6

Connections with Partial Differential Equations

6.1 Introduction

There are two ways to compute a derivative security price: (1) use Monte Carlo simulation to generate paths of the underlying security or securities under the risk-neutral measure and use these paths to estimate the risk-neutral expected discounted payoff; or (2) numerically solve a partial differential equation. This chapter addresses the second of these methods by showing how to connect the risk-neutral pricing problem to partial differential equations. Section 6.2 explains the concept of stochastic differential equations, which is used to model asset prices. Solutions to stochastic differential equations have the Markov property, as is discussed in Section 6.3. Because of this, related to each stochastic differential equation there are two partial differential equations, one that includes discounting and one that does not. These partial differential equations and their derivations are the subject of Section 6.4. Section 6.5 shows how these ideas can be applied to interest rate models to compute bond prices and the prices of derivatives on bonds. The discussion of Sections 6.2–6.5 concerns one-dimensional processes. The multidimensional theory is outlined in Section 6.6, and a representative example that uses this theory, pricing and hedging an Asian option, is presented in that section.

6.2 Stochastic Differential Equations

A *stochastic differential equation* is an equation of the form

$$dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u). \tag{6.2.1}$$

Here $\beta(u,x)$ and $\gamma(u,x)$ are given functions, called the *drift* and *diffusion*, respectively. In addition to this equation, an *initial condition* of the form $X(t) = x$, where $t \geq 0$ and $x \in \mathbb{R}$, is specified. The problem is then to find a stochastic process $X(T)$, defined for $T \geq t$, such that

$$X(t) = x, \tag{6.2.2}$$

$$X(T) = X(t) + \int_t^T \beta(u, X(u))\,du + \int_t^T \gamma(u, X(u))\,dW(u). \tag{6.2.3}$$

Under mild conditions on the functions $\beta(u,x)$ and $\gamma(u,x)$, there exists a unique process $X(T)$, $T \geq t$, satisfying (6.2.2) and (6.2.3). However, this process can be difficult to determine explicitly because it appears on both the left- and right-hand sides of equation (6.2.3).

The solution $X(T)$ at time T will be $\mathcal{F}(T)$-measurable (i.e., $X(T)$ only depends on the path of the Brownian motion up to time T.) In fact, since the initial condition $X(t) = x$ is specified, all that is really needed to determine $X(T)$ is the path of the Brownian motion between times t and T.

Although stochastic differential equations are, in general, difficult to solve, a *one-dimensional linear stochastic differential equation* can be solved explicitly. This is a stochastic differential equation of the form

$$dX(u) = \big(a(u) + b(u)X(u)\big)\,du + \big(\gamma(u) + \sigma(u)X(u)\big)\,dW(u), \tag{6.2.4}$$

where $a(u)$, $b(u)$, $\sigma(u)$, and $\gamma(u)$ are nonrandom functions of time. Indeed, this equation can even be solved when $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ are adapted random processes (see Exercise 6.1), although it is then no longer of the form (6.2.1). In order to guarantee that the solution to (6.2.1) has the Markov property discussed in Section 6.3 below, the only randomness we permit on the right-hand side of (6.2.1) is the randomness inherent in the solution $X(u)$ and in the driving Brownian motions $W(u)$. There cannot be additional randomness such as would occur if any of the processes $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ appearing in (6.2.4) were themselves random. The next two examples are special cases of (6.2.4) in which $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ are nonrandom.

Example 6.2.1 (Geometric Brownian motion). The stochastic differential equation for geometric Brownian motion is

$$dS(u) = \alpha S(u)\,du + \sigma S(u)\,dW(u).$$

In the notation of (6.2.1), $\beta(u,x) = \alpha x$ and $\gamma(u,x) = \sigma x$. We know the formula for the solution to this stochastic differential equation when the initial time is zero and the initial position is $S(0)$, namely

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}.$$

Similarly, for $T \geq t$,

$$S(T) = S(0)\exp\left\{\sigma W(T) + \left(\alpha - \frac{1}{2}\sigma^2\right)T\right\}.$$

Dividing $S(T)$ by $S(t)$, we obtain

$$\frac{S(T)}{S(t)} = \exp\left\{\sigma\big(W(T) - W(t)\big) + \left(\alpha - \frac{1}{2}\sigma^2\right)(T - t)\right\}.$$

If the initial condition is given at time t rather than at time zero and is $S(t) = x$, then this last equation becomes

$$S(T) = x\exp\left\{\sigma\big(W(T) - W(t)\big) + \left(\alpha - \frac{1}{2}\sigma^2\right)(T - t)\right\}.$$

As expected, when we use the initial condition $S(t) = x$, then $S(T)$ depends only on the path of the Brownian motion between times t and T.

Example 6.2.2 (Hull-White interest rate model). Consider the stochastic differential equation

$$dR(u) = \big(a(u) - b(u)R(u)\big)\,du + \sigma(u)\,d\widetilde{W}(u),$$

where $a(u)$, $b(u)$, and $\sigma(u)$ are nonrandom positive functions of the time variable u and $\widetilde{W}(u)$ is a Brownian motion under a risk-neutral measure $\widetilde{\mathbb{P}}$. In this case, we use the dummy variable r rather than x, and $\beta(u,r) = a(u) - b(u)r$, $\gamma(u,r) = \sigma(u)$. Let us take the initial condition $R(t) = r$. We can solve the stochastic differential equation by first using the stochastic differential equation to compute

$$\begin{aligned} d\left(e^{\int_0^u b(v)dv}R(u)\right) &= e^{\int_0^u b(v)dv}\big(b(u)R(u)\,du + dR(u)\big) \\ &= e^{\int_0^u b(v)dv}\big(\alpha(u)\,du + \sigma(u)\,d\widetilde{W}(u)\big). \end{aligned}$$

Integrating both sides from t to T and using the initial condition $R(t) = r$, we obtain the formula

$$e^{\int_0^T b(v)dv}R(T) = re^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)\,dv}\alpha(u)\,du + \int_t^T e^{\int_0^u b(v)\,dv}\sigma(u)\,d\widetilde{W}(u),$$

which we can solve for $R(T)$:

$$R(T) = re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)\,dv}\alpha(u)\,du + \int_t^T e^{-\int_u^T b(v)\,dv}\sigma(u)\,d\widetilde{W}(u).$$

This is an explicit formula for the solution $R(T)$. The right-hand side of the final equation does not involve the interest rate process $R(u)$ apart from the initial condition $R(t) = r$; it contains only this initial condition, an integral with respect to time, and an Itô integral of given functions. Note also that the Brownian motion path between times t and T only enters this formula.

Recall from Theorem 4.4.9 that the Itô integral $\int_t^T e^{-\int_u^T b(v)dv}\sigma(u)\,d\widetilde{W}(u)$ of the nonrandom integrand $e^{-\int_u^T b(v)dv}\sigma(u)$ is normally distributed with mean zero and variance $\int_t^T e^{-2\int_u^T b(v)dv}\sigma^2(u)\,du$. The other terms appearing in the formula above for $R(T)$ are nonrandom. Therefore, under the risk-neutral measure $\widetilde{\mathbb{P}}$, $R(T)$ is normally distributed with mean

$$re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)\,dv}\alpha(u)\,du$$

and variance

$$\int_t^T e^{-2\int_u^T b(v)dv}\sigma^2(u)\,du.$$

In particular, there is a positive probability that $R(T)$ is negative. This is one of the principal objections to the Hull-White model. □

Example 6.2.3 (Cox-Ingersoll-Ross interest rate model). In the Cox-Ingersoll-Ross (CIR) model, the interest rate is given by the stochastic differential equation

$$dR(u) = \big(a - bR(u)\big)\,du + \sigma\sqrt{R(u)}\,d\widetilde{W}(u), \tag{6.2.5}$$

where a, b, and σ are positive constants. Suppose an initial condition $R(t) = r$ is given. Although there is no formula for $R(T)$, there is one and only one solution to this differential equation starting from the given initial condition. This solution can be approximated by Monte Carlo simulation, and many of its properties can be determined, even though we do not have an explicit formula for it. For instance, in Example 4.4.11, the mean and variance of $R(T)$ were computed when the initial time is $t = 0$ and the initial interest rate is $R(0)$.

Unlike the interest rate in the Hull-White model, the interest rate in the Cox-Ingersoll-Ross model cannot take negative values. When the interest rate approaches zero, the term $\sigma\sqrt{R(u)}\,d\widetilde{W}(u)$ also approaches zero. With the volatility disappearing, the behavior of the interest rate near zero depends on the drift term $a - bR(u)$, and this is $a > 0$ when $R(u) = 0$. The positive drift prevents the interest rate from crossing zero into negative territory.

More information about the solution to (6.2.5) is provided in Exercise 6.6 and Remark 6.9.1 following that exercise. □

6.3 The Markov Property

Consider the stochastic differential equation (6.2.1). Let $0 \le t \le T$ be given, and let $h(y)$ be a Borel-measurable function. Denote by

$$g(t,x) = \mathbb{E}^{t,x}h(X(T)) \tag{6.3.1}$$

the expectation of $h(X(T))$, where $X(T)$ is the solution to (6.2.1) with initial condition $X(t) = x$. (We assume that $\mathbb{E}^{t,x}|h(X(T))| < \infty$.) Note that there is nothing random about $g(t,x)$; it is an ordinary (actually, Borel-measurable) function of the two dummy variables t and x.

If we do not have an explicit formula for the distribution of $X(T)$, we could compute $g(t,x)$ numerically by beginning at $X(t) = x$ and simulating the stochastic differential equation. One way to do this would be to use the *Euler method*, a particular type of Monte Carlo method: choose a small positive step size δ, and then set

$$X(t+\delta) = x + \beta(t,x)\delta + \gamma(t,x)\sqrt{\delta}\,\epsilon_1,$$

where ϵ_1 is a standard normal random variable. Then set

$$X(t+2\delta) = X(t+\delta) + \beta\big(t+\delta, X(t+\delta)\big)\delta + \gamma\big(t+\delta, X(t+\delta)\big)\sqrt{\delta}\,\epsilon_2,$$

where ϵ_2 is a standard normal random variable independent of ϵ_1. By this device, one eventually determines a value for $X(T)$ (assuming δ is chosen so that $\frac{T-t}{\delta}$ is an integer). This gives one realization of $X(T)$ (corresponding to one ω). Now repeat this process many times and compute the average of $h(X(T))$ over all these simulations to get an approximate value for $g(t,x)$. Note that if one were to begin with a different time t and initial value x, one would get a different answer (i.e., the answer is a function of t and x). This dependence on t and x is emphasized by the notation $E^{t,x}$ in (6.3.1).

Theorem 6.3.1. *Let $X(u)$, $u \geq 0$, be a solution to the stochastic differential equation (6.2.1) with initial condition given at time* 0. *Then, for* $0 \leq t \leq T$,

$$\mathbb{E}\big[h(X(T))\big|\mathcal{F}(t)\big] = g\big(t, X(t)\big). \tag{6.3.2}$$

While the details of the proof of Theorem 6.3.1 are quite technical and will not be given, the intuitive content is clear. Suppose the process $X(u)$ begins at time zero, being generated by the stochastic differential equation (6.2.1), and one watches it up to time t. Suppose now that one is asked, based on this information, to compute the conditional expectation of $h(X(T))$, where $T \geq t$. Then one should pretend that the process is starting at time t at its current position, generate the solution to the stochastic differential equation corresponding to this initial condition, and compute the expected value of $h(X(T))$ generated in this way. In other words, replace $X(t)$ by a dummy x in order to hold it constant, compute $g(t,x) = \mathbb{E}^{t,x}h(X(T))$, and after computing this function put the random variable $X(t)$ back in place of the dummy x. This is the procedure set forth in the Independence Lemma, Lemma 2.3.4, and it is applicable here because the value of $X(T)$ is determined by the value of $X(t)$, which is $\mathcal{F}(t)$-measurable, and the increments of the Brownian motion between times t and T, which are independent of $\mathcal{F}(t)$.

Notice in the discussion above that although one watches the stochastic process $X(u)$ for $0 \leq u \leq t$, the only relevant piece of information when computing $\mathbb{E}\big[h(X(T))\big|\mathcal{F}(t)\big]$ is the value of $X(t)$. This means that $X(t)$ is a Markov process (see Definition 2.3.6). We highlight this fact as a corollary.

Corollary 6.3.2. *Solutions to stochastic differential equations are Markov processes.*

6.4 Partial Differential Equations

The Feynman-Kac Theorem below relates stochastic differential equations and partial differential equations. When this partial differential equation is solved (usually numerically), it produces the function $g(t,x)$ of (6.3.1). The Euler method described in the previous section for determining this function converges slowly and gives the function value for only one pair (t,x). Numerical algorithms for solving equation (6.4.1) below converge quickly in the case of one-dimensional x being considered here and give the function $g(t,x)$ for all values of (t,x) simultaneously. The relationship between geometric Brownian motion and the Black-Scholes-Merton partial differential equation is a special case of the relationship between stochastic differential equations and partial differential equations developed in the following theorems.

Theorem 6.4.1 (Feynman-Kac). *Consider the stochastic differential equation*

$$dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u). \tag{6.2.1}$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0,T]$ be given. Define the function

$$g(t,x) = \mathbb{E}^{t,x} h(X(T)). \tag{6.3.1}$$

(We assume that $\mathbb{E}^{t,x}|h(X(T))| < \infty$ for all t and x.) Then $g(t,x)$ satisfies the partial differential equation

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0 \tag{6.4.1}$$

and the terminal condition

$$g(T,x) = h(x) \text{ for all } x. \tag{6.4.2}$$

The proof of the Feynman-Kac Theorem depends on the following lemma.

Lemma 6.4.2. *Let $X(u)$ be a solution to the stochastic differential equation (6.2.1) with initial condition given at time 0. Let $h(y)$ be a Borel-measurable function, fix $T > 0$, and let $g(t,x)$ be given by (6.3.1). Then the stochastic process*

$$g(t, X(t)), \quad 0 \le t \le T,$$

is a martingale.

PROOF: Let $0 \le s \le t \le T$ be given. Theorem 6.3.1 implies

$$\mathbb{E}\big[h(X(T))\big|\mathcal{F}(s)\big] = g(s, X(s)),$$

$$\mathbb{E}\big[h(X(T))\big|\mathcal{F}(t)\big] = g(t, X(t)).$$

Take conditional expectations of the second equation, using iterated conditioning and the first equation, to obtain

$$\begin{aligned}\mathbb{E}\big[g(t,X(t))\big|\mathcal{F}(s)\big] &= \mathbb{E}\big[\mathbb{E}\big[h(X(T))\big|\mathcal{F}(t)\big]\big|\mathcal{F}(s)\big]\\ &= \mathbb{E}\big[h(X(T))\big|\mathcal{F}(s)\big]\\ &= g(s,X(s)).\end{aligned}$$ $\square$

Outline of Proof of Theorem 6.4.1: Let $X(t)$ be the solution to the stochastic differential equation (6.2.1) starting at time zero. Since $g(t,X(t))$ is a martingale, the net dt term in the differential $dg(t,X(t))$ must be zero. If it were positive at any time, then $g(t,X(t))$ would have a tendency to rise at that time; if it were negative, $g(t,X(t))$ would have a tendency to fall. Omitting the argument $(t,X(t))$ in several places below, we compute

$$\begin{aligned}dg(t,X(t)) &= g_t\,dt + g_x\,dX + \frac{1}{2}g_{xx}\,dX\,dX\\ &= g_t\,dt + \beta g_x\,dt + \gamma g_x\,dW + \frac{1}{2}\gamma^2 g_{xx}\,dt\\ &= \left[g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx}\right]dt + \gamma g_x\,dW.\end{aligned}$$

Setting the dt term to zero and putting back the argument $(t,X(t))$, we obtain

$$g_t(t,X(t)) + \beta(t,X(t))g_x(t,X(t)) + \frac{1}{2}\gamma^2(t,X(t))g_{xx}(t,X(t)) = 0$$

along every path of X. Therefore,

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0$$

at every point (t,x) that can be reached by $(t,X(t))$. For example, if $X(t)$ is a geometric Brownian motion, then (6.4.1) must hold for every $t\in[0,T)$ and every $x>0$. On the other hand, if $X(t)$ is a Hull-White interest rate process, which can take any positive or negative value, then (6.4.1) must hold for every $t\in[0,T)$ and every $x\in\mathbb{R}$. $\square$

The general principle behind the proof of the Feynman-Kac theorem is:

1. find the martingale,
2. take the differential, and
3. set the dt term equal to zero.

This gives a partial differential equation, which can then be solved numerically. We illustrate this three-step procedure in the following theorem and subsequent examples.

Theorem 6.4.3 (Discounted Feynman-Kac). *Consider the stochastic differential equation*

$$dX(u) = \beta(u,X(u))\,du + \gamma(u,X(u))\,dW(u). \qquad (6.2.1)$$

Let $h(y)$ be a Borel-measurable function and let r be constant. Fix $T > 0$, and let $t \in [0,T]$ be given. Define the function

$$f(t,x) = \mathbb{E}^{t,x}\big[e^{-r(T-t)}h(X(T))\big]. \tag{6.4.3}$$

(We assume that $\mathbb{E}^{t,x}|h(X(T))| < \infty$ for all t and x.) Then $f(t,x)$ satisfies the partial differential equation

$$f_t(t,x) + \beta(t,x)f_x(t,x) + \frac{1}{2}\gamma^2(t,x)f_{xx}(t,x) = rf(t,x) \tag{6.4.4}$$

and the terminal condition

$$f(T,x) = h(x) \text{ for all } x. \tag{6.4.5}$$

Outline of Proof: Let $X(t)$ be the solution to the stochastic differential equation (6.2.1) starting at time zero. Then

$$f(t,X(t)) = \mathbb{E}\big[e^{-r(T-t)}h(X(T))\big|\mathcal{F}(t)\big].$$

However, it is not the case that $f(t,X(t))$ is a martingale. Indeed, if $0 \le s \le t \le T$, then

$$\begin{aligned}\mathbb{E}\big[f(t,X(t))\big|\mathcal{F}(s)\big] &= \mathbb{E}\big[\mathbb{E}\big[e^{-r(T-t)}h(X(T))\big|\mathcal{F}(t)\big]\big|\mathcal{F}(s)\big]\\ &= \mathbb{E}\big[e^{-r(T-t)}h(X(T))\big|\mathcal{F}(s)\big],\end{aligned}$$

which is not the same as

$$f(s,X(s)) = \mathbb{E}\big[e^{-r(T-s)}h(X(T))\big|\mathcal{F}(s)\big]$$

because of the differing discount terms. The difficulty here is that in order to get the martingale property from iterated conditioning, we need the random variable being estimated not to depend on t, the time of the conditioning. To achieve this, we "complete the discounting," observing that

$$e^{-rt}f(t,X(t)) = \mathbb{E}\big[e^{-rT}h(X(T))\big|\mathcal{F}(t)\big].$$

We may now apply iterated conditioning to show that $e^{-rt}f(t,X(t))$ is a martingale. The differential of this martingale is

$$\begin{aligned}d\big(e^{-rt}f(t,X(t))\big) &= e^{-rt}\Big[-rf\,dt + f_t\,dt + f_x\,dX + \frac{1}{2}f_{xx}\,dX\,dX\Big]\\ &= e^{-rt}\Big[-rf + f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx}\Big]\,dt + e^{-rt}\gamma f_x\,dW.\end{aligned}$$

Setting the dt term equal to zero, we obtain (6.4.4). □

Example 6.4.4 (Options on a geometric Brownian motion). Let $h(S(T))$ be the payoff at time T of a derivative security whose underlying asset is the geometric Brownian motion

$$dS(u) = \alpha S(u)\,du + \sigma S(u)\,dW(u). \tag{6.4.6}$$

We may rewrite this as

$$dS(u) = rS(u)\,du + \sigma S(u)\,d\widetilde{W}(u), \tag{6.4.7}$$

where $\widetilde{W}(u)$ is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Here we assume that σ and the interest rate r are constant. According to the risk-neutral pricing formula (5.2.31), the price of the derivative security at time t is

$$V(t) = \widetilde{\mathbb{E}}\big[e^{-r(T-t)}h(S(T))\big|\mathcal{F}(t)\big]. \tag{6.4.8}$$

Because the stock price is Markov and the payoff is a function of the stock price alone, there is a function $v(t,x)$ such that $V(t) = v(t, S(t))$. Moreover, the function $v(t,x)$ must satisfy the discounted partial differential equation (6.4.4). This is the Black-Scholes-Merton equation

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x) = rv(t,x). \tag{6.4.9}$$

When the underlying asset is a geometric Brownian motion, this is the right pricing equation for a European call, a European put, a forward contract, and any other option that pays off some function of $S(T)$ at time T.

Note that to derive (6.4.9) we use the discounted partial differential equation (6.4.4) when the stochastic differential equation for the underlying process is (6.4.7) rather than (6.4.6) (i.e., we have $rxv_x(t,x)$ in (6.4.9) rather than $\alpha xv_x(t,x)$). This is because we are computing the conditional expectation in (6.4.8) under the risk-neutral measure $\widetilde{\mathbb{P}}$ and hence must use the differential equation that represents $S(u)$ in terms of $\widetilde{W}(u)$, the Brownian motion under $\widetilde{\mathbb{P}}$. In other words, we are using the Discounted Feynman-Kac Theorem with $\widetilde{W}(u)$ replacing $W(u)$ and $\widetilde{\mathbb{P}}$ replacing $\mathbb{P}$. □

In the previous example, if σ were a function of time and stock price (i.e., $\sigma(t,x)$), then the stock price would no longer be a geometric Brownian motion and the Black-Scholes-Merton formula would no longer apply. However, one can still solve for the option price by solving the partial differential equation (6.4.9), where now the constant σ^2 is replaced by $\sigma^2(t,x)$:

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2(t,x)x^2v_{xx}(t,x) = rv(t,x). \tag{6.4.10}$$

This equation is not difficult to solve numerically.

It has been observed in markets that if one assumes a constant volatility, the parameter σ that makes the theoretical option price given by (6.4.9) agree with the market price, the so-called *implied volatility*, is different for options having different strikes. In fact, this implied volatility is generally a convex function of the strike price. One refers to this phenomenon as the *volatility smile*.

One simple model with nonconstant volatility is the *constant elasticity of variance* (CEV) model, in which $\sigma(t,x) = \sigma x^{\delta-1}$ depends on x but not t. The parameter $\delta \in (0,1)$ is chosen so that the model gives a good fit to option prices across different strikes at a single expiration date. For this model, the stock price is governed by the stochastic differential equation

$$dS(t) = rS(t)\,dt + \sigma S^{\delta}(t)\,d\widetilde{W}(t).$$

The volatility $\sigma S^{\delta-1}(t)$ is a decreasing function of the stock price.

When one wishes to account for different volatilities implied by options expiring at different dates as well as different strikes, one needs to allow σ to depend on t as well as x. This function $\sigma(t,x)$ is called the *volatility surface* (see Exercise 6.10).

6.5 Interest Rate Models

The simplest models for fixed income markets begin with a stochastic differential equation for the interest rate, e.g.,

$$dR(t) = \beta(t, R(t))\,dt + \gamma(t, R(t))\,d\widetilde{W}(t), \tag{6.5.1}$$

where $\widetilde{W}(t)$ is a Brownian motion under a risk-neutral probability measure $\widetilde{\mathbb{P}}$. In these models, one begins with a risk-neutral measure $\widetilde{\mathbb{P}}$ and uses the risk-neutral pricing formula to price all assets. This guarantees that discounted asset prices are martingales under the risk-neutral measure, and hence there is no arbitrage. The issue of *calibration* of these models (i.e., choosing the model and the model parameters so that they give a good fit to market prices) is not discussed in this text.

Models for the interest rate $R(t)$ are sometimes called *short-rate models* because $R(t)$ is the interest rate for short-term borrowing. When the interest rate is determined by only one stochastic differential equation, as is the case in this section, the model is said to have *one factor*. The primary shortcoming of one-factor models is that they cannot capture complicated yield curve behavior; they tend to produce parallel shifts in the yield curve but not changes in its slope or curvature.

The *discount process* is as given in (5.2.17),

$$D(t) = e^{-\int_0^t R(s)ds},$$

and we denote the *money market account price process* to be

$$\frac{1}{D(t)} = e^{\int_0^t R(s)ds}.$$

This is thc value at time t of one unit of currency invested in the money market account at time zero and continuously rolled over at the short-term interest rate $R(s)$, $s \geq 0$. As discussed following (5.2.18), we have the differential formulas

$$dD(t) = -R(t)D(t)\,dt, \quad d\Big(\frac{1}{D(t)}\Big) = \frac{R(t)}{D(t)}\,dt.$$

A *zero-coupon bond* is a contract promising to pay a certain "face" amount, which we take to be 1, at a fixed maturity date T. Prior to that, the bond makes no payments. The risk-neutral pricing formula (5.2.30) says that the discounted price of this bond should be a martingale under the risk-neutral measure. In other words, for $0 \leq t \leq T$, the price of the bond $B(t,T)$ should satisfy

$$D(t)B(t,T) = \widetilde{\mathbb{E}}\big[D(T)\big|\mathcal{F}(t)\big]. \tag{6.5.2}$$

(Note that $B(T,T) = 1$.) This gives us the *zero-coupon bond pricing formula*

$$B(t,T) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(s)ds}\middle|\,\mathcal{F}(t)\right], \tag{6.5.3}$$

which we take as a definition. Once zero-coupon bond prices have been computed, we can define the *yield* between times t and T to be

$$Y(t,T) = -\frac{1}{T-t}\log B(t,T)$$

or, equivalently,

$$B(t,T) = e^{-Y(t,T)(T-t)}.$$

The yield $Y(t,T)$ is the constant rate of continuously compounding interest between times t and T that is consistent with the bond price $B(t,T)$. The 30-year rate at time t is $Y(t,30+t)$; this is an example of a long rate. Notice that once we adopt a model (6.5.1) for the short rate, the long rate is determined by the formulas above; we may not model the long rate separately.

Since R is given by a stochastic differential equation, it is a Markov process and we must have

$$B(t,T) = f(t,R(t))$$

for some function $f(t,r)$ of the dummy variables t and r. This is a slight step beyond the way we have used the Markov property previously because the random variable $e^{-\int_t^T R(s)ds}$ being estimated in (6.5.3) depends on the path segment $R(s)$, $t \leq s \leq T$, not just on $R(T)$. However, the only relevant part of the path of R before time t is its value at time t, and so the bond price $B(t,T)$ must be a function of time t and $R(t)$.

To find the partial differential equation for the unknown function $f(t,r)$, we find a martingale, take its differential, and set the dt term equal to zero. The martingale in this case is $D(t)B(t,T) = D(t)f(t,R(t))$. Its differential is

$$\begin{aligned}
d\big(D(t)f(t,R(t))\big) &= f(t,R(t))\,dD(t) + D(t)\,df(t,R(t))\\
&= D(t)\left[-Rf\,dt + f_t\,dt + f_r dR + \frac{1}{2}f_{rr}\,dR\,dR\right]\\
&= D(t)\left[-Rf + f_t + \beta f_r + \frac{1}{2}\gamma^2 f_{rr}\right]dt + D(t)\gamma f_r\,d\widetilde{W}.
\end{aligned}$$

Setting the dt term equal to zero, we obtain the partial differential equation

$$f_t(t,r) + \beta(t,r)f_r(t,r) + \frac{1}{2}\gamma^2(t,r)f_{rr}(t,r) = rf(t,r). \tag{6.5.4}$$

We also have the terminal condition

$$f(T,r) = 1 \text{ for all } r \tag{6.5.5}$$

because the value of the bond at maturity is its face value 1.

Example 6.5.1 (Hull-White interest rate model). In the Hull-White model, the evolution of the interest rate is given by

$$dR(t) = \big(a(t) - b(t)R(t)\big)\,dt + \sigma(t)\,d\widetilde{W}(t),$$

where $a(t)$, $b(t)$, and $\sigma(t)$ are nonrandom positive functions of time. The partial differential equation (6.5.4) for the zero-coupon bond price becomes

$$f_t(t,r) + \big(a(t) - b(t)r\big)f_r(t,r) + \frac{1}{2}\sigma^2(t)f_{rr}(t,r) = rf(t,r). \tag{6.5.6}$$

We initially guess and subsequently verify that the solution has the form

$$f(t,r) = e^{-rC(t,T)-A(t,T)}$$

for some nonrandom functions $C(t,T)$ and $A(t,T)$ to be determined. These are functions of $t \in [0,T]$; the maturity T is fixed. In this case, the yield

$$Y(t,T) = -\frac{1}{T-t}\log f(t,r) = \frac{1}{T-t}\Big(rC(t,T) + A(t,T)\Big)$$

is an *affine* function of r (i.e., a number times r plus another number). The Hull-White model is a special case of a class of models called *affine yield models.*

Furthermore,

$$\begin{aligned}
f_t(t,r) &= \big(-rC'(t,T) - A'(t,T)\big)f(t,r),\\
f_r(t,r) &= -C(t,T)f(t,r),\\
f_{rr}(t,r) &= C^2(t,T)f(t,r),
\end{aligned}$$

where $C'(t,T) = \frac{\partial}{\partial t}C(t,T)$ and $A'(t,T) = \frac{\partial}{\partial t}A(t,T)$. Substitution into the partial differential equation (6.5.6) gives

$$\begin{aligned}
&\Big[\big(-C'(t,T) + b(t)C(t,T) - 1\big)r\\
&\quad -A'(t,T) - a(t)C(t,T) + \frac{1}{2}\sigma^2(t)C^2(t,T)\Big]f(t,r) = 0.
\end{aligned} \tag{6.5.7}$$

Because this equation must hold for all r, the term that multiplies r in this equation must be zero. Otherwise, changing the value of r would change the value of the left-hand side of (6.5.7), and hence it could not always be equal to zero. This gives us an ordinary differential equation in t:

$$C'(t,T) = b(t)C(t,T) - 1. \tag{6.5.8}$$

Setting this term equal to zero in (6.5.7), we now see that

$$A'(t,T) = -a(t)C(t,T) + \frac{1}{2}\sigma^2(t)C^2(t,T). \tag{6.5.9}$$

The terminal condition (6.5.5) must hold for all r, and this implies that $C(T,T) = A(T,T) = 0$. Equations (6.5.8) and (6.5.9) and these terminal conditions provide enough information to determine the functions $A(t,T)$ and $C(t,T)$ for $0 \le t \le T$. They are

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)dv} ds, \tag{6.5.10}$$

$$A(t,T) = \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T) \right) ds. \tag{6.5.11}$$

It is clear that these formulas give functions that satisfy $C(T,T) = A(T,T) = 0$. The verification that these formulas provide the unique solutions to (6.5.8) and (6.5.9) is Exercise 6.3.

In conclusion, we have derived an explicit formula for the price of a zero-coupon bond as a function of the interest rate in the Hull-White model. It is

$$B(t,T) = e^{-R(t)C(t,T)-A(t,T)}, \quad 0 \le t \le T,$$

where $C(t,T)$ and $A(t,T)$ are given by (6.5.10) and (6.5.11). □

Example 6.5.2 (Cox-Ingersoll-Ross interest rate model). In the CIR model, the evolution of the interest rate is given by

$$dR(t) = \big(a - bR(t)\big)\,dt + \sigma\sqrt{R(t)}\,d\widetilde{W}(t),$$

where a, b, and σ are positive constants. The partial differential equation (6.5.4) for the bond price becomes

$$f_t(t,r) + (a - br)f_r(t,r) + \frac{1}{2}\sigma^2 r f_{rr}(t,r) = rf(t,r). \tag{6.5.12}$$

Again, we initially guess and subsequently verify that the solution has the form

$$f(t,r) = e^{-rC(t,T)-A(t,T)}.$$

The Cox-Ingersoll-Ross model is another example of an affine yield model. Substitution into the differential equation (6.5.12) gives

$$\left[\left(-C'(t,T)+bC(t,T)+\frac{1}{2}\sigma^2C^2(t,T)-1\right)r \\ -A'(t,T)-aC(t,T)\right]f(t,r)=0. \qquad (6.5.13)$$

We can again conclude that the term multiplying r must be zero and then conclude that the other term must also be zero, thereby obtaining two ordinary differential equations in t:

$$C'(t,T)=bC(t,T)+\frac{1}{2}\sigma^2C^2(t,T)-1, \qquad (6.5.14)$$

$$A'(t,T)=-aC(t,T). \qquad (6.5.15)$$

The solutions to these equations satisfying the terminal conditions $C(T,T)=A(T,T)=0$ are

$$C(t,T)=\frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t))+\frac{1}{2}b\sinh(\gamma(T-t))}, \qquad (6.5.16)$$

$$A(t,T)=-\frac{2a}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma\cosh(\gamma(T-t))+\frac{1}{2}b\sinh(\gamma(T-t))}\right], \qquad (6.5.17)$$

where $\gamma=\frac{1}{2}\sqrt{b^2+2\sigma^2}$, $\sinh u=\frac{e^u-e^{-u}}{2}$, and $\cosh u=\frac{e^u+e^{-u}}{2}$. The verification of this assertion is Exercise 6.4. □

Example 6.5.3 (Option on a bond). Consider the general short-rate model (6.5.1). Let $0\le t\le T_1<T_2$ be given. In this example, the fixed time T_2 is the maturity date for a zero-coupon bond. The fixed time T_1 is the expiration date for a European call on this bond. We wish to determine the value of this call at time t.

Suppose we have solved for the function $f(t,r)$ satisfying the partial differential equation (6.5.4) together with the terminal condition (6.5.5). This gives us the price of the zero-coupon bond as a function of time and the underlying interest rate.

According to the risk-neutral pricing formula (5.2.31) and the Markov property, the value of the call at time t is

$$\begin{aligned} c(t,R(t)) &= \widetilde{\mathbb{E}}\left[e^{-\int_t^{T_1}R(s)ds}\Big(f(T_1,R(T_1))-K\Big)^+\Big|\mathcal{F}(t)\right] \\ &= \frac{1}{D(t)}\widetilde{\mathbb{E}}\left[D(T_1)\Big(f(T_1,R(T_1))-K\Big)^+\Big|\mathcal{F}(t)\right] \end{aligned}$$

for some function $c(t,r)$ of the dummy variables t and r. The discounted call price

$$D(t)c(t,R(t))=\widetilde{\mathbb{E}}\left[D(T_1)\Big(f(T_1,R(T_1))-K\Big)^+\Big|\mathcal{F}(t)\right],\ 0\le t\le T_1,$$

is a martingale. The differential of the discounted call price is

$$
\begin{aligned}
d\big(D(t)c(t,R(t))\big) &= c(t,R(t))\,dD(t) + D(t)\,dc(t,R(t)) \\
&= D\left[-Rc\,dt + c_t\,dt + c_r\,dR + \frac{1}{2}c_{rr}\,dR\,dR\right] \\
&= D\left[-Rc + c_t + \beta c_r + \frac{1}{2}\gamma^2 c_{rr}\right] dt + D\gamma c_r\,d\widetilde{W}.
\end{aligned}
$$

Setting the dt term to zero, we obtain the partial differential equation

$$c_t(t,r) + \beta(t,r)c_r(t,r) + \frac{1}{2}\gamma^2(t,r)c_{rr}(t,r) = rc(t,r).$$

This is the same partial differential equation that governs $f(t,r)$. However, $c(t,r)$ and $f(t,r)$ have different terminal conditions. The terminal condition for $c(t,r)$ is

$$c(T_1,r) = (f(T_1,r) - K)^+ \text{ for all } r.$$

One can use these conditions to numerically determine the call price function $c(t,r)$. □

6.6 Multidimensional Feynman-Kac Theorems

The Feynman-Kac and Discounted Feynman-Kac Theorems, Theorems 6.4.1 and 6.4.3, have multidimensional versions. The number of differential equations and the number of Brownian motions entering those differential equations can both be larger than one and do not need to be the same. We illustrate the general situation by working out the details for two stochastic differential equations driven by two Brownian motions.

Let $W(t) = (W_1(t), W_2(t))$ be a two-dimensional Brownian motion (i.e., a vector of two independent, one-dimensional Brownian motions). Consider two stochastic differential equations

$$
\begin{aligned}
dX_1(u) &= \beta_1(u,X_1(u),X_2(u))\,du + \gamma_{11}(u,X_1(u),X_2(u))\,dW_1(u) \\
&\quad + \gamma_{12}(u,X_1(u),X_2(u))\,dW_2(u), \\
dX_2(u) &= \beta_2(u,X_1(u),X_2(u))\,du + \gamma_{21}(u,X_1(u),X_2(u))\,dW_1(u) \\
&\quad + \gamma_{22}(u,X_1(u),X_2(u))\,dW_2(u).
\end{aligned}
$$

The solution to this pair of stochastic differential equations, starting at $X_1(t) = x_1$ and $X_2(t) = x_2$, depends on the specified initial time t and the initial positions x_1 and x_2. Regardless of the initial condition, the solution is a Markov process.

Let a Borel-measurable function $h(y_1, y_2)$ be given. Corresponding to the initial condition t, x_1, x_2, where $0 \le t \le T$, we define

$$g(t,x_1,x_2) = \mathbb{E}^{t,x_1,x_2} h(X_1(T), X_2(T)), \tag{6.6.1}$$

$$f(t,x_1,x_2) = \mathbb{E}^{t,x_1,x_2}\left[e^{-r(T-t)}h(X_1(T), X_2(T))\right]. \tag{6.6.2}$$

Then

$$g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} \\ + \frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) g_{x_1 x_1} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) g_{x_1 x_2} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) g_{x_2 x_2} = 0, \tag{6.6.3}$$

$$f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} \\ + \frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) f_{x_1 x_1} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) f_{x_1 x_2} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) f_{x_2 x_2} = rf. \tag{6.6.4}$$

Of course, these functions also satisfy the terminal conditions

$$g(T, x_1, x_2) = f(T, x_1, x_2) = h(x_1, x_2) \text{ for all } x_1 \text{ and } x_2.$$

Equations (6.6.3) and (6.6.4) are derived by starting the pair of processes X_1, X_2 at time zero, observing that the processes $g(t, X_1(t), X_2(t))$ and $e^{-rt} f(t, X_1(t), X_2(t))$ are martingales, taking their differentials, and setting the dt terms equal to zero. When taking the differentials, one uses the fact that W_1 and W_2 are independent. We leave the details to the reader in Exercise 6.5. This exercise also provides the counterparts of (6.6.3) and (6.6.4) when W_1 and W_2 are correlated Brownian motions.

Example 6.6.1 (Asian option). We show by example how the Discounted Feynman-Kac Theorem can be used to find prices and hedges, even for path-dependent options. The option we choose for this example is an Asian option. A more detailed discussion of this option is presented in Section 7.5. The payoff we consider is

$$V(T) = \left(\frac{1}{T} \int_0^T S(u)\, du - K \right)^+,$$

where $S(u)$ is a geometric Brownian motion, the expiration time T is fixed and positive, and K is a positive strike price. In terms of the Brownian motion $\widetilde{W}(u)$ under the risk-neutral measure $\widetilde{\mathbb{P}}$, we may write the stochastic differential equation for $S(u)$ as

$$dS(u) = rS(u)\, du + \sigma S(u)\, d\widetilde{W}(u). \tag{6.6.5}$$

Because the payoff depends on the whole path of the stock price via its integral, at each time t prior to expiration it is not enough to know just the stock price in order to determine the value of the option. We must also know the integral of the stock price,

$$Y(t) = \int_0^t S(u)\, du,$$

up to the current time t. Similarly, it is not enough to know just the integral $Y(t)$. We must also know the current stock price $S(t)$. Indeed, for the same value of $Y(t)$, the Asian option is worth more for high values of $S(t)$ than for low values because the high values of $S(t)$ make it more likely that the option will have a high payoff.

For the process $Y(u)$, we have the stochastic differential equation

$$dY(u) = S(u)\,du. \tag{6.6.6}$$

Because the pair of processes $(S(u), Y(u))$ is given by the pair of stochastic differential equations (6.6.5) and (6.6.6), the pair of processes $(S(u), Y(u))$ is a two-dimensional Markov process.

Note that $Y(u)$ alone is not a Markov process because its stochastic differential equation involves the process $S(u)$. However, the pair $(S(u), Y(u))$ is Markov because the pair of stochastic differential equations for these processes involves only these processes (and, of course, the driving Brownian motion $\widetilde{W}(u)$).

If we use (6.6.5) and (6.6.6) to generate the processes $S(u)$ and $Y(u)$ starting with initial values $S(0) > 0$ and $Y(0) = 0$ at time zero, then the payoff of the Asian option at expiration time T is $V(T) = (\frac{1}{T}Y(T) - K)^+$. According to the risk-neutral pricing formula (5.2.31), the value of the Asian option at times prior to expiration is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}Y(T) - K\right)^+ \middle| \mathcal{F}(t)\right], \quad 0 \le t \le T.$$

Because the pair of processes $(S(u), Y(u))$ is Markov, this can be written as some function of the time variable t and the values at time t of these processes. In other words, there is a function $v(t, x, y)$ such that

$$v\big(t, S(t), Y(t)\big) = V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}Y(T) - K\right)^+ \middle| \mathcal{F}(t)\right].$$

Note that this function must satisfy the terminal condition

$$v(T, x, y) = \left(\frac{y}{T} - K\right)^+ \text{ for all } x \text{ and } y. \tag{6.6.7}$$

Using iterated conditioning, it is easy to see that the discounted option value $e^{-rt}v(t, S(t), Y(t))$ is martingale. Its differential is

$$\begin{aligned} &d\big(e^{-rt}v(t, S(t), Y(t))\big) \\ &= e^{-rt}\bigg[-rv\,dt + v_t\,dt + v_x\,dS + v_y\,dY + \frac{1}{2}v_{xx}\,dS\,dS + v_{xy}\,dS\,dY \\ &\qquad + \frac{1}{2}v_{yy}\,dY\,dY\bigg] \end{aligned}$$

$$
\begin{aligned}
&= e^{-rt}\left[-rv\,dt + v_t\,dt + v_x(rS\,dt + \sigma S\,d\widetilde{W}) + v_y S\,dt + \frac{1}{2}\sigma^2 S^2 v_{xx}\,dt\right]\\
&= e^{-rt}\bigg[-rv(t,S(t),Y(t)) + v_t(t,S(t),Y(t)) + rS(t)v_x(t,S(t),Y(t))\\
&\qquad\qquad + S(t)v_y(t,S(t),Y(t)) + \frac{1}{2}\sigma^2 S^2(t) v_{xx}(t,S(t),Y(T))\bigg]\,dt\\
&\qquad + e^{-rt}\sigma S(t) v_x(t,S(t),Y(t))\,d\widetilde{W}(t). \qquad (6.6.8)
\end{aligned}
$$

Because the discounted option price is a martingale, the dt term in this differential must be zero. We obtain the partial differential equation

$$
v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y). \quad (6.6.9)
$$

This is an example of the Discounted Feynman-Kac Theorem, a special case of equation (6.6.4). In particular, (6.6.8) simplifies to

$$
d\big(e^{-rt}v(t,S(t),Y(t))\big) = e^{-rt}\sigma S(t) v_x(t,S(t),Y(t))\,d\widetilde{W}(t). \quad (6.6.10)
$$

Recall from (5.2.27) that the discounted value of a portfolio satisfies the equation

$$
d\big(e^{-rt}X(t)\big) = e^{-rt}\sigma S(t)\Delta(t)\,d\widetilde{W}(t). \quad (6.6.11)
$$

If we sell the Asian option at time zero for $v(0,S(0),0)$ and use this as the initial capital for a hedging portfolio (i.e., take $X(0) = v(0,S(0),0)$), and at each time t use the portfolio process $\Delta(t) = v_x(t,S(t),Y(t))$, then we will have

$$
d\big(e^{-rt}X(t)\big) = d\big(e^{-rt}v(t,S(t),Y(t))\big)
$$

for all times t, and hence

$$
X(T) = v(T,S(T),Y(T)) = \left(\frac{1}{T}Y(T) - K\right)^+.
$$

This procedure hedges a short position in the Asian option. We have obtained the usual formula that the number of shares held to hedge a short position in the option is the derivative of the option value with respect to the underlying stock price. However, the Asian option price is the solution to a partial differential equation that contains a term $xv_y(t,x,y)$ that does not appear in the partial differential equation for the price of a European option. □

6.7 Summary

When the underlying price of an asset is given by a stochastic differential equation, the asset price is Markov and the price of any non-path-dependent derivative security based on that asset is given by a partial differential equation. In order to price path-dependent securities, one first seeks to determine the variables on which the path-dependent payoff depends and then introduce one or more additional stochastic differential equations in order to have a system of such equations that describes the relevant variables. If this can be done, then again the price of the derivative security is given by a partial differential equation.

This leads to the following four-step procedure for finding the pricing differential equation and for constructing a hedge for a derivative security.

1. Determine the variables on which the derivative security price depends. In addition to time t, these are the underlying asset price $S(t)$ and possibly other stochastic processes. We call these stochastic processes the *state processes*. One must be able to represent the derivative security payoff in terms of these state processes.
2. Write down a system of stochastic differential equations for the state processes. Be sure that, except for the driving Brownian motions, the only random processes appearing on the right-hand sides of these equations are the state processes themselves. This ensures that the vector of state processes is Markov.
3. The Markov property guarantees that the derivative security price at each time is a function of time and the state processes at that time. The discounted option price is a martingale under the risk-neutral measure. Compute the differential of the discounted option price, set the dt term equal to zero, and obtain thereby a partial differential equation.
4. The terms multiplying the Brownian motion differentials in the discounted derivative security price differential must be matched by the terms multiplying the Brownian motion differentials in the evolution of the hedging portfolio value; see (5.4.27). Matching these terms determines the hedge for a short position in the derivative security.

6.8 Notes

Conditions for the existence and uniqueness of solutions to stochastic differential equations are provided by Karatzas and Shreve [101], Chapter 5, Section 2, who also show in Chapter 5, Section 4, that solutions to stochastic differential equations have the Markov property. This is based on work of Stroock and Varadhan [151]. The ideas behind the Feynman-Kac Theorem, although not the presentation we give here, trace back to Feynman [65] and Kac [99].

Hull and White presented their interest rate model in [88], in which they generalized a model of Vasicek [154] to allow time-varying coefficients. The origin of the Cox-Ingersoll-Ross model is [41], where one can find a closed-form formula for the distribution of the interest rate in the model. These are examples of *affine-yield models*, a class identified by Duffie and Kan [58]. They are sometimes called *multifactor CIR models*.

Example 6.6.1 obtains a partial differential equation for the price of an Asian option but does not address computational issues. In the form given here, the equation is difficult to handle numerically. Večer [156] and Rogers and Shi [139] present transformations of this equation that are numerically more stable. See also Andreasen [4] for an application of the change-of-numéraire idea of Chapter 9 to discretely sampled Asian options. The transformation of Večer and its use for both continuously sampled and discretely sampled Asian options is presented in Section 7.5.

The Heston stochastic volatility model of Exercise 6.7 is taken from Heston [84]. Exercise 6.10 on implying the volatility surface comes from Dupire [61]. The same idea for binomial trees was worked out by Derman et al. [50], [51].

6.9 Exercises

Exercise 6.1. Consider the stochastic differential equation

$$dX(u) = \big(a(u) + b(u)X(u)\big)\,du + \big(\gamma(u) + \sigma(u)X(u)\big)\,dW(u), \tag{6.2.4}$$

where $W(u)$ is a Brownian motion relative to a filtration $\mathcal{F}(u)$, $u \geq 0$, and we allow $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ to be processes adapted to this filtration. Fix an initial time $t \geq 0$ and an initial position $x \in \mathbb{R}$. Define

$$Z(u) = \exp\left\{\int_t^u \sigma(v)\,dW(v) + \int_t^u \left(b(v) - \frac{1}{2}\sigma^2(v)\right)dv\right\},$$
$$Y(u) = x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)}\,dv + \int_t^u \frac{\gamma(v)}{Z(v)}\,dW(v).$$

(i) Show that $Z(t) = 1$ and

$$dZ(u) = b(u)Z(u)\,du + \sigma(u)Z(u)\,dW(u),\ \ u \geq t.$$

(ii) By its very definition, $Y(u)$ satisfies $Y(t) = x$ and

$$dY(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}\,du + \frac{\gamma(u)}{Z(u)}\,dW(u),\ \ u \geq t.$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation (6.2.4) and satisfies the initial condition $X(t) = x$.

Exercise 6.2 (No-arbitrage derivation of bond-pricing equation). In Section 6.5, we began with the stochastic differential equation (6.5.1) for the interest rate under the risk-neutral measure $\widetilde{\mathbb{P}}$, used the risk-neutral pricing formula (6.5.3) to consider a zero-coupon bond maturing at time T whose price $B(t,T)$ at time t before maturity is a function $f(t, R(t))$ of the time and the interest rate, and derived the partial differential equation (6.5.4) for the function $f(t,r)$. In this exercise, we show how to derive this partial differential equation from no-arbitrage considerations rather than by using the risk-neutral pricing formula.

Suppose the interest rate is given by a stochastic differential equation

$$dR(t) = \alpha(t, R(t))\,dt + \gamma(t, R(t))\,dW(t), \tag{6.9.1}$$

where $W(t)$ is a Brownian motion under a probability measure $\mathbb{P}$ not assumed to be risk-neutral. Assume further that, for each T, the T-maturity zero-coupon bond price is a function $f(t, R(t), T)$ of the current time t, the current interest rate $R(t)$, and the maturity of the bond T. We do *not* assume that this bond price is given by the risk-neutral pricing formula (6.5.3).

Assume for the moment that $f_r(t, r, T) \neq 0$ for all values of r and $0 \le t \le T$, so we can define

$$\beta(t, r, T) = -\frac{1}{f_r(t, r, T)}\left[-rf(t, r, T) + f_t(t, r, T) + \frac{1}{2}\gamma^2(t, r) f_{rr}(t, r, T)\right], \tag{6.9.2}$$

and then have

$$f_t(t, r, T) + \beta(t, r, T) f_r(t, r, T) + \frac{1}{2}\gamma^2(t, r) f_{rr}(t, r, T) = rf(t, r, T). \tag{6.9.3}$$

Equation (6.9.3) will reduce to (6.5.4) for the function $f(t, r, T)$ if we can show that $\beta(t, r, T)$ does not depend on T.

(i) Consider two maturities $0 < T_1 < T_2$, and consider a portfolio that at each time $t \le T_1$ holds $\Delta_1(t)$ bonds maturing at time T_1 and $\Delta_2(t)$ bonds maturing at time T_2, financing this by investing or borrowing at the interest rate $R(t)$. Show that the value of this portfolio satisfies

$$\begin{aligned}
&d\big(D(t)X(t)\big)\\
&= \Delta_1(t)D(t)\bigg[-R(t)f(t, R(t), T_1) + f_t(t, R(t), T_1)\\
&\qquad + \alpha(t, R(t)) f_r(t, R(t), T_1) + \frac{1}{2}\gamma^2(t, R(t)) f_{rr}(t, R(t), T_1)\bigg]\,dt\\
&\quad + \Delta_2(t)D(t)\bigg[-R(t)f(t, R(t), T_2) + f_t(t, R(t), T_2)\\
&\qquad + \alpha(t, R(t)) f_r(t, R(t), T_2) + \frac{1}{2}\gamma^2(t, R(t)) f_{rr}(t, R(t), T_2)\bigg]\,dt\\
&\quad + D(t)\gamma(t, R(t))\big[\Delta_1(t) f_r(t, R(t), T_1) + \Delta_2(t) f_r(t, R(t), T_2)\big]\,dW(t)\\
&= \Delta_1(t)D(t)\big[\alpha(t, R(t)) - \beta(t, R(t), T_1)\big] f_r(t, R(t), T_1)\,dt\\
&\quad + \Delta_2(t)D(t)\big[\alpha(t, R(t)) - \beta(t, R(t), T_2)\big] f_r(t, R(t), T_2)\,dt\\
&\quad + D(t)\gamma(t, R(t))\big[\Delta_1(t) f_r(t, R(t), T_1) + \Delta_2(t) f_r(t, R(t), T_2)\big]\,dW(t).
\end{aligned} \tag{6.9.4}$$

(ii) Denote

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

and

$$S(t) = \text{sign}\left\{ \left[\beta(t, R(t), T_2) - \beta(t, R(t), T_1)\right] f_r(t, R(t), T_1) f_r(t, R(t), T_2) \right\}.$$

Show that the portfolio processes $\Delta_1(t) = S(t) f_r(t, R(t), T_2)$ and $\Delta_2(t) = -S(t) f_r(t, R(t), T_1)$ result in arbitrage unless $\beta(t, R(t), T_1) = \beta(t, R(t), T_2)$. Since T_1 and T_2 are arbitrary, we conclude that $\beta(t, r, T)$ does not depend on T.

(iii) Now let a maturity $T > 0$ be given and consider a portfolio $\Delta(t)$ that invests only in the bond of maturity T, financing this by investing or borrowing at the interest rate $R(t)$. Show that the value of this portfolio satisfies

$$\begin{aligned} &d\big(D(t)X(t)\big) \\ &= \Delta(t)D(t)\big[- R(t)f(t, R(t), T) + f_t(t, R(t), T) \\ &\qquad + \alpha(t, R(t)) f_r(t, R(t), T) + \frac{1}{2}\gamma^2(t, R(t)) f_{rr}(t, R(t), T)\big]\, dt \\ &\qquad + D(t)\Delta(t)\gamma(t, R(t)) f_r(t, R(t), T)\, dW(t). \end{aligned} \tag{6.9.5}$$

Show that if $f_r(t, r, T) = 0$, then there is an arbitrage unless

$$f_t(t, r, T) + \frac{1}{2}\gamma^2(t, r) f_{rr}(t, r, T) = r f(t, r, T). \tag{6.9.6}$$

In other words, if $f_r(t, r, T) = 0$, then (6.9.3) must hold no matter how we choose $\beta(t, r, T)$.

In conclusion, we have shown that if trading in the zero-coupon bonds presents no arbitrage opportunity, then for all t, r, and T such that $f_r(t, r, T) \neq 0$, we can define $\beta(t, r)$ by (6.9.2) because the right-hand side of (6.9.2) does not depend on T. We then have

$$f_t(t, r, T) + \beta(t, r) f_r(t, r, T) + \frac{1}{2}\gamma^2(t, r) f_{rr}(t, r, T) = r f(t, r, T), \tag{6.9.7}$$

which is (6.5.4) for the T-maturity bond. If $f_r(t, r, T) = 0$, then (6.9.6) holds, so (6.9.7) must still hold, no matter how $\beta(t, r)$ is defined. If we now change to a measure $\widetilde{\mathbb{P}}$ under which

$$\widetilde{W}(t) = W(t) + \int_0^t \frac{1}{\gamma(u, R(u))} \big[\alpha(u, R(u)) - \beta(u, R(u))\big]\, du$$

is a Brownian motion, then (6.9.1) can be rewritten as (6.5.1). The probability measure $\widetilde{\mathbb{P}}$ is risk-neutral.

Exercise 6.3 (Solution of Hull-White model). This exercise solves the ordinary differential equations (6.5.8) and (6.5.9) to produce the solutions $C(t,T)$ and $A(t,T)$ given in (6.5.10) and (6.5.11).

(i) Use equation (6.5.8) with s replacing t to show that

$$\frac{d}{ds}\left[e^{-\int_0^s b(v)dv}C(s,T)\right] = -e^{-\int_0^s b(v)dv}.$$

(ii) Integrate the equation in (i) from $s=t$ to $s=T$, and use the terminal condition $C(T,T)$ to obtain (6.5.10).

(iii) Replace t by s in (6.5.9), integrate the resulting equation from $s=t$ to $s=T$, use the terminal condition $A(T,T)=0$, and obtain (6.5.11).

Exercise 6.4 (Solution of Cox-Ingersoll-Ross model). This exercise solves the ordinary differential equations (6.5.14) and (6.5.15) to produce the solutions $C(t,T)$ and $A(t,T)$ given in (6.5.16) and (6.5.17).

(i) Define the function

$$\varphi(t) = \exp\left\{\frac{1}{2}\sigma^2\int_t^T C(u,T)\,du\right\}.$$

Show that

$$C(t,T) = -\frac{2\varphi'(t)}{\sigma^2\varphi(t)}, \tag{6.9.8}$$

$$C'(t,T) = -\frac{2\varphi''(t)}{\sigma^2\varphi(t)} + \frac{1}{2}\sigma^2C^2(t,T). \tag{6.9.9}$$

(ii) Use the equation (6.5.14) to show that

$$\varphi''(t) - b\varphi'(t) - \frac{1}{2}\sigma^2\varphi(t) = 0. \tag{6.9.10}$$

This is a constant-coefficient linear ordinary differential equation. All solutions are of the form

$$\varphi(t) = a_1e^{\lambda_1 t} + a_2e^{\lambda_2 t},$$

where λ_1 and λ_2 are solutions of the so-called *characteristic equation* $\lambda^2 - b\lambda - \frac{1}{2}\sigma^2 = 0$, and a_1 and a_2 are constants.

(iii) Show that $\varphi(t)$ must be of the form

$$\varphi(t) = \frac{c_1}{\frac{1}{2}b+\gamma}e^{-(\frac{1}{2}b+\gamma)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma}e^{-(\frac{1}{2}b-\gamma)(T-t)} \tag{6.9.11}$$

for some constants c_1 and c_2, where $\gamma = \frac{1}{2}\sqrt{b^2+2\sigma^2}$.

(iv) Show that

$$\varphi'(t) = c_1 e^{-(\frac{1}{2}b+\gamma)(T-t)} - c_2 e^{-(\frac{1}{2}b-\gamma)(T-t)}. \qquad (6.9.12)$$

Use the fact that $C(T,T) = 0$ to show that $c_1 = c_2$.

(v) Show that

$$\begin{aligned}\varphi(t) &= c_1 e^{-\frac{1}{2}b(T-t)} \left[\frac{\frac{1}{2}b-\gamma}{\frac{1}{4}b^2-\gamma^2} e^{-\gamma(T-t)} - \frac{\frac{1}{2}b+\gamma}{\frac{1}{4}b^2-\gamma^2} e^{\gamma(T-t)} \right] \\ &= \frac{2c_1}{\sigma^2} e^{-\frac{1}{2}b(T-t)} \left[b\sinh(\gamma(T-t)) + 2\gamma\cosh(\gamma(T-t)) \right], \\ \varphi'(t) &= -2c_1 e^{-\frac{1}{2}b(T-t)} \sinh(\gamma(T-t)).\end{aligned}$$

Conclude that $C(t,T)$ is given by (6.5.16).

(vi) From (6.5.15) and (6.9.8), we have

$$A'(t,T) = \frac{2a\varphi'(t)}{\sigma^2\varphi(t)}.$$

Replace t by s in this equation, integrate from $s = t$ to $s = T$, and show that $A(t,T)$ is given by (6.5.17).

Exercise 6.5 (Two-dimensional Feynman-Kac).

(i) With $g(t,x_1,x_2)$ and $f(t,x_1,x_2)$ defined by (6.6.1) and (6.6.2), show that $g(t,X_1(t),X_2(t))$ and $e^{-rt}f(t,X_1(t),X_2(t))$ are martingales.

(ii) Assuming that W_1 and W_2 are independent Brownian motions, use the Itô-Doeblin formula to compute the differentials of $g(t,X_1(t),X_2(t))$ and $e^{-rt}f(t,X_1(t),X_2(t))$, set the dt term to zero, and thereby obtain the partial differential equations (6.6.3) and (6.6.4).

(iii) Now consider the case that $dW_1(t)\,dW_2(t) = \rho\,dt$, where ρ is a constant. Compute the differentials of $g(t,X_1(t),X_2(t))$ and $e^{-rt}f(t,X_1(t),X_2(t))$, set the dt term to zero, and obtain the partial differential equations

$$\begin{aligned} g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} &+ \left(\frac{1}{2}\gamma_{11}^2 + \rho\gamma_{11}\gamma_{12} + \frac{1}{2}\gamma_{12}^2\right) g_{x_1x_1} \\ &+ (\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21} + \gamma_{12}\gamma_{22}) g_{x_1x_2} \\ &+ \left(\frac{1}{2}\gamma_{21}^2 + \rho\gamma_{21}\gamma_{22} + \frac{1}{2}\gamma_{22}^2\right) g_{x_2x_2} = 0, \qquad (6.9.13) \\ f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} &+ \left(\frac{1}{2}\gamma_{11}^2 + \rho\gamma_{11}\gamma_{12} + \frac{1}{2}\gamma_{12}^2\right) f_{x_1x_1} \\ &+ (\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21} + \gamma_{12}\gamma_{22}) f_{x_1x_2} \\ &+ \left(\frac{1}{2}\gamma_{21}^2 + \rho\gamma_{21}\gamma_{22} + \frac{1}{2}\gamma_{22}^2\right) f_{x_2x_2} = rf. \qquad (6.9.14) \end{aligned}$$

Exercise 6.6 (Moment-generating function for Cox-Ingersoll-Ross process).

(i) Let $W_1,\ldots,W_d$ be independent Brownian motions and let a and σ be positive constants. For $j = 1,\ldots,d$, let $X_j(t)$ be the solution of the *Ornstein-Uhlenbeck* stochastic differential equation

$$dX_j(t) = -\frac{b}{2}X_j(t)\,dt + \frac{1}{2}\sigma\,dW_j(t). \tag{6.9.15}$$

Show that

$$X_j(t) = e^{-\frac{1}{2}bt}\left[X_j(0) + \frac{\sigma}{2}\int_0^t e^{\frac{1}{2}bu}\,dW_j(u)\right]. \tag{6.9.16}$$

Show further that for fixed t, the random variable $X_j(t)$ is normal with

$$\mathbb{E}X_j(t) = e^{-\frac{1}{2}bt}X_j(0), \quad \text{Var}(X_j(t)) = \frac{\sigma^2}{4b}\left[1 - e^{-bt}\right]. \tag{6.9.17}$$

(Hint: Use Theorem 4.4.9.)

(ii) Define

$$R(t) = \sum_{j=1}^{d} X_j^2(t), \tag{6.9.18}$$

and show that

$$dR(t) = \big(a - bR(t)\big)\,dt + \sigma\sqrt{R(t)}\,dB(t), \tag{6.9.19}$$

where $a = \frac{d\sigma^2}{4}$ and

$$B(t) = \sum_{j=1}^{d}\int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\,dW_j(s) \tag{6.9.20}$$

is a Brownian motion. In other words, $R(t)$ is a Cox-Ingersoll-Ross interest rate process (Example 6.5.2). (Hint: Use Lévy's Theorem, Theorem 4.6.4, to show that $B(t)$ is a Brownian motion.)

(iii) Suppose $R(0) > 0$ is given, and define

$$X_j(0) = \sqrt{\frac{R(0)}{d}}.$$

Show then that $X_1(t), \ldots, X_d(t)$ are independent, identically distributed, normal random variables, each having expectation

$$\mu(t) = e^{-\frac{1}{2}bt}\sqrt{\frac{R(0)}{d}}$$

and variance

$$v(t) = \frac{\sigma^2}{4b}\left[1 - e^{-bt}\right].$$

(iv) Part (iii) shows that $R(t)$ given by (6.9.18) is the sum of squares of independent, identically distributed, normal random variables and hence has a *noncentral* χ^2 *distribution*, the term "noncentral" referring to the fact that $\mu(t) = \mathbb{E}X_j(t)$ is not zero. To compute the moment-generating function of $R(t)$, first compute the moment-generating function

$$\mathbb{E}\exp\left\{uX_j^2(t)\right\} = \frac{1}{\sqrt{1-2v(t)u}}\exp\left\{\frac{u\mu^2(t)}{1-2v(t)u}\right\} \text{ for all } u < \frac{1}{2v(t)}. \tag{6.9.21}$$

(Hint: You will need to complete a square, first deriving and then using the equation

$$ux^2 - \frac{1}{2v(t)}\big(x-\mu(t)\big)^2 = -\frac{1-2v(t)u}{2v(t)}\left(x - \frac{\mu(t)}{1-2v(t)u}\right)^2 + \frac{u\mu^2(t)}{1-2v(t)u}.$$

The integral from $-\infty$ to ∞ of the normal density with mean $\mu(t)/(1-2v(t)u)$ and variance $v(t)/(1-2v(t)u)$,

$$\sqrt{\frac{1-2v(t)u}{2\pi v(t)}}\exp\left\{-\frac{1-2v(t)u}{2v(t)}\left(x-\frac{\mu(t)}{1-2v(t)u}\right)^2\right\},$$

is equal to 1.)

(v) Show that $R(t)$ given by (6.9.19) has moment-generating function

$$\begin{aligned}\mathbb{E}e^{uR(t)} &= \left(\frac{1}{1-2v(t)u}\right)^{d/2}\exp\left\{\frac{e^{-bt}uR(0)}{1-2v(t)u}\right\}\\ &= \left(\frac{1}{1-2v(t)u}\right)^{2a/\sigma^2}\exp\left\{\frac{e^{-bt}uR(0)}{1-2v(t)u}\right\} \text{ for all } u < \frac{1}{2v(t)}.\end{aligned} \tag{6.9.22}$$

Remark 6.9.1 (Cox-Ingersoll-Ross process hitting zero). Although we have derived (6.9.22) under the assumption that d is a positive integer, the second line of (6.9.22) is expressed in terms of only the parameters a, b, and σ entering (6.9.19), and this formula is valid for all $a > 0$, $b > 0$, and $\sigma > 0$. When $d \geq 2$ (i.e., $a \geq \frac{1}{2}\sigma^2$), the multidimensional process $\big(X_1(t), \ldots, X_d(t)\big)$ never hits the origin in $\mathbb{R}^d$, and hence $R(t)$ is never zero. In fact, $R(t)$ is never zero if and only if $a \geq \frac{1}{2}\sigma^2$. If $0 < a < \frac{1}{2}\sigma^2$, then $R(t)$ hits zero repeatedly but after each hit becomes positive again.

Exercise 6.7 (Heston stochastic volatility model). Suppose that under a risk-neutral measure $\widetilde{\mathbb{P}}$ a stock price is governed by

$$dS(t) = rS(t)\,dt + \sqrt{V(t)}\,S(t)\,d\widetilde{W}_1(t), \tag{6.9.23}$$

where the interest rate r is constant and the volatility $\sqrt{V(t)}$ is itself a stochastic process governed by the equation

$$dV(t) = \big(a - bV(t)\big)\,dt + \sigma\sqrt{V(t)}\,d\widetilde{W}_2(t). \tag{6.9.24}$$

The parameters a, b, and σ are positive constants, and $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are correlated Brownian motions under $\widetilde{\mathbb{P}}$ with

$$d\widetilde{W}_1(t)\, d\widetilde{W}_2(t) = \rho\, dt$$

for some $\rho \in (-1,1)$. Because the two-dimensional process $(S(t), V(t))$ is governed by the pair of stochastic differential equations (6.9.23) and (6.9.24), it is a two-dimensional Markov process.

So long as trading takes place only in the stock and money market account, this model is incomplete. One can create a one-parameter family of risk-neutral measures by changing the dt term in (6.9.24) without affecting (6.9.23).

At time t, the risk-neutral price of a call expiring at time $T \geq t$ in this stochastic volatility model is $\widetilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t)]$. Because of the Markov property, there is a function $c(t, s, v)$ such that

$$c(t, S(t), V(t)) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T) - K)^+ \middle| \mathcal{F}(t)\right], \quad 0 \leq t \leq T. \tag{6.9.25}$$

This problem shows that the function $c(t, s, v)$ satisfies the partial differential equation

$$c_t + rsc_s + (a - bv)c_v + \frac{1}{2}s^2vc_{ss} + \rho\sigma svc_{sv} + \frac{1}{2}\sigma^2vc_{vv} = rc \tag{6.9.26}$$

in the region $0 \leq t < T$, $s \geq 0$, and $v \geq 0$. The function $c(t, s, v)$ also satisfies the boundary conditions

$$c(T, s, v) = (s - K)^+ \text{ for all } s \geq 0, v \geq 0, \tag{6.9.27}$$
$$c(t, 0, v) = 0 \text{ for all } 0 \leq t \leq T, v \geq 0, \tag{6.9.28}$$
$$c(t, s, 0) = \left(s - e^{-r(T-t)}K\right)^+ \text{ for all } 0 \leq t \leq T, s \geq 0, \tag{6.9.29}$$
$$\lim_{s\to\infty} \frac{c(t, s, v)}{s - K} = 1 \text{ for all } 0 \leq t \leq T, v \geq 0, \tag{6.9.30}$$
$$\lim_{v\to\infty} c(t, s, v) = s \text{ for all } 0 \leq t \leq T, s \geq 0. \tag{6.9.31}$$

In this problem, we shall be concerned only with (6.9.27).

(i) Show that $e^{-rt}c(t, S(t), V(t))$ is a martingale under $\widetilde{\mathbb{P}}$, and use this fact to obtain (6.9.26).

(ii) Suppose there are functions $f(t, x, v)$ and $g(t, x, v)$ satisfying

$$f_t + \left(r + \frac{1}{2}v\right) f_x + (a - bv + \rho\sigma v)f_v + \frac{1}{2}vf_{xx} + \rho\sigma vf_{xv} + \frac{1}{2}\sigma^2vf_{vv} = 0, \tag{6.9.32}$$

$$g_t + \left(r - \frac{1}{2}v\right) g_x + (a - bv)g_v + \frac{1}{2}vg_{xx} + \rho\sigma vg_{xv} + \frac{1}{2}\sigma^2vg_{vv} = 0, \tag{6.9.33}$$

in the region $0 \leq t < T$, $-\infty < x < \infty$, and $v \geq 0$. Show that if we define

$$c(t,s,v) = sf(t,\log s,v) - e^{-r(T-t)}Kg(t,\log s,v), \tag{6.9.34}$$

then $c(t,s,v)$ satisfies the partial differential equation (6.9.26).

(iii) Suppose a pair of processes $(X(t),V(t))$ is governed by the stochastic differential equations

$$dX(t) = \left(r + \frac{1}{2}V(t)\right)dt + \sqrt{V(t)}\,dW_1(t), \tag{6.9.35}$$

$$dV(t) = \big(a - bV(t) + \rho\sigma V(t)\big)\,dt + \sigma\sqrt{V(t)}\,dW_2(t), \tag{6.9.36}$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under some probability measure $\mathbb{P}$ with $dW_1(t)\,dW_2(t) = \rho\,dt$. Define

$$f(t,x,v) = \mathbb{E}^{t,x,v}\mathbb{I}_{\{X(T)\geq\log K\}}. \tag{6.9.37}$$

Show that $f(t,x,v)$ satisfies the partial differential equation (6.9.32) and the boundary condition

$$f(T,x,v) = \mathbb{I}_{\{x\geq\log K\}} \text{ for all } x\in\mathbb{R}, v\geq 0. \tag{6.9.38}$$

(iv) Suppose a pair of processes $(X(t),V(t))$ is governed by the stochastic differential equations

$$dX(t) = \left(r - \frac{1}{2}V(t)\right)dt + \sqrt{V(t)}\,dW_1(t), \tag{6.9.39}$$

$$dV(t) = \big(a - bV(t)\big)\,dt + \sigma\sqrt{V(t)}\,dW_2(t), \tag{6.9.40}$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under some probability measure $\mathbb{P}$ with $dW_1(t)\,dW_2(t) = \rho\,dt$. Define

$$g(t,x,v) = \mathbb{E}^{t,x,v}\mathbb{I}_{\{X(T)\geq\log K\}}. \tag{6.9.41}$$

Show that $g(t,x,v)$ satisfies the partial differential equation (6.9.33) and the boundary condition

$$g(T,x,v) = \mathbb{I}_{\{x\geq\log K\}} \text{ for all } x\in\mathbb{R}, v\geq 0. \tag{6.9.42}$$

(v) Show that with $f(t,x,v)$ and $g(t,x,v)$ as in (iii) and (iv), the function $c(t,x,v)$ of (6.9.34) satisfies the boundary condition (6.9.27).

Remark 6.9.2. In fact, with $f(t,x,v)$ and $g(t,x,v)$ as in (iii) and (iv), the function $c(t,x,v)$ of (6.9.34) satisfies all the boundary conditions (6.9.27)–(6.9.31) and is the function appearing on the left-hand side of (6.9.25).

Exercise 6.8 (Kolmogorov backward equation). Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))\, du + \gamma(u, X(u))\, dW(u).$$

We assume that, just as with a geometric Brownian motion, if we begin a process at an arbitrary initial positive value $X(t) = x$ at an arbitrary initial time t and evolve it forward using this equation, its value at each time $T > t$ could be any positive number but cannot be less than or equal to zero. For $0 \le t < T$, let $p(t, T, x, y)$ be the transition density for the solution to this equation (i.e., if we solve the equation with the initial condition $X(t) = x$, then the random variable $X(T)$ has density $p(t, T, x, y)$ in the y variable). We are assuming that $p(t, T, x, y) = 0$ for $0 \le t < T$ and $y \le 0$.

Show that $p(t, T; x, y)$ satisfies the *Kolmogorov backward equation*

$$-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y). \tag{6.9.43}$$

(Hint: We know from the Feynman-Kac Theorem, Theorem 6.4.1, that, for any function $h(y)$, the function

$$g(t, x) = \mathbb{E}^{t,x} h\big(X(T)\big) = \int_0^\infty h(y)p(t, T, x, y)dy \tag{6.9.44}$$

satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0. \tag{6.9.45}$$

Use (6.9.44) to compute g_t, g_x, and g_{xx}, and then argue that the only way (6.9.45) can hold regardless of the choice of the function $h(y)$ is for $p(t, T, x, y)$ to satisfy the Kolmogorov backward equation.)

Exercise 6.9 (Kolmogorov forward equation). (Also called the *Fokker-Planck equation*). We begin with the same stochastic differential equation,

$$dX(u) = \beta\big(u, X(u)\big)\, du + \gamma\big(u, X(u)\big)\, dW(u), \tag{6.9.46}$$

as in Exercise 6.8, use the same notation $p(t, T, x, y)$ for the transition density, and again assume that $p(t, T, x, y) = 0$ for $0 \le t < T$ and $y \le 0$. In this problem, we show that $p(t, T, x, y)$ satisfies the *Kolmogorov forward equation*

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\left(\beta(t, y)p(t, T, x, y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T, y)p(t, T, x, y)\right). \tag{6.9.47}$$

In contrast to the Kolmogorov backward equation, in which T and y were held constant and the variables were t and x, here t and x are held constant and the variables are y and T. The variables t and x are sometimes called the *backward variables*, and T and y are called the *forward variables*.

(i) Let b be a positive constant and let $h_b(y)$ be a function with continuous first and second derivatives such that $h_b(x) = 0$ for all $x \leq 0$, $h_b'(x) = 0$ for all $x \geq b$, and $h_b(b) = h_b'(b) = 0$. Let $X(u)$ be the solution to the stochastic differential equation with initial condition $X(t) = x \in (0, b)$, and use Itô's formula to compute $dh_b\big(X(u)\big)$.

(ii) Let $0 \leq t < T$ be given, and integrate the equation you obtained in (i) from t to T. Take expectations and use the fact that $X(u)$ has density $p(t, u, x, y)$ in the y-variable to obtain

$$\begin{aligned}\int_0^b h_b(y)p(t,T,x,y)dy = h_b(x) &+ \int_t^T \int_0^b \beta(u,y)p(t,u,x,y)h_b'(y)dy\,du \\ &+\frac{1}{2}\int_t^T \int_0^b \gamma^2(u,y)p(t,u,x,y)h_b''(y)dy.\end{aligned} \tag{6.9.48}$$

(iii) Integrate the integrals $\int_0^b \cdots dy$ on the right-hand side of (6.9.48) by parts to obtain

$$\begin{aligned}&\int_0^b h_b(y)p(t,T,x,y)dy \\ &= h_b(x) - \int_t^T \int_0^b \frac{\partial}{\partial y}\left[\beta(u,y)p(t,u,x,y)\right] h_b(y)dy\,du \\ &\quad+\frac{1}{2}\int_t^T \int_0^b \frac{\partial^2}{\partial y^2}\left[\gamma^2(u,y)p(t,u,x,y)\right] h_b(y)dy\,du.\end{aligned} \tag{6.9.49}$$

(iv) Differentiate (6.9.49) with respect to T to obtain

$$\begin{aligned}\int_0^b h_b(y)\bigg[&\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(T,y)p(t,T,x,y)\right) \\ &-\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)p(t,T,x,y)\right)\bigg]dy = 0.\end{aligned} \tag{6.9.50}$$

(v) Use (6.9.50) to show that there cannot be numbers $0 < y_1 < y_2$ such that

$$\begin{aligned}&\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\big(\beta(T,y)p(t,T,x,y)\big) \\ &\quad-\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big) > 0 \text{ for all } y \in (y_1, y_2).\end{aligned}$$

Similarly, there cannot be numbers $0 < y_1 < y_2$ such that

$$\begin{aligned}&\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\big(\beta(T,y)p(t,T,x,y)\big) \\ &\quad-\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big) < 0 \text{ for all } y \in [y_1, y_2].\end{aligned}$$

This is as much as you need to do for this problem. It is now obvious that if

$$\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\big(\beta(T,y)p(t,T,x,y)\big) - \frac{1}{2}\frac{\partial}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big)$$

is a continuous function of y, then this expression must be zero for every $y > 0$, and hence $p(t,T,x,y)$ satisfies the Kolmogorov forward equation stated at the beginning of this problem.

Exercise 6.10 (Implying the volatility surface). Assume that a stock price evolves according to the stochastic differential equation

$$dS(u) = rS(u)\,dt + \sigma\big(u,S(u)\big)S(u)\,d\widetilde{W}(u),$$

where the interest rate r is constant, the volatility $\sigma(u,x)$ is a function of time and the underlying stock price, and $\widetilde{W}$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. This is a special case of the stochastic differential equation (6.9.46) with $\beta(u,x) = rx$ and $\gamma(u,x) = \sigma(u,x)x$. Let $\tilde{p}(t,T,x,y)$ denote the transition density.

According to Exercise 6.9, the transition density $\tilde{p}(t,T,x,y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial T}\tilde{p}(t,T,x,y) = -\frac{\partial}{\partial y}\left(ry\tilde{p}(t,T,x,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(t,T,x,y)\right). \tag{6.9.51}$$

Let

$$c(0,T,x,K) = e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy \tag{6.9.52}$$

denote the time-zero price of a call expiring at time T, struck at K, when the initial stock price is $S(0) = x$. Note that

$$c_T(0,T,x,K) = -rc(0,T,x,K) + e^{-rT}\int_K^\infty (y-K)\tilde{p}_T(0,T,x,y)dy. \tag{6.9.53}$$

(i) Integrate once by parts to show that

$$-\int_K^\infty (y-K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right)dy = \int_K^\infty ry\tilde{p}(0,T,x,y)dy. \tag{6.9.54}$$

You may assume that

$$\lim_{y\to\infty}(y-K)ry\tilde{p}(0,T,x,y) = 0. \tag{6.9.55}$$

(ii) Integrate by parts and then integrate again to show that

$$\begin{aligned}\frac{1}{2}\int_K^\infty (y-K)&\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)dy \\ &= \frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K).\end{aligned} \tag{6.9.56}$$

You may assume that

$$\lim_{y\to\infty}(y-K)\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)=0, \tag{6.9.57}$$

$$\lim_{y\to\infty}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)=0. \tag{6.9.58}$$

(iii) Now use (6.9.53), (6.9.52), (6.9.51), (6.9.54), (6.9.56), and Exercise 5.9 of Chapter 5 in that order to obtain the equation

$$\begin{aligned}
&c_T(0,T,x,K)\\
&=e^{-rT}rK\int_K^\infty \tilde{p}(0,T,x,y)dy+\frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K)\\
&=-rKc_K(0,T,x,K)+\frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K).
\end{aligned} \tag{6.9.59}$$

This is the end of the problem. Note that under the assumption that $c_{KK}(0,T,x,K)\neq 0$, (6.9.59) can be solved for the volatility term $\sigma^2(T,K)$ in terms of the quantities $c_T(0,T,x,K)$, $c_K(0,T,x,K)$, and $c_{KK}(0,T,x,K)$, which can be inferred from market prices.

7

Exotic Options

7.1 Introduction

The European calls and puts considered thus far in this text are sometimes called *vanilla* or even *plain vanilla* options. Their payoffs depend only on the final value of the underlying asset. Options whose payoffs depend on the path of the underlying asset are called *path-dependent* or *exotic*.

In this chapter, we present three types of exotic options on a geometric Brownian motion asset and work out a detailed analysis for one option of each type. The types considered are *barrier options*, *lookback options*, and *Asian options*. In each case, we work out the standard partial differential equation governing the option price. The first two options have explicit pricing formulas, which are based on the reflection principle for Brownian motion. Such a formula for Asian options is not known. However, for the Asian option there is a change-of-numéraire argument that reduces the pricing partial differential equation to a simple form that can easily be solved numerically. We present this argument in Subsection 7.5.3.

7.2 Maximum of Brownian Motion with Drift

In this section, we derive the joint density for a Brownian motion with drift and its maximum to date. This density is used in Sections 7.3 and 7.4 to obtain explicit pricing formulas for a barrier option and a lookback option. To derive this formula, we begin with a Brownian motion $\widetilde{W}(t)$, $0 \leq t \leq T$, defined on a probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. Under $\widetilde{\mathbb{P}}$, the Brownian motion $\widetilde{W}(t)$ has zero drift (i.e., it is a martingale). Let α be a given number, and define

$$\widehat{W}(t) = \alpha t + \widetilde{W}(t), \quad 0 \leq t \leq T. \tag{7.2.1}$$

This Brownian motion $\widehat{W}(t)$ has drift α under $\widetilde{\mathbb{P}}$. We further define

$$\widehat{M}(T) = \max_{0 \le t \le T} \widehat{W}(t). \tag{7.2.2}$$

Because $\widehat{W}(0) = 0$, we have $\widehat{M}(T) \ge 0$. We also have $\widehat{W}(T) \le \widehat{M}(T)$. Therefore, the pair of random variables $\big(\widehat{M}(t), \widehat{W}(T)\big)$ takes values in the set $\{(m, w); w \le m, m \ge 0\}$ shown in Figure 7.2.1.

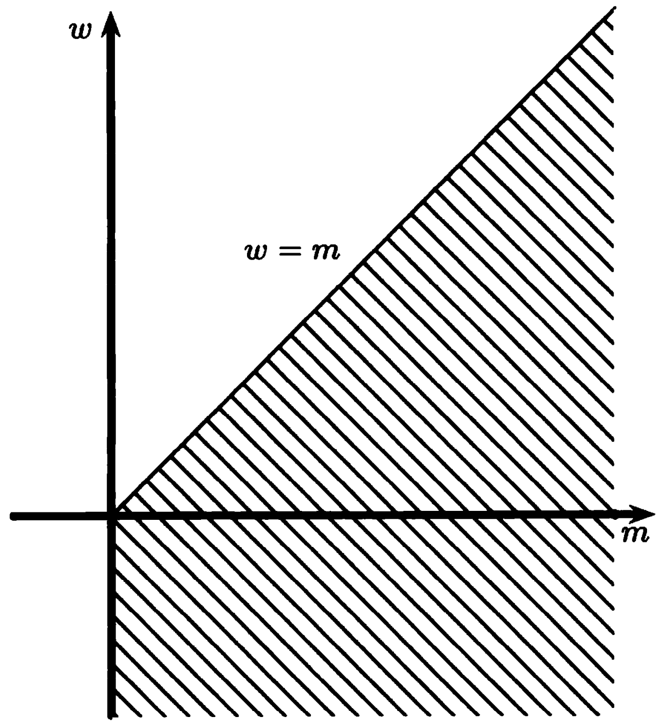

Fig. 7.2.1. Range of $\big(\widehat{M}(T), \widehat{W}(T)\big)$.

Theorem 7.2.1. *The joint density under* $\widetilde{\mathbb{P}}$ *of the pair* $\big(\widehat{M}(T), \widehat{W}(T)\big)$ *is*

$$\widetilde{f}_{\widehat{M}(T),\widehat{W}(T)}(m, w) = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, \quad w \le m, \ m \ge 0, \tag{7.2.3}$$

and is zero for other values of m *and* w.

PROOF: We define the exponential martingale

$$\widehat{Z}(t) = e^{-\alpha \widetilde{W}(t) - \frac{1}{2}\alpha^2 t} = e^{-\alpha \widehat{W}(t) + \frac{1}{2}\alpha^2 t}, \quad 0 \le t \le T,$$

and use $\widehat{Z}(T)$ to define a new probability measure $\widehat{\mathbb{P}}$ by

$$\widehat{\mathbb{P}}(A) = \int_A Z(T)\, d\widetilde{\mathbb{P}} \text{ for all } A \in \mathcal{F}.$$

According to Girsanov's Theorem, Theorem5.2.3, $\widehat{W}(t)$ is a Brownian motion (with zero drift) under $\widehat{\mathbb{P}}$. Theorem 3.7.3 gives us the joint density of $\big(\widehat{M}(T), \widehat{W}(T)\big)$ under $\widehat{\mathbb{P}}$, which is

$$\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2m-w)^2}, \quad w \le m,\ m \ge 0, \tag{7.2.4}$$

and is zero for other values of m and w. To work out the density of $(\widehat{M}(T), \widehat{W}(T))$ under $\widetilde{\mathbb{P}}$, we use Lemma 5.2.1, which implies

$$\begin{aligned}
&\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m, \widehat{W}(T) \le w\} \\
&\qquad = \widetilde{\mathbb{E}}\big[\mathbb{I}_{\{\widehat{M}(T)\le m, \widehat{W}(T)\le w\}}\big] \\
&\qquad = \widehat{\mathbb{E}}\left[\frac{1}{\widehat{Z}(T)}\mathbb{I}_{\{\widehat{M}(T)\le m, \widehat{W}(T)\le w\}}\right] \\
&\qquad = \widehat{\mathbb{E}}\left[e^{\alpha\widehat{W}(T)-\frac{1}{2}\alpha^2 T}\mathbb{I}_{\{\widehat{M}(T)\le m, \widehat{W}(T)\le w\}}\right] \\
&\qquad = \int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y-\frac{1}{2}\alpha^2 T}\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(x,y)\,dx\,dy.
\end{aligned}$$

Therefore, the density of $(\widehat{M}(T), \widehat{W}(T))$ under $\widetilde{\mathbb{P}}$ is

$$\frac{\partial^2}{\partial m\,\partial w}\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m, \widehat{W}(T) \le w\} = e^{\alpha w-\frac{1}{2}\alpha^2 T}\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(m,w). \tag{7.2.5}$$

When $w \le m$ and $m \ge 0$, this is formula (7.2.3). For other values of m and w, we obtain zero because $\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(m,w)$ is zero. □

Corollary 7.2.2. *We have*

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\} = N\left(\frac{m-\alpha T}{\sqrt{T}}\right) - e^{2\alpha m}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right), \quad m \ge 0, \tag{7.2.6}$$

and the density under $\widetilde{\mathbb{P}}$ of the random variable $\widehat{M}(T)$ is

$$\widetilde{f}_{\widehat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right), \ m \ge 0, \tag{7.2.7}$$

and is zero for $m < 0$.

PROOF: We integrate the density (7.2.3) over the region in Figure 7.2.2 to compute

$$\begin{aligned}
&\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\} \\
&= \int_0^m \int_w^m \frac{2(2\mu-w)}{T\sqrt{2\pi T}} e^{\alpha w-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2\mu-w)^2}\,d\mu\,dw \\
&\quad + \int_{-\infty}^0 \int_0^m \frac{2(2\mu-w)}{T\sqrt{2\pi T}} e^{\alpha w-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2\mu-w)^2}\,d\mu\,dw
\end{aligned}$$

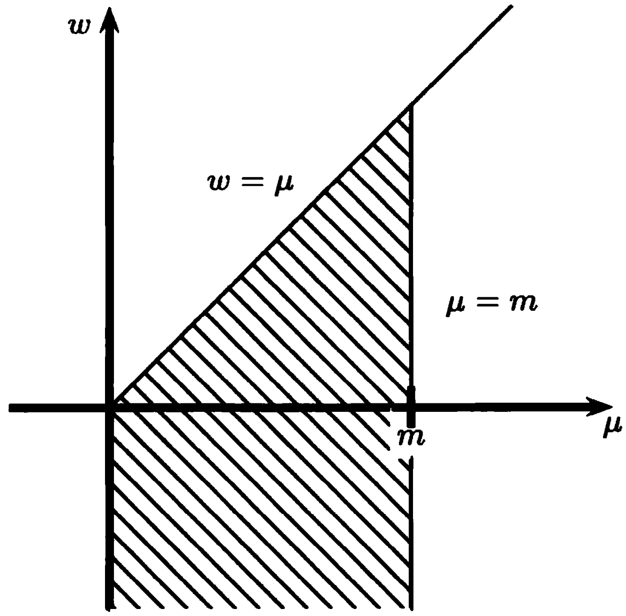

Fig. 7.2.2. The region $\widehat{M}(T) \le m$.

$$= -\int_0^m \frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu - w)^2} \Big|_{\mu=w}^{\mu=m} dw$$
$$- \int_{-\infty}^0 \frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu - w)^2} \Big|_{\mu=0}^{\mu=m} dw$$
$$= -\frac{1}{\sqrt{2\pi T}} \int_0^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_0^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2} dw$$
$$- \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^0 e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^0 e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2} d$$
$$= -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2} dw.$$

We complete the squares. Observe that

$$-\frac{1}{2T}(w - 2m - \alpha T)^2 = -\frac{(2m-w)^2}{2T} + \alpha w - 2\alpha m - \frac{1}{2}\alpha^2 T,$$
$$-\frac{1}{2T}(w - \alpha T)^2 = -\frac{w^2}{2T} + \alpha w - \frac{1}{2}\alpha^2 T.$$

Therefore,

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\}$$
$$= -\frac{e^{2\alpha m}}{\sqrt{2\pi T}} \int_{-\infty}^m e^{-\frac{1}{2T}(w - 2m - \alpha T)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{-\frac{1}{2T}(w-\alpha T)^2} dw.$$

We make the change of variable $y = \frac{w-2m-\alpha T}{\sqrt{T}}$ in the first integral and $y = \frac{w-\alpha T}{\sqrt{T}}$ in the second, thereby obtaining

$$\begin{aligned}\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\} &= -\frac{e^{2\alpha m}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-m-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\,dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\,du \\ &= -e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + N\left(\frac{m-\alpha T}{\sqrt{T}}\right).\end{aligned}$$

This establishes (7.2.6).

To obtain the density (7.2.7), we differentiate (7.2.6) with respect to m:

$$\begin{aligned}&\frac{d}{dm}\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\} \\ &= N'\left(\frac{m-\alpha T}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{T}}\right) - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) \\ &\qquad - e^{2\alpha m} N'\left(\frac{-m-\alpha T}{\sqrt{T}}\right)\left(-\frac{1}{\sqrt{T}}\right) \\ &= \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + \frac{e^{2\alpha m}}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(-m-\alpha T)^2}.\end{aligned}$$

The exponent in the third term is

$$\begin{aligned}2\alpha m - \frac{(-m-\alpha T)^2}{2T} &= \frac{4\alpha m}{2T} - \frac{m^2 + 2\alpha m T + \alpha^2 T^2}{2T} \\ &= -\frac{m^2 - 2\alpha m T + \alpha^2 T^2}{2T} \\ &= -\frac{(m-\alpha T)^2}{2T},\end{aligned}$$

which is the exponent in the first term. Combining the first and third terms, we obtain (7.2.7). □

7.3 Knock-out Barrier Options

There are several types of barrier options. Some "knock out" when the underlying asset price crosses a barrier (i.e., they become worthless). If the underlying asset price begins below the barrier and must cross above it to cause the knock-out, the option is said to be *up-and-out.* A *down-and-out* option has the barrier below the initial asset price and knocks out if the asset price falls below the barrier. Other options "knock in" at a barrier (i.e., they pay off zero unless they cross a barrier). Knock-in options also fall into two classes, *up-and-in* and *down-and-in.* The payoff at expiration for barrier options is typically either that of a put or a call. More complex barrier options require the asset price to not only cross a barrier but spend a certain amount of time across the barrier in order to knock in or knock out.

In this section, we treat an up-and-out call on a geometric Brownian motion. The methodology we develop works equally well for up-and-in, down-and-out, and down-and-in puts and calls.

7.3.1 Up-and-Out Call

Our underlying risky asset is geometric Brownian motion

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),$$

where $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Consider a European call, expiring at time T, with strike price K and up-and-out barrier B. We assume $K < B$; otherwise, the option must knock out in order to be in the money and hence could only pay off zero. The solution to the stochastic differential equation for the asset price is

$$S(t) = S(0)e^{\sigma\widetilde{W}(t)+(r-\frac{1}{2}\sigma^2)t} = S(0)e^{\sigma\widehat{W}(t)}, \tag{7.3.1}$$

where $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$, and

$$\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right).$$

We define $\widehat{M}(T) = \max_{0\le t\le T}\widehat{W}(t)$, so

$$\max_{0\le t\le T} S(t) = S(0)e^{\sigma\widehat{M}(T)}.$$

The option knocks out if and only if $S(0)e^{\sigma\widehat{M}(T)} > B$; if $S(0)e^{\sigma\widehat{M}(T)} \le B$, the option pays off

$$\big(S(T) - K\big)^+ = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)^+.$$

In other words, the payoff of the option is

$$\begin{aligned}
V(T) &= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)^+ \mathbb{I}_{\{S(0)e^{\sigma\widehat{M}(T)}\le B\}}\\
&= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)\mathbb{I}_{\{S(0)e^{\sigma\widehat{W}(T)}\ge K,\, S(0)e^{\sigma\widehat{M}(T)}\le B\}}\\
&= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)\mathbb{I}_{\{\widehat{W}(T)\ge k,\, \widehat{M}(T)\le b\}},
\end{aligned} \tag{7.3.2}$$

where

$$k = \frac{1}{\sigma}\log\frac{K}{S(0)}, \quad b = \frac{1}{\sigma}\log\frac{B}{S(0)}. \tag{7.3.3}$$

7.3.2 Black-Scholes-Merton Equation

The price of an up-and-out call satisfies a Black-Scholes-Merton equation that has been modified to account for the barrier. This equation can be used to solve for the price. In this particular case, we do not need to find the price this way because it can be computed analytically (see Subsection 7.3.3). However, we providc the equation and its derivation because this methodology works in situations where analytical solutions cannot be obtained.

Theorem 7.3.1. *Let $v(t,x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t and $S(t)=x$. Then $v(t,x)$ satisfies the Black-Scholes-Merton partial differential equation*

$$v_t(t,x)+rxv_x(t,x)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x)=rv(t,x) \tag{7.3.4}$$

in the rectangle $\{(t,x); 0\le t<T, 0\le x\le B\}$ and satisfies the boundary conditions

$$v(t,0)=0,\quad 0\le t\le T, \tag{7.3.5}$$
$$v(t,B)=0,\quad 0\le t<T, \tag{7.3.6}$$
$$v(T,x)=(x-K)^+,\quad 0\le x\le B. \tag{7.3.7}$$

The lower boundary condition (7.3.5) follows as in the usual Black-Scholes-Merton framework: If the asset price begins at zero, it stays there and the option expires out of the money. The upper boundary condition follows from the fact that when the geometric Brownian $S(t)$ hits the level B, it immediately rises above B. In fact, because it has nonzero quadratic variation, the asset price $S(t)$ oscillates, rising and falling across the level B infinitely many times immediately after hitting it. The option price is zero when the asset price hits B because the option is on the verge of knocking out. The only exception to this is if the level B is first reached at the expiration time T, for then there is no time left for the knock-out. In this case, the option price is given by the terminal condition (7.3.7). In particular, the function $v(t,x)$ is not continuous at the corner of its domain where $t=T$ and $x=B$. It is continuous everywhere else in the rectangle $\{(t,x); 0\le t\le T, 0\le x\le B\}$.

Exercise 7.8 outlines the steps to verify the Black-Scholes-Merton equation by direct computation, starting with the analytical formula (7.3.20) obtained in Subsection 7.3.3. Here we derive this partial differential equation (7.3.4) by the simpler but more generally applicable argument used previously: (1) find the martingale, (2) take the differential, and (3) set the dt term equal to zero.

Let us begin with an initial asset price $S(0)\in(0,B)$. We then define the option payoff $V(T)$ by (7.3.2). The price of the option at time t between initiation and expiration is given by the risk-neutral pricing formula

$$V(t)=\widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)\Big|\mathcal{F}(t)\right],\quad 0\le t\le T. \tag{7.3.8}$$

The usual iterated conditioning argument (e.g., (5.3.3)) shows that

$$e^{-rt}V(t)=\widetilde{\mathbb{E}}\left[e^{-rT}V(T)\big|\mathcal{F}(t)\right],\quad 0\le t\le T, \tag{7.3.9}$$

is a martingale. We would like to use the Markov property as we did in Example 6.4.4 to say that $V(t)=v\big(t,S(t)\big)$, where $v(t,x)$ is the function in Theorem 7.3.1. However, this equation does not hold for all values of t along all paths. Recall that $v(t,S(t))$ is the value of the option under the assumption that it has not knocked out prior to t, whereas $V(t)$ is the value of the option without any assumption. In particular, if the underlying asset price rises above the barrier B and then returns below the barrier by time t, then $V(t)$ will be zero because the option has knocked out, but $v\big(t, S(t)\big)$ will be strictly positive because $v(t,x)$ given by (7.3.20) is strictly positive for all values of $0 \le t < T$ and $0 < x < B$. The process $V(t)$ is path-dependent and remembers that the option has knocked out. The process $v\big(t, S(t)\big)$ is not path-dependent, and when $S(t) < B$, it gives the price of the option under the assumption that it has not knocked out, *even if that assumption is incorrect.*

We resolve this annoyance by defining ρ to be the first time t at which the asset price reaches the barrier B. In other words, ρ is chosen in a path-dependent way so that $S(t) < B$ for $0 \le t \le \rho$ and $S(\rho) = B$. Since the asset price almost surely exceeds the barrier immediately after reaching it, we may regard ρ as the time of knock-out. If the asset price does not reach the barrier before expiration, we set $\rho = \infty$. If the asset price first reaches the barrier at time T, then $\rho = T$ but knock-out does not occur because there is no time left for the asset price to exceed the barrier. However, the probability that the asset price first reaches the barrier at time T is zero, so this anomaly does not matter.

The random variable ρ is a *stopping time* because it chooses its value based on the path of the asset price up to time ρ. Stopping times in the binomial model were defined in Definition 4.3.1 of Volume I. The Optional Sampling Theorem, Theorem 4.3.2 of Volume I, asserts that a martingale stopped at a stopping time is still a martingale. The same is true in continuous time. In particular, the process

$$e^{-r(t\wedge\rho)}V(t\wedge\rho) = \begin{cases} e^{-rt}V(t) & \text{if } 0 \le t \le \rho, \\ e^{-r\rho}V(\rho) & \text{if } \rho < t \le T, \end{cases} \tag{7.3.10}$$

is a $\widetilde{\mathbb{P}}$-martingale. Before t gets to ρ, this is just the martingale $e^{-rt}V(t)$. Once t gets to ρ, although the time parameter t can march on, the value of the process is frozen at $e^{-r\rho}V(\rho)$. A process that does not move is trivially a martingale. The only way the martingale property could be ruined would be if ρ "looked ahead" when deciding to stop the process. If ρ stopped at a time because the process was about to go up and let the process continue if it was about to go down, the stopped process would have a downward tendency. So long as ρ makes the decision to stop at the current time based only on the path up to and perhaps including the current time, the act of stopping a martingale at time ρ preserves the martingale property.

Lemma 7.3.2. *We have*

$$V(t) = v\big(t, S(t)\big), \quad 0 \le t \le \rho. \tag{7.3.11}$$

In particular, $e^{-rt}v\big(t, S(t)\big)$ *is a* $\widetilde{\mathbb{P}}$*-martingale up to time* ρ*, or, put another way, the* stopped process

$$e^{-r(t\wedge\rho)}v\big(t\wedge\rho, S(t\wedge\rho)\big), \quad 0 \le t \le T, \tag{7.3.12}$$

is a martingale under $\widetilde{\mathbb{P}}$.

SKETCH OF PROOF: Because $v(t, S(t))$ is the value of the up-and-out call under the assumption that it has not knocked out before time t, and for $t \le \rho$ this assumption is correct, we have (7.3.11) for $t \le \rho$. From (7.3.11), we conclude that the process in (7.3.12) is the $\widetilde{\mathbb{P}}$-martingale (7.3.10). □

PROOF OF THEOREM 7.3.1: We compute the differential

$$\begin{aligned} d\big(e^{-rt}v(t,S(t))\big) &= e^{-rt}\Big[-rv(t,S(t))\,dt + v_t(t,S(t))\,dt + v_x(t,S(t))\,dS(t) \\ &\qquad + \frac{1}{2}v_{xx}(t,S(t))\,dS(t)\,dS(t)\Big] \\ &= e^{-rt}\Big[-rv(t,S(t)) + v_t(t,S(t)) + rS(t)v_x(t,S(t)) \\ &\qquad + \frac{1}{2}\sigma^2 S^2(t)v_{xx}(t,S(t))\Big]\,dt \\ &\quad + e^{-rt}\sigma S(t)v_x(t,S(t))\,d\widetilde{W}(t). \end{aligned} \tag{7.3.13}$$

The dt term must be zero for $0 \le t \le \rho$, (i.e., before the option knocks out). But since $(t, S(t))$ can reach any point in $\{(t,x); 0 \le t < T, 0 \le x \le B\}$ before the option knocks out, the Black-Scholes-Merton equation (7.3.4) must hold for every $t \in [0,T)$ and $x \in [0,B]$. □

Remark 7.3.3. From Theorem 7.3.1 and its proof, we see how to construct a hedge, at least theoretically. Setting the dt term in (7.3.13) equal to zero, we obtain

$$d\big(e^{-rt}v(t,S(t))\big) = e^{-rt}\sigma S(t)v_x(t,S(t))\,d\widetilde{W}(t), \quad 0 \le t \le \rho. \tag{7.3.14}$$

The discounted value of a portfolio that at each time t holds $\Delta(t)$ shares of the underlying asset is given by (see (5.2.27))

$$d\big(e^{-rt}X(t)\big) = e^{-rt}\sigma S(t)\Delta(t)\,d\widetilde{W}(t).$$

At least theoretically, if an agent begins with a short position in the up-and-out call and with initial capital $X(0) = v(0, S(0))$, then the usual delta-hedging formula

$$\Delta(t) = v_x(t, S(t)) \tag{7.3.15}$$

will cause her portfolio value $X(t)$ to track the option value $v(t, S(t))$ up to the time ρ of knock-out or up to expiration T, whichever comes first.

In practice, the delta hedge is impossible to implement if the option has not knocked out and the underlying asset price approaches the barrier near expiration of the option. The function $v(T,x)$ is discontinuous at $x = B$, jumping from $B - K$ to 0 at that point. For t near T and x just below B, the function $v(t,x)$ is approaching a discontinuity and has large negative delta $v_x(t,x)$ and large negative gamma $v_{xx}(t,x)$ values. Near expiration near the barrier, the delta-hedging formula (7.3.15) requires the agent to take a large short position in the underlying asset and to make large adjustments in the position (because of the large negative gamma) whenever the asset price moves. The Black-Scholes-Merton model assumes the bid–ask spread is zero, and here that assumption is a poor model of reality. The delta-hedging formula calls for such a large amount of trading that the bid–ask spread becomes significant. The common industry practice is to price and hedge the up-and-out call as if the barrier were at a level slightly higher than B. In this way, the large delta and gamma values of the option occur in the region above the contractual barrier B, and the hedging position will be closed out upon knock-out at the contractual barrier before the asset price reaches this region. □

7.3.3 Computation of the Price of the Up-and-Out Call

The risk-neutral price at time zero of the up-and-out call with payoff $V(T)$ given by (7.3.2) is $V(0) = \widetilde{\mathbb{E}}[e^{-rT}V(T)]$. We use the density formula (7.2.3) to compute this. If $k \geq 0$, we must integrate over the region $\{(m,w); k \leq w \leq m \leq b\}$. On the other hand, if $k < 0$, we integrate over the region $\{(m,w); k \leq w \leq m, 0 \leq m \leq b\}$. In both cases, the region can be described as $\{(m,w); k \leq w \leq b, w^+ \leq m \leq b\}$; see Figure 7.3.1. We assume here that $S(0) \leq B$ so that $b > 0$. Otherwise, the region over which we integrate has zero area, and the time-zero value of the call is zero rather than the integral computed below. We also assume $S(0) > 0$ so that b and k are finite.

When $0 < S(0) \leq B$, the time-zero value of the up-and-out call is

$$
\begin{aligned}
V(0) &= \int_k^b \int_{w^+}^b e^{-rT}\big(S(0)e^{\sigma w} - K\big)\frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\, dm\, dw \\
&= -\int_k^b e^{-rT}\big(S(0)e^{\sigma w} - K\big)\frac{1}{\sqrt{2\pi T}}\, e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\Big|_{m=w^+}^{m=b} dw \\
&= \frac{1}{\sqrt{2\pi T}} \int_k^b \big(S(0)e^{\sigma w} - K\big)e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2}\, dw \\
&\quad - \frac{1}{\sqrt{2\pi T}} \int_k^b \big(S(0)e^{\sigma w} - K\big)e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2}\, dw \\
&= S(0)I_1 - KI_2 - S(0)I_3 + KI_4,
\end{aligned}
$$

where

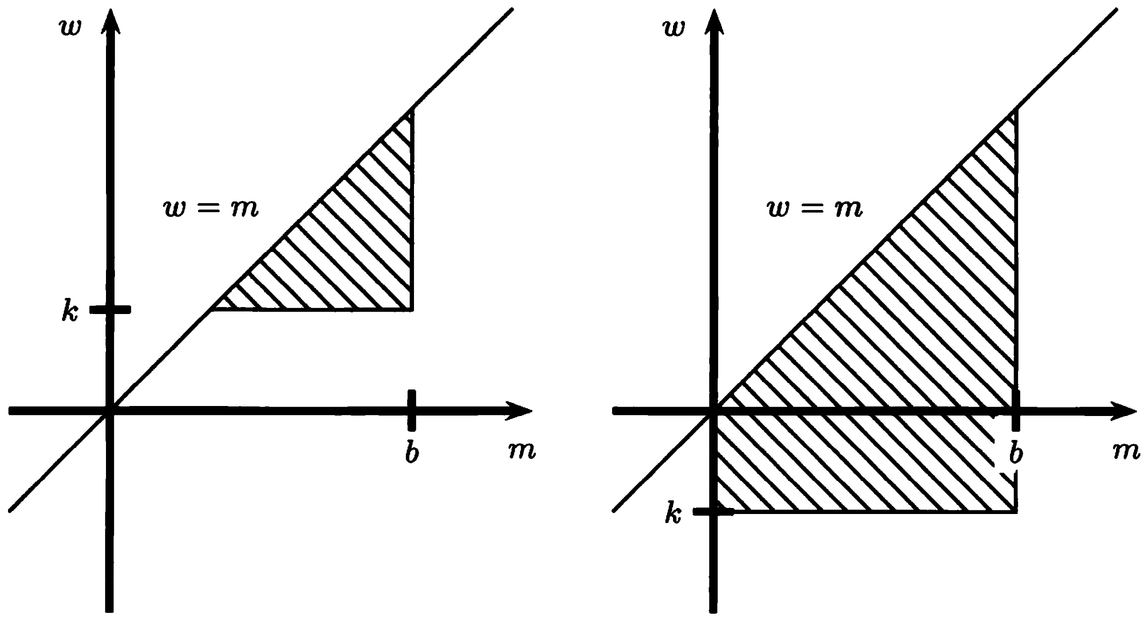

Fig. 7.3.1. Regions of integration for $k \geq 0$ and $k < 0$.

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2}\, dw,\\
I_2 &= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2}\, dw,\\
I_3 &= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2}\, dw,\\
&= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{2}{T}b^2 + \frac{2}{T}bw - \frac{1}{2T} w^2}\, dw,\\
I_4 &= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2}\, dw\\
&= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{2}{T}b^2 + \frac{2}{T}bw - \frac{1}{2T} w^2}\, dw.
\end{aligned}
$$

Each of these integrals is of the form

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta + \gamma w - \frac{1}{2T} w^2}\, dw &= \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-\frac{1}{2T}(w-\gamma T)^2 + \frac{1}{2}\gamma^2 T + \beta}\, dw\\
&= e^{\frac{1}{2}\gamma^2 T + \beta} \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{T}}(k-\gamma T)}^{\frac{1}{\sqrt{T}}(b-\gamma T)} e^{-\frac{1}{2}y^2}\, dy, \quad (7.3.16)
\end{aligned}
$$

where we have made the change of variable $y = \frac{w-\gamma T}{\sqrt{T}}$. Using the standard cumulative normal distribution property $N(z) = 1 - N(-z)$ and (7.3.3), we continue, writing

$$\begin{aligned}
\frac{1}{\sqrt{2\pi T}} &\int_k^b e^{\beta+\gamma w-\frac{w^2}{2T}}\, dw \\
&= e^{\frac{1}{2}\gamma^2 T+\beta}\left[N\left(\frac{b-\gamma T}{\sqrt{T}}\right) - N\left(\frac{k-\gamma T}{\sqrt{T}}\right)\right] \\
&= e^{\frac{1}{2}\gamma^2 T+\beta}\left[N\left(\frac{-k+\gamma T}{\sqrt{T}}\right) - N\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right] \\
&= e^{\frac{1}{2}\gamma^2 T+\beta}\left[N\left(\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K}+\gamma\sigma T\right]\right)\right. \\
&\qquad\qquad \left. -N\left(\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{B}+\gamma\sigma T\right]\right)\right]. \qquad (7.3.17)
\end{aligned}$$

Set

$$\delta_\pm(\tau, s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]. \qquad (7.3.18)$$

The integral I_1 is of the form (7.3.17) with $\beta = -rT - \frac{1}{2}\alpha^2 T$ and $\gamma = \alpha + \sigma$, so $\frac{1}{2}\gamma^2 T + \beta = 0$ and $\gamma\sigma = r + \frac{1}{2}\sigma^2$. Therefore,

$$I_1 = N\left(\delta_+\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_+\left(T, \frac{S(0)}{B}\right)\right).$$

The integral I_2 is of the form (7.3.17) with $\beta = -rT - \frac{1}{2}\alpha^2 T$ and $\gamma = \alpha$, so $\frac{1}{2}\gamma^2 T + \beta = -rT$ and $\gamma\sigma = r - \frac{1}{2}\sigma^2$. Therefore,

$$I_2 = e^{-rT}\left[N\left(\delta_-\left(T, \frac{S(0)}{K}\right)\right) - N\left(d_-\left(T, \frac{S(0)}{B}\right)\right)\right].$$

For I_3, we have $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2b^2}{T}$ and $\gamma = \alpha + \sigma + \frac{2b}{T}$, so

$$\begin{aligned}
\frac{1}{2}\gamma^2 T + \beta &= \log\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}-1}, \\
\gamma\sigma T &= \left(r + \frac{1}{2}\sigma^2\right)T + \log\left(\frac{B}{S(0)}\right)^2.
\end{aligned}$$

Therefore,

$$I_3 = \left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}-1}\left[N\left(\delta_+\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta_+\left(T, \frac{B}{S(0)}\right)\right)\right].$$

Finally, for I_4, we have $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2b^2}{T}$ and $\gamma = \alpha + \frac{2b}{T}$, so

$$\begin{aligned}
\frac{1}{2}\gamma^2 T + \beta &= -rT + \log\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}, \\
\gamma\sigma T &= \left(r - \frac{1}{2}\sigma^2\right)T + \log\left(\frac{B}{S(0)}\right)^2.
\end{aligned}$$

Therefore,

$$I_4 = e^{-rT}\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T,\frac{B^2}{KS(0)}\right)\right) - N\left(\delta_-\left(T,\frac{B}{S(0)}\right)\right)\right].$$

Putting all this together, under the assumption $0 < S(0) \le B$, we have the up-and-out call price formula

$$\begin{aligned}
&V(0)\\
&= S(0)\left[N\left(\delta_+\left(T,\frac{S(0)}{K}\right)\right) - N\left(\delta_+\left(T,\frac{S(0)}{B}\right)\right)\right]\\
&\quad -e^{-rT}K\left[N\left(\delta_-\left(T,\frac{S(0)}{K}\right)\right) - N\left(\delta_-\left(T,\frac{S(0)}{B}\right)\right)\right]\\
&\quad -B\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(T,\frac{B^2}{KS(0)}\right)\right) - N\left(\delta_+\left(T,\frac{B}{S(0)}\right)\right)\right]\\
&\quad +e^{-rT}K\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T,\frac{B^2}{KS(0)}\right)\right) - N\left(\delta_-\left(T,\frac{B}{S(0)}\right)\right)\right].
\end{aligned} \tag{7.3.19}$$

Now let $t \in [0,T)$ be given, and assume the underlying asset price at time t is $S(t) = x$. As above, we assume $0 < x \le B$. If the call has not knocked out prior to time t, its price at time t is obtained by replacing T by the time to expiration $\tau = T - t$ and replacing $S(0)$ by x in (7.3.19). This gives us the call price as a function $v(t,x)$ of the two variables t and x:

$$\begin{aligned}
v(t,x) &= x\left[N\left(\delta_+\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\right]\\
&\quad -e^{-r\tau}K\left[N\left(\delta_-\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_-\left(\tau,\frac{x}{B}\right)\right)\right]\\
&\quad -B\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right) - N\left(\delta_+\left(\tau,\frac{B}{x}\right)\right)\right]\\
&\quad +e^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau,\frac{B^2}{Kx}\right)\right) - N\left(\delta_-\left(\tau,\frac{B}{x}\right)\right)\right],\\
&\qquad 0 \le t < T,\ 0 < x \le B.
\end{aligned} \tag{7.3.20}$$

Formula (7.3.20) was derived under the assumption that $\tau > 0$ (i.e., $t < T$) and $0 < x \le B$. For $0 \le t \le T$ and $x > B$, we have $v(t,x) = 0$ because the option knocks out when the asset price exceeds the barrier B. Indeed, if the asset price reaches the barrier before expiration, then it will immediately exceed the barrier almost surely, and so $v(t,B) = 0$ for $0 \le t < T$. However, $v(T,B) = B - K$. We also have $v(t,0) = 0$ because geometric Brownian motion starting at 0 stays at zero, and hence the call expires out of the money. Finally, if the option does not knock out prior to expiration, then its payoff is that of a European call (i.e., $v(T,x) = (x-K)^+$). In summary, $v(t,x)$ satisfies the boundary conditions (7.3.5)–(7.3.7). Formula (7.3.6) can be obtained by substitution of $x = B$ in (7.3.20), but for $x > B$, the right-hand side of (7.3.20) is not $v(t,x) = 0$. Formula (7.3.20) was derived under the assumption $0 < x \leq B$, and it is incorrect if $x > B$. Formulas (7.3.5) and (7.3.7) cannot be obtained by substitution of $x = 0$ and $t = T$ ($\tau = 0$) into (7.3.20) because this leads to zeroes in denominators, but it can be shown that (7.3.20) gives these formulas as limits as $x \downarrow 0$ and $\tau \downarrow 0$; see Exercise 7.2.

7.4 Lookback Options

An option whose payoff is based on the maximum that the underlying asset price attains over some interval of time prior to expiration is called a *lookback option*. In this section we price a *floating strike lookback option*. The payoff of this option is the difference between the maximum asset price over the time between initiation and expiration and the asset price at expiration. The discussion of this option introduces a new type of differential, a differential that is neither dt nor $d\widetilde{W}(t)$.

7.4.1 Floating Strike Lookback Option

We begin with a geometric Brownian motion asset price, which may be written as in (7.3.1) as

$$S(t) = S(0)e^{\sigma\widehat{W}(t)}, \tag{7.4.1}$$

where, as in Subsection 7.3.1, $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$ and

$$\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right).$$

With

$$\widehat{M}(t) = \max_{0 \leq u \leq t} \widehat{W}(u), \quad 0 \leq t \leq T, \tag{7.4.2}$$

we may write the maximum of the asset price up to time t as

$$Y(t) = \max_{0 \leq u \leq t} S(u) = S(0)e^{\sigma\widehat{M}(t)}. \tag{7.4.3}$$

The lookback option considered in this section pays off

$$V(T) = Y(T) - S(T) \tag{7.4.4}$$

at expiration time T. This payoff is nonnegative because $Y(T) \geq S(T)$.

Let $t \in [0,T]$ be given. At time t, the risk-neutral price of the lookback option is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(Y(T) - S(T))\Big|\mathcal{F}(t)\right]. \tag{7.4.5}$$

Because the pair of processes $\big(S(t), Y(t)\big)$ has the Markov property (see Exercise 7.3), there must exist a function $v(t,x,y)$ such that

$$V(t) = v\big(t, S(t), Y(t)\big).$$

In Subsection 7.4.2, we characterize this function by the Black-Scholes-Merton equation. In Subsection 7.4.3, we compute it explicitly.

7.4.2 Black-Scholes-Merton Equation

Theorem 7.4.1. *Let $v(t,x,y)$ denote the price at time t of the floating strike lookback option under the assumption that $S(t) = x$ and $Y(t) = y$. Then $v(t,x,y)$ satisfies the Black-Scholes-Merton partial differential equation*

$$v_t(t,x,y) + rxv_x(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y) \tag{7.4.6}$$

in the region $\{(t,x,y); 0 \le t < T,\ 0 \le x \le y\}$ and satisfies the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}y, \quad 0 \le t \le T,\ y \ge 0, \tag{7.4.7}$$
$$v_y(t,y,y) = 0, \quad 0 \le t \le T,\ y > 0, \tag{7.4.8}$$
$$v(T,x,y) = y - x, \quad 0 \le x \le y. \tag{7.4.9}$$

Iterated conditioning implies that $e^{-rt}V(t) = e^{-rt}v\big(t, S(t), Y(t)\big)$, where $V(t)$ is given by (7.4.5), is a martingale under $\widetilde{\mathbb{P}}$. We compute its differential and set the dt term equal to zero to obtain (7.4.6). However, when we do this, the term $dY(t)$ appears. This is different from the term $dS(t)$, because $S(t)$ has nonzero quadratic variation, whereas $Y(t)$ has zero quadratic variation. This is because $Y(t)$ is continuous and nondecreasing in t. Let $0 = t_0 < t_1 < \cdots < t_m = T$ be a partition of $[0,T]$. Then

$$\begin{aligned}
&\sum_{j=1}^{m} \big(Y(t_j) - Y(t_{j-1})\big)^2 \\
&\le \max_{j=1,\dots,m} \big(Y(t_j) - Y(t_{j-1})\big) \sum_{j=1}^{m} \big(Y(t_j) - Y(t_{j-1})\big) \\
&= \max_{j=1,\dots,m} \big(Y(t_j) - Y(t_{j-1})\big) \cdot \big(Y(T) - Y(0)\big),
\end{aligned} \tag{7.4.10}$$

and $\max_{j=1,\dots,m}\big(Y(t_j) - Y(t_{j-1})\big)$ has limit zero as $\max_{j=1,\dots,m}(t_j - t_{j-1})$ goes to zero because $Y(t)$ is continuous. We conclude that $Y(t)$ accumulates zero quadratic variation on $[0,T]$, a fact we record by writing

$$dY(t)\,dY(t) = 0. \tag{7.4.11}$$

This argument works because $Y(t_j) - Y(t_{j-1})$ is nonnegative, and hence we do not need to take the absolute value of these terms in (7.4.10). This argument shows that on any interval in which a function is continuous and nondecreasing, it will accumulate zero quadratic variation.

On the other hand, $dY(t)$ is not a dt term: there is no process $\Theta(t)$ such that $dY(t) = \Theta(t)\,dt$. In other words, we cannot write $Y(t)$ as

$$Y(t) = Y(0) + \int_0^t \Theta(u)\,du. \tag{7.4.12}$$

If we could, then $\Theta(u)$ would be zero whenever u is in a "flat spot" of $Y(t)$, which occurs whenever $S(t)$ drops below its maximum to date (see Figure 7.4.1). Figure 7.4.1 suggests that there are time intervals in which $Y(t)$ is strictly increasing, but in fact no such interval exists. Such an interval can occur only if $S(t)$ is strictly increasing on the interval, and if there were such an interval, then $S(t)$ would accumulate zero quadratic variation on the interval (see the argument in the previous paragraph). This is not the case because $dS(t)\,dS(t) = \sigma S^2(t)\,dt$ is positive for all t. Thus, despite the suggestion of Figure 7.4.1, the lengths of the "flat spots" of $Y(t)$ on any time interval $[0,T]$ sum to T. Therefore, if (7.4.12) were to hold, we would need to have $\Theta(u) = 0$ for Lebesgue almost every u in $[0,T]$. This would result in $Y(t) = Y(0)$ for $0 \leq t \leq T$. But in fact $Y(t) > Y(0)$ for all $t > 0$. We conclude that $Y(t)$ cannot be represented in the form (7.4.12); $dY(t)$ is not a dt term.

The paths of $Y(t)$ increase over time, but they do so on a set of times having zero Lebesgue measure. Each time interval $[0,T]$ contains a sequence of subintervals whose lengths sum to T, and on each of these subintervals, $Y(t)$ is constant. The particular subintervals depend on the path, but regardless of the path, the lengths of these subintervals sum to T. A similar situation is described in Appendix A, Section A.3. In the case discussed there, $T = 1$ and the subintervals are explicitly exhibited. Their union is the Cantor set. It is verified that although the lengths of these subintervals sum to 1, there are uncountably many points not contained in these intervals. The function $F(x)$ described in Section A.3 increases, but only on the complement of the Cantor set. Furthermore, $F(x)$ is continuous. Functions of this kind are said to be *singularly continuous.*

Fortunately, we can work with the differential of $Y(t)$. We have already argued that $dY(t)\,dY(t) = 0$. Similarly, we have

$$dY(t)\,dS(t) = 0 \tag{7.4.13}$$

(see Exercise 7.4). We now provide the proof of Theorem 7.4.1.

PROOF OF THEOREM 7.4.1: We use the Itô-Doeblin formula and (7.4.11) and (7.4.13) to differentiate the martingale $e^{-rt}v\big(t, S(t), Y(t)\big)$ to obtain

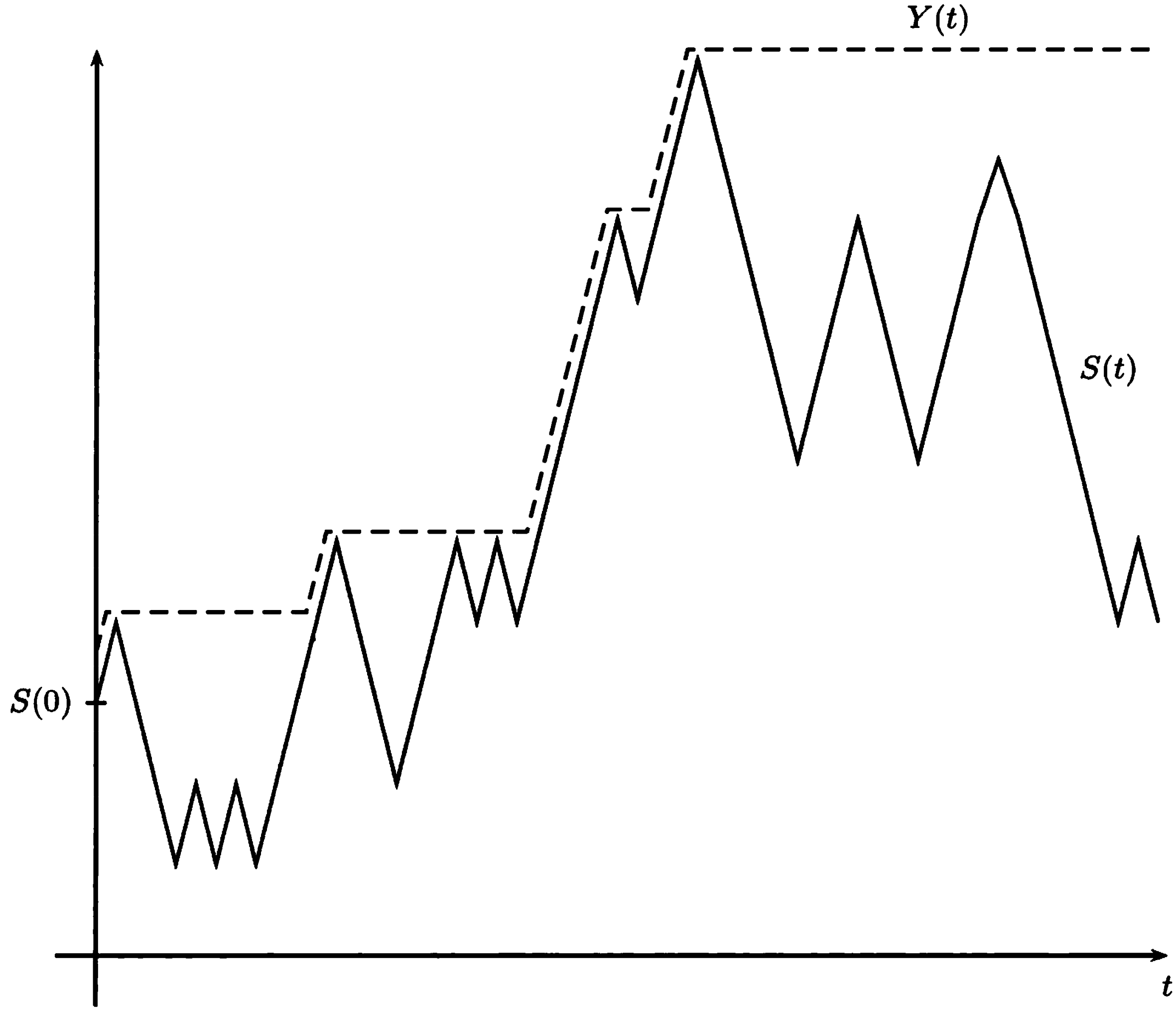

Fig. 7.4.1. Geometric Brownian motion and its maximum to date.

$$
\begin{aligned}
&d\left(e^{-rt}v\big(t,S(t),Y(t)\big)\right)\\
&= e^{-rt}\Bigg[-rv\big(t,S(t),Y(t)\big)\,dt + v_t\big(t,S(t),Y(t)\big)\,dt\\
&\qquad +v_x\big(t,S(t),Y(t)\big)\,dS(t) + \frac{1}{2}v_{xx}\big(t,S(t),Y(t)\big)\,dS(t)\,dS(t)\\
&\qquad +v_y\big(t,S(t),Y(t)\big)\,dY(t)\Bigg]\\
&= e^{-rt}\Bigg[-rv\big(t,S(t),Y(t)\big) + v_t\big(t,S(t),Y(t)\big) + rS(t)v_x\big(t,S(t),Y(t)\big)\\
&\qquad +\frac{1}{2}\sigma^2S^2(t)v_{xx}\big(t,S(t),Y(t)\big)\Bigg]\,dt\\
&\quad +e^{-rt}\sigma S(t)v_x\big(t,S(t),Y(t)\big)\,d\widetilde{W}(t)\\
&\quad +e^{-rt}v_y\big(t,S(t),Y(t)\big)\,dY(t). \qquad (7.4.14)
\end{aligned}
$$

In order to have a martingale, the dt term must be zero, and this gives us the Black-Scholes-Merton equation (7.4.6). The new feature is that the term

$e^{-rt}v_y(t,S(t),Y(t))\,dY(t)$ must also be zero. It cannot be canceled by the dt term nor by the $d\widetilde{W}(t)$ term because it is fundamentally different from both of these terms. The $dY(t)$ term is naturally zero on the "flat spots" of $Y(t)$ (i.e., when $S(t) < Y(t)$). However, at the times when $Y(t)$ increases, which are the times when $S(t) = Y(t)$, the term $e^{-rt}v_y\big(t,S(t),Y(t)\big)$ must be zero because $dY(t)$ is "positive." This gives us the boundary condition (7.4.8).

The boundary condition (7.4.9) is the payoff of the option. If at any time t we have $S(t) = 0$, then we will have $S(T) = 0$. Furthermore, Y will be constant on $[t,T]$; if $Y(t) = y$, then $Y(T) = y$ and the price of the option at time t is this value discounted from T back to t. This gives us the boundary condition (7.4.7). □

Remark 7.4.2. The proof of Theorem 7.4.1 shows that

$$d\left(e^{-rt}v\big(t,S(t),Y(t)\big)\right) = e^{-rt}\sigma S(t)v_x\big(t,S(t),Y(t)\big)\,d\widetilde{W}(t).$$

Just as in Remark 7.3.3, this equation implies that the delta-hedging formula (7.3.15) works. In contrast to the situation in Remark 7.3.3, here the function $v(t,x,y)$ is continuous and we have no problems with large delta and gamma values. □

7.4.3 Reduction of Dimension

The price of the floating strike lookback option has a linear scaling property:

$$v(t,\lambda x,\lambda y) = \lambda v(t,x,y) \text{ for all } \lambda > 0. \tag{7.4.15}$$

This is because scaling both $S(t)$ and $Y(t)$ by the same positive constant at a time t prior to expiration results in the payoff $Y(T) - S(T)$ being scaled by the same constant. In particular, if we know the function of two variables

$$u(t,z) = v(t,z,1), \quad 0 \le t \le T,\ 0 \le z \le 1, \tag{7.4.16}$$

then we can easily determine the function of three variables $v(t,x,y)$ by the formula

$$v(t,x,y) = yv\left(t,\frac{x}{y},1\right) = yu\left(t,\frac{x}{y}\right), \quad 0 \le t \le T,\ 0 \le x \le y,\ y > 0. \tag{7.4.17}$$

From (7.4.17), we can compute the partial derivatives:

$$\begin{aligned} v_t(t,x,y) &= yu_t\left(t,\frac{x}{y}\right), \\ v_x(t,x,y) &= yu_z\left(t,\frac{x}{y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right) \;=\; u_z\left(t,\frac{x}{y}\right), \end{aligned}$$

$$v_{xx}(t,x,y) = u_{zz}\left(t,\frac{x}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{y}u_{zz}\left(t,\frac{x}{y}\right),$$

$$\begin{aligned} v_y(t,x,y) &= u\left(t,\frac{x}{y}\right) + yu_z\left(t,\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right) \\ &= u\left(t,\frac{x}{y}\right) - \frac{x}{y}u_z\left(t,\frac{x}{y}\right). \end{aligned}$$

Substitution into the Black-Scholes-Merton equation (7.4.6) yields

$$\begin{aligned} 0 &= -rv(t,x,y) + v_t(t,x,y) + rxv_x(t,x,y) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) \\ &= y\left[-ru\left(t,\frac{x}{y}\right) + u_t\left(t,\frac{x}{y}\right) + r\left(\frac{x}{y}\right)u_z\left(t,\frac{x}{y}\right)\right. \\ &\qquad \left. + \frac{1}{2}\sigma^2\left(\frac{x}{y}\right)^2 u_{zz}\left(t,\frac{x}{y}\right)\right]. \end{aligned}$$

Canceling y and making the change of variable $z = \frac{x}{y}$, we see that $u(t,z)$ satisfies the Black-Scholes-Merton equation

$$u_t(t,z) + rzu_z(t,z) + \frac{1}{2}\sigma^2z^2u_{zz}(t,z) = ru(t,z), \quad 0 \le t < T,\ 0 < z < 1. \tag{7.4.18}$$

Boundary conditions for $u(t,z)$ can be obtained from the boundary conditions (7.4.7)–(7.4.9) for $v(t,x,y)$. In particular,

$$e^{-r(T-t)}y = v(t,0,y) = yu(t,0)$$

implies

$$u(t,0) = e^{-r(T-t)}, \quad 0 \le t \le T. \tag{7.4.19}$$

Furthermore,

$$0 = v_y(t,y,y) = u(t,1) - u_z(t,1)$$

implies

$$u(t,1) = u_z(t,1), \quad 0 \le t < T. \tag{7.4.20}$$

Finally,

$$y - x = v(T,x,y) = yu\left(T,\frac{x}{y}\right)$$

implies

$$u(T,z) = 1 - z, \quad 0 \le z \le 1. \tag{7.4.21}$$

Equation (7.4.18) and the boundary conditions (7.4.19)–(7.4.21) uniquely determine the function $u(t,z)$. As a consequence, we see that the Black-Scholes-Merton equation and boundary conditions in Theorem 7.4.1 uniquely determine the function $v(t,x,y)$.

7.4.4 Computation of the Price of the Lookback Option

In this subsection, we compute the function $v(t,x,y)$ of Theorem 7.4.1. We do this for $0 \le t < T$ and $0 < x \le y$. Because $Y(t) \ge S(t)$ for all t, we do not need to compute values of $v(t,x,y)$ for $x > y$. The reader is invited in Exercise 7.5 to compute the partial derivatives of $v(t,x,y)$ and verify that the Black-Scholes-Merton equation and boundary conditions in Theorem 7.4.1 are satisfied.

For $0 \le t < T$ and $\tau = T - t$, we observe that

$$Y(T) = S(0)e^{\sigma \widehat{M}(t)} e^{\sigma(\widehat{M}(T) - \widehat{M}(t))} = Y(t)e^{\sigma(\widehat{M}(T)-\widehat{M}(t))}.$$

If $\max_{t \le u \le T} \widehat{W}(u) > \widehat{M}(t)$ (i.e., if $\widehat{W}$ attains a new maximum in $[t,T]$), then

$$\widehat{M}(T) - \widehat{M}(t) = \max_{t \le u \le T} \widehat{W}(u) - \widehat{M}(t).$$

On the other hand, if $\max_{t \le u \le T} \widehat{W}(u) \le \widehat{M}(t)$, then $\widehat{M}(T) = \widehat{M}(t)$ and

$$\widehat{M}(T) - \widehat{M}(t) = 0.$$

In either case, we have

$$\begin{aligned}
\widehat{M}(T) - \widehat{M}(t) &= \left[\max_{t \le u \le T} \widehat{W}(u) - \widehat{M}(t)\right]^+ \\
&= \left[\max_{t \le u \le T} \left(\widehat{W}(u) - \widehat{W}(t)\right) - \left(\widehat{M}(t) - \widehat{W}(t)\right)\right]^+.
\end{aligned}$$

Multiplying this equation by σ and using (7.4.1) and (7.4.3), we obtain

$$\sigma\left(\widehat{M}(T) - \widehat{M}(t)\right) = \left[\max_{t \le u \le T} \sigma\left(\widehat{W}(u) - \widehat{W}(t)\right) - \log \frac{Y(t)}{S(t)}\right]^+. \tag{7.4.22}$$

Therefore, $V(t)$ in (7.4.5) is

$$\begin{aligned}
V(t) = e^{-r\tau}\widetilde{\mathbb{E}}&\left[Y(t)\exp\left\{\left[\max_{t \le u \le T} \sigma\left(\widehat{W}(u) - \widehat{W}(t)\right) - \log\frac{Y(t)}{S(t)}\right]^+\right\}\Bigg|\mathcal{F}(t)\right] \\
&- e^{rt}\widetilde{\mathbb{E}}\left[e^{-rT}S(T)\big|\mathcal{F}(t)\right].
\end{aligned} \tag{7.4.23}$$

Because the discounted asset price is a martingale under $\widetilde{\mathbb{P}}$, the second term in (7.4.23) is $-e^{rt}e^{-rt}S(t) = S(t)$. For the first term, we can "take out what is known" (see Theorem 2.3.2(ii)) to obtain

$$e^{-r\tau}Y(t)\widetilde{\mathbb{E}}\left[\exp\left\{\left[\max_{t \le u \le T} \sigma\left(\widehat{W}(u) - \widehat{W}(t)\right) - \log\frac{Y(t)}{S(t)}\right]^+\right\}\Bigg|\mathcal{F}(t)\right]. \tag{7.4.24}$$

Because $Y(t)$ and $S(t)$ are $\mathcal{F}(t)$-measurable and $\max_{t\le u\le T}\sigma\big(\widehat{W}(u)-\widehat{W}(t)\big)$ is independent of $\mathcal{F}(t)$, we can use the Independence Lemma, Lemma 2.3.4, to write the conditional expectation in (7.4.24) as $g\big(S(t),Y(t)\big)$, where

$$g(x,y)=\widetilde{\mathbb{E}}\exp\left\{\left[\max_{t\le u\le T}\sigma\big(\widehat{W}(u)-\widehat{W}(t)\big)-\log\frac{y}{x}\right]^+\right\}. \tag{7.4.25}$$

Note that the expectation in (7.4.25) is no longer conditioned on $\mathcal{F}(t)$. Putting this all together, we have

$$V(t)=e^{-r\tau}Y(t)g\big(S(t),Y(t)\big)-S(t)$$

or, equivalently,

$$v(t,x,y)=e^{-r\tau}yg(x,y)-x. \tag{7.4.26}$$

It remains to compute the function $g(x,y)$. Because

$$\max_{t\le u\le T}\sigma\big(\widehat{W}(u)-\widehat{W}(t)\big)=\sigma\max_{t\le u\le T}\big(\widehat{W}(u)-\widehat{W}(t)\big),$$

and $\max_{t\le u\le T}\big(\widehat{W}(u)-\widehat{W}(t)\big)$ has the same unconditional distribution under $\widetilde{\mathbb{P}}$ as $\max_{0\le u\le\tau}\big(\widehat{W}(u)-\widehat{W}(0)\big)=\widehat{M}(\tau)$, the function $g(x,y)$ of (7.4.25) can also be written as

$$\begin{aligned}g(x,y)&=\widetilde{\mathbb{E}}\exp\left\{\left[\sigma\widehat{M}(\tau)-\log\frac{y}{x}\right]^+\right\}\\&=\widetilde{\mathbb{P}}\left\{\widehat{M}(\tau)\le\frac{1}{\sigma}\log\frac{y}{x}\right\}+\frac{x}{y}\widetilde{\mathbb{E}}\left[e^{\sigma\widehat{M}(\tau)}\mathbb{I}_{\{\widehat{M}(\tau)\ge\frac{1}{\sigma}\log\frac{y}{x}\}}\right].\end{aligned} \tag{7.4.27}$$

We compute both terms on the right-hand side of (7.4.27).

In order to compute the first term on the right-hand side of (7.4.27), we use (7.2.6) with T replaced by τ and m replaced by $\frac{1}{\sigma}\log\frac{y}{x}$. With these replacements, the arguments of N appearing on the right-hand side of (7.2.6) are

$$\begin{aligned}\frac{1}{\sqrt{\tau}}\left[\frac{1}{\sigma}\log\frac{y}{x}-\alpha\tau\right]&=\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{y}{x}-\left(r-\frac{1}{2}\sigma^2\right)\tau\right]\\&=-\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{y}+\left(r-\frac{1}{2}\sigma^2\right)\tau\right]\\&=-\delta_-\left(\tau,\frac{x}{y}\right),\\\frac{1}{\sqrt{\tau}}\left[-\frac{1}{\sigma}\log\frac{y}{x}-\alpha\tau\right]&=\frac{1}{\sigma\sqrt{\tau}}\left[-\log\frac{y}{x}-\left(r-\frac{1}{2}\sigma^2\right)\tau\right]\\&=-\delta_-\left(\tau,\frac{y}{x}\right),\end{aligned}$$

where $\delta_\pm(\tau, s)$ is defined by (7.3.18). The term $e^{2\alpha m}$ appearing on the right-hand side of (7.2.6) becomes

$$\exp\left\{\frac{2\alpha}{\sigma}\log\frac{y}{x}\right\} = \exp\left\{\left(\frac{2r}{\sigma^2}-1\right)\log\frac{y}{x}\right\} = \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}.$$

It follows from (7.2.6) that

$$\widetilde{\mathbb{P}}\left\{\widehat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x}\right\} = N\left(-\delta_-\left(\tau, \frac{x}{y}\right)\right) - \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau, \frac{y}{x}\right)\right). \tag{7.4.28}$$

The second term on the right-hand side of (7.4.27) is computed using the density for $\widehat{M}(\tau)$ under $\widetilde{\mathbb{P}}$ given by (7.2.7) with τ replacing T. Indeed,

$$\begin{aligned}
&\frac{x}{y}\widetilde{\mathbb{E}}\left[e^{\sigma\widehat{M}(\tau)}\mathbb{I}_{\{\widehat{M}(\tau)\ge\frac{1}{\sigma}\log\frac{y}{x}\}}\right]\\
&\quad= \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} e^{\sigma m}\widetilde{f}_{\widehat{M}(\tau)}(m)\,dm\\
&\quad= \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} \frac{2}{\sqrt{2\pi\tau}}e^{\sigma m-\frac{1}{2\tau}(m-\alpha\tau)^2}\,dm\\
&\qquad\quad -\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} 2\alpha e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm.
\end{aligned} \tag{7.4.29}$$

We compute the first integral on the right-hand side of (7.4.29). Because

$$\begin{aligned}
&r\tau - \frac{1}{2\tau}(m-\alpha\tau-\sigma\tau)^2\\
&\quad= r\tau - \frac{1}{2\tau}(m-\alpha\tau)^2 + \sigma(m-\alpha\tau) - \frac{1}{2}\sigma^2\tau\\
&\quad= r\tau - \frac{1}{2\tau}(m-\alpha\tau)^2 + \sigma m - \left(r-\frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}\sigma^2\tau\\
&\quad= \sigma m - \frac{1}{2\tau}(m-\alpha\tau)^2,
\end{aligned}$$

we may write the first term on the right-hand side of (7.4.29) as

$$\begin{aligned}
&\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} \frac{2}{\sqrt{2\pi\tau}}e^{\sigma m-\frac{1}{2\tau}(m-\alpha\tau)^2}\,dm\\
&\qquad= \frac{2xe^{r\tau}}{y\sqrt{2\pi\tau}}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} e^{-\frac{1}{2\tau}(m-\alpha\tau-\sigma\tau)^2}\,dm.
\end{aligned} \tag{7.4.30}$$

We make the change of variable $\xi = \frac{\alpha\tau+\sigma\tau-m}{\sqrt{\tau}}$, so the lower limit of integration $\frac{1}{\sigma}\log\frac{y}{x}$ becomes

$$\frac{1}{\sqrt{\tau}}\left(\alpha\tau+\sigma\tau-\frac{1}{\sigma}\log\frac{y}{x}\right) = \frac{1}{\sigma\sqrt{\tau}}\left(\log\frac{x}{y}+r\tau+\frac{1}{2}\sigma^2\tau\right) = \delta_+\left(\tau, \frac{x}{y}\right).$$

With this change of variable in the integral on the right-hand side of (7.4.30), we obtain the following formula for the first term on the right-hand side of (7.4.29):

$$\begin{aligned}\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\frac{2}{\sqrt{2\pi\tau}}e^{\sigma m-\frac{1}{2\tau}(m-\alpha\tau)^2}\,dm &= \frac{2xe^{r\tau}}{y\sqrt{2\pi}}\int_{-\infty}^{\delta_+(\tau,\frac{x}{y})}e^{-\frac{1}{2}\xi^2}\,d\xi\\ &= \frac{2xe^{r\tau}}{y}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right). \end{aligned}\tag{7.4.31}$$

The second term on the right-hand side of (7.4.29) requires a reversal of the order of integration over the region shown in Figure 7.4.2. Because $\sigma+2\alpha=\frac{2r}{\sigma}$, this term is

$$\begin{aligned}&-\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{x}{y}}^{\infty}2\alpha e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm\\ &\quad=-\frac{2\alpha x}{y\sqrt{2\pi}}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\int_{-\infty}^{\frac{1}{\sqrt{\tau}}(-m-\alpha\tau)}e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}\,d\xi\,dm\\ &\quad=-\frac{2\alpha x}{y\sqrt{2\pi}}\int_{-\infty}^{-\delta_-(\tau,\frac{y}{x})}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau}e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}\,dm\,d\xi. \end{aligned}\tag{7.4.32}$$

The inner integral in (7.4.32) can be evaluated. Indeed,

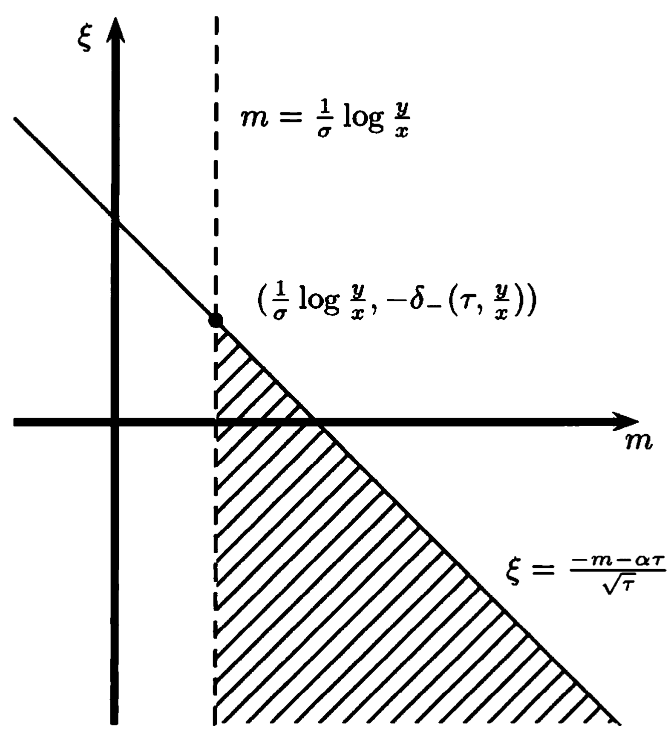

Fig. 7.4.2. Reversal of integration in (7.4.32).

$$\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau} e^{\frac{2rm}{\sigma}-\frac{\xi^2}{2}}\,dm = \frac{\sigma}{2r}\,e^{\frac{2rm}{\sigma}-\frac{\xi^2}{2}}\Bigg|_{m=\frac{1}{\sigma}\log\frac{y}{x}}^{m=-\xi\sqrt{\tau}-\alpha\tau}$$
$$= \frac{\sigma}{2r}e^{\frac{2r}{\sigma}(-\xi\sqrt{\tau}-\alpha\tau)-\frac{1}{2}\xi^2} - \frac{\sigma}{2r}e^{\frac{2r}{\sigma^2}\log\frac{y}{x}-\frac{1}{2}\xi^2}.$$

But

$$\begin{aligned}
\frac{2r}{\sigma}(-\xi\sqrt{\tau}-\alpha\tau)-\frac{\xi^2}{2} &= -\frac{\xi^2}{2}-\frac{2r\xi\sqrt{\tau}}{\sigma}-\frac{2r\alpha\tau}{\sigma}\\
&= -\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+\frac{2r^2\tau}{\sigma^2}-\frac{2r\alpha\tau}{\sigma}\\
&= -\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+\frac{2r\tau}{\sigma^2}(r-\sigma\alpha)\\
&= -\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+r\tau
\end{aligned}$$

and

$$e^{\frac{2r}{\sigma^2}\log\frac{y}{x}-\frac{\xi^2}{2}} = \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}} e^{-\frac{\xi^2}{2}}.$$

Therefore, the inner integral in (7.4.32) is

$$\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau} e^{\frac{2rm}{\sigma}-\frac{\xi^2}{2}}\,dm = \frac{\sigma}{2r}e^{r\tau-\frac{1}{2}(\xi+\frac{2r\sqrt{\tau}}{\sigma})^2} - \frac{\sigma}{2r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}} e^{-\frac{\xi^2}{2}}.$$

We continue (7.4.32), making this substitution for the inner integral:

$$\begin{aligned}
-\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} & 2\alpha e^{(\sigma+2\alpha)m} N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm\\
&= -\frac{\alpha\sigma x}{ry\sqrt{2\pi}}\int_{-\infty}^{-\delta_-(\tau,\frac{y}{x})} e^{r\tau-\frac{1}{2}(\xi+\frac{2r\sqrt{\tau}}{\sigma})^2}\,d\xi\\
&\quad+\frac{\alpha\sigma}{r\sqrt{2\pi}}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}\int_{-\infty}^{-\delta_-(\tau,\frac{y}{x})} e^{-\frac{\xi^2}{2}}\,d\xi\\
&= -\frac{\alpha\sigma x e^{r\tau}}{ry\sqrt{2\pi}}\int_{-\infty}^{-\delta_-(\tau,\frac{y}{x})} e^{-\frac{1}{2}(\xi+\frac{2r\sqrt{\tau}}{\sigma})^2}\,d\xi\\
&\quad+\frac{\alpha\sigma}{r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right).
\end{aligned} \tag{7.4.33}$$

In the first integral on the right-hand side of (7.4.33), we make the change of variable $\eta = \xi + \frac{2r\sqrt{\tau}}{\sigma}$, and the upper limit of integration becomes

$$
\begin{aligned}
-\delta_-\left(\tau,\frac{y}{x}\right)+\frac{2r\sqrt{\tau}}{\sigma} &= \frac{1}{\sigma\sqrt{\tau}}\left[-\log\frac{y}{x}-\left(r-\frac{1}{2}\sigma^2\right)\tau+2r\tau\right]\\
&= \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{y}+\left(r+\frac{1}{2}\sigma^2\right)\tau\right]\\
&= \delta_+\left(\tau,\frac{x}{y}\right).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&-\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} 2\alpha e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm\\
&= -\frac{\alpha\sigma x}{ry}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)+\frac{\alpha\sigma}{r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right).
\end{aligned}
\tag{7.4.34}
$$

We put all the pieces together. The function $v(t,x,y)$ for $0\le t<T$ and $0<x\le y$ is given by (7.4.26), where $g(x,y)$ is given by (7.4.27). We have computed both terms on the right-hand side of (7.4.27). The first term is given by (7.4.28), and the second term is itself the sum of the two terms in (7.4.29). These two terms are given by (7.4.31) and (7.4.34). Furthermore, the term $\frac{\alpha\sigma}{r}$ appearing in these formulas is equal to $1-\frac{\sigma^2}{2r}$. We conclude that

$$
\begin{aligned}
v(t,x,y) = e^{-r\tau}y\Bigg[&N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right)-\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)\\
&+2\left(\frac{x}{y}\right)e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)\\
&-\left(1-\frac{\sigma^2}{2r}\right)\left(\frac{x}{y}\right)e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)\\
&+\left(1-\frac{\sigma^2}{2r}\right)\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)\Bigg]-x.
\end{aligned}
$$

Simplification results in the formula

$$
\begin{aligned}
v(t,x,y) = &\left(1+\frac{\sigma^2}{2r}\right)xN\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)+e^{-r\tau}yN\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right)\\
&-\frac{\sigma^2}{2r}e^{-r\tau}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}}xN\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)-x,\ 0\le t<T,\ 0<x\le y.
\end{aligned}
\tag{7.4.35}
$$

The function u related to v by (7.4.16) satisfies

$$
\begin{aligned}
u\left(t,\frac{x}{y}\right) = &\left(1+\frac{\sigma^2}{2r}\right)\left(\frac{x}{y}\right)N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)+e^{-r\tau}N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right)\\
&-\frac{\sigma^2}{2r}e^{-r\tau}\left(\frac{x}{y}\right)^{1-\frac{2r}{\sigma^2}}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)-\frac{x}{y}.
\end{aligned}
$$

Making the change of variable $z = \frac{x}{y}$, we obtain

$$u(t,z) = \left(1 + \frac{\sigma^2}{2r}\right) zN\big(\delta_+(\tau,z)\big) + e^{-r\tau}N\big(-\delta_-(\tau,z)\big)$$
$$-\frac{\sigma^2}{2r}e^{-r\tau}z^{1-\frac{2r}{\sigma^2}}N\big(-\delta_-(\tau,z^{-1})\big) - z,\ 0 \le t < T,\ 0 < z \le 1. \tag{7.4.36}$$

7.5 Asian Options

An *Asian option* is one whose payoff includes a time average of the underlying asset price. The average may be over the entire time period between initiation and expiration or may be over some period of time that begins later than the initiation of the option and ends with the option's expiration. The average may be from continuous sampling,

$$\frac{1}{T}\int_0^T S(t)\,dt,$$

or may be from discrete sampling,

$$\frac{1}{m}\sum_{j=1}^{m} S(t_j),$$

where $0 < t_1 < t_2 \cdots < t_m = T$. The primary reason to base an option payoff on an average asset price is to make it more difficult for anyone to significantly affect the payoff by manipulation of the underlying asset price.

The price of Asian options is not known in closed form. Therefore, in this section we discuss two ways to derive partial differential equations for Asian option prices. The first of these was briefly presented in Example 6.6.1. The other method for computing Asian option prices is Monte Carlo simulation.

7.5.1 Fixed-Strike Asian Call

Once again, we begin with a geometric Brownian motion $S(t)$ given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \tag{7.5.1}$$

where $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Consider a *fixed-strike Asian call* whose payoff at time T is

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)^+, \tag{7.5.2}$$

where the strike price K is a nonnegative constant. The price at times t prior to the expiration time T of this call is given by the risk-neutral pricing formula

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)\Big|\,\mathcal{F}(t)\right], \quad 0 \le t \le T. \tag{7.5.3}$$

The usual iterated conditioning argument shows that

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}\left[e^{-rT}V(T)\big|\,\mathcal{F}(t)\right], \quad 0 \le t \le T,$$

is a martingale under $\widetilde{\mathbb{P}}$. This is the quantity we wish to compute. In the next two subsections, we describe two different ways to undertake this.

7.5.2 Augmentation of the State

The Asian option payoff $V(T)$ in (7.5.2) is *path-dependent*. The price of the option at time t depends not only on t and $S(t)$, but also on the path that the asset price has followed up to time t. In particular, we cannot invoke the Markov property to claim that $V(t)$ is a function of t and $S(t)$ because $V(T)$ is not a function of T and $S(T)$; $V(T)$ depends on the whole path of S.

To overcome this difficulty, we *augment* the state $S(t)$ by defining a second process

$$Y(t) = \int_0^t S(u)\,du. \tag{7.5.4}$$

The stochastic differential equation for $Y(t)$ is thus

$$dY(t) = S(t)\,dt. \tag{7.5.5}$$

Because the pair of processes $(S(t), Y(t))$ is governed by the pair of stochastic differential equations (7.5.1) and (7.5.5), they constitute a two-dimensional Markov process (Corollary 6.3.2). Furthermore, the call payoff $V(T)$ is a function of T and the final value $\big(S(T), Y(T)\big)$ of this process. Indeed, $V(T)$ depends only on T and $Y(T)$, by the formula

$$V(T) = \left(\frac{1}{T}Y(T) - K\right)^+. \tag{7.5.6}$$

This implies that there must exist some function $v(t,x,y)$ such that the Asian call price (7.5.3) is given as

$$\begin{aligned} v\big(t, S(t), Y(t)\big) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}Y(T) - K\right)^+\Bigg|\,\mathcal{F}(t)\right] \\ &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)\Big|\,\mathcal{F}(t)\right]. \end{aligned} \tag{7.5.7}$$

The function $v(t,x,y)$ satisfies a partial differential equation. This equation and three boundary conditions are provided in the next theorem. However, in order to numerically solve this equation, it would normally be necessary to also specify the behavior of $v(t,x,y)$ as x approaches ∞ and y approaches either ∞ or $-\infty$. This can be avoided by the method discussed in Subsection 7.5.3; see Remark 7.5.4 below.

Theorem 7.5.1. *The Asian call price function $v(t,x,y)$ of (7.5.7) satisfies the partial differential equation*

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y),$$
$$0 \le t < T,\ x \ge 0,\ y \in \mathbb{R}, \qquad (7.5.8)$$

and the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+,\ 0 \le t < T,\ y \in \mathbb{R}, \qquad (7.5.9)$$
$$\lim_{y \downarrow -\infty} v(t,x,y) = 0,\ 0 \le t < T,\ x \ge 0, \qquad (7.5.10)$$
$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+,\ x \ge 0,\ y \in \mathbb{R}. \qquad (7.5.11)$$

PROOF: Using the stochastic differential equations (7.5.1) and (7.5.5) and noting that $dS(t)\,dY(t) = dY(t)\,dY(t) = 0$, we take the differential of the $\widetilde{\mathbb{P}}$-martingale $e^{-rt}V(t) = e^{-rt}v\big(t,S(t),Y(t)\big)$. This differential is

$$\begin{aligned}
&d\left(e^{-rt}v\big(t,S(t),Y(t)\big)\right)\\
&= e^{-rt}\left[-rv\,dt + v_t\,dt + v_x\,dS + v_y\,dY + \frac{1}{2}v_{xx}\,dS\,dS\right]\\
&= e^{-rt}\left[-rv + v_t + rSv_x + Sv_y + \frac{1}{2}\sigma^2S^2v_{xx}\right]dt\\
&\qquad + e^{-rt}\sigma Sv_x\,d\widetilde{W}(t). \qquad (7.5.12)
\end{aligned}$$

In order for this to be a martingale, the dt term must be zero, which implies

$$\begin{aligned}
v_t\big(t,S(t),Y(t)\big) &+ rS(t)v_x\big(t,S(t),Y(t)\big) + S(t)v_y\big(t,S(t),Y(t)\big)\\
&+\frac{1}{2}\sigma^2S^2(t)v_{xx}\big(t,S(t),Y(t)\big) = rv\big(t,S(t),Y(t)\big).
\end{aligned}$$

Replacing $S(t)$ by the dummy variable x and $Y(t)$ by the dummy variable y, we obtain (7.5.8).

We note that $S(t)$ must always be nonnegative, and so (7.5.8) holds for $x \ge 0$. If $S(t) = 0$ and $Y(t) = y$ for some value of t, then $S(u) = 0$ for all $u \in [t,T]$, and so $Y(u)$ is constant on $[t,T]$. Therefore, $Y(T) = y$, and the value of the Asian call at time t is $\left(\frac{y}{T} - K\right)^+$, discounted from T back to t. This gives us the boundary condition (7.5.9).

In contrast, it is not the case that if $Y(t) = 0$ for some time t, then $Y(u) = 0$ for all $u \ge 0$. Therefore, we cannot easily determine the value of

$v(t, x, 0)$, and we do not provide a condition on the boundary $y = 0$. Indeed, at least mathematically there is no problem with allowing y to be negative. If at time t we set $Y(t) = y$, then $Y(T)$ is defined by (7.5.5). In integrated form, this formula is

$$Y(T) = y + \int_t^T S(u)\, du. \tag{7.5.13}$$

Even if y is negative, this makes sense, and in this case we could still have $Y(T) > 0$ or even $\frac{1}{T}Y(T) - K > 0$, so that the call expires in the money. When using the differential equations (7.5.1) and (7.5.5) to describe the "state" processes $S(t)$ and $Y(t)$, there is no reason to require that $Y(t)$ be nonnegative. (We still require that $S(t)$ be nonnegative because $x = 0$ is a natural boundary for $S(t)$.) For this reason, we do not restrict the values of y in the partial differential equation (7.5.8). The natural boundary for y is $y = -\infty$. If $Y(t) = y$, $S(t) = x$, and holding x fixed we let $y \to -\infty$, then $Y(T)$ approaches $-\infty$ (see (7.5.13)), the probability that the call expires in the money approaches zero, and the option price approaches zero. The natural boundary for y is $y = -\infty$, and the boundary condition there is (7.5.10).

The boundary condition (7.5.11) is just the payoff of the call. □

Remark 7.5.2. After we set the dt term in (7.5.12) equal to zero, we see that

$$d\left(e^{-rt} v\big(t, S(t), Y(t)\big)\right) = e^{-rt}\sigma S(t) v_x\big(t, S(t), Y(t)\big)\, d\widetilde{W}(t). \tag{7.5.14}$$

The discounted value of a portfolio that at each time t holds $\Delta(t)$ shares of the underlying asset is given by (see (5.2.27))

$$d\big(e^{-rt} X(t)\big) = e^{-rt}\sigma S(t)\Delta(t)\, d\widetilde{W}(t). \tag{7.5.15}$$

To hedge a short position in the Asian call, an agent should equate these two differentials, which leads to the delta-hedging formula

$$\Delta(t) = v_x\big(t, S(t), Y(t)\big).$$

7.5.3 Change of Numéraire

In this subsection we present a partial differential equation whose solution leads to Asian option prices. We work this out for both continuous and discrete averaging. The derivation of this equation involves a *change of numéraire*, a concept discussed systematically in Chapter 9. In this section, we derive formulas under the assumption that the interest rate r is not zero. The case $r = 0$ is treated in Exercise 7.8.

We first consider the case of an Asian call with payoff

$$V(T) = \left(\frac{1}{c}\int_{T-c}^{T} S(t)\, dt - K\right)^+, \tag{7.5.16}$$

where c is a constant satisfying $0 < c \le T$ and K is a nonnegative constant. If $c = T$, this is the Asian call (7.5.2) considered in Subsection 7.5.2. Here we also admit the possibility that the averaging is over less than the full time between initiation and expiration of the call.

To price this call, we create a portfolio process whose value at time T is

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(u)\,du - K.$$

We begin with a nonrandom function of time $\gamma(t)$, $0 \le t \le T$, which will be the number of shares of the risky asset held by our portfolio. There will be no Brownian motion term in $\gamma(t)$, and because of this it will satisfy $d\gamma(t)\,d\gamma(t) = d\gamma(t)\,dS(t) = 0$. This implies that

$$d\big(\gamma(t)S(t)\big) = \gamma(t)\,dS(t) + S(t)\,d\gamma(t), \tag{7.5.17}$$

which further implies

$$\begin{aligned} d\big(e^{r(T-t)}\gamma(t)S(t)\big) &= e^{r(T-t)}d\big(\gamma(t)S(t)\big) - re^{r(T-t)}\gamma(t)S(t)\,dt \\ &= e^{r(T-t)}\gamma(t)\,dS(t) + e^{r(T-t)}S(t)\,d\gamma(t) \\ &\qquad -re^{r(T-t)}\gamma(t)S(t)\,dt. \end{aligned} \tag{7.5.18}$$

Rearranging terms in (7.5.18), we obtain

$$e^{r(T-t)}\gamma(t)\big(dS(t) - rS(t)\,dt\big) = d\big(e^{r(T-t)}\gamma(t)S(t)\big) - e^{r(T-t)}S(t)\,d\gamma(t). \tag{7.5.19}$$

An agent who holds $\gamma(t)$ shares of the risky asset at each time t and finances this by investing or borrowing at the interest rate r will have a portfolio whose value evolves according to the equation

$$\begin{aligned} dX(t) &= \gamma(t)\,dS(t) + r\big(X(t) - \gamma(t)S(t)\,\big)dt \\ &= rX(t)\,dt + \gamma(t)\big(dS(t) - rS(t)\,dt\big). \end{aligned} \tag{7.5.20}$$

Using this equation and (7.5.19), we obtain

$$\begin{aligned} d\big(e^{r(T-t)}X(t)\big) &= -re^{r(T-t)}X(t)\,dt + e^{r(T-t)}\,dX(t) \\ &= e^{r(T-t)}\gamma(t)\big(S(t) - rS(t)\,dt\big) \\ &= d\big(e^{r(T-t)}\gamma(t)S(t)\big) - e^{r(T-t)}S(t)\,d\gamma(t). \end{aligned} \tag{7.5.21}$$

To study the Asian call with payoff (7.5.16), we take $\gamma(t)$ to be

$$\gamma(t) = \begin{cases} \dfrac{1}{rc}\big(1 - e^{-rc}\big), & 0 \le t \le T-c, \\ \dfrac{1}{rc}\big(1 - e^{-r(T-t)}\big), & T-c \le t \le T, \end{cases} \tag{7.5.22}$$

and we take the initial capital to be

$$X(0) = \frac{1}{rc}(1 - e^{-rc})S(0) - e^{-rT}K. \tag{7.5.23}$$

In the time interval $[0, T - c]$, the process $\gamma(t)$ mandates a buy-and-hold strategy. At time zero, we buy $\frac{1}{rc}(1 - e^{-rc})$ shares of the risky asset, which costs $\frac{1}{rc}(1 - e^{-rc})S(0)$. Our initial capital is insufficient to do this, and we must borrow $e^{-rT}K$ from the money market account. For $0 \leq t \leq T - c$, the value of our holdings in the risky asset is $\frac{1}{rc}(1 - e^{-rc})S(t)$ and we owe $e^{-r(T-t)}K$ to the money market account. Therefore,

$$X(t) = \frac{1}{rc}(1 - e^{-rc})S(t) - e^{-r(T-t)}K, \quad 0 \leq t \leq T - c. \tag{7.5.24}$$

In particular,

$$X(T - c) = \frac{1}{rc}(1 - e^{-rc})S(T - c) - e^{-rc}K. \tag{7.5.25}$$

For $T - c \leq t \leq T$, we have $d\gamma(t) = -\frac{1}{c}e^{-r(T-t)}$ and we compute $X(t)$ by first integrating (7.5.21) from $T - c$ to t and using (7.5.25) and (7.5.22) to obtain

$$\begin{aligned}
&e^{r(T-t)}X(t) \\
&= e^{rc}X(T - c) + \int_{T-c}^{t} d\left(e^{r(T-u)}\gamma(u)S(u)\right) - \int_{T-c}^{t} e^{r(T-u)}S(u)\,d\gamma(u) \\
&= \frac{1}{rc}e^{rc}(1 - e^{-rc})S(T - c) - K + e^{r(T-t)}\gamma(t)S(t) \\
&\qquad - \frac{1}{rc}e^{rc}(1 - e^{-rc})S(T - c) + \frac{1}{c}\int_{T-c}^{t} S(u)\,du \\
&= -K + e^{r(T-t)}\gamma(t)S(t) + \frac{1}{c}\int_{T-c}^{t} S(u)\,du.
\end{aligned}$$

Therefore,

$$X(t) = \frac{1}{rc}\left(1 - e^{-r(T-t)}\right)S(t) + e^{-r(T-t)}\frac{1}{c}\int_{T-c}^{t} S(u)\,du - e^{-r(T-t)}K,$$
$$T - c \leq t \leq T. \tag{7.5.26}$$

In particular,

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(u)\,du - K, \tag{7.5.27}$$

as desired, and

$$V(T) = X^{+}(T) = \max\{X(T), 0\}. \tag{7.5.28}$$

The price of the Asian call at time t prior to expiration is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)\middle|\mathcal{F}(t)\right] = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}X^{+}(T)\middle|\mathcal{F}(t)\right]. \tag{7.5.29}$$

The calculation of the right-hand side of (7.5.29) uses a change-of-numéraire argument, which we now exlain. Let us define

$$Y(t) = \frac{X(t)}{S(t)} = \frac{e^{-rt}X(t)}{e^{-rt}S(t)}.$$

This is the value of the portfolio denominated in units of the risky asset rather than in dollars. We have changed the numéraire, the unit of account, from dollars to the risky asset.

We work out the differential of $Y(t)$. Note first that

$$d\big(e^{-rt}S(t)\big) = -re^{-rt}S(t)\,dt + e^{-rt}\,dS(t) = \sigma e^{-rt}S(t)\,d\widetilde{W}(t). \tag{7.5.30}$$

Therefore,

$$\begin{aligned}
&d\left[\big(e^{-rt}S(t)\big)^{-1}\right]\\
&= -\big(e^{-rt}S(t)\big)^{-2}d\big(e^{-rt}S(t)\big) + \big(e^{-rt}S(t)\big)^{-3}d\big(e^{-rt}S(t)\big)\,d\big(e^{-rt}S(t)\big)\\
&= -\big(e^{-rt}S(t)\big)^{-2}\sigma\big(e^{-rt}S(t)\big)\,d\widetilde{W}(t) + \big(e^{-rt}S(t)\big)^{-3}\big(e^{-rt}S(t)\big)^{2}\sigma^2\,dt\\
&= -\sigma\big(e^{-rt}S(t)\big)^{-1}\,d\widetilde{W}(t) + \sigma^2\big(e^{-rt}S(t)\big)^{-1}\,dt.
\end{aligned}$$

On the other hand, (7.5.20) and (7.5.30) imply

$$\begin{aligned}
d\big(e^{-rt}X(t)\big) &= -re^{-rt}X(t)\,dt + e^{-rt}\,dX(t)\\
&= \gamma(t)e^{-rt}\big(dS(t) - rS(t)\big)\,dt\\
&= \gamma(t)\sigma e^{-rt}S(t)\,d\widetilde{W}(t).
\end{aligned}$$

Itô's product rule implies

$$\begin{aligned}
dY(t) &= d\left[\big(e^{-rt}X(t)\big)\big(e^{-rt}S(t)\big)^{-1}\right]\\
&= e^{-rt}X(t)\,d\left[\big(e^{-rt}S(t)\big)^{-1}\right] + \big(e^{-rt}S(t)\big)^{-1}\,d\big(e^{-rt}X(t)\big)\\
&\qquad + d\big(e^{-rt}X(t)\big)\,d\left[\big(e^{-rt}S(t)\big)^{-1}\right]\\
&= -\sigma Y(t)\,d\widetilde{W}(t) + \sigma^2 Y(t)\,dt + \sigma\gamma(t)\,d\widetilde{W}(t) - \sigma^2\gamma(t)\,dt\\
&= \sigma\big[\gamma(t) - Y(t)\big]\big[d\widetilde{W}(t) - \sigma\,dt\big].
\end{aligned} \tag{7.5.31}$$

The process $Y(t)$ is not a martingale under $\widetilde{\mathbb{P}}$ because its differential (7.5.31) has a dt term. However, we can change measure so that $Y(t)$ is a martingale, and this will simplify (7.5.31). We set

$$\widetilde{W}^S(t) = \widetilde{W}(t) - \sigma t \tag{7.5.32}$$

and then have

$$dY(t) = \sigma\big[\gamma(t) - Y(t)\big]\, d\widetilde{W}^S(t). \tag{7.5.33}$$

According to Girsanov's Theorem, Theorem 5.2.3, we can change the measure so that $\widetilde{W}^S(t)$, $0 \le t \le T$, is a Brownian motion. In this situation, $-\sigma$ plays the role of Θ in Theorem 5.2.3, and $\widetilde{W}$ and $\widetilde{\mathbb{P}}$ play the roles of W and $\mathbb{P}$. The Radon-Nikodým derivative process of (5.2.11) is

$$Z(t) = \exp\left\{\sigma\widetilde{W}(t) - \frac{1}{2}\sigma^2 t\right\}.$$

In other words,

$$Z(t) = \frac{e^{-rt}S(t)}{S(0)}. \tag{7.5.34}$$

Under the probability measure $\widetilde{\mathbb{P}}^S$ defined by

$$\widetilde{\mathbb{P}}^S(A) = \int_A Z(T)\, d\widetilde{\mathbb{P}} \text{ for all } A \in \mathcal{F},$$

$\widetilde{W}^S(t)$ is a Brownian motion and $Y(t)$ is a martingale.

Under the probability measure $\widetilde{\mathbb{P}}^S$, the process $Y(t)$ is Markov. It is given by the stochastic differential equation (7.5.33), and because $\gamma(t)$ is nonrandom, the term multiplying $d\widetilde{W}^S(t)$ in (7.5.33) is a function of t and $Y(t)$ and has no source of randomness other than $Y(t)$. Equation (7.5.33) is a stochastic differential equation of the type (6.2.1), and solutions to such equations are Markov (see Corollary 6.3.2).

We return to the option price $V(t)$ of (7.5.29) and use Lemma 5.2.2 to write (7.5.29) as

$$\begin{aligned}
V(t) &= e^{rt}\widetilde{\mathbb{E}}\big[e^{-rT}X^+(T)\big|\mathcal{F}(t)\big]\\
&= \frac{S(t)}{e^{-rt}S(t)}\widetilde{\mathbb{E}}\left[e^{-rT}S(T)\left(\frac{e^{-rT}X(T)}{e^{-rT}S(T)}\right)^+ \middle|\, \mathcal{F}(t)\right]\\
&= \frac{S(t)}{Z(t)}\widetilde{\mathbb{E}}\left[Z(T)Y^+(T)\big|\, \mathcal{F}(t)\right]\\
&= S(t)\widetilde{\mathbb{E}}^S\left[Y^+(T)\ \big|\mathcal{F}(t)\right],
\end{aligned} \tag{7.5.35}$$

where $\widetilde{\mathbb{E}}^S[\cdots|\mathcal{F}(t)]$ denotes conditional expectation under the probability measure $\widetilde{\mathbb{P}}^S$. Because Y is Markov under $\widetilde{\mathbb{P}}^S$, there must be some function $g(t,y)$ such that

$$g\big(t, Y(t)\big) = \widetilde{\mathbb{E}}^S\left[Y^+(T)\big|\, \mathcal{F}(t)\right]. \tag{7.5.36}$$

From (7.5.36), we see that

$$g\big(T, Y(T)\big) = \widetilde{\mathbb{E}}^S\left[Y^+(T)\big|\, \mathcal{F}(T)\right] = Y^+(T). \tag{7.5.37}$$

We note that $Y(T) = \frac{X(T)}{S(T)}$ can take any value since the numerator $X(T)$, given by (7.5.27), can be either positive or negative, and the denominator $S(T)$ can be any positive number. Therefore, (7.5.37) leads to the boundary condition

$$g(T,y) = y^+, \quad y \in \mathbb{R}. \tag{7.5.38}$$

The usual iterated conditioning argument shows that the right-hand side of (7.5.36) is a martingale under $\widetilde{\mathbb{P}}^S$, and so the differential of $g\big(t, Y(t)\big)$ should have only a $d\widetilde{W}^S(t)$ term. This differential is

$$\begin{aligned} dg\big(t,Y(t)\big) &= g_t\big(t,Y(t)\big)\,dt + g_y\big(t,Y(t)\big)\,dY(t) \\ &\quad + \frac{1}{2} g_{yy}\big(t,Y(t)\big)\,dY(t)\,dY(t) \\ &= \left[g_t\big(t,Y(t)\big) + \frac{1}{2}\sigma^2\big(\gamma(t) - Y(t)\big)^2 g_{yy}\big(t,Y(t)\big)\right] dt \\ &\quad + \sigma\big(\gamma(t) - Y(t)\big) g_y\big(t,Y(t)\big)\,d\widetilde{W}^S(t). \end{aligned}$$

We conclude that $g(t,y)$ satisfies the partial differential equation

$$g_t(t,y) + \frac{1}{2}\sigma^2\big(\gamma(t) - y\big)^2 g_{yy}(t,y) = 0, \quad 0 \le t < T,\ y \in \mathbb{R}. \tag{7.5.39}$$

We summarize this discussion with the following theorem.

Theorem 7.5.3 (Večeř). *For $0 \le t \le T$, the price $V(t)$ at time t of the continuously averaged Asian call with payoff (7.5.16) at time T is*

$$V(t) = S(t)\, g\left(t, \frac{X(t)}{S(t)}\right), \tag{7.5.40}$$

where $g(t,y)$ satisfies (7.5.39) and $X(t)$ is given by (7.5.24) and (7.5.26). The boundary conditions for $g(t,y)$ are (7.5.38) and

$$\lim_{y \to -\infty} g(t,y) = 0, \quad \lim_{y \to \infty} \big[g(t,y) - y\big] = 0, \quad 0 \le t \le T. \tag{7.5.41}$$

Remark 7.5.4 (Boundary conditions). Let $0 \le t \le T$ be given. The first boundary condition in (7.5.41) can be derived from the fact that when $Y(t)$ is very negative, the probability that $Y(T)$ also is negative is near one and therefore the probability that $Y^+(T) = 0$ is near one. This causes $g(t, Y(t))$ in (7.5.36) to be near zero. The second boundary condition in (7.5.41) is a consequence of that fact that when $Y(t)$ is large, then the probability that $Y(T) > 0$ is near one. Therefore, $g\big(t, Y(t)\big)$ given by (7.5.36) is approximately equal to $\widetilde{\mathbb{E}}^S\big[Y(T)\big|\mathcal{F}(t)\big]$, and because $Y(T)$ is a martingale under $\widetilde{\mathbb{P}}^S$, this conditional expectation is $Y(t)$.

It is easier to derive these boundary conditions at $y = \pm\infty$ for $g(t,y)$ than it is to derive the boundary conditions for $v(t,x,y)$ in Theorem 7.5.1 because

$v(t,x,y)$ has two variables, x and y, that can become large. For example, it is not at all clear how $v(t,x,y)$ behaves as $x \to \infty$ and $y \to -\infty$. The reduction of the Asian option pricing problem provided by Theorem 7.5.3 reduces the dimensionality of the problem and simplifies the boundary conditions. It also removes a so-called "degeneracy" in equation (7.5.8) created by the absence of the $v_{yy}(t,x,y)$ term. This degeneracy complicates the numerical solution of (7.5.8). □

In the remainder of this subsection, we adapt the arguments just given to treat a *discretely sampled Asian call.* Assume we are given times $0 = t_0 < t_1 < t_2 \cdots < t_m = T$ and the Asian call payoff is

$$V(T) = \left(\frac{1}{m}\sum_{j=1}^{m} S(t_j) - K\right)^{+}. \tag{7.5.42}$$

We wish to create a portfolio process so that

$$X(T) = \frac{1}{m}\sum_{j=1}^{m} S(t_j) - K.$$

In place of (7.5.22), we define

$$\gamma(t_j) = \frac{1}{m}\sum_{i=j}^{m} e^{-r(T-t_i)}, \quad j = 0, 1, \ldots, m. \tag{7.5.43}$$

Then

$$\gamma(t_j) = \gamma(t_{j-1}) - \frac{1}{m}e^{-r(T-t_{j-1})}, \quad j = 1, \ldots, m, \tag{7.5.44}$$

and $\gamma(T) = \gamma(t_m) = \frac{1}{m}$. We complete the definition of $\gamma(t)$ by setting

$$\gamma(t) = \gamma(t_j), \quad t_{j-1} < t \leq t_j. \tag{7.5.45}$$

This defines $\gamma(t)$ for all $t \in [0,T]$. In this situation, (7.5.21) still holds, but now $d\gamma(t) = 0$ in each subinterval (t_{j-1}, t_j). Integrating (7.5.21) from t_{j-1} to t_j and using (7.5.44) and the fact that $\gamma(t) = \gamma(t_j)$ for $t \in (t_{j-1}, t_j]$, we obtain

$$\begin{aligned}
&e^{r(T-t_j)}X(t_j) - e^{r(T-t_{j-1})}X(t_{j-1}) \\
&= \gamma(t_j)\left[e^{r(T-t_j)}S(t_j) - e^{r(T-t_{j-1})}S(t_{j-1})\right] \\
&= \gamma(t_j)e^{r(T-t_j)}S(t_j) - \left(\gamma(t_{j-1}) - \frac{1}{m}e^{-r(T-t_{j-1})}\right)e^{r(T-t_{j-1})}S(t_{j-1}) \\
&= \gamma(t_j)e^{r(T-t_j)}S(t_j) - \gamma(t_{j-1})e^{r(T-t_{j-1})}S(t_{j-1}) + \frac{1}{m}S(t_{j-1}).
\end{aligned}$$

Summing this equation from $j = 1$ to $j = k$, we see that

$$\begin{aligned}
&e^{r(T-t_k)}X(t_k) - e^{rT}X(0)\\
&= \gamma(t_k)e^{r(T-t_k)}S(t_k) - \gamma(0)e^{rT}S(0) + \frac{1}{m}\sum_{j=1}^{k} S(t_{j-1})\\
&= \gamma(t_k)e^{r(T-t_k)}S(t_k) + \frac{1}{m}\sum_{i=1}^{k-1} S(t_i) + \left(-\gamma(0)e^{rT} + \frac{1}{m}\right) S(0).
\end{aligned}$$

We set

$$X(0) = e^{-rT}\left[\gamma(0)e^{rT} - \frac{1}{m}\right] S(0) - e^{-rT}K,$$

so this equation becomes

$$e^{r(T-t_k)}X(t_k) = \gamma(t_k)e^{r(T-t_k)}S(t_k) + \frac{1}{m}\sum_{i=1}^{k-1} S(t_i) - K$$

or, equivalently,

$$X(t_k) = \gamma(t_k)S(t_k) + e^{-r(T-t_k)}\frac{1}{m}\sum_{i=1}^{k-1} S(t_i) - e^{-r(T-t_k)}K. \tag{7.5.46}$$

In particular,

$$X(T) = X(t_m) = \frac{1}{m}\sum_{i=1}^{m} S(t_i) - K \tag{7.5.47}$$

as desired.

To determine $X(t)$ for $t_k \le t \le t_{k+1}$, we integrate (7.5.21) from t_k to t to obtain

$$\begin{aligned}
e^{r(T-t)}X(t) &= e^{r(T-t_k)}X(t_k) + \gamma(t_{k+1})\left[e^{r(T-t)}S(t) - e^{r(T-t_k)}S(t_k)\right]\\
&= \gamma(t_k)e^{r(T-t_k)}S(t_k) + \frac{1}{m}\sum_{i=1}^{k-1} S(t_i) - K + \gamma(t_{k+1})e^{r(T-t)}S(t)\\
&\quad - \left(\gamma(t_k) - \frac{1}{m}e^{-r(T-t_k)}\right) e^{r(T-t_k)}S(t_k)\\
&= \gamma(t_{k+1})e^{r(T-t)}S(t) + \frac{1}{m}\sum_{i=1}^{k} S(t_i) - K.
\end{aligned}$$

Therefore,

$$X(t) = \gamma(t_{k+1})S(t) + e^{-r(T-t)}\frac{1}{m}\sum_{i=1}^{k} S(t_i) - e^{-r(T-t)}K, \quad t_k \le t \le t_{k+1}. \tag{7.5.48}$$

We now proceed with the change of numéraire as before. This leads again to Theorem 7.5.3 for the discretely sampled Asian call with payoff (7.5.42).

The price at time t is given by (7.5.40), where $g(t, x)$ satisfies (7.5.39) with boundary conditions (7.5.38) and (7.5.41). The only difference is that now the nonrandom function $\gamma(t)$ appearing in (7.5.39) is given by (7.5.43) and (7.5.45) and the process $X(t)$ in (7.5.40) is given by (7.5.48).

7.6 Summary

Three specific exotic options on a geometric Brownian motion have been considered: an up-and-out barrier call, a lookback call, and an Asian call. In each case, the discounted option price is a martingale under the risk-neutral measure, and this leads to a partial differential equation of the Black-Scholes-Merton type. However, the lookback call and the Asian call equations have an additional state variable in this equation.

For the barrier call and the lookback call, the option price was computed explicitly. The Asian option pricing problem was transformed by a change of numéraire to an equation with a single state variable. This transformation was done both for the continuously sampled and the discretely sampled Asian options.

7.7 Notes

There are scores of different exotic options, and the search for explicit pricing formulas can lead to complex computations. Analysis of many exotic options is provided by Zhang [167] and Haug [80]. Papers by a variety of authors who treat exotic options, including some of those cited below, have been collected by Lipton [110]. Exotic options are prevalent in foreign exchange markets. Analysis of several instruments appearing in these markets is provided by Hakala and Wystup [76]. Many exotic pricing formulas can be derived from the formulas for distributions related to Brownian motion collected by Borodin and Salminen [18].

The analysis of barrier options presented here follows Rubinstein and Reiner [142]. Monte Carlo simulation of barrier options normally obtains the price for the case when barrier crossing is checked only at discrete times. Broadie, Glasserman and Kou [22] provide a correction term to adjust this result to obtain the price for an option in which the barrier is monitored continuously. The problem of large delta and gamma values for barrier options near expiration near the barrier can be ameliorated by placing an a priori constraint on the hedging strategy and pricing this constraint into the option; see Schmock, Shreve, and Wystup [148].

The change-of-numéraire approach to Asian options, explained in Subsection 7.5.3, is due to Večeř [155], [156]. This methodology was extended to jump processes by Večeř and Xu [157]. Other partial differential equations for pricing Asian options are provided by Andreasen [4], Lipton [109], and Rogers and Shi [139].

Geman and Yor [71] obtain a closed-form formula for a Laplace transform of the Asian option price. Fu, Madan, and Wang [67] compare Monte Carlo and Laplace transform methods for Asian option pricing.

7.8 Exercises

Exercise 7.1 (Black-Scholes-Merton equation for the up-and-out call). This exercise shows by direct calculation that the function $v(t,x)$ of (7.3.20) satisfies the Black-Scholes-Merton equation (7.3.4).

(i) Recall that $\tau = T - t$, so $\frac{d\tau}{dt} = -1$. Show that $\delta_\pm(\tau, s)$ given by (7.3.18) satisfies

$$\frac{\partial}{\partial t}\delta_\pm(\tau, s) = -\frac{1}{2\tau}\delta_\pm\left(\tau, \frac{1}{s}\right). \tag{7.8.1}$$

(ii) Show that for any positive constant c,

$$\frac{\partial}{\partial x}\delta_\pm\left(\tau, \frac{x}{c}\right) = \frac{1}{x\sigma\sqrt{\tau}}, \quad \frac{\partial}{\partial x}\delta_\pm\left(\tau, \frac{c}{x}\right) = -\frac{1}{x\sigma\sqrt{\tau}}. \tag{7.8.2}$$

(iii) Show that

$$\frac{N'\big(\delta_+(\tau,s)\big)}{N'\big(\delta_-(\tau,s)\big)} = \frac{e^{-r\tau}}{s}$$

and hence

$$e^{-r\tau}N'\big(\delta_-(\tau,s)\big) = sN'\big(\delta_+(\tau,s)\big). \tag{7.8.3}$$

(iv) Show that

$$\frac{N'\big(\delta_\pm(\tau,s)\big)}{N'\big(\delta_\pm(\tau,s^{-1})\big)} = s^{-\left(\frac{2r}{\sigma^2}\pm 1\right)}$$

and hence

$$N'\big(\delta_\pm(\tau,s^{-1})\big) = s^{\frac{2r}{\sigma^2}\pm 1}N'\big(\delta_\pm(\tau,s)\big). \tag{7.8.4}$$

(v) Show that

$$\delta_+(\tau,s) - \delta_-(\tau,s) = \sigma\sqrt{\tau}. \tag{7.8.5}$$

(vi) Show that

$$\delta_\pm(\tau,s) - \delta_\pm(\tau,s^{-1}) = \frac{2}{\sigma\sqrt{\tau}}\log s. \tag{7.8.6}$$

(vii) Show that

$$N''(y) = -yN'(y). \tag{7.8.7}$$

(viii) Use (i) to compute $v_t(t,x)$ and (7.8.3)–(7.8.5) to simplify it, obtaining

$$\begin{aligned}
&v_t(t,x)\\
&= -\frac{x\sigma}{2\sqrt{\tau}}N'\left(\delta_+\left(\tau,\frac{x}{K}\right)\right) - \frac{x(B-K)}{B\sigma\tau\sqrt{\tau}}\log\frac{x}{B}N'\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\\
&\quad+\frac{B\sigma}{2\sqrt{\tau}}\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}N'\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right)\\
&\quad-re^{-r\tau}K\left[N\left(\delta_-\left(\tau,\frac{x}{K}\right)\right)-N\left(\delta_-\left(\tau,\frac{x}{B}\right)\right)\right]\\
&\quad+re^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau,\frac{B^2}{Kx}\right)\right)-N\left(\delta_-\left(\tau,\frac{B}{x}\right)\right)\right]. \qquad (7.8.8)
\end{aligned}$$

(ix) Use (ii) to compute $v_x(t,x)$ and (7.8.3) and (7.8.4) to simplify it, obtaining

$$\begin{aligned}
&v_x(t,x)\\
&= \left[N\left(\delta_+\left(\tau,\frac{x}{K}\right)\right)-N\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\right] - \frac{2(B-K)}{B\sigma\sqrt{\tau}}N'\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\\
&\quad+\frac{2r}{\sigma^2}\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}-1}\left[N\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right)-N\left(\delta_+\left(\tau,\frac{B}{x}\right)\right)\right]\\
&\quad+\frac{e^{-r\tau}K}{B}\left(-\frac{2r}{\sigma^2}+1\right)\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\\
&\qquad\qquad\times\left[N\left(\delta_-\left(\tau,\frac{B^2}{Kx}\right)\right)-N\left(\delta_-\left(\tau,\frac{B}{x}\right)\right)\right]. \qquad (7.8.9)
\end{aligned}$$

(x) Use (ii) and (7.8.9) to compute $v_{xx}(t,x)$ and (7.8.3) and (7.8.4) to simplify it, obtaining

$$\begin{aligned}
&v_{xx}(t,x)\\
&= \frac{1}{x\sigma\sqrt{\tau}}N'\left(\delta_+\left(\tau,\frac{x}{K}\right)\right) - \frac{1}{B\sigma\sqrt{\tau}}\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}-2}N'\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right)\\
&\quad+\frac{2(B-K)}{xB\sigma\sqrt{\tau}}\left(\frac{2r}{\sigma^2}+\frac{1}{\sigma^2\tau}\log\frac{x}{B}\right)N'\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\\
&\quad-\frac{2r}{B\sigma^2}\left(\frac{2r}{\sigma^2}+1\right)\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}-2}\\
&\qquad\qquad\times\left[N\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right)-N\left(\delta_+\left(\tau,\frac{B}{x}\right)\right)\right]\\
&\quad-\frac{e^{-r\tau}K}{B^2}\left(\frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}+1\right)\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}-1}\\
&\qquad\qquad\times\left[N\left(d_-\left(\tau,\frac{B^2}{Kx}\right)\right)-N\left(\delta_-\left(\tau,\frac{B}{x}\right)\right)\right]. \qquad (7.8.10)
\end{aligned}$$

(xi) Now verify that $v(t,x)$ satisfies the Black-Scholes-Merton equation (7.3.4).

Exercise 7.2 (Boundary conditions for the up-and-out call). In this exercise, it is verified that the up-and-out call price $v(t,x)$ given by (7.3.20) satisfies the boundary condition (7.3.6). Furthermore, the limit as $x \downarrow 0$ satisfies (7.3.5) and the limit as $t \uparrow T$ satisfies (7.3.7).

(i) Verify by direct substitution into (7.3.20) that (7.3.6) is satisfied.

(ii) Show that, for any positive constant c,

$$\lim_{x \downarrow 0} \delta_{\pm}\left(\tau, \frac{x}{c}\right) = -\infty, \quad \lim_{x \downarrow 0} \delta_{\pm}\left(\tau, \frac{c}{x}\right) = \infty. \tag{7.8.11}$$

Use this to show that for any $p \in \mathbb{R}$ and positive constants c_1 and c_2, we have

$$\lim_{x \downarrow 0} x^p \left[N\left(\delta_{\pm}\left(\tau, \frac{x}{c_1}\right)\right) - N\left(\delta_{\pm}\left(\tau, \frac{x}{c_2}\right)\right)\right] = 0, \tag{7.8.12}$$

$$\lim_{x \downarrow 0} x^p \left[N\left(\delta_{\pm}\left(\tau, \frac{c_1}{x}\right)\right) - N\left(\delta_{\pm}\left(\tau, \frac{c_2}{x}\right)\right)\right] = 0. \tag{7.8.13}$$

If $p \geq 0$, (7.8.12) and (7.8.13) are immediate consequences of (7.8.11). However, if $p < 0$, one should first use L'Hôpital's rule and then show that

$$\lim_{x \downarrow 0} x^p \exp\left\{-\frac{1}{2}\delta_{\pm}^2\left(\tau, \frac{x}{c_i}\right)\right\} = 0, \ \lim_{x \downarrow 0} x^p \exp\left\{-\frac{1}{2}\delta_{\pm}^2\left(\tau, \frac{c_i}{x}\right)\right\} = 0. \tag{7.8.14}$$

To establish (7.8.14), you may wish to prove and use the inequality

$$\frac{1}{2}a^2 - b^2 \leq (a+b)^2 \text{ for all } a, b \in \mathbb{R}. \tag{7.8.15}$$

Conclude that $\lim_{x \downarrow 0} v(t,x) = 0$ for $0 \leq t < T$.

(iii) Show that, for any positive c,

$$\lim_{\tau \downarrow 0} \delta_{\pm}(\tau, c) = \begin{cases} -\infty & \text{if } 0 < c < 1, \\ 0 & \text{if } c = 1, \\ \infty & \text{if } c > 1. \end{cases} \tag{7.8.16}$$

Use this to show that $\lim_{\tau \downarrow 0} v(t,x) = (x-K)^+$ for $0 < x < B$.

Exercise 7.3 (Markov property for geometric Brownian motion and its maximum to date). Recall the geometric Brownian motion $S(t)$ of (7.4.1) and its maximum-to-date process $Y(t)$ of (7.4.3). According to Definition 2.3.6, in order to show that the pair of processes $\big(S(t), Y(t)\big)$ is Markov, we must show that whenever $0 \leq t \leq T$ and $f(x,y)$ is a function, there exists another function $g(x,y)$ such that

$$\mathbb{E}\big[f\big(S(T), Y(T)\big)\big|\mathcal{F}(t)\big] = g\big(S(t), Y(t)\big). \tag{7.8.17}$$

Use the Independence Lemma, Lemma 2.3.4, to show that such a function $g(x,y)$ exists.

Exercise 7.4 (Cross variation of geometric Brownian motion and its maximum to date). Let $S(t)$ be the geometric Brownian motion (7.4.1) and let $Y(t)$ be the maximum-to-date process (7.4.3). Let T be fixed and let $0 = t_0 < t_1 < \dots t_m = T$ be a partition of $[0, T]$. Show that as the number of partition points m approaches infinity and the length of the longest subinterval $\max_{j=1,\dots,m} t_j - t_{j-1}$ approaches zero, the sum

$$\sum_{j=1}^{m} \big(Y(t_j) - Y(t_{j-1})\big)\big(S(t_j) - S(t_{j-1})\big)$$

has limit zero.

Exercise 7.5 (Black-Scholes-Merton equation for lookback option). We wish to verify by direct computation that the function $v(t, x, y)$ of (7.4.35) satisfies the Black-Scholes-Merton equation (7.4.6). As we saw in Subsection 7.4.3, this is equivalent to showing that the function u defined by (7.4.36) satisfies the Black-Scholes-Merton equation (7.4.18). We verify that $u(t, z)$ satisfies (7.4.18) in the following steps. Let $0 \le t < T$ be given, and define $\tau = T - t$.

(i) Use (7.8.1) to compute $u_t(t, z)$, and use (7.8.3) and (7.8.4) to simplify the result, thereby showing that

$$\begin{aligned} u_t(t, z) = re^{-r\tau} N\big(-\delta_-(\tau, z)\big) - \frac{1}{2}\sigma^2 e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N\big(-\delta_-(\tau, z^{-1})\big) \\ -\frac{\sigma z}{\sqrt{\tau}} N'\big(\delta_+(\tau, z)\big). \end{aligned} \tag{7.8.18}$$

(ii) Use (7.8.2) to compute $u_z(t, z)$, and use (7.8.3) and (7.8.4) to simplify the result, thereby showing that

$$\begin{aligned} u_z(t, z) = \left(1 + \frac{\sigma^2}{2r}\right) N\big(\delta_+(\tau, z)\big) \\ + \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N\big(-\delta_-(\tau, z^{-1})\big) - 1. \end{aligned} \tag{7.8.19}$$

(iii) Use (7.8.19) and (7.8.2) to compute $u_{zz}(t, z)$, and use (7.8.3) and (7.8.4) to simplify the result, thereby showing that

$$u_{zz}(t, z) = \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N\big(-\delta_-(\tau, z^{-1})\big) + \frac{2}{z\sigma\sqrt{\tau}} N'\big(\delta_+(\tau, z)\big). \tag{7.8.20}$$

(iv) Verify that $u(t, z)$ satisfies the Black-Scholes-Merton equation (7.4.18).

(v) Verify that $u(t, z)$ satisfies the boundary condition (7.4.20).

Exercise 7.6 (Boundary conditions for lookback option). The lookback option price $v(t, x, y)$ of (7.4.35) must satisfy the boundary conditions

(7.4.7)–(7.4.9). As we saw in Subsection 7.4.3, this is equivalent to the function $u(t,z)$ of (7.4.16) given by (7.4.36),

$$u(t,z) = \left(1 + \frac{\sigma^2}{2r}\right) zN\big(\delta_+(\tau,z)\big) + e^{-r\tau}N\big(-\delta_-(\tau,z)\big)$$
$$-\frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N\big(-\delta_-(\tau,z^{-1})\big) - z,\ 0 \le t < T,\ 0 < z \le 1,$$

satisfying the boundary conditions (7.4.19)–(7.4.21). This function was shown to satisfy boundary condition (7.4.20) in Exercise 7.5(v). Here we verify by direct computation that the limit of $u(t,z)$ as $z \downarrow 0$ satisfies (7.4.19) and the limit of $u(t,z)$ as $t \uparrow T$ ($\tau \downarrow 0$) satisfies (7.4.21).

(i) If you have not worked Exercise 7.2, then verify (7.8.11), the second equality in (7.8.14) and (7.8.16).
(ii) Use (7.8.11) and the second part of (7.8.14) to show that $\lim_{z\downarrow 0} u(t,z) = e^{-r\tau}$ for $0 \le t < T$.
(iii) Use (7.8.16) to show that $\lim_{\tau\downarrow 0} u(t,z) = 1 - z$ for $0 < z \le 1$.

Exercise 7.7 (Zero-strike Asian call). Consider a zero-strike Asian call whose payoff at time T is

$$V(T) = \frac{1}{T}\int_0^T S(u)\,du.$$

(i) Suppose at time t we have $S(t) = x \ge 0$ and $\int_0^t S(u)\,du = y \ge 0$. Use the fact that $e^{-ru}S(u)$ is a martingale under $\widetilde{\mathbb{P}}$ to compute

$$e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)\,du \,\middle|\, \mathcal{F}(t)\right].$$

Call your answer $v(t,x,y)$.
(ii) Verify that the function $v(t,x,y)$ you obtained in (i) satisfies the Black-Scholes-Merton equation (7.5.8) and the boundary conditions (7.5.9) and (7.5.11) of Theorem 7.5.1. (We do not try to verify (7.5.10) because the computation of $v(t,x,y)$ outlined here works only for $y \ge 0$.)
(iii) Determine explicitly the process $\Delta(t) = v_x\big(t, S(t), Y(t)\big)$, and observe that it is not random.
(iv) Use the Itô-Doeblin formula to show that if you begin with initial capital $X(0) = v\big(0, S(0), 0\big)$ and at each time you hold $\Delta(t)$ shares of the underlying asset, investing or borrowing at the interest rate r in order to do this, then at time T the value of your portfolio will be

$$X(T) = \frac{1}{T}\int_0^T S(u)\,du.$$

Exercise 7.8. Consider the continuously sampled Asian option of Subsection 7.5.3, but assume now that the interest rate is $r = 0$. Find an initial capital $X(0)$ and a nonrandom function $\gamma(t)$ to replace (7.5.22) so that

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(u)\,du - K \tag{7.5.27}$$

still holds. Give the formula for the resulting process $X(t)$, $0 \le t \le T$, to replace (7.5.24) and (7.5.26). With this function $\gamma(t)$ and process $X(t)$, Theorem 7.5.3 still holds.

Exercise 7.9. Let $g(t,y)$ be the function in Theorem 7.5.3. Then the value of the Asian option at time t is $V(t) = v(t, S(t), X(t))$, where $v(t,s,x) = sg(t,y)$ and $y = \frac{x}{s}$. The process $S(t)$ is given by (7.5.1). For the sake of specificity, we consider the case of continuous sampling with $r \neq 0$, so $\gamma(t)$ is given by (7.5.22) and $X(t)$ is given by (7.5.24) and (7.5.26).

(i) Verify the derivative formulas

$$\begin{aligned}
v_t(t,s,x) &= sg_t(t,y),\\
v_s(t,s,x) &= g(t,y) - yg_y(t,y),\\
v_x(t,s,x) &= g_y(t,y),\\
v_{ss}(t,s,x) &= \frac{y^2}{s} g_{yy}(t,y),\\
v_{sx}(t,s,x) &= -\frac{y}{s} g_{yy}(t,y),\\
v_{xx}(t,s,x) &= \frac{1}{s} g_{yy}(t,y).
\end{aligned}$$

(ii) Show that $e^{-rt}v\big(t, S(t), X(t)\big)$ is a martingale under $\widetilde{\mathbb{P}}$ by computing its differential, writing the differential in terms of dt and $d\widetilde{W}$, and verifying that the dt term is zero. (Hint: Use the fact that $g(t,y)$ satisfies (7.5.39).)

(iii) Suppose we begin with initial capital $v(0, S(0), X(0))$ and at each time t take a position $\Delta(t)$ in the risky asset, investing or borrowing at the interest rate r in order to finance this. We want to do this so that the portfolio value at the final time is $\left(\frac{1}{c}\int_{T-c}^{T} S(u)\,du - K\right)^{+}$. Give a formula for $\Delta(t)$ in terms of the function v and the processes $S(t)$ and $X(t)$. (Warning: The process $X(t)$ appearing in Theorem 7.5.3 and in this problem is not the value of the hedging portfolio. For example, $X(0)$ is given by (7.5.23), and this is different from $v(0, S(0), X(0))$, the initial value of the hedging portfolio.)

8

American Derivative Securities

8.1 Introduction

European option contracts specify an expiration date, and if the option is to be exercised at all, the exercise must occur on the expiration date. An option whose owner can choose to exercise at any time up to and including the expiration date is called *American*. Because of this early exercise feature, such an option is at least as valuable as its European counterpart. Sometimes the difference in value is negligible or even zero, and then American and European options are close or exact substitutes. We shall see in this chapter that the early exercise feature for a call on a stock paying no dividends is worthless; American and European calls on such a stock have the same price. In other cases, most notably put options, the value of this early exercise feature, the so-called *early exercise premium*, can be substantial. An intermediate option between American and European is *Bermudan*, an option that permits early exercise but only on a contractually specified finite set of dates.

Because an American option can be exercised at any time prior to its expiration, it can never be worth less than the payoff associated with immediate exercise. This is called the *intrinsic value* of the option.

In contrast to the case for a European option, whose discounted price process is a martingale under the risk-neutral measure, the discounted price process of an American option is a supermartingale under this measure. The holder of this option may fail to exercise at the optimal exercise date, and in this case the discounted option price has a tendency to fall; hence, the supermartingale property. During any period of time in which it is not optimal to exercise, however, the discounted price process behaves as a martingale.

To price an American option, just as with a European option, we could imagine selling the option in exchange for some initial capital and then consider how to use this capital to hedge the short position in the option. In this case, we would need to be ready to pay off the option at all times prior to the expiration date because we do not know when it will be exercised. We could determine when, from our point of view, is the worst time for the owner to exercise the option. From the owner's point of view, this would be the *optimal exercise time*, and we shall call it that. We could then compute the initial capital we need in order to be hedged against exercise at the optimal exercise time. Finally, we could show how to invest this capital so that we are hedged even if the owner exercises at a nonoptimal time. In the subsequent sections, we do all these things but begin the analysis at a different point than for European options. We define the price of American options using a risk-neutral pricing formula and then show that this price is the smallest initial capital that permits construction of the hedge just described.

For the binomial model, the program described above was carried out in Chapter 4 of Volume I. Here we revisit these matters in a continuous-time setting. We treat first the perpetual American put (Section 8.3), which is not actually traded. The analysis of this option provides lessons that we apply in the subsequent sections. In Section 8.4, we discuss the finite-expiration American put, an option that is traded. Section 8.5 treats the American call. In the case of a non-dividend-paying stock, we show that the American and European calls have the same price. However, if the stock pays dividends, these prices can differ. We show how to compute the American call price in this latter case.

8.2 Stopping Times

Throughout this chapter, we need the concept of stopping times. These were defined and discussed in the binomial model in Section 4.3 of Volume I. A stopping time is a random variable τ that takes values in $[0, \infty]$. The stopping times we shall encounter are the times at which an American option is exercised. The decision of an agent to exercise this option may depend on all the information available at that time but may not depend on future information. We provide a mathematical formulation of this property in Definition 8.2.1 below. Before stating this definition, we seek to motivate it.

In the N-period model of Volume I, where the filtration is generated by coin tossing and there are only finitely many dates, we defined a stopping time to be a random variable τ taking values $0, 1, \dots, N$ or ∞ and having the property that if $\tau(\omega_1 \dots \omega_n \omega_{n+1} \dots \omega_N) = n$, then $\tau(\omega_1 \dots \omega_n \omega'_{n+1} \dots \omega'_N) = n$ for all $\omega'_{n+1} \dots \omega'_N$. This condition guarantees that the decision to stop at time n does not depend on the coin tosses that come after time n.

One way to try to capture this same idea in continuous time is to require that for each nonrandom $t \geq 0$, the set $\{\tau = t\} = \{\omega \in \Omega; \tau(\omega) = t\}$ should be in $\mathcal{F}(t)$ (i.e., the agent stops (exercises the option) at time t based on the information available at time t). However, we shall be interested in sets of ωs of the form $\{\omega \in \Omega; T_1 \leq \tau(\omega) \leq T_2\}$, and these cannot be gotten by taking countable unions of sets of the form $\{\omega \in \Omega; \tau(\omega) = t\}$. Therefore, we impose the slightly stronger condition of Definition 8.2.1 below.

Definition 8.2.1. *A* stopping time τ *is a random variable taking values in* $[0, \infty]$ *and satisfying*

$$\{\tau \leq t\} \in \mathcal{F}(t) \text{ for all } t \geq 0. \tag{8.2.1}$$

Remark 8.2.2. Let $t \geq 0$ be given. Note that (8.2.1) and the properties of σ-algebras imply that $\left\{\tau > t - \frac{1}{n}\right\} = \{\tau \leq t - \frac{1}{n}\}^c \in \mathcal{F}\left(t - \frac{1}{n}\right)$ for all positive integers n. Since every set in $\mathcal{F}\left(t - \frac{1}{n}\right)$ is also in $\mathcal{F}(t)$, we conclude that $\left\{\tau > t - \frac{1}{n}\right\}$ is in $\mathcal{F}(t)$ for every n, and hence

$$\{\tau = t\} = \{\tau \leq t\} \cap \left(\bigcap_{n=1}^{\infty} \left\{\tau > t - \frac{1}{n}\right\}\right)$$

is also in $\mathcal{F}(t)$. In other words, by Definition 8.2.1, a stopping time τ has the property that the decision to stop at time t must be based on information available at time t.

Example 8.2.3 (First passage time for a continuous process). Let $X(t)$ be an adapted process with continuous paths, let m be a number, and set

$$\tau_m = \min\{t \geq 0; X(t) = m\}. \tag{8.2.2}$$

This is the first time the process $X(t)$ reaches the level m. If $X(t)$ never reaches the level m, then we interpret τ_m to be ∞. Intuitively, τ_m must be a stopping time because the value of τ_m is determined by the path of $X(t)$ up to time τ_m. An agent can exercise an option the first time the underlying asset price reaches a level; this exercise strategy does not require information about the underlying price movements after the exercise time.

We use Definition 8.2.1 and the properties of σ-algebras to show mathematically that τ_m is a stopping time. Let $t \geq 0$ be given. We need to show that $\{\tau \leq t\}$ is in $\mathcal{F}(t)$.

If $t = 0$, then $\{\tau \leq t\} = \{\tau = 0\}$ is either Ω or $\emptyset$, depending on whether $X(0) = m$ or $X(0) \neq m$. In either case, $\{\tau \leq 0\} \in \mathcal{F}(0)$.

We consider the case $t > 0$. Suppose $\omega \in \Omega$ satisfies $\tau(\omega) \leq t$. Then there is some number $s \leq t$ such that $X(s, \omega) = m$, where we indicate explicitly the dependence of X on ω. For each positive integer n, there is an open interval of time containing s for which the process X is in $\left(m - \frac{1}{n}, m + \frac{1}{n}\right)$. In this interval, there is a rational number $q \leq s \leq t$. Therefore, ω is in the set

$$A = \bigcap_{n=1}^{\infty} \bigcup_{0 \leq q \leq t, q \text{ rational}} \left\{m - \frac{1}{n} < X(q) < m + \frac{1}{n}\right\}.$$

We have shown that $\{\tau \leq t\} \subset A$.

On the other hand, if $\omega \in A$, then for every positive integer n there is a rational number $q_n \leq t$ such that

$$m - \frac{1}{n} < X(q_n, \omega) < m + \frac{1}{n}.$$

The infinite sequence $\{q_n\}_{n=1}^{\infty}$ must have an accumulation point s in the closed, bounded interval $[0, t]$. In other words, there must exist a number $s \in [0, t]$ and a subsequence $\{q_{n_k}\}_{k=1}^{n}$ such that $\lim_{k \to \infty} q_{n_k} = s$. But

$$m - \frac{1}{n_k} < X(q_{n_k}, \omega) < m + \frac{1}{n_k} \text{ for all } k = 1, 2, \ldots.$$

Letting $k \to \infty$ in these inequalities and using the fact that X has continuous paths, we see that $X(s, \omega) = m$. It follows that $\tau(\omega) \leq t$. We have shown that $A \subset \{\tau \leq t\}$. Therefore $A = \{\tau \leq t\}$.

Because X is adapted to the filtration, for each positive integer n and rational $q \in [0, t]$, the set

$$\left\{m - \frac{1}{n} < X(q) < m + \frac{1}{n}\right\}$$

is in $\mathcal{F}(q)$ and hence in the larger σ-algebra $\mathcal{F}(t)$. Because there are only countably many rational numbers q in $[0, t]$, they can be arranged in a sequence, and the union

$$B_n = \bigcup_{0 \leq q \leq t, q \text{ rational}} \left\{m - \frac{1}{n} < X(q) < m + \frac{1}{n}\right\}$$

is really a union of a sequence of sets in $\mathcal{F}(t)$. The set B_n must therefore also be in $\mathcal{F}(t)$. Because B_n is in $\mathcal{F}(t)$ for every positive integer n, the intersection $\cap_{n=1}^{\infty} B_n = A$ is also in $\mathcal{F}(t)$. We have already shown that $A = \{\tau \leq t\}$. We conclude that $\{\tau \leq t\} \in \mathcal{F}(t)$. □

Suppose now that we have an adapted process $X(t)$ and a stopping time τ. We define the *stopped process* $X(t \wedge \tau)$, where $\wedge$ denotes the minimum of two quantities (i.e., $t \wedge \tau = \min\{t, \tau\}$). The stopped process $X(t \wedge \tau)$ agrees with $X(t)$ up to time τ, and thereafter it is frozen at the value of $X(\tau)$. See Figure 8.2.1.

Theorem 8.2.4 (Optional sampling). *A martingale stopped at a stopping time is a martingale. A supermartingale (or submartingale) stopped at a stopping time is a supermartingale (or submartingale, respectively).*

While the proof of Theorem 8.2.4 is technical and will not be given here, the intuition is clear. If $M(t)$ is a martingale, then the stopped process $M(t \wedge \tau)$ agrees with $M(t)$ before time τ and thus is also a martingale. After time τ, the stopped process is frozen (i.e., it no longer changes with time), and this is a trivial martingale. A martingale goes neither up nor down "on average." After being frozen, a process goes neither up nor down, path-by-path. The only way the martingale property could be violated is if the stopping decision

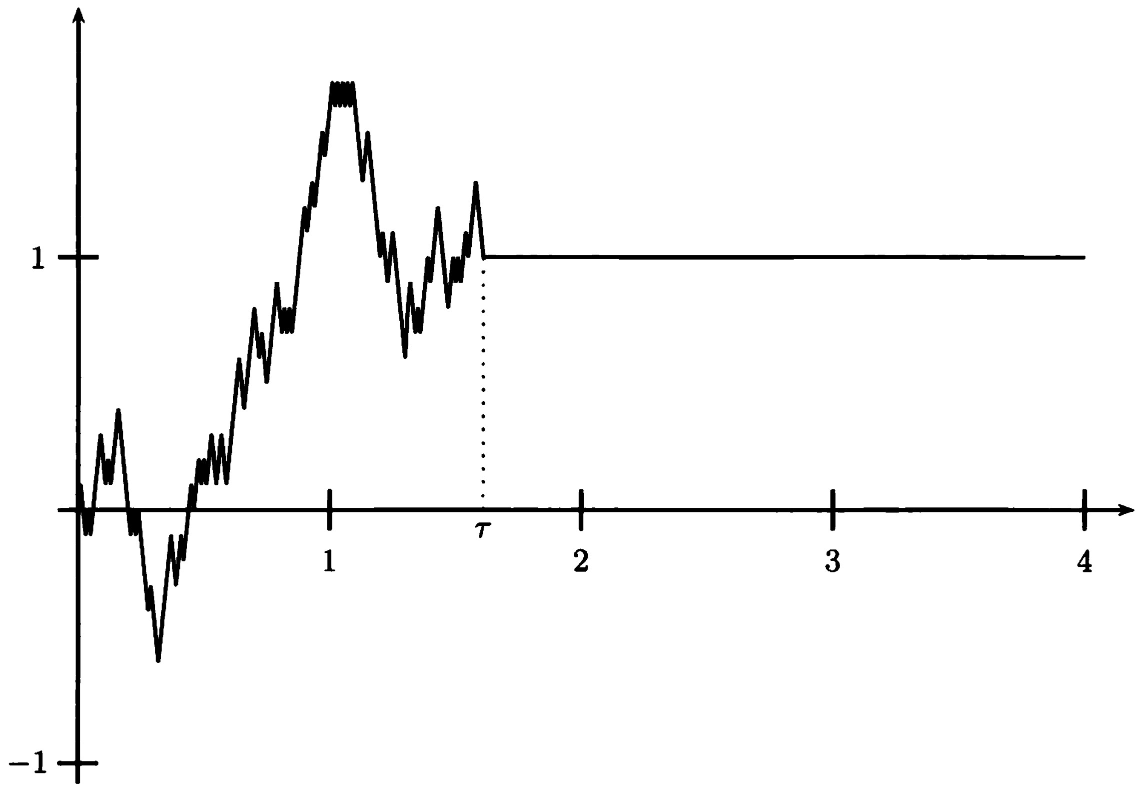

Fig. 8.2.1. A stopped process.

looked ahead. Suppose that a martingale is stopped (frozen) if it will go up in the near future but is allowed to continue if it will go down. Then stopping introduces a downward bias by removing the upward possibility. Figure 8.2.2 shows a martingale in a discrete-time model under the assumption that the probability of H (an up move) is $\tilde{p} = \frac{1}{2}$ and the probability of T (a down move) is $\tilde{q} = \frac{1}{2}$. Figures 8.2.2–8.2.4 are taken from Section 4.3 of Volume I, where the martingale in Figure 8.2.2 is a discounted stock price under a risk-neutral measure. Figure 8.2.3 shows a random time ρ that is not a stopping time; this random time ρ causes stopping at time 0 if there is an H on the first toss (an up move) but lets the process continue if there is a T on the first toss. Similarly, if there is a T on the first toss and an H on the second toss, ρ stops the martingale at time 1 but lets it continue to time 2 if there is a T on the first toss and an H on the second toss. The stopped martingale is shown in Figure 8.2.4, and it is not a martingale. For example,

$$\widetilde{\mathbb{E}} M_{2\wedge\rho} = \frac{1}{4}(4 + 4 + 1.60 + 0.64) = 2.56 < M_0 = 4,$$

whereas the expectation of a martingale does not change over time. Our definition of stopping time rules out this kind of stopping.

Similar intuition applies to supermartingales. A stopped supermartingale is a supermartingale before being frozen, and after being frozen it is a martingale, which is a special case of a supermartingale. The situation with submartingales is analogous. Again, the stopping must be done at a stopping time.

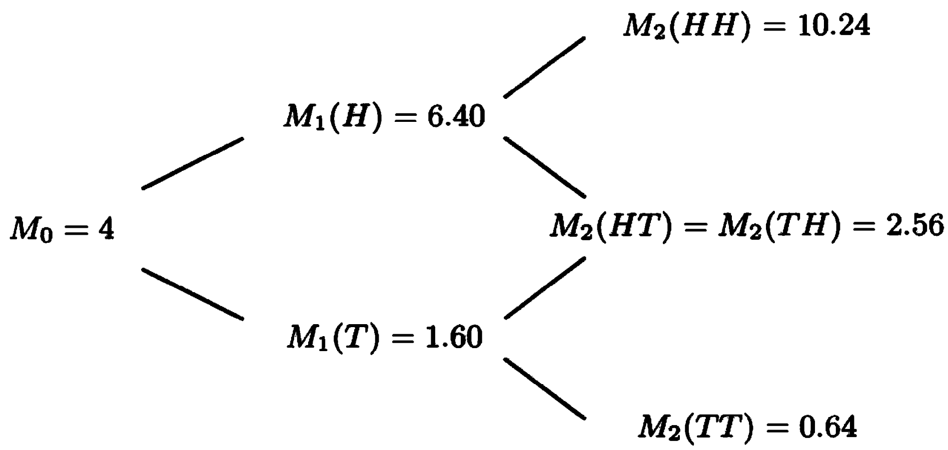

Fig. 8.2.2. Martingale under $\tilde{p} = \tilde{q} = \frac{1}{2}$.

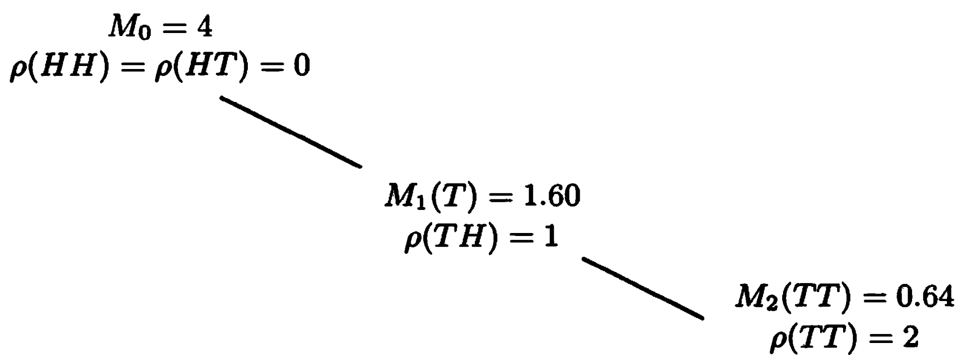

Fig. 8.2.3. Non-stopping time ρ.

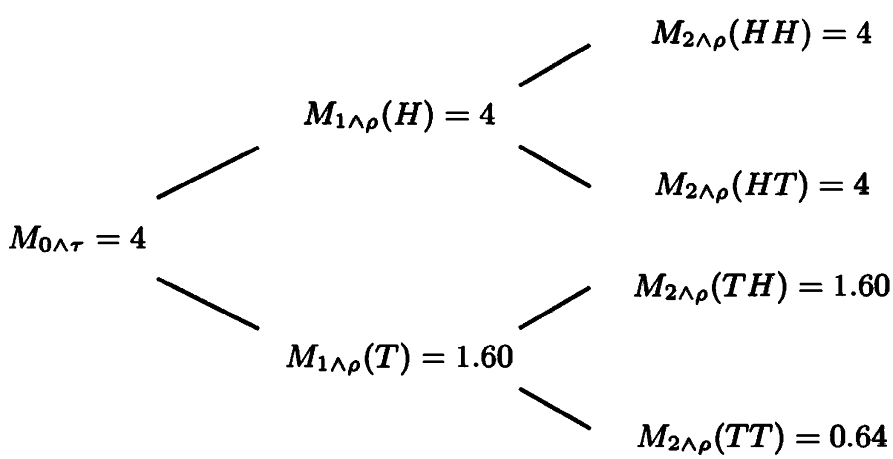

Fig. 8.2.4. Martingale stopped at the non-stopping time ρ.

Looking ahead to make the stopping decision can ruin the supermartingale (respectively, submartingale) property.

8.3 Perpetual American Put

The simplest interesting American option is the *perpetual American put.* It is interesting because the optimal exercise policy is not obvious, and it is simple because this policy can be determined explicitly. Although this is not a traded option, we begin our discussion with it in order to present in a simple context the ideas behind the subsequent analysis of more realistic options.

The underlying asset in most of this chapter (except in Subsection 8.5.2, where the asset pays dividends) has the price process $S(t)$ given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \tag{8.3.1}$$

where the interest rate r and the volatility σ are strictly positive constants and $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. The perpetual American put pays $K - S(t)$ if it is exercised at time t. This is its intrinsic value.

Definition 8.3.1. *Let $\mathcal{T}$ be the set of all stopping times. The* price of the perpetual American put *is defined to be*

$$v_*(x) = \max_{\tau\in\mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\right], \tag{8.3.2}$$

where $x = S(0)$ in (8.3.2) is the initial stock price. In the event that $\tau = \infty$, we interpret $e^{-r\tau}(K - S(\tau))$ to be zero.

The idea behind Definition 8.3.1 is that the owner of the perpetual American put can choose an exercise time τ, subject only to the condition that she may not look ahead to determine when to exercise. The mathematical formulation of this "not look ahead" restriction is that τ must be a stopping time. The price of the option at time zero is the risk-neutral expected payoff of the option, discounted from the exercise time back to time zero. If the option is never exercised, its payoff is zero. This explains the term under the expectation on the right-hand side of (8.3.2). The owner of the option should choose the exercise strategy that maximizes this expected payoff, discounted back to time zero, and thus we define the price of the option to be the maximum over $\tau \in \mathcal{T}$ of the discounted expected payoffs.

This risk-neutral pricing definition of the perpetual American put price appears to differ from the construction of the price of a European call in Section 4.5. There we took the price to be the initial capital required by an agent holding a short position in the option in order for this agent to hedge the short position (i.e., invest in the stock and money market account in such a way that at expiration of the option the resulting portfolio value is the payoff of the option). It turns out that $v_*(x)$ defined above is the initial capital required for an agent to hedge a short position in the American put *regardless of the exercise strategy τ used by the owner of the put*; see Corollaries 8.3.6 and 8.3.7.

The owner of the perpetual American put can exercise at any time. In particular, there is no expiration date after which the put can no longer be exercised. This makes every date like every other date; the time remaining to expiration is always the same (i.e., infinity). Because every date is like every other date, it is reasonable to expect that the optimal exercise policy depends only on the value of $S(t)$ and not on the time variable t. The owner of the put should exercise as soon as $S(t)$ falls "far enough" below K. In other words, it is reasonable to expect that the optimal exercise policy is of the form

"Exercise the put as soon as $S(t)$ falls to the level L_*."

We have two questions to answer:

(i) What is the value of L_* and how do we know it corresponds to optimal exercise?
(ii) What is the value of the put?

For the perpetual American put, we can base the answers to these questions on explicit computations.

8.3.1 Price Under Arbitrary Exercise

Theorem 8.3.2 (Laplace transform for first passage time of drifted Brownian motion). *Let $\widetilde{W}(t)$ be a Brownian motion under a probability measure $\widetilde{\mathbb{P}}$, let μ be a real number, and let m be a positive number. Define $X(t) = \mu t + \widetilde{W}(t)$, and set*

$$\tau_m = \min\{t \geq 0; X(t) = m\},$$

so that τ_m is the stopping time of Example 8.2.3. If $X(t)$ never reaches the level m, then we interpret τ_m to be ∞. Then

$$\widetilde{\mathbb{E}} e^{-\lambda \tau_m} = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})} \text{ for all } \lambda > 0, \tag{8.3.3}$$

where we interpret $e^{-\lambda \tau_m}$ to be zero if $\tau_m = \infty$.

PROOF: Define $\sigma = -\mu + \sqrt{\mu^2 + 2\lambda}$ so that $\sigma > 0$ and

$$\begin{aligned}
\sigma\mu + \frac{1}{2}\sigma^2 &= -\mu^2 + \mu\sqrt{\mu^2 + 2\lambda} + \frac{1}{2}\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)^2 \\
&= -\mu^2 + \mu\sqrt{\mu^2 + 2\lambda} + \frac{1}{2}\mu^2 - \mu\sqrt{\mu^2 + 2\lambda} + \frac{1}{2}\mu^2 + \lambda \\
&= \lambda.
\end{aligned}$$

Then

$$e^{\sigma X(t)-\lambda t} = e^{\sigma\mu t+\sigma\widetilde{W}(t)-\sigma\mu t-\frac{1}{2}\sigma^2 t} = e^{\sigma\widetilde{W}(t)-\frac{1}{2}\sigma^2 t},$$

which is a martingale under $\widetilde{\mathbb{P}}$ (its differential has a $d\widetilde{W}(t)$ term and no dt term). According to Theorem 8.2.4 (optional sampling), the stopped martingale

$$M(t) = e^{\sigma\widetilde{W}(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)}$$

is also a martingale. Therefore, for each positive integer n,

$$\begin{aligned}
1 &= M(0) = \widetilde{\mathbb{E}}M(n)\\
&= \widetilde{\mathbb{E}}\left[e^{\sigma X(n\wedge\tau_m)-\lambda(n\wedge\tau_m)}\right]\\
&= \widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m\le n\}}\right] + \widetilde{\mathbb{E}}\left[e^{\sigma X(n)-\lambda n}\mathbb{I}_{\{\tau_m>n\}}\right]. \qquad (8.3.4)
\end{aligned}$$

The nonnegative random variables $e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m\le n\}}$ increase with n, and their limit is $e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m<\infty\}}$. In other words,

$$0 \le e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m\le 1\}} \le e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m\le 2\}} \le \dots \text{ almost surely,}$$

and

$$\lim_{n\to\infty} e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m\le n\}} = e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m<\infty\}} \text{ almost surely.}$$

The Monotone Convergence Theorem, Theorem 1.4.5, implies

$$\lim_{n\to\infty}\widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m\le n\}}\right] = \widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m<\infty\}}\right]. \qquad (8.3.5)$$

On the other hand, the random variable $e^{\sigma X(n)-\lambda n}\mathbb{I}_{\{\tau_m>n\}}$ satisfies

$$0 \le e^{\sigma X(n)-\lambda n}\mathbb{I}_{\{\tau_m>n\}} \le e^{\sigma m-\lambda n} \le e^{\sigma m} \text{ almost surely}$$

because $X(n) \le m$ for $n < \tau_m$ and σ is positive. Because λ is positive, we have

$$\lim_{n\to\infty} e^{\sigma X(n)-\lambda n}\mathbb{I}_{\{\tau_m>n\}} \le \lim_{n\to\infty} e^{\sigma m-\lambda n} = 0.$$

According to the Dominated Convergence Theorem, Theorem 1.4.9,

$$\lim_{n\to\infty}\mathbb{E}\left[e^{\sigma X(n)-\lambda n}\mathbb{I}_{\{\tau_m>n\}}\right] = 0. \qquad (8.3.6)$$

Taking the limit in (8.3.4) and using (8.3.5) and (8.3.6), we obtain

$$1 = \widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbb{I}_{\{\tau_m<\infty\}}\right]$$

or, equivalently,

$$\widetilde{\mathbb{E}}\left[e^{-\lambda\tau_m}\mathbb{I}_{\{\tau_m<\infty\}}\right] = e^{-\sigma m} = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})} \text{ for all } \lambda > 0. \qquad (8.3.7)$$

This is (8.3.3) when we interpret $e^{-\lambda\tau_m}$ to be zero if $\tau_m = \infty$. □

Remark 8.3.3. We used the strict positivity of λ to derive (8.3.7), but now that we have it, we can take the limit as $\lambda \downarrow 0$. The random variables $e^{-\lambda\tau_m}\mathbb{I}_{\{\tau_m<\infty\}}$ are nonnegative and increase to $\mathbb{I}_{\{\tau_m<\infty\}}$ as $\lambda \downarrow 0$, and the Monotone Convergence Theorem allows us to conclude that

$$\widetilde{\mathbb{P}}\{\tau_m < \infty\} = \widetilde{\mathbb{E}}\mathbb{I}_{\{\tau_m<\infty\}} = \lim_{\lambda\downarrow 0} e^{-m(-\mu+\sqrt{\mu^2+2\lambda})} = e^{m\mu-m|\mu|}.$$

If $\mu \geq 0$, the drift in $X(t)$ is zero or upward, toward level m, and $\widetilde{\mathbb{P}}\{\tau_m < \infty\} = 1$; the level $X(t)$ is reached with probability one. On the other hand, if $\mu < 0$, the drift in $X(t)$ is downward, away from level m, and $\widetilde{\mathbb{P}}\{\tau_m < \infty\} = e^{-2m|\mu|} < 1$; there is a positive probability of never reaching m. □

The solution to (8.3.1) is

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}. \tag{8.3.8}$$

Suppose the owner of the perpetual American put sets a positive level $L < K$ and resolves to exercise the put the first time the stock price falls to L. If the initial stock price is at or below L, she exercises immediately (at time zero). The value of the put in this case is $v_L\big(S(0)\big) = K - S(0)$. If the initial stock price is above L, she exercises at the stopping time

$$\tau_L = \min\{t \geq 0; S(t) = L\}, \tag{8.3.9}$$

where τ_L is set equal to ∞ if the stock price never reaches the level L. At the time of exercise, the put pays $K - S(\tau_L) = K - L$. Discounting this back to time zero and taking the risk-neutral expected value, we compute the value of the put under this exercise strategy to be

$$v_L\big(S(0)\big) = (K - L)\widetilde{\mathbb{E}}e^{-r\tau_L} \text{ for all } S(0) \geq L. \tag{8.3.10}$$

On those paths where $\tau_L = \infty$, we interpret $e^{-r\tau_L}$ to be zero. (Recall our assumption at the beginning of this section that r is strictly positive.) Although not explicitly indicated by the notation, the distribution of τ_L depends on the initial stock price $S(0)$, so the right-hand side (8.3.10) is a function of $S(0)$.

Lemma 8.3.4. *The function $v_L(x)$ is given by the formula*

$$v_L(x) = \begin{cases} K - x, & 0 \leq x \leq L, \\ (K - L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & x \geq L. \end{cases} \tag{8.3.11}$$

PROOF: We only need to establish the second line of (8.3.11). If $x = L$, then $\tau_L = 0$ and (8.3.10) implies $v_L(x) = K - L$.

We consider the case $S(0) = x > L$. The stopping time τ_L is the first time

$$S(t) = x\exp\left\{\sigma\widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)\right\}$$

reaches the level L. But $S(t) = L$ if and only if

$$-\widetilde{W}(t) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)t = \frac{1}{\sigma}\log\frac{x}{L}.$$

We now apply Theorem 8.3.2 with $X(t)$ in that theorem replaced by $-\widetilde{W}(t) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)t$ (the processes $\widetilde{W}(t)$ and $-\widetilde{W}(t)$ are both Brownian motions under $\widetilde{\mathbb{P}}$), with λ replaced by r, with μ replaced by $-\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$, and with m replaced by $\frac{1}{\sigma}\log\frac{x}{L}$, which is positive. With these replacements, τ_m in Theorem 8.3.2 is τ_L and

$$\begin{aligned}\mu^2 + 2\lambda &= \frac{1}{\sigma^2}\left(r^2 - r\sigma^2 + \frac{1}{4}\sigma^4\right) + 2r \\ &= \frac{1}{\sigma^2}\left(r^2 + r\sigma^2 + \frac{1}{4}\sigma^4\right) \\ &= \frac{1}{\sigma^2}\left(r + \frac{1}{2}\sigma^2\right)^2.\end{aligned}$$

Therefore,

$$-\mu + \sqrt{\mu^2 + 2\lambda} = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma}\left(r + \frac{1}{2}\sigma^2\right) = \frac{2r}{\sigma}.$$

Equation (8.3.3) implies

$$\widetilde{\mathbb{E}}^{-r\tau_L} = \exp\left\{-\frac{1}{\sigma}\log\frac{x}{L}\cdot\frac{2r}{\sigma}\right\} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}.$$

The second line in (8.3.11) follows. □

8.3.2 Price Under Optimal Exercise

Figure 8.3.1 shows the function $v_L(x)$ for three different values of L. The function $v_{L_1}(x)$ in that figure actually lies below the intrinsic value $K - x$ for x between L_1 and L_2. If the initial stock price is between L_1 and L_2, then the strategy of exercising the first time the stock price falls to L_1 is obviously a poor one; it would be better to exercise at time zero and receive the intrinsic value. The function $v_{L_2}(x)$ agrees with the intrinsic value for $0 \le x \le L_2$ and follows the indicated curve for $x \ge L_2$. The function $v_{L_*}(x)$ agrees with the intrinsic value for $0 \le x \le L_*$ and follows the indicated curve for $x \ge L_*$. For $x \ge L_*$, the function $v_{L_*}(x)$ is strictly larger than the function $v_{L_2}(x)$, and hence the strategy of exercising the first time the stock price falls to L_* is better than exercising the first time the stock price falls to L_2.

As Figure 8.3.1 suggests, for any value of L smaller than L_*, the function $v_L(x)$ agrees with the intrinsic value for $0 \le x \le L$, lies below the intrinsic

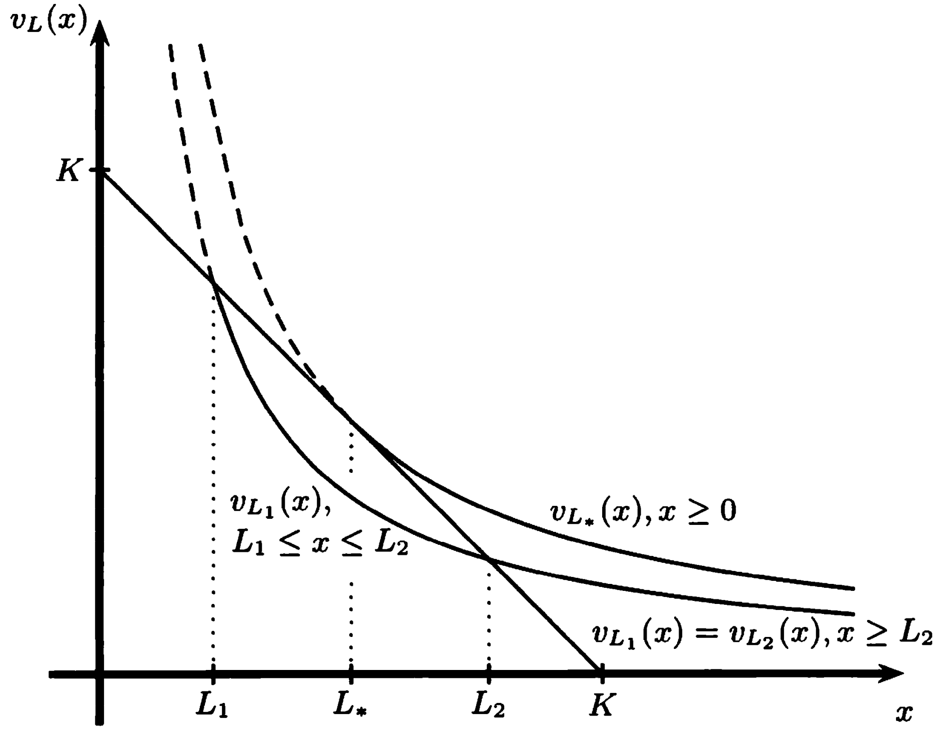

Fig. 8.3.1. $(K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$ for three values of L.

value immediately to the right of L, and lies below $v_{L_*}(x)$ everywhere to the right of L. For any value of L larger than L_*, the function $v_L(x)$ agrees with the intrinsic value for $0 \le x \le L$ and lies below $v_{L_*}(x)$ for all $x \ge L_*$. Thus, among those exercise policies of the form

"Exercise the put as soon as $S(t)$ falls to the level L,"

the best one is obtained by choosing $L = L_*$. We expect therefore that $v_{L_*}(x)$ is the price of the put $v_*(x)$ of Definition 8.3.1. We prove this below.

We must first determine the value of L_*. We note that

$$v_L(x) = (K-L)L^{\frac{2r}{\sigma^2}}x^{-\frac{2r}{\sigma^2}} \text{ for all } x \ge L.$$

From Figure 8.3.1, we know that L_* is the value of L that maximizes this quantity when we hold x fixed. We thus define

$$g(L) = (K-L)L^{\frac{2r}{\sigma^2}}$$

and seek the value of L that maximizes this function over $L \ge 0$. Because $\frac{2r}{\sigma^2}$ is strictly positive, we have $g(0) = 0$ and $\lim_{L\to\infty} g(L) = -\infty$. Moreover,

$$g'(L) = -L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}(K-L)L^{\frac{2r}{\sigma^2}-1} = -\frac{2r+\sigma^2}{\sigma^2}L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}KL^{\frac{2r}{\sigma^2}-1}.$$

Setting this equal to zero, we solve for

$$L_* = \frac{2r}{2r+\sigma^2}K. \tag{8.3.12}$$

This is a number between 0 and K. Furthermore,

$$g(L_*) = \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{\frac{2r}{\sigma^2}} K^{\frac{2r+\sigma^2}{\sigma^2}}$$

is strictly positive. Therefore, the graph of $y = g(L)$ must be as shown in Figure 8.3.2, and L_* given by (8.3.12) is the point where $g(L)$ attains its maximum.

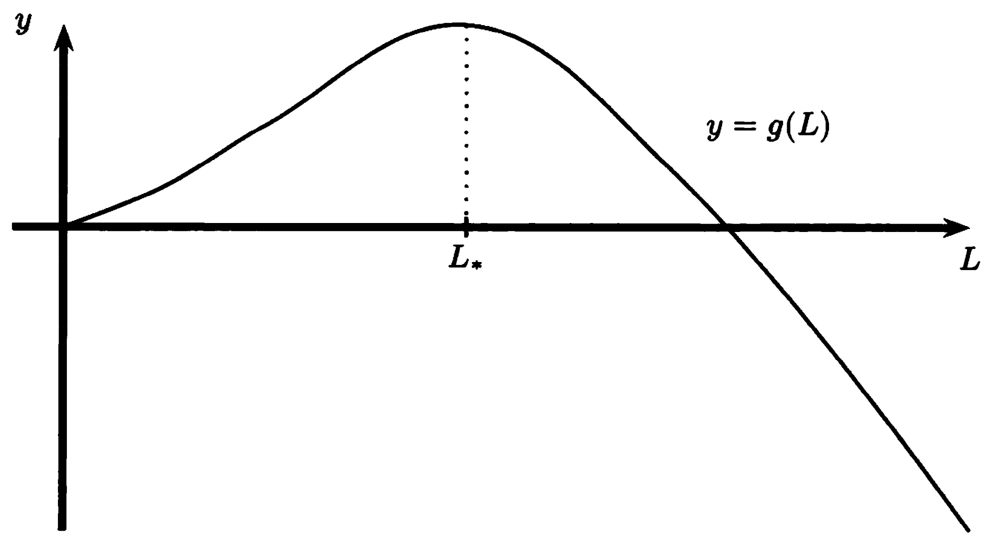

Fig. 8.3.2. Graph of $g(L)$.

8.3.3 Analytical Characterization of the Put Price

We have

$$v_{L_*}(x) = \begin{cases} K - x, & 0 \le x \le L_*, \\ (K - L_*)\left(\dfrac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L_*, \end{cases} \tag{8.3.13}$$

so that

$$v'_{L_*}(x) = \begin{cases} -1, & 0 \le x \le L_*, \\ -(K - L_*)\dfrac{2r}{\sigma^2 x}\left(\dfrac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L_*. \end{cases} \tag{8.3.14}$$

If we evaluate the second line in (8.3.14) at $x = L_*$, we get the right-hand derivative

$$v'_{L_*}(L_*+) = -\frac{2r}{\sigma^2 L_*}(K - L_*) = -\frac{2rK}{\sigma^2 L_*} + \frac{2r}{\sigma^2} = -\frac{2r}{\sigma^2}\cdot\frac{2r+\sigma^2}{2r} + \frac{2r}{\sigma^2} = -1,$$

which agrees with the left-hand derivative $v'_{L_*}(L_*-) = -1$ provided by the first line in (8.3.14). The derivative of $v_{L_*}(x)$ is continuous at $x = L_*$. This is known as *smooth pasting.* The two parts of the definition of $v_{L_*}(x)$ fit together at $x = L_*$ so that both $v_{L_*}(x)$ and $v'_{L_*}(x)$ are continuous. This is because the graph of the function $y = (K - L_*)(\frac{x}{L_*})^{-\frac{2r}{\sigma^2}}$ is tangent to the line $y = K - x$ at $x = L_*$, as one can see from Figure 8.3.1. In fact, we could have used the smooth pasting condition to solve for L_* (see Exercise 8.1).

The second derivative of $v(x)$ has a jump at $x = L_*$, and hence is undefined at this point. Indeed,

$$v''_{L_*}(x) = \begin{cases} 0, & 0 \le x < L_*, \\ (K - L_*)\dfrac{2r(2r+\sigma^2)}{\sigma^4 x^2}\left(\dfrac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x > L_*. \end{cases} \tag{8.3.15}$$

The left-hand and right-hand second derivatives at $x = L_*$ are $v(L_*-) = 0$ and $v''(L_*+) = (K - L_*)\frac{2r(2r+\sigma^2)}{\sigma^4 L_*^2} > 0$.

For $x > L_*$, we can verify by direct computation that

$$\begin{aligned} & rv_{L_*}(x) - rxv'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) \\ & \quad = (K - L_*)\left(r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}} = 0. \end{aligned} \tag{8.3.16}$$

On the other hand, for $0 \le x < L_*$,

$$rv_{L_*}(x) - rxv'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) = r(K - x) + rx = rK. \tag{8.3.17}$$

In particular, we see that $v_{L_*}(x)$ satisfies the so-called *linear complementarity conditions*

$$v(x) \ge (K - x)^+ \text{ for all } x \ge 0, \tag{8.3.18}$$

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \ge 0 \text{ for all } x \ge 0, \text{ and} \tag{8.3.19}$$

$$\text{for each } x \ge 0, \text{ equality holds in either (8.3.18) or (8.3.19).} \tag{8.3.20}$$

The point L_* is slightly problematical in (8.3.19) since $v''_{L_*}(L_*)$ is undefined. However, if we replace $v''_{L_*}(L_*)$ in (8.3.19) by either $v''_{L_*}(L_*-)$ or $v''_{L_*}(L_*+)$, the inequality holds.

The linear complementarity conditions (8.3.18)–(8.3.20) determine the function $v_{L_*}(x)$. More precisely, the function $v_{L_*}(x)$ given by (8.3.13) is the only bounded continuous function having a continuous derivative that satisfies these conditions; see Exercise 8.3.

8.3.4 Probabilistic Characterization of the Put Price

Theorem 8.3.5. *Let $S(t)$ be the stock price given by (8.3.1) and let τ_{L_*} be given by (8.3.9) with $L = L_*$. Then $e^{-rt}v_{L_*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(t\wedge\tau_{L_*})}v_{L_*}(S(t\wedge\tau_{L_*}))$ is a martingale.*

PROOF: Fortunately, the Itô-Doeblin formula applies to functions whose second derivatives have jumps, provided the first derivative is continuous (see Exercise 4.20 for a discussion related to this). We may thus compute

$$\begin{aligned}
d&\big[e^{-rt}v_{L_*}(S(t))\big]\\
&= e^{-rt}\left[-rv_{L_*}(S(t))\,dt + v'_{L_*}(S(t))\,dS(t) + \frac{1}{2}v''_{L_*}(S(t))\,dS(t)\,dS(t)\right]\\
&= e^{-rt}\left[-rv_{L_*}(S(t)) + rS(t)v'_{L_*}(S(t)) + \frac{1}{2}\sigma^2S^2(t)v''_{L_*}(S(t))\right]dt\\
&\qquad + e^{-rt}\sigma S(t)v'_{L_*}(S(t))\,d\widetilde{W}(t).
\end{aligned}$$

Because of (8.3.16) and (8.3.17), the dt term in this expression is either 0 or $-rK$, depending on whether $S(t) > L_*$ or $S(t) < L_*$. If $S(t) = L_*$, $v''_{L_*}(S(t))$ is undefined, but the probability $S(t) = L_*$ is zero so this does not matter. We thus have

$$d\big[e^{-rt}v_{L_*}(S(t))\big] = -e^{-rt}rK\mathbb{I}_{\{S(t)<L^*\}}\,dt + e^{-rt}\sigma S(t)v'_{L_*}(S(t))\,d\widetilde{W}(t). \tag{8.3.21}$$

Because the dt term in (8.3.21) is less than or equal to zero, $e^{-rt}v_{L_*}(S(t))$ is a supermartingale; when $S(t) < L_*$ it has a downward tendency. If the initial stock price is above L_*, then prior to the time τ_{L_*} when the stock price first reaches L_*, the dt term in (8.3.21) is zero and hence $e^{-r(t\wedge\tau_{L_*})}v(S(t\wedge\tau_{L_*}))$ is a martingale. Indeed, integration of (8.3.21) yields

$$e^{-r(t\wedge\tau_{L_*})}v_{L_*}(S(t\wedge\tau_{L_*})) = v_{L_*}(0) + \int_0^{t\wedge\tau_{L_*}} e^{-ru}\sigma S(u)v'_{L_*}(S(u))\,d\widetilde{W}(u).$$

Itô integrals are martingales, and hence the Itô integral above stopped at the stopping time τ_{L_*} is a martingale. □

Corollary 8.3.6. *Recall that $\mathcal{T}$ is the set of all stopping times, not just those of the form (8.3.9). We have*

$$v_{L_*}(x) = \max_{\tau\in\mathcal{T}}\widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right],$$

where $x = S(0)$ is the initial stock price. In other words, $v_{L_*}(x)$ is the perpetual American put price of Definition 8.3.1.

PROOF: Because $e^{-rt}v_{L_*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$, we have from Theorem 8.2.4 (optional sampling) that, for every stopping time $\tau \in \mathcal{T}$,

$$v_{L_*}(x) = v_{L_*}(S(0)) \geq \widetilde{\mathbb{E}}\left[e^{-r(t\wedge\tau)} v_{L_*}(S(t\wedge\tau))\right]. \tag{8.3.22}$$

Because $v_{L_*}(S(t\wedge\tau))$ is bounded, we may let $t\to\infty$ in (8.3.22), using the Dominated Convergence Theorem, Theorem 1.4.9, to conclude that

$$v_{L_*}(x) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau} v_{L_*}(S(\tau))\right] \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right],$$

where we have gotten the last inequality from (8.3.18). Because this inequality holds for every $\tau\in\mathcal{T}$, we have

$$v_{L_*}(x) \geq \max_{\tau\in\mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

On the other hand, if we replace τ by τ_{L_*}, we obtain equality in (8.3.22) because $e^{-r(t\wedge\tau_{L_*})}v(S(t\wedge\tau_{L_*}))$ is a martingale under $\widetilde{\mathbb{P}}$. Letting $t\to\infty$ and using the Dominated Convergence Theorem, we obtain

$$v_{L_*}(x) = \widetilde{\mathbb{E}}\left[e^{-r\tau_{L_*}} v_{L_*}(S(\tau_{L_*}))\right].$$

Since

$$e^{-r\tau_{L_*}} v_{L_*}(S(\tau_{L_*})) = e^{-r\tau_{L_*}} v_{L_*}(L_*) = e^{-r\tau_{L_*}}(K - L_*) = e^{-r\tau_{L_*}}(K - S(\tau_{L_*}))$$

if $\tau_{L_*} < \infty$ (and is interpreted to be zero if $\tau_{L_*} = \infty$), we see that

$$v_{L_*}(x) = \widetilde{\mathbb{E}}\left[e^{-r\tau_{L_*}}(K - S(\tau_{L_*}))\right]. \tag{8.3.23}$$

It follows that

$$v_{L_*}(x) \leq \max_{\tau\in\mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right]. \qquad \square$$

Discounted European option prices are martingales under the risk-neutral probability measure. Discounted American option prices are martingales up to the time they should be exercised. If they are not exercised when they should be, they tend downward. Since a martingale is a special case of a supermartingale, and processes that tend downward are supermartingales, discounted American option prices are supermartingales. An agent who is short an American option can hedge that short position in the usual way during the time the discounted option price is a martingale. If the option is not exercised when it should be, then the agent can continue the hedge and take money off the table. The following corollary illustrates this for the perpetual American put of this section.

Corollary 8.3.7. *Consider an agent with initial capital $X(0) = v_{L_*}(S(0))$, the initial perpetual American put price. Suppose this agent uses the portfolio process $\Delta(t) = v'_{L_*}(S(t))$ and consumes cash at rate $C(t) = rK\mathbb{I}_{\{S(t)<L^*\}}$ (i.e., consumes cash at rate rK whenever $S(t) < L^*$). Then the value $X(t)$ of the agent's portfolio agrees with the option price $v_{L_*}(S(t))$ for all times t until the option is exercised. In particular, $X(t) \geq (K - S(t))^+$ for all t until the option is exercised, so the agent can pay off a short option position regardless of when the option is exercised.*

PROOF: The differential of the agent's portfolio value process is

$$dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt - C(t)\,dt,$$

so the differential of the discounted portfolio value process is

$$\begin{aligned} d\big(e^{-rt}X(t)\big) &= e^{-rt}\big(-rX(t)\,dt + dX(t)\big)\\ &= e^{-rt}\big(\Delta(t)\,dS(t) - r\Delta(t)S(t)\,dt - C(t)\,dt\big)\\ &= e^{-rt}\big(\Delta(t)\sigma S(t)\,d\widetilde{W}(t) - C(t)\,dt\big). \end{aligned} \tag{8.3.24}$$

Substituting $\Delta(t) = v'_{L_*}(S(t))$ and $C(t) = rK\mathbb{I}_{\{S(t)<L^*\}}$ into (8.3.24) and comparing it to (8.3.21), we see that $d(e^{-rt}X(t)) = d\big[e^{-rt}v_{L_*}(S(t))\big]$. Integrating both sides of this equation and using the initial equality $X(0) = v_{L_*}(S(0))$, we obtain $X(t) = v_{L_*}(S(t))$ for all t prior to exercise. □

Remark 8.3.8. During any period in which $S(t) < L^*$, the agent in Corollary 8.3.7 has stock position $\Delta(t) = v'_{L_*}(S(t)) = -1$ (i.e., is short one share of stock) and has a total portfolio value $X(t) = v_{L_*}(S(t)) = K - S(t)$. Therefore, the agent has K invested in the money market. If the owner of the put exercises, the agent in Corollary 8.3.7 receives a share of stock, which covers his short position, and pays out K from his money market account. If the owner of the put does not exercise, the agent holds his position and consumes the interest from the money market investment (i.e., consumes cash at rate rK per unit time). □

The argument in Corollary 8.3.7 applies generally. In a complete market, whenever some discounted price process is a supermartingale, it is possible to construct a hedging portfolio whose value tracks the price process. This portfolio may sometimes consume. In the case of the perpetual American put, the supermartingale property for the discounted put price follows from (8.3.19). If, in addition, the price process dominates some intrinsic value (see (8.3.18) for the perpetual American put), then a short position in the American option with that intrinsic value can be hedged. There are always two conditions on the price of any American option, corresponding to (8.3.18) and (8.3.19). These conditions guarantee that the price is sufficient to satisfy the seller of the put.

However, conditions (8.3.18) and (8.3.19) alone are not enough to determine the price of the perpetual American put. There can be functions that satisfy these conditions but are strictly greater than the price $v_{L_*}(x)$ we constructed in (8.3.13) (see Exercise 8.2). There must be some additional condition that guarantees that the price is satisfactory for the purchaser of the put. One version of this condition for the perpetual American put is (8.3.20). Condition (8.3.20) guarantees that there exists an exercise strategy that permits the owner of the put to capture the full value of the put. It says that if we divide the half-line $[0,\infty)$ into two sets, the *stopping set*

$$\mathcal{S} = \{x \geq 0; v_{L_*}(x) = (K - x)^+\} \tag{8.3.25}$$

and the *continuation set*

$$\mathcal{C} = \{x \geq 0; v_{L_*}(x) > (K - x)^+\}, \tag{8.3.26}$$

then equality holds in (8.3.19) for $x \in \mathcal{C}$. If the initial stock price is in $\mathcal{S}$, then the owner of the put can get full value by exercising it immediately. On the other hand, if the initial stock price is in $\mathcal{C}$, then the put is more valuable than its intrinsic value, and the owner of the put can capture this extra value by waiting until the stock price enters $\mathcal{S}$ to exercise, if it ever does enter $\mathcal{S}$. The time of entry into the set $\mathcal{S}$ is in fact τ_{L_*} in Theorem 8.3.5. We saw in (8.3.23) that

$$v\big(S(0)\big) = \widetilde{\mathbb{E}}\left[e^{-r\tau^*} v\big(S(\tau^*)\big)\right] = \widetilde{\mathbb{E}}\left[e^{-r\tau^*}\big(K - S(\tau^*)\big)\right].$$

In conclusion, the three linear complementarity conditions have counterparts that can be stated probabilistically rather than analytically (i.e., without writing conditions on the derivatives of $v(x)$). Let $V(t) = e^{-rt}v(S(t))$ be the value of the perpetual American put. The stochastic process $V(t)$ satisfies the following three conditions:

(i) $V(t) \geq \big(K - S(t)\big)^+$ for all $t \geq 0$,
(ii) $e^{-rt}V(t)$ is a supermartingale under $\widetilde{\mathbb{P}}$, and
(iii) there exists a stopping time τ_* such that

$$V(0) = \widetilde{\mathbb{E}}\left[e^{-r\tau_*}\big(K - S(\tau_*)\big)^+\right].$$

These three conditions determine the value of $V(0)$.

8.4 Finite-Expiration American Put

In this section, we consider an American put on a stock whose price is the geometric Brownian motion (8.3.1), but now the put has a finite expiration time T.

Definition 8.4.1. *Let $0 \leq t \leq T$ and $x \geq 0$ be given. Assume $S(t) = x$. Let $\mathcal{F}_u^{(t)}$, $t \leq u \leq T$, denote the σ-algebra generated by the process $S(v)$ as v ranges over $[t, u]$, and let $\mathcal{T}_{t,T}$ denote the set of stopping times for the filtration $\mathcal{F}_u^{(t)}$, $t \leq u \leq T$, taking values in $[t, T]$ or taking the value ∞. In other words, $\{\tau \leq u\} \in \mathcal{F}_u^{(t)}$ for every $u \in [t, T]$; a stopping time in $\mathcal{T}_{t,T}$ makes the decision to stop at a time $u \in [t, T]$ based only on the path of the stock price between* *times t and u. The* price at time t of the American put expiring at time T *is defined to be*[1]

$$v(t,x) = \max_{\tau \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}} \left[e^{-r(\tau-t)} \big(K - S(\tau)\big) \Big| S(t) = x \right]. \tag{8.4.1}$$

In the event that $\tau = \infty$, we interpret $e^{-r\tau}(K - S(\tau))$ to be zero. This is the case when the put expires unexercised.

In Subsection 8.4.1 we present without proof the primary analytical properties of the finite-expiration American put price $v(t,x)$. These are time-dependent versions of the properties developed in Section 8.3 for the perpetual American put. In Subsection 8.4.2, we show that the only function possessing the analytical properties presented in Subsection 8.4.1 is $v(t,x)$ defined by (8.4.1).

8.4.1 Analytical Characterization of the Put Price

The finite-expiration American put price function $v(t,x)$ satisfies the *linear complementarity conditions* (cf. (8.3.18)–(8.3.20))

$$v(t,x) \geq (K-x)^+ \text{ for all } t \in [0,T],\ x \geq 0, \tag{8.4.2}$$

$$rv(t,x) - v_t(t,x) - rxv_x(t,x) - \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0 \quad \text{for all } t \in [0,T),\ x \geq 0, \text{ and} \tag{8.4.3}$$

$$\text{for each } t \in [0,T) \text{ and } x \geq 0, \text{ equality holds in either (8.4.2) or (8.4.3).} \tag{8.4.4}$$

As with the perpetual American put, the owner of the finite-expiration American put should wait until the stock price falls to a certain level at or below K before exercising, but now this level $L(T-t)$ depends on the time to expiration $T-t$. The level L_* of (8.3.12) for the perpetual American put is $\lim_{T\to\infty} L(T)$. At the other extreme, $L(0) = K$; at expiration, one should exercise the put if the stock price is below K, one should not exercise if the stock price is above K, and one is indifferent between exercising and not exercising if the stock price is equal to K. No formula is known for the function $L(T-t)$, but this function can be determined numerically from the analytic characterization of the put price provided in the next subsection. It is known that $L(T)$ decreases wtih increasing T, as shown in Figure 8.4.1. The set $\{(t,x); 0 \leq t \leq T, x \geq 0\}$ can be divided into two regions, the *stopping set*

$$\mathcal{S} = \{(t,x); v(t,x) = (K-x)^+\} \tag{8.4.5}$$

and the *continuation set*

[1] Here we use $v(t,x)$ rather than $v_*(x)$ as in Section 8.3 to denote the put price because in this section we do not consider functions of t and x other than the put price itself.

$$\mathcal{C} = \{(t, x); v(t, x) > (K - x)^+\}. \tag{8.4.6}$$

The graph of the function $x = L(T - t)$ forms the boundary between $\mathcal{C}$ and $\mathcal{S}$ and belongs to $\mathcal{S}$. Because of (8.4.4), equality holds in (8.4.3) for (t, x) in $\mathcal{C}$, $t \neq T$. For (t, x) in $\mathcal{S}$, strict inequality holds in (8.4.3) except on the curve $x = L(T - t)$, where equality holds in (8.4.3). Because $v(t, x) = (K - x)^+ = K - x$ for $0 \leq x \leq L(T - t)$, we have (see Figure 8.4.1)

$$rv(t, x) - v_t(t, x) - rxv_x(t, x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rK \text{ for } x \in \mathcal{C}.$$

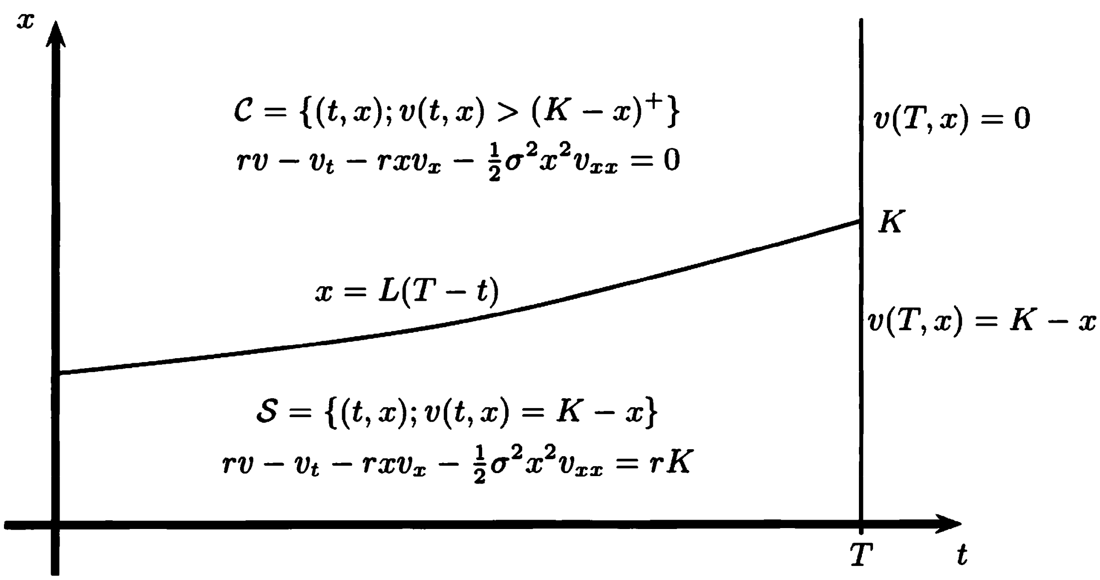

Fig. 8.4.1. Finite-expiration American put.

Because $v(t, x) = K - x$ for $0 \leq x \leq L(T - t)$, we also have the left-hand derivative $v_x(t, x-) = -1$ on the curve $x = L(T - t)$. The put price $v(t, x)$ satisfies the *smooth-pasting* condition that $v_x(t, x)$ is continuous, even at $x = L(T - t)$. In other words,

$$v_x(t, x+) = v_x(t, x-) = -1 \text{ for } x = L(T - t),\ 0 \leq t < T. \tag{8.4.7}$$

The smooth-pasting condition does not hold at $t = T$. Indeed,

$$L(0) = K \text{ and } v(T, x) = (K - x)^+, \tag{8.4.8}$$

so $v_x(T, x-) = -1$, whereas $v_x(T, x+) = 0$ for $x = L(0)$. Also, $v_t(t, x)$ and $v_{xx}(t, x)$ are not continuous along the curve $x = L(T - t)$.

The equations

$$rv(t, x) - v_t(t, x) - rxv_x(t, x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = 0,\ x \geq L(T - t),$$
$$v(t, x) = K - x,\ 0 \leq x \leq L(T - t),$$

together with the smooth-pasting condition (8.4.7), the terminal condition (8.4.8), and the asymptotic condition

$$\lim_{x\to\infty} v(t,x) = 0, \tag{8.4.9}$$

determine the function $v(t,x)$. Using these equations, one can set up a finite-difference scheme to simultaneously compute $v(t,x)$ and $L(T-t)$.

8.4.2 Probabilistic Characterization of the Put Price

Theorem 8.4.2. *Let $S(u)$, $t \le u \le T$, be the stock price of (8.3.1) starting at $S(t) = x$ and with the stopping set $\mathcal{S}$ defined by (8.4.5). Let*

$$\tau_* = \min\{u \in [t,T]; \big(u, S(u)\big) \in \mathcal{S}\}, \tag{8.4.10}$$

where we interpret τ_* to be ∞ if $(u, S(u))$ doesn't enter $\mathcal{S}$ for any $u \in [t,T]$. Then $e^{-ru}v(u,S(u))$, $t \le u \le T$, is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(u\wedge\tau_*)}v(u, S(u\wedge\tau_*))$, $t \le u \le T$, is a martingale.

PROOF: The Itô-Doeblin formula applies to $e^{-ru}v(u,S(u))$, even though $v_u(u,x)$ and $v_{xx}(u,x)$ are not continuous along the curve $x = L(T-u)$ because the process $S(u)$ spends zero time on this curve. All that is needed for the Itô-Doeblin formula to apply is that $v_x(u,x)$ be continuous (see Exercise 4.20 for a discussion related to this), and this follows from the smooth-pasting condition (8.4.7). We may thus compute

$$\begin{aligned}
&d\big[e^{-ru}v\big(u,S(u)\big)\big]\\
&= e^{-ru}\Big[-rv\big(u,S(u)\big)\,du + v_u\big(u,S(u)\big)\,du + v_x\big(u,S(u)\big)\,dS(u)\\
&\qquad\qquad + \frac{1}{2}v_{xx}\big(u,S(u)\big)\,dS(u)\,dS(u)\Big]\\
&= e^{-ru}\Big[-rv\big(u,S(u)\big) + v_u\big(u,S(u)\big) + rS(u)v_x\big(u,S(u)\big)\\
&\qquad\qquad + \frac{1}{2}\sigma^2S^2(u)v_{xx}\big(u,S(u)\big)\Big]\,du + e^{-ru}\sigma S(u)v_x\big(u,S(u)\big)\,d\widetilde{W}(u).
\end{aligned} \tag{8.4.11}$$

According to Figure 8.4.1, the du term in (8.4.11) is $-e^{-ru}rK\mathbb{I}_{\{S(u)<L(T-u)\}}$. This is nonpositive, and so $e^{-ru}v(u,S(u))$ is a supermartingale under $\widetilde{\mathbb{P}}$. In fact, starting from $u = t$ and up until time τ_*, we have $S(u) > L(T-u)$, so the du term is zero. Therefore, the stopped process $e^{-r(u\wedge\tau_*)}v(u\wedge\tau_*, S(u\wedge\tau_*))$, $t \le u \le T$, is a martingale. □

Corollary 8.4.3. *Consider an agent with initial capital $X(0) = v(0,S(0))$, the initial finite-expiration put price. Suppose this agent uses the portfolio process $\Delta(u) = v_x(u,S(u))$ and consumes cash at rate $C(u) = rK\mathbb{I}_{\{S(u)<L(T-u)\}}$*

per unit time. Then $X(u) = v(u, S(u))$ for all times u between $u = 0$ and the time the option is exercised or expires. In particular, $S(u) \geq (K - S(u))^+$ for all times u until the option is exercised or expires, so the agent can pay off a short option position regardless of when the option is exercised.

PROOF: The differential of the agent's discounted portfolio value is given by (8.3.24). Substituting for $\Delta(u)$ and $C(u)$ in this equation and comparing it to (8.4.11), we see that $d(e^{-ru}X(u)) = d[e^{-ru}v(u, S(u))]$. Integrating this equation and using $X(0) = v(0, S(0))$, we obtain $X(t) = v(t, S(t))$ for all times t prior to exercise or expiration. □

Remark 8.4.4. The proofs of Theorem 8.4.2 and Corollary 8.4.3 use the analytic characterization of the American put price captured in Figure 8.4.1 plus the smooth-pasting condition that guarantees that $v_x(t, x)$ is continuous even on the curve $x = L(T-t)$ so that the Itô-Doeblin formula can be applied. Here we show that the only function $v(t, x)$ satisfying these conditions is the function $v(t, x)$ defined by (8.4.1). To do this, we first fix t with $0 \leq t \leq T$. The supermartingale property for $e^{-rt}v\big(t, S(t)\big)$ of Theorem 8.4.2 and Theorem 8.2.4 (optional sampling) implies that

$$e^{-r(t\wedge\tau)}v\big(t\wedge\tau, S(t\wedge\tau)\big) \geq \widetilde{\mathbb{E}}\left[e^{-r(T\wedge\tau)}v\big(T\wedge\tau, S(T\wedge\tau)\big)\Big|\,\mathcal{F}(t)\right].$$

For $\tau \in \mathcal{T}_{t,T}$, we have $t \wedge \tau = t$, whereas $T \wedge \tau = \tau$ if $\tau < \infty$ and $T \wedge \tau = T$ if $\tau = \infty$. Therefore, for $\tau \in \mathcal{T}_{t,T}$,

$$\begin{aligned} e^{-rt}v\big(t, S(t)\big) &\geq \widetilde{\mathbb{E}}\left[e^{-r\tau}v\big(\tau, S(\tau)\big)\mathbb{I}_{\{\tau<\infty\}} + e^{-rT}v\big(T, S(T)\big)\mathbb{I}_{\{\tau=\infty\}}\,\big|\,\mathcal{F}(t)\right] \\ &\geq \widetilde{\mathbb{E}}\left[e^{-r\tau}v\big(\tau, S(\tau)\big)\,\big|\,\mathcal{F}(t)\right], \end{aligned} \tag{8.4.12}$$

where, as usual, we interpret $e^{-r\tau}v\big(\tau, S(\tau)\big) = 0$ if $\tau = \infty$. Inequality (8.4.2) and the fact that $(K - S(t))^+ \geq K - S(t)$ imply that

$$\widetilde{\mathbb{E}}\left[e^{-r\tau}v\big(\tau, S(\tau)\big)\,\big|\,\mathcal{F}(t)\right] \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\,\big|\,\mathcal{F}(t)\right]. \tag{8.4.13}$$

Putting (8.4.12) and (8.4.13) together, we conclude that

$$e^{-rt}v\big(t, S(t)\big) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\,\big|\,\mathcal{F}(t)\right]. \tag{8.4.14}$$

Because $S(t)$ is a Markov process, the right-hand side of (8.4.14) is a function of t and $S(t)$. In particular, if we denote the value of $S(t)$ by x, we may rewrite (8.4.14) as

$$e^{-rt}v(t, x) = \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\,\big|\,S(t) = x\right]. \tag{8.4.15}$$

Since (8.4.15) holds for any $\tau \in \mathcal{T}_{t,T}$, we conclude that

$$v(t, x) \geq \max_{\tau\in\mathcal{T}_{t,T}} \widetilde{\mathbb{E}}\left[e^{-r(\tau-t)}\big(K - S(\tau)\big)\Big|\,S(t) = x\right]. \tag{8.4.16}$$

For the reverse inequality, we recall from Theorem 8.4.2 that the stopped process $e^{-r(t\wedge\tau^*)}v(t\wedge\tau_*, S(t\wedge\tau_*))$ is a martingale, where τ_* defined by (8.4.10) is such that $v(\tau_*, S(\tau_*)) = K - S(\tau_*)$ if $\tau_* < \infty$. Replacing τ by τ_* in (8.4.12), we make the first inequality into an equality. If $\tau^* = \infty$, we have $(T, S(T)) \in \mathcal{C}$ (i.e., $S(T) > K$), so $v(T, S(T))\mathbb{I}_{\{\tau_*=\infty\}} = 0$. This makes the second inequality in (8.4.12) into an equality. Finally, because $v(\tau, S(\tau)) = K - S(\tau)$ on $\mathbb{I}_{\{\tau<\infty\}}$, the inequality in (8.4.13) is an equality, and hence (8.4.15) becomes

$$v(t,x) = \widetilde{\mathbb{E}}\left[e^{-r(\tau_*-t)}\big(K - S(\tau_*)\big)\Big| S(t) = x\right]. \tag{8.4.17}$$

Equation (8.4.17) shows that equality must hold in (8.4.16), and this is (8.4.1).
□

8.5 American Call

In this section, we treat the American call, first on the usual geometric Brownian motion asset of (8.3.1) and then on a variation of this asset that pays dividends at discrete dates. In the first case, presented in Subsection 8.5.1, we see that the American call price is the same as the European call price. In the second case, presented in Subsection 8.5.2, we provide a recursion formula for computing the American call price.

8.5.1 Underlying Asset Pays No Dividends

We begin with a case slightly more general than a call option. Consider a stock whose price process $S(t)$ is given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \tag{8.5.1}$$

where the interest rate r and the volatility σ are strictly positive and $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$.

Lemma 8.5.1. *Let $h(x)$ be a nonnegative, convex function of $x \geq 0$ satisfying $h(0) = 0$. Then the discounted intrinsic value $e^{-rt}h(S(t))$ of the American derivative security that pays $h(S(t))$ upon exercise is a submartingale.*

Proof: Because $h(x)$ is convex, for $0 \leq \lambda \leq 1$ and $0 \leq x_1 \leq x_2$, we have

$$h\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda)h(x_1) + \lambda h(x_2). \tag{8.5.2}$$

See Figure 8.5.1 for the case of a call payoff, $h(x) = (x - K)^+$.

Taking $x_1 = 0$, $x_2 = x$, and using the fact that $h(0) = 0$, we obtain from (8.5.2) that

$$h(\lambda x) \leq \lambda h(x) \text{ for all } x \geq 0,\ 0 \leq \lambda \leq 1. \tag{8.5.3}$$

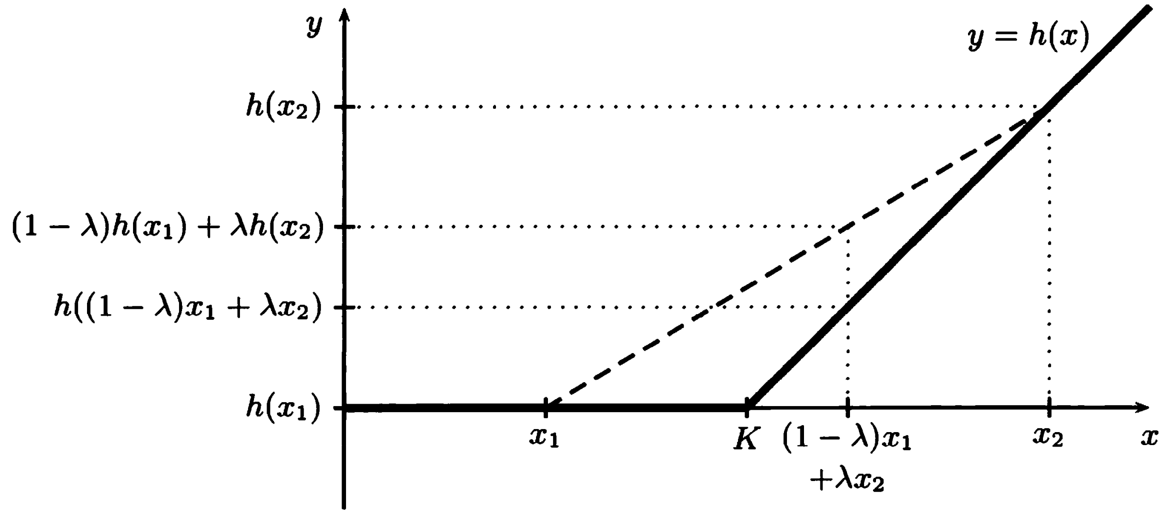

Fig. 8.5.1. The convex function $h(x) = (x - K)^+$.

For $0 \le u \le t \le T$, we have $0 \le e^{-r(t-u)} \le 1$, and (8.5.3) implies

$$\widetilde{\mathbb{E}}\left[e^{-r(t-u)}h\big(S(t)\big)\Big|\mathcal{F}(u)\right] \ge \widetilde{\mathbb{E}}\left[h\left(e^{-r(t-u)}S(t)\right)\Big|\mathcal{F}(u)\right]. \tag{8.5.4}$$

The conditional Jensen's inequality (Theorem 2.3.2(v)) implies

$$\begin{aligned}\widetilde{\mathbb{E}}\left[h\left(e^{-r(t-u)}S(t)\right)\Big|\mathcal{F}(u)\right] &\ge h\left(\widetilde{\mathbb{E}}\left[e^{-r(t-u)}S(t)\Big|\mathcal{F}(u)\right]\right)\\ &= h\left(e^{ru}\widetilde{\mathbb{E}}\left[e^{-rt}S(t)\big|\mathcal{F}(u)\right]\right).\end{aligned} \tag{8.5.5}$$

Because $e^{-rt}S(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we have

$$h\left(e^{ru}\widetilde{\mathbb{E}}\left[e^{-rt}S(t)\big|\mathcal{F}(u)\right]\right) = h\big(e^{ru}e^{-ru}S(u)\big) = h\big(S(u)\big). \tag{8.5.6}$$

Putting (8.5.4)–(8.5.6) together, we conclude that

$$\widetilde{\mathbb{E}}\left[e^{-r(t-u)}h\big(S(t)\big)\Big|\mathcal{F}(u)\right] \ge h\big(S(u)\big) \tag{8.5.7}$$

or, equivalently,

$$\widetilde{\mathbb{E}}\left[e^{-rt}h\big(S(t)\big)\big|\mathcal{F}(u)\right] \ge e^{-ru}h\big(S(u)\big). \tag{8.5.8}$$

This is the submartingale property for $e^{-rt}h(S(t))$. □

Theorem 8.5.2. *Let $h(x)$ be a nonnegative, convex function of $x \ge 0$ satisfying $h(0) = 0$. Then the price of the American derivative security expiring at time T and having intrinsic value $h(S(t))$, $0 \le t \le T$, is the same as the price of the European derivative security paying $h(S(T))$ at expiration T.*

PROOF: Replacing t by T in (8.5.7), we obtain

$$\widetilde{\mathbb{E}}\left[e^{-r(T-u)}h(S(T))\Big|\mathcal{F}(u)\right] \geq h(S(u)),\ 0 \leq u \leq T.$$

In other words, the European derivative security price always dominates the intrinsic value of the American derivative security. This shows that the option to exercise early is worthless, and the price of the American derivative security agrees with the price of the European security. □

Corollary 8.5.3. *The price of an American call on an asset not paying a dividend is the same as the price of the European call on the same asset with the same expiration.*

PROOF: Take $h(x) = (x - K)^+$ in Theorem 8.5.2. □

The idea behind Corollary 8.5.3 is that the discounted process $e^{-rt}(S(t) - K)^+$ is a submartingale under $\widetilde{\mathbb{P}}$ and hence tends to rise. Therefore, it is optimal to wait until expiration before deciding whether to exercise. There are two factors that contribute to the submartingale property for $e^{-rt}(S(t)-K)^+$. One is the discounting of the strike. In fact, $e^{-rt}(S(t)-K)$ (without the $^+$) is a submartingale because $e^{-rt}S(t)$ is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$ and $-e^{-rt}K$ increases as t increases (throughout this chapter, we assume a strictly positive interest rate r). When we reinstate the $^+$, we are taking a convex function of a submartingale and, because of Jensen's inequality, this reinforces the upward trend.

The previous argument does not apply to the American put, whose discounted intrinsic value $e^{-rt}(K - S(t))$ (without the $^+$) is a supermartingale ($e^{-rt}K$ falls and $-e^{-rt}S(t)$ is a martingale). Jensen's inequality creates an upward trend that competes with this supermartingale property, and the analysis becomes complicated.

If the underlying asset pays a dividend, the case considered in the next subsection, the argument above no longer applies to the American call. In this case, $e^{-rt}S(t)$ is a supermartingale and tends to fall because of the dividend outflow.

8.5.2 Underlying Asset Pays Dividends

In this subsection, we consider an American call on an asset whose price process is a geometric Brownian motion governed by (8.5.1) between dividend payment dates. We assume there are times $0 < t_1 < t_2 < \cdots < t_n < T$, and at each time t_j the dividend paid is $a_jS(t_j-)$, where $S(t_j-)$ denotes the asset price just prior to the dividend payment. The asset price $S(t_j)$ after the dividend payment is the asset price before the dividend payment less the dividend payment:

$$S(t_j) = S(t_j-) - a_jS(t_j-) = (1 - a_j)S(t_j-). \tag{8.5.9}$$

We assume that each a_j, $j = 1, \ldots, n$, is a number between 0 and 1. We set $t_0 = 0$, but this is not a dividend payment date. We also assume that T is not a dividend payment date, although it is not difficult to modify the analysis given below to handle the case when T is a dividend payment date.

We shall see that it is not optimal to exercise an American call on this asset except possibly immediately before a dividend payment. The price of the call will be seen to satisfy the Black-Scholes-Merton partial differential equation between dividend payment dates. At dividend payment dates, the price of the call is the maximum of the call's intrinsic value and the price of the call after the dividend is paid and the stock price is reduced by the amount of the payment. These observations lead to a recursive algorithm for determining the price, and that is developed in this subsection.

The asset price process in this section was considered in Subsection 5.5.4. For $t_j \leq t < t_{j+1}$, we have

$$S(t) = S(t_j)\exp\left\{\sigma\big(\widetilde{W}(t) - \widetilde{W}(t_j)\big) + \left(r - \frac{1}{2}\sigma^2\right)(t - t_j)\right\},$$

which implies

$$S(t_{j+1}-) = S(t_j)\exp\left\{\sigma\big(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\big) + \left(r - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\} \tag{8.5.10}$$

and

$$\begin{aligned} &S(t_{j+1}) \\ &= (1 - a_{j+1})S(t_j)\exp\left\{\sigma\big(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\big) + \left(r - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\}. \end{aligned} \tag{8.5.11}$$

We also have

$$S(T) = S(t_n)\exp\left\{\sigma\big(\widetilde{W}(T) - \widetilde{W}(t_n)\big) + \left(r - \frac{1}{2}\sigma^2\right)(T - t_n)\right\}. \tag{8.5.12}$$

We consider an American call expiring at time T with strike price K. For $t_n \leq t \leq T$, the discounted asset price $e^{-rt}S(t)$ is a martingale under $\widetilde{\mathbb{P}}$, and Lemma 8.5.1 can be invoked to show that $e^{-rt}(S(t) - K)^+$ is a submartingale. Therefore,

$$\widetilde{\mathbb{E}}\left[e^{-rT}(S(T) - K)^+ \Big| \mathcal{F}(t)\right] \geq e^{-rt}(S(t) - K)^+,\ t_n \leq t \leq T. \tag{8.5.13}$$

This shows that, for all $t \in [t_n, T]$, the price of the European call at time t,

$$c_n\big(t, S(t)\big) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T) - K)^+ \Big| \mathcal{F}(t)\right],$$

is greater than the intrinsic value of the American call, $(S(t)-K)^+$. Consequently, the early exercise feature of the American call is worthless, and the prices at time t of the European and American calls agree for $t_n \le t \le T$. This price is given by the Black-Scholes-Merton formula

$$c_n(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big), \tag{8.5.14}$$

where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right].$$

Although one cannot simply substitute $x=0$ into (8.5.14), we have $c(t,0)=0$; see equation (4.5.17) and Exercise 4.9. Formula (8.5.14) can be determined by computing the conditional expectation in (8.5.13) under the condition $S(t) = x$. In the case $t=t_n$, using (8.5.12), this leads to

$$\begin{aligned}
&c_n(t_n,x)\\
&= \widetilde{\mathbb{E}}\left[e^{-r(T-t_n)}\left(x\exp\left\{\sigma\big(\widetilde{W}(T)-\widetilde{W}(t_n)\big)+\left(r-\frac{1}{2}\sigma^2\right)(T-t_n)\right\}-K\right)^+\right].
\end{aligned} \tag{8.5.15}$$

The function $c_n(t,x)$ also satisfies the Black-Scholes-Merton differential equation

$$\frac{\partial}{\partial t}c_n(t,x) + rx\frac{\partial}{\partial x}c_n(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}c_n(t,x) = rc_n(t,x),\ \ t_n \le t < T,\ \ x \ge 0, \tag{8.5.16}$$

and the terminal condition

$$c_n(t,x) = (x-K)^+,\ \ x \ge 0. \tag{8.5.17}$$

The function $c_n(t_n,x)$ is convex in x. This is well-known, but we establish it here anyway to demonstrate a method we need later. To show convexity in x, we show that, whenever $0 \le x_1 \le x_2$ and $0 \le \lambda \le 1$, we have

$$c_n\big(t_n,(1-\lambda)x_1+\lambda x_2\big) \le (1-\lambda)c_n(t_n,x_1) + \lambda c_n(t_n,x_2). \tag{8.5.18}$$

We begin with the observation that, for any number α, the function $(\alpha x-K)^+$ is convex in x, and therefore

$$\left(x\exp\left\{\sigma\big(\widetilde{W}(T)-\widetilde{W}(t_n)\big)+\left(r-\frac{1}{2}\sigma^2\right)(T-t_n)\right\}-K\right)^+$$

is convex in x. It follows that

$$
\begin{aligned}
&c_n\big(t_n,(1-\lambda)x_1+\lambda x_2\big)\\
&=\widetilde{\mathbb{E}}\left[e^{-r(T-t_n)}\Bigg(\big((1-\lambda)x_1+\lambda x_2\big)\exp\Big\{\sigma\big(\widetilde{W}(T)-\widetilde{W}(t_n)\big)\right.\\
&\qquad\left.+\left(r-\frac{1}{2}\sigma^2\right)\Big\}-K\Bigg)^+\right]\\
&\le(1-\lambda)\widetilde{\mathbb{E}}\left[e^{-r(T-t_n)}\left(x_1\exp\left\{\sigma\big(\widetilde{W}(T)-\widetilde{W}(t_n)\big)+\left(r-\frac{1}{2}\sigma^2\right)\right\}-K\right)^+\right]\\
&\quad+\lambda\widetilde{\mathbb{E}}\left[e^{-r(T-t_n)}\left(x_2\exp\left\{\sigma\big(\widetilde{W}(T)-\widetilde{W}(t_n)\big)+\left(r-\frac{1}{2}\sigma^2\right)\right\}-K\right)^+\right]\\
&=(1-\lambda)c_n(t_n,x_1)+\lambda c_n(t_n,x_2).
\end{aligned}
\tag{8.5.19}
$$

This proves (8.5.18).

At time t_n, immediately before the dividend payment, the owner of the American call has two choices. She can exercise the option and receive $S(t_n-)-K$, or she can decline to exercise, permit the dividend to be paid (not to her) and the asset price to fall to $S(t_n)=(1-a_n)S(t_n-)$, and have an option valued at $c_n(t_n,(1-a_n)S(t_n-))$. The optimal decision is to exercise if $S(t_n-)-K>c_n(t_n,(1-a_n)S(t_n-))$ and to decline to exercise if $S(t_n-)-K<c_n(t_n,(1-a_n)S(t_n-))$. If $S(t_n-)-K=c_n(t_n,(1-a_n)S(t_n-))$, it does not matter whether she exercises or declines to exercise. Therefore, the call value at time t_n immediately before the dividend is paid is $h_n(S(t_n)-)$, where

$$h_n(x)=\max\{x-K,c_n(t_n,(1-a_n)x)\},\ x\ge 0. \tag{8.5.20}$$

We show that $h_n(x)$ satisfies the assumptions of Lemma 8.5.1. It is clear that $h_n(x)\ge 0$ for all $x\ge 0$ because $c_n(t_n,(1-a_n)x)\ge 0$ for all $x\ge 0$. It is also clear that $h_n(0)=0$ because $c_n(t_n,(1-a_n)0)=0$. To establish the convexity of $h_n(x)$, we recall from (8.5.18) that $c_n(t_n,x)$ is convex in x. For $0\le x_1\le x_2$ and $0\le\lambda\le 1$, we replace x_1 in (8.5.18) by $(1-a_n)x_1$ and replace x_2 by $(1-a_n)x_2$ to obtain

$$c_n\big(t_n,(1-a_n)((1-\lambda)x_1+\lambda x_2)\big)\le(1-\lambda)c_n\big(t_n,(1-a_n)x_1\big)+\lambda c_n\big(t_n,(1-a_n)x_2\big).$$

This shows that $c_n(t,(1-a_n)x)$ is a convex function of x. The maximum of two convex functions is convex (see Exercise 8.7), and therefore $h_n(x)$ defined by (8.5.20) is convex.

Starting from time t, where $t_{n-1}\le t<t_n$, the owner of the American call can exercise at any time $u\in[t,t_n)$, and if she does, she receives $S(u)-K$. If she does not exercise prior to t_n, then at time t_n, immediately before the dividend payment, she owns a call whose value we have just determined to be $h_n(S(t_n-))$. Therefore, for $t_{n-1}\le t<t_n$, the American call expiring at time T has the same price as the American call expiring immediately before the dividend payment at date t_n and paying $h_n(S(t_n-))$ upon expiration.

Because the underlying asset evolves as a geometric Brownian motion after the dividend is paid at time t_{n-1} until the dividend is paid at time t_n, Lemma 8.5.1 implies that $e^{-rt}h_n(S(t))$ is a submartingale for $t_{n-1} \le t < t_n$. In particular,

$$\widetilde{\mathbb{E}}\left[e^{-r(u-t)}h_n\big(S(u)\big)\Big|\mathcal{F}(t)\right] \ge h_n\big(S(t)\big),\ t_{n-1} \le t \le u < t_n,$$

and letting $u \uparrow t_n$, we obtain

$$\widetilde{\mathbb{E}}\left[e^{-r(t_n-t)}h_n\big(S(t_n-)\big)\Big|\mathcal{F}(t)\right] \ge h_n\big(S(t)\big). \tag{8.5.21}$$

By the definition of $h_n(x)$,

$$h_n\big(S(t)\big) \ge S(t) - K. \tag{8.5.22}$$

This shows that the value of the European call expiring at time t_n immediately before the dividend is paid and paying $h_n(S(t_n-))$ upon expiration, which is the left-hand side of (8.5.21), is greater than or equal to the intrinsic value of the American call, which is the right-hand side of (8.5.22). Therefore, the option to exercise the American call before time t_n is worthless, and the American call value is the same as the value of the European call just described.

Because $S(t)$ is a Markov process, there is some function $c_{n-1}(t,x)$ such that the left-hand side of (8.5.21), the European call value, is

$$c_{n-1}\big(t,S(t)\big) = \widetilde{\mathbb{E}}\left[e^{-r(t_n-t)}h_n\big(S(t_n-)\big)\Big|\mathcal{F}(t)\right]. \tag{8.5.23}$$

The function $c_{n-1}(t,x)$ can be determined by computing the conditional expectation in (8.5.23) under the condition $S(t)=x$. In the case $t=t_{n-1}$, using (8.5.10), this leads to

$$\begin{aligned} &c_{n-1}(t_{n-1},x)\\ &= \widetilde{\mathbb{E}}\Bigg[e^{-r(t_n-t_{n-1})} &(8.5.24)\\ &\qquad \times h_n\left(x\exp\left\{\sigma\big(\widetilde{W}(t_n)-\widetilde{W}(t_{n-1})\big)+\left(r-\frac{1}{2}\sigma^2\right)(t_n-t_{n-1})\right\}\right)\Bigg]. \\ & &(8.5.25)\end{aligned}$$

The function $c_{n-1}(t,x)$ also satisfies the Black-Scholes-Merton differential equation

$$\frac{\partial}{\partial t}c_{n-1}(t,x) + rx\frac{\partial}{\partial x}c_{n-1}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}c_{n-1}(t,x) = rc_{n-1}(t,x),$$
$$t_{n-1} \le t < t_n,\ x \ge 0, \tag{8.5.26}$$

and the terminal condition

$$c_{n-1}(t_n, x) = h_n(t_n, x), \ x \geq 0. \tag{8.5.27}$$

We repeat this process, defining

$$h_{n-1}(x) = \max\left\{x - K, c_{n-1}\big(t_{n-1}, (1-a_{n-1})x\big)\right\}, \ x \geq 0.$$

We can show as above that $h_{n-1}(x)$ satisfies the hypotheses of Lemma 8.5.1, and we continue.

In conclusion, we obtain an algorithm for the American call price on an asset paying dividends at the dates $t_1, t_2, \ldots, t_n$. Solve recursively for $j = n, n-1, \ldots, 0$, the partial differential equation

$$\frac{\partial}{\partial t} c_{j-1}(t,x) + rx\frac{\partial}{\partial x} c_{j-1}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} c_{j-1}(t,x) = rc_{j-1}(t,x),$$
$$t_{j-1} \leq t < t_j, \ x \geq 0, \tag{8.5.28}$$

with the terminal condition

$$c_{j-1}(t_j, x) = h_j(x), \ x \geq 0. \tag{8.5.29}$$

The functions $c_n(t,x)$ and $h_n(x)$ needed to get started are given by (8.5.14) and (8.5.20), and the function $h_{j-1}(x)$ needed for the next step is given by

$$h_{j-1}(x) = \max\left\{x - K, c_{j-1}\big(t_{j-1}, (1-a_{j-1})x\big)\right\}, \ x \geq 0. \tag{8.5.30}$$

For $t_{j-1} \leq t < t_j$, if $S(t) = x$, then $c_{j-1}(t,x)$ is the American call price. Within each interval $[t_{j-1}, t_j)$, the American call price is actually the price of a European call expiring at t_j. The optimal exercise time is immediately prior to the dividend payment at the smallest time t_j for which $S(t_j-) - K$ exceeds $c_j(t_j, (1-a_j)S(t_j-))$. If there is no t_j for which this condition is satisfied, then optimal exercise takes place at time T if $S(T) > K$, and otherwise the option should be allowed to expire unexercised.

8.6 Summary

This chaper discusses American puts and calls. To do this, we introduce the notions of stopping times and optional sampling in Section 8.2. The value of an American option can then be defined as the maximum over all stopping times of the discounted, risk-neutral payoff of the option evaluated at the stopping time. We do this for the perpetual American put in Section 8.3 and for the finite-horizon American put in Section 8.4. This definition of option value gives the no-arbitrage price. Starting with initial capital given by this definition, a person holding a short position in the option can hedge in such a way that, regardless of when the option is exercised, he will be able to pay off the short position. Furthermore, this definition of American option price is the smallest initial capital that permits such hedging. In particular, there is an optimal stopping time, and if the option owner exercises at this time, she captures the full value of the option.

The American put has an analytical characterization, which we present as linear complementarity conditions in Subsections 8.3.3 and 8.4.1. According to this characterization, there are two regions in the space of time and stock prices (t, x), one in which it is optimal to exercise the put (the stopping set) and another in which it is optimal not to exercise (the continuation set). The put price $v(t, x)$ and its first derivative $v_x(t, x)$ are continuous across the boundary between these two regions (smooth pasting), and this fact tells us that $v_x(t, x) = -1$ on this boundary. Using this smooth-pasting condition, one can solve numerically for the American put price.

The American call on a stock that pays no dividends has the same price as the corresponding European call; see Section 8.5.1. If the stock pays dividends, the American call can be more valuable than the European call. In Section 8.5.2, we work out an algorithm for the American call price when dividends are paid at discrete dates.

8.7 Notes

The use of stopping times with martingales was pioneered by Doob [53], who provided Theorem 8.2.4. A modern treatment can be found in many texts, including Chung [35] and Williams [161] in discrete time and Karatzas and Shreve [101] in continuous time.

The perpetual American put problem was first solved by McKean [119], who also wrote down the analytic characterization of the finite-horizon American put price. The fact that this analytic characterization determines the finite-horizon American put price follows from the optimal-stopping theory developed by van Moerbeke [153]. For the particular case of the American put, a simpler derivation of this fact is provided by Jacka [93], and this is presented in Section 2.7 of Karatzas and Shreve [102]. Although the price of the American put cannot be computed explicitly, it is possible to give a variety of characterizations of the early exercise premium, the difference between the American put price and the corresponding European put price; see Carr, Jarrow, and Myneni [27], Jacka [93], and Kim [103].

The probabilistic characterization of the American put price is due to Bensoussan [9] and Karatzas [100]. This is also reported in Section 2.5 of Karatzas and Shreve [102]. A survey of all these things, and a wealth of other references, are provided by Myneni [127]. Merton [122] observed that an American call on a stock paying no dividends has the same value as a European call.

There are two principal ways to compute option prices numerically: finite-difference schemes and Monte Carlo simulation. A finite-difference scheme for the American put is described in Wilmott, Howison, and Dewynne [165].

Monte Carlo methods are more difficult to develop because one must simultaneously determine the price of the put and determine the boundary between the stopping and continuation sets. A novel method to deal with this was recently provided by Longstaff and Schwartz [112] and Tsitsiklis and Van Roy [152]. Results on convergence of a modification of the Longstaff-Schwartz algorithm can be found in Clément, Lamberton, and Protter [37] and Glasserman and Yu [75]. Papers that use binomial trees and analytic approximations are listed in Section 2.8 of Karatzas and Shreve [102].

8.8 Exercises

Exercise 8.1 (Determination of L_* by smooth pasting). Consider the function $v_L(x)$ in (8.3.11). The first line in formula (8.3.11) implies that the left-hand derivative of $v_L(x)$ at $x = L$ is $v_L(L-) = -1$. Use the second line in formula (8.3.11) to compute the right-hand derivative $v_L'(L+)$. Show that the smooth-pasting condition

$$v_{L_*}'(L_*-) = v_{L_*}'(L_*+)$$

is satisfied only by L_* given by (8.3.12).

Exercise 8.2. Consider two perpetual American puts on the geometric Brownian motion (8.3.1). Suppose the puts have different strike prices, K_1 and K_2, where $0 < K_1 < K_2$. Let $v_1(x)$ and $v_2(x)$ denote their respective prices, as determined in Section 8.3.2. Show that $v_2(x)$ satisfies the first two linear complementarity conditions,

$$v_2(x) \geq (K_1 - x)^+ \text{ for all } x \geq 0, \tag{8.8.1}$$

$$rv_2(x) - rxv_2'(x) - \frac{1}{2}\sigma^2 x^2 v_2''(x) \geq 0 \text{ for all } x \geq 0, \tag{8.8.2}$$

for the perpetual American put price with strike K_1 but that $v_2(x)$ does not satisfy the third linear complementarity condition:

$$\text{for each } x \geq 0, \text{ equality holds in either (8.8.1) or (8.8.2) or both.} \tag{8.8.3}$$

Exercise 8.3 (Solving the linear complementarity conditions). Suppose $v(x)$ is a bounded continuous function having a continuous derivative and satisfying the linear complementarity conditions (8.3.18)–(8.3.20). This exercise shows that $v(x)$ must be the function $v_{L_*}(x)$ given by (8.3.13) with L_* given by (8.3.12). We assume that K is strictly positive.

(i) First consider an interval of x-values in which $v(x)$ satisfies (8.3.19) with equality, i.e., where

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) = 0. \tag{8.8.4}$$

Equation (8.8.4) is a linear, second-order ordinary differential equation, and it has two solutions of the form x^p, the solutions differing because of different values of p. Substitute x^p into (8.8.4) and show that the only values of p that cause x^p to satisfy (8.8.4) are $p = -\frac{2r}{\sigma^2}$ and $p = 1$.

(ii) The functions $x^{-\frac{2r}{\sigma^2}}$ and x are said to be linearly independent solutions of (8.8.4), and every function that satisfies (8.8.4) on an interval must be of the form

$$f(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx$$

for some constants A and B. Use this fact and the fact that both $v(x)$ and $v'(x)$ are continuous to show that there cannot be an interval $[x_1, x_2]$, where $0 < x_1 < x_2 < \infty$, such that $v(x)$ satisfies (8.3.19) with equality on $[x_1, x_2]$ and satisfies (8.3.18) with equality for x at and immediately to the left of x_1 and for x at and immediately to the right of x_2 unless $v(x)$ is identically zero on $[x_1, x_2]$.

(iii) Use the fact that $v(0)$ must equal K to show that there cannot be a number $x_2 > 0$ such that $v(x)$ satisfies (8.3.19) with equality on $[0, x_2]$.

(iv) Explain why $v(x)$ cannot satisfy (8.3.19) with equality for all $x \geq 0$.

(v) Explain why $v(x)$ cannot satisfy (8.3.18) with equality for all $x \geq 0$.

(vi) From (iv) and (v) and (8.3.20), we see that $v(x)$ sometimes satisfies (8.3.18) with equality and sometimes does not satisfy (8.3.18) with equality, in which case it must satisfy (8.3.19) with equality. From (ii) and (iii) we see that the region in which $v(x)$ does not satisfy (8.3.18) with equality and satisfies (8.3.19) with equality is not an interval $[x_1, x_2]$, where $0 \leq x_1 < x_2 < \infty$, nor can this region be a union of disjoint intervals of this form. Therefore, it must be a half-line $[x_1, \infty)$, where $x_1 > 0$. In the region $[0, x_1]$, $v(x)$ satisfies (8.3.18) with equality. Show that x_1 must equal L_* given by (8.3.12) and $v(x)$ must be $v_{L_*}(x)$ given by (8.3.13).

Exercise 8.4. It was asserted at the end of Subsection 8.3.3 and established in Exercise 8.3 that $v_{L_*}(x)$ given by (8.3.13) is the only *bounded* continuous function having a continuous derivative and satisfying the linear complementarity conditions (8.3.18)–(8.3.20). There are, however, unbounded functions that satisfy these conditions. Let $0 < L < K$ be given, and assume that

$$\frac{2r}{2r + \sigma^2} K > L. \tag{8.8.5}$$

(i) Show that, for any constants A and B, the function

$$f(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx \tag{8.8.6}$$

satisfies the differential equation

$$rf(x) - rxf'(x) - \frac{1}{2}\sigma^2 x^2 f''(x) = 0 \text{ for all } x \geq 0. \tag{8.8.7}$$

(ii) Show that the constants A and B can be chosen so that

$$f(L) = K - L, \quad f'(L) = -1. \tag{8.8.8}$$

(iii) With the constants A and B you chose in (ii), show that $f(x) \geq (K - x)^+$ for all $x \geq L$.

(iv) Define

$$v(x) = \begin{cases} K - x, & 0 \leq x \leq L, \\ f(x), & x \geq L. \end{cases}$$

Show that $v(x)$ satisfies the linear complementarity conditions (8.3.18)–(8.3.20), but $v(x)$ is not the function $v_{L_*}(x)$ given by (8.3.13).

(v) Every solution of the differential equation (8.8.7) is of the form (8.8.6). In order to have a bounded solution, we must have $B = 0$. Show that in order to have $B = 0$, we must have $L = \frac{2r}{2r+\sigma^2}K$, and in this case $v(x)$ agrees with the function $v_{L_*}(x)$ of (8.3.13).

Exercise 8.5 (Perpetual American put paying dividends). Consider a perpetual American put on a geometric Brownian motion asset price paying dividends at a constant rate $a > 0$. The differential of this asset is

$$dS(t) = (r - a)S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \tag{8.8.9}$$

where $\widetilde{W}(t)$ is a Brownian motion under a risk-neutral measure $\widetilde{\mathbb{P}}$. (Equation (8.8.9) can be obtained by computing the differential in (5.5.8).)

(i) Suppose we adopt the strategy of exercising the put the first time the asset price is at or below L. What is the risk-neutral expected discounted payoff of this strategy? Write this as a function $v_L(x)$ of the initial asset price x. (Hint: Define the positive constant

$$\gamma = \frac{1}{\sigma^2}\left(r - a - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\left(r - a - \frac{1}{2}\sigma^2\right)^2 + 2r}$$

and write $v_L(x)$ using γ.)

(ii) Determine L_*, the value of L that maximizes the risk-neutral expected discounted payoff computed in (i).

(iii) Show that, for any initial asset price $S(0) = x$, the process $e^{-rt}v_{L_*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$. Show that if $S(0) = x > L_*$ and $e^{-rt}v_{L_*}(S(t))$ is stopped the first time the asset price reaches L_*, then the stopped supermartingale is a martingale. (Hint: Show that

$$r + (r - a)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma + 1) = 0.) \tag{8.8.10}$$

(iv) Show that, for any initial asset price $S(0) = x$,

$$v_{L_*}(x) = \max_{\tau \in \mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right]. \tag{8.8.11}$$

Exercise 8.6. There is a second part to Theorem 8.2.4 (optional sampling), which says the following.

Theorem 8.8.1 (Optional sampling – Part II). *Let $X(t)$, $t \geq 0$, be a submartingale, and let τ be a stopping time. Then $\mathbb{E}X(t \wedge \tau) \leq \mathbb{E}X(t)$. If $X(t)$ is a supermartingale, then $\mathbb{E}X(t \wedge \tau) \geq \mathbb{E}X(t)$. If $X(t)$ is a martingale, then $\mathbb{E}X(t \wedge \tau) = \mathbb{E}X(t)$.*

The proof is technical and is omitted. The idea behind the statement about submartingales is the following. Submartingales tend to go up. Since $t \wedge \tau \leq t$, we would expect this upward trend to result in the inequality $\mathbb{E}X(t \wedge \tau) \leq \mathbb{E}X(t)$. When τ is a stopping time, this intuition is correct. Once we have Theorem 8.8.1 for submartingales, we easily obtain it for supermartingales by using the fact that the negative of a supermartingale is a submartingale. Since a martingale is both a submartingale and a supermartingale, we obtain the equality $\mathbb{E}X(t \wedge \tau) = \mathbb{E}X(t)$ for martingales.

Use Theorem 8.8.1 and Lemma 8.5.1 to show in the context of Subsection 8.5.1 that

$$\widetilde{\mathbb{E}}\left[e^{-rT}\big(S(T) - K\big)^+\right] = \max_{\tau \in \mathcal{T}_{0,T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(S(\tau) - K\big)^+\right], \tag{8.8.12}$$

where as usual we interpret $e^{-r\tau}(S(\tau) - K)^+$ to be zero if $\tau = \infty$. The right-hand side is the American call price analogous to Definition 8.4.1 for the American put price. The left-hand side is the European call price.

Exercise 8.7. A function $f(x)$ defined for $x \geq 0$ is said to be *convex* if, for every $0 \leq x_1 \leq x_2$ and every $0 \leq \lambda \leq 1$, the inequality

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$$

holds. Suppose $f(x)$ and $g(x)$ are convex functions defined for $x \geq 0$. Show that

$$h(x) = \max\{f(x), g(x)\}$$

is also convex.