h = 0.5, col = 'red')
q
# Question 2

### Version 2----
df2 <- read.csv('Q2Data2.csv', header = TRUE)
df2 <- df2[complete.cases(df2),]
df2[,5:7] <- as.data.frame(sapply(df2[,5:7],as.numeric))


growth <- function(x)x/lag(x)-1

Q2 <- df2 %>%
    group_by(country) %>%
    filter(1960 %in% year) %>%
    filter(year >= 1960) %>%
    mutate(rgdpcap_Y2Y = growth(rgdpcap) * 100)


AvgGrowth <- filter(Q2, year <= 2000) %>%
    summarise(mean = mean(rgdpcap_Y2Y, na.rm = TRUE))

USA1960 <- filter(Q2, year == 1960 & countrycode == 'USA') %>%
    select(country, rgdpcap)
USA1960 <- as.numeric(USA1960[[2]])

GDPRatio <- mutate(Q2, xaxis = rgdpcap / USA1960) %>%
    filter(year == 1960) %>%
    select(country,xaxis) %>%
    arrange(country)

Q2A <- cbind(as.data.frame(AvgGrowth), as.data.frame(GDPRatio[,2]))

png("Q2A.png", width = 465, height = 225, units='mm', res = 300)
ggplot(data = Q2A, aes(x = xaxis)) +
    geom_point(aes(y = mean, size = 1.2)) +
    geom_smooth(aes(y = mean), method = 'lm', formula = y~x, size = 1.2) +
    theme_bw() +
    labs(x = 'Relative GDP per capita to USA in 1960', y = 'Annual Growth Rate (%)') +
    theme(legend.position = "none",
          axis.text = element_text(size=22),
          axis.title = element_text(size=26),
          plot.title=element_text(size = 26))
dev.off()
cor.test(Q2A[['mean']], Q2A[['xaxis']])
#2b ----

year1960 <-  df2 %>%
    group_by(country) %>%
    filter(1960 %in% year) %>%
    filter(year == 1960 & countrycode != 'USA') %>%
    select(country,rgdpcap)
Hpercentile <- filter(year1960, rgdpcap > quantile(year1960[[2]], .50))
Lpercentile <- filter(year1960, rgdpcap <= quantile(year1960[[2]], .5))

Q2B_Highgrowth <- Q2 %>%
    filter(country %in% Hpercentile[[1]]) %>%
    group_by(country) %>%
    filter(year <= 2000) %>%
    summarise(mean = mean(rgdpcap_Y2Y, na.rm = TRUE)) %>%
    mutate(percentile = 'Top 50 Percentile for Output Level in 1960')

Q2B_Highratio <- Q2%>%
    filter(country %in% Hpercentile[[1]]) %>%
    mutate(xaxis = rgdpcap/USA1960) %>%
    filter(year == 1960) %>%
    select(country, xaxis) %>%
    arrange(country)

Q2B_Lowgrowth <- Q2 %>%
    filter(country %in% Lpercentile[[1]]) %>%
    filter(year <= 2000) %>%
    group_by(country) %>%
    summarise(mean = mean(rgdpcap_Y2Y, na.rm = TRUE)) %>%
    mutate(percentile = 'Bottom 50 Percentile for Output Level in 1960')

Q2B_Lowratio <- Q2%>%
    filter(country %in% Lpercentile[[1]]) %>%
    mutate(xaxis = rgdpcap/USA1960) %>%
    filter(year == 1960) %>%
    select(country, xaxis) %>%
    arrange(country)

Q2B <- cbind(as.data.frame(rbind(Q2B_Highgrowth, Q2B_Lowgrowth)),
            as.data.frame(rbind(Q2B_Highratio[,2], Q2B_Lowratio[,2])))

png("Q2B.png", width = 465, height = 225, units='mm', res = 300)
ggplot(data = Q2B, aes(x = xaxis)) +
    geom_point(aes(y = mean, size = 1.2)) +
    geom_smooth(aes(y = mean), method = 'lm', formula = y~x) +
```

```
    facet_grid(col = vars(percentile)) +
    theme_bw() +
    labs(x = 'Relative GDP per capita to USA in 1960', y = 'Annual Growth Rate (%)') +
    theme(legend.position="none",
          axis.text = element_text(size=22),
          axis.title = element_text(size=26),
          plot.title=element_text(size = 26),
          strip.text.x = element_text(size = 18))
dev.off()

test <- filter(Q2B, percentile == 'Top 50 Percentile for Output Level in 1960')

cor.test(test[['mean']], test[['xaxis']])

###Q2C ----

Q2C <- df2 %>%
    group_by(country) %>%
    filter(1975 %in% year) %>%
    filter(year >= 1975) %>%
    mutate(rgdpcap_Y2Y = growth(rgdpcap) * 100)

AvgGrowthC <- filter(Q2C, year <= 2015) %>%
    summarise(bleh= mean(rgdpcap_Y2Y, na.rm = TRUE))

USA1975 <- filter(Q2C, year == 1975 & countrycode == 'USA') %>%
    select(country, rgdpcap)

GDPRatioC <- mutate(Q2C, xaxis = rgdpcap / USA1975[[2]]) %>%
    filter(year == 1975) %>%
    select(country, xaxis) %>%
    arrange(country)

Q2CDF <- cbind(as.data.frame(AvgGrowthC), xaxis = as.data.frame(GDPRatioC)[,2])

png("Q2C.png", width = 465, height = 225, units='mm', res = 300)
ggplot(data = Q2CDF, aes(x = xaxis)) +
    geom_point(aes(y = bleh, size = 1.2)) +
    geom_smooth(aes(y = bleh), method = 'lm', formula = y~x) +
    theme_bw() +
    labs(x = 'Relative GDP per capita to USA in 1975', y = 'Annual Growth Rate (%)') +
    theme(legend.position = "none",
          axis.text = element_text(size=22),
          axis.title = element_text(size=26),
          plot.title=element_text(size = 26))
dev.off()
cor.test(Q2CDF[['bleh']], Q2CDF[['xaxis']])

### Q2D OPEC countries
OPEC <- c('DZA', 'AGO', 'COD', 'ECU', 'GIN', 'GAB', 'IRN', 'IRQ', 'KWT', 'NGA',
          'QAT', 'SAU', 'ARE', 'VEN')


Q2D <-  df2 %>%
    group_by(countrycode) %>%
    filter(1970 %in% year) %>%
    filter(year >= 1970 & year <= 2010) %>%
    mutate(rgdpcap_Y2Y = growth(rgdpcap) * 100)

AvgGrowthD <- summarise(Q2D, bleh= mean(rgdpcap_Y2Y, na.rm = TRUE))

USA1970 <- filter(Q2D, year == 1970 & countrycode == 'USA') %>%
    select(countrycode, rgdpcap)

GDPRatioD <- mutate(Q2D, xaxis = rgdpcap / USA1970[[2]]) %>%
    filter(year == 1970) %>%
    select(countrycode, xaxis) %>%
    arrange(countrycode)

Q2DDF <- cbind(as.data.frame(AvgGrowthD), xaxis = as.data.frame(GDPRatioD)[,2])

OPEC_2D <- filter(Q2DDF, countrycode %in% OPEC) %>%
    mutate(type = 'OPEC')

Others <- filter(Q2DDF, !(countrycode %in% OPEC)) %>%
    mutate(type = 'Not OPEC')

Final_DF <- rbind(as.data.frame(OPEC_2D), as.data.frame(Others))
#Final_DF['type'] <- as.factor(Final_DF['type'])

png("Q2D.png", width = 465, height = 225, units='mm', res = 300)
ggplot(data = Final_DF, aes(x = xaxis)) +
    geom_point(aes(y = bleh, size = 1.2)) +
    geom_smooth(aes(y = bleh), method = 'lm', formula = y~x)+
    facet_grid(col = vars(type)) +
    theme_bw() +
    labs(x = 'Relative GDP per capita to USA in 1970', y = 'Annual Growth Rate (%)') +
    theme(legend.position = "none",
          axis.text = element_text(size=22),
          axis.title = element_text(size=26),
          plot.title=element_text(size = 26),
          strip.text.x = element_text(size = 18))
```

```
dev.off()
cor.test(OPEC_2D[['bleh']], OPEC_2D[['xaxis']])
cor.test(Others[['bleh']], Others[['xaxis']])



//*********
//Intro formalities
//Question 5

clear all
capture drop _all
//Log file
capture log close
log using output.log, replace

// Options
set linesize 255
set more off
set maxiter 100
// Timer
timer on 1

import delimited "D:/Users/Ziqiu/OneDrive/Documents/Masters Courses/EC640 Macroeconomics/A2/CDataQ", encoding(ISO-8859-9)


generate time=tq(1961q1)+_n-1
format %tq time
tsset time
gen y = 0
gen t = 0

matrix param          = (0.05, 0.36, 0.025, 0.02, 0.3, 0.8)
matrix colnames param = s     alpha delta   n    g rho
dsgenl          ( y = z*k^({alpha}) )                                 ///
( t = {g}*e) ///
( i = y - c - t) ///
( c = (1-{s})*(y-t) )                                     ///
        ( F.k = (1-{delta})/(1+{n})*k+(1/(1+{n}))*{s}*y-{s}*t/(1+{n}) )     ///
( ln(F.z) = {rho}*ln(z))     ///
( ln(F.e) = {rho}*ln(e))  ///
,observed(y t) unobserved(c i) endostate(k) exostate(z e) solve noidencheck from(param)


estat steady, compact

estat stable

//Approximate state transition matrix.
estat transition

// Policy matrix
estat policy, compact

//*********
// IRFS

irf set solowirf
irf create imp_res, replace
irf graph irf, irf(imp_res) impulse(z) response(t z k y c i) byopts(yrescale) legend( nobox region(lstyle(areastyle)) ) xtitle("Quarters") ytitle("Leve
graph export "5f1.png", replace width(4000) // save graph

irf graph irf, irf(imp_res) impulse(e) response(t z k y c i) byopts(yrescale) legend( nobox region(lstyle(none)) ) xtitle("Quarters") ytitle("Level") g
graph export "5f2.png", replace width(4000) // save graph

//**************************
// Close and save log file
log close
timer off 1
timer list 1
```