4

Stochastic Calculus

4.1 Introduction

This chapter defines Itô integrals and develops their properties. These are used to model the value of a portfolio that results from trading assets in continuous time. The calculus used to manipulate these integrals is based on the Itô-Doeblin formula of Section 4.4 and differs from ordinary calculus. This difference can be traced to the fact that Brownian motion has a nonzero quadratic variation and is the source of the volatility term in the Black-Scholes-Merton partial differential equation. The Black-Scholes-Merton equation is presented in Section 4.5. This is in the spirit of Sections 1.1 and 1.2 of Volume I in which we priced options by determining the portfolio that would hedge a short position. In particular, there is no discussion of risk-neutral pricing in this chapter. That topic is taken up in Chapter 5.

Section 4.6 extends stochastic calculus to multiple processes. Section 4.7 discusses the Brownian bridge, which plays a useful role in Monte Carlo methods for pricing. We do not treat Monte Carlo methods in this text; we include the Brownian bridge only because it is a natural application of the stochastic calculus developed in the earlier sections.

4.2 Itô's Integral for Simple Integrands

We fix a positive number T and seek to make sense of

$$\int_0^T \Delta(t)\,dW(t). \tag{4.2.1}$$

The basic ingredients here are a Brownian motion $W(t)$, $t \geq 0$, together with a filtration $\mathcal{F}(t)$, $t \geq 0$, for this Brownian motion. We will let the integrand $\Delta(t)$ be an adapted stochastic process. Our reason for doing this is that $\Delta(t)$ will eventually be the position we take in an asset at time t, and this typically depends on the price path of the asset up to time t. Anything that depends on the path of a random process is itself random. Requiring $\Delta(t)$ to be adapted means that we require $\Delta(t)$ to be $\mathcal{F}(t)$-measurable for each $t \geq 0$. In other words, the information available at time t is sufficient to evaluate $\Delta(t)$ at that time. When we are standing at time 0 and t is strictly positive, $\Delta(t)$ is unknown to us. It is a random variable. When we get to time t, we have sufficient information to evaluate $\Delta(t)$; its randomness has been resolved.

Recall that increments of the Brownian motion after time t are independent of $\mathcal{F}(t)$, and since $\Delta(t)$ is $\mathcal{F}(t)$-measurable, it must also be independent of these future Brownian increments. Positions we take in assets may depend on the price history of those assets, but they must be independent of the future increments of the Brownian motion that drives those prices.

The problem we face when trying to assign meaning to the Itô integral (4.2.1) is that Brownian motion paths cannot be differentiated with respect to time. If $g(t)$ is a differentiable function, then we can define

$$\int_0^T \Delta(t)\, dg(t) = \int_0^T \Delta(t) g'(t)\, dt,$$

where the right-hand side is an ordinary (Lebesgue) integral with respect to time. This will not work for Brownian motion.

4.2.1 Construction of the Integral

To define the integral (4.2.1), Itô devised the following way around the nondifferentiability of the Brownian paths. We first define the Itô integral for simple integrands $\Delta(t)$ and then extend it to nonsimple integrands as a limit of the integral of simple integrands. We describe this procedure.

Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$; i.e.,

$$0 = t_0 \leq t_1 \leq \cdots \leq t_n = T.$$

Assume that $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$. Such a process $\Delta(t)$ is a *simple process.*

Figure 4.2.1 shows a single path of a simple process $\Delta(t)$. We shall always choose these simple processes, as shown in this figure, to take a value at a partition time t_j and then hold it up to but not including the next partition time t_{j+1}. Although it is not apparent from Figure 4.2.1, the path shown depends on the same ω on which the path of the Brownian motion $W(t)$ (not shown) depends. If one were to choose a different ω, there would be a different path of the Brownian motion and possibly a different path of $\Delta(t)$. However, the value of $\Delta(t)$ can depend only on the information available at time t. Since there is no information at time 0, the value of $\Delta(0)$ must be the same for all paths, and hence the first piece of $\Delta(t)$, for $0 \leq t < t_1$, does not really depend on ω. The value of $\Delta(t)$ on the second interval, $[t_1, t_2)$, can depend on observations made during the first time interval $[0, t_1)$.

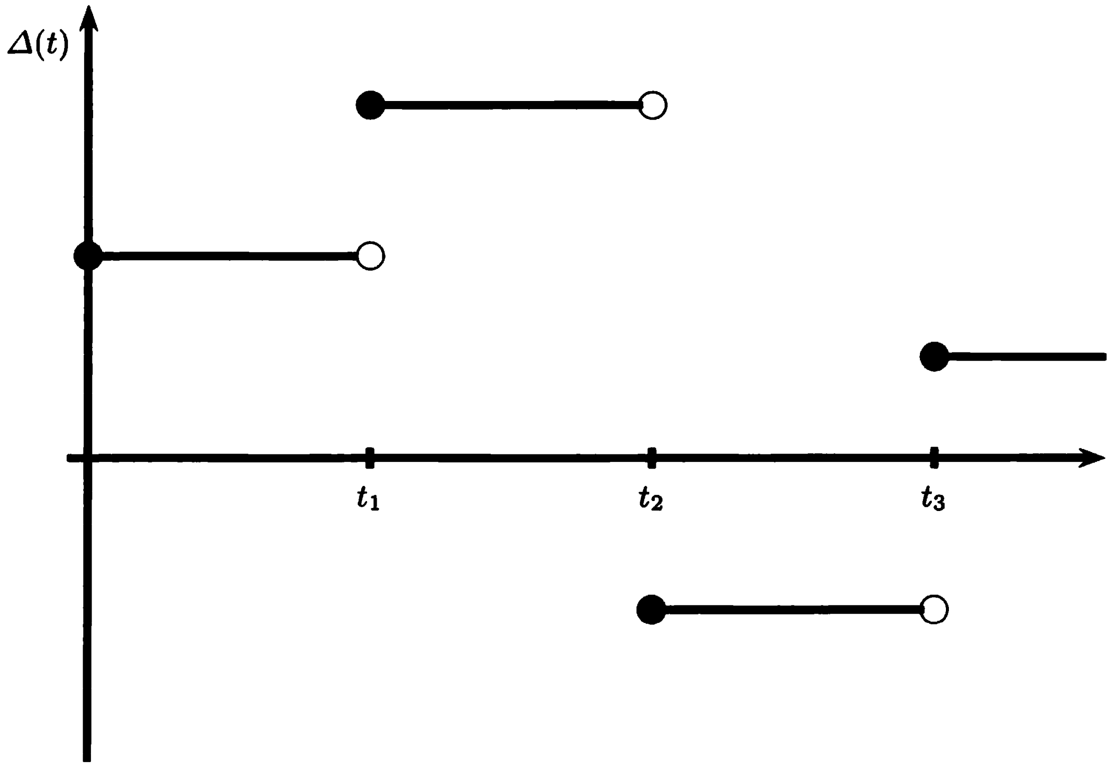

Fig. 4.2.1. A path of a simple process.

We shall think of the interplay between the simple process $\Delta(t)$ and the Brownian motion $W(t)$ in (4.2.1) in the following way. Regard $W(t)$ as the price per share of an asset at time t. (Since Brownian motion can take negative as well as positive values, it is not a good model of the price of a limited-liability asset such as a stock. For the sake of this illustration, we ignore that issue.) Think of $t_0, t_1, \ldots, t_{n-1}$ as the *trading dates* in the asset, and think of $\Delta(t_0), \Delta(t_1), \ldots, \Delta(t_{n-1})$ as the position (number of shares) taken in the asset at each trading date and held to the next trading date. The gain from trading at each time t is given by

$$
\begin{aligned}
I(t) &= \Delta(t_0)[W(t) - W(t_0)] = \Delta(0)W(t), \quad 0 \le t \le t_1, \\
I(t) &= \Delta(0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \le t \le t_2, \\
I(t) &= \Delta(0)W(t_1) + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t) - W(t_2)], \\
&\qquad\qquad t_2 \le t \le t_3,
\end{aligned}
$$

and so on. In general, if $t_k \le t \le t_{k+1}$, then

$$
I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]. \tag{4.2.2}
$$

The process $I(t)$ in (4.2.2) is the Itô integral of the simple process $\Delta(t)$, a fact that we write as

$$I(t) = \int_0^t \Delta(u)\,dW(u).$$

In particular, we can take $t = t_n = T$, and (4.2.2) provides a definition for the Itô integral (4.2.1). We have managed to define this integral not only for the upper limit of integration T but also for every upper limit of integration t between 0 and T.

4.2.2 Properties of the Integral

The Itô integral (4.2.2) is defined as the gain from trading in the martingale $W(t)$. A martingale has no tendency to rise or fall, and hence it is to be expected that $I(t)$, thought of as a process in its upper limit of integration t, also has no tendency to rise or fall. We formalize this observation by the next theorem and proof.

Theorem 4.2.1. *The Itô integral defined by (4.2.2) is a martingale.*

PROOF: Let $0 \le s \le t \le T$ be given. We shall assume that s and t are in different subintervals of the partition Π (i.e., there are partition points t_ℓ and t_k such that $t_\ell < t_k$, $s \in [t_\ell, t_{\ell+1})$, and $t \in [t_k, t_{k+1})$). If s and t are in the same subinterval, the following proof simplifies. Equation (4.2.2) may be rewritten as

$$\begin{aligned} I(t) = &\sum_{j=0}^{\ell-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \Delta(t_\ell)\big[W(t_{\ell+1}) - W(t_\ell)\big] \\ &+ \sum_{j=\ell+1}^{k-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \Delta(t_k)\big[W(t) - W(t_k)\big]. \quad (4.2.3) \end{aligned}$$

We must show that $\mathbb{E}\big[I(t)\big|\mathcal{F}(s)\big] = I(s)$. We take the conditional expectation of each of the four terms on the right-hand side of (4.2.3). Every random variable in the first sum $\sum_{j=0}^{\ell-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big]$ is $\mathcal{F}(s)$-measurable because the latest time appearing in this sum is t_ℓ and $t_\ell \le s$. Therefore,

$$\mathbb{E}\left[\sum_{j=0}^{\ell-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big]\,\middle|\,\mathcal{F}(s)\right] = \sum_{j=0}^{\ell-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big]. \tag{4.2.4}$$

For the second term on the right-hand side of (4.2.3), we "take out what is known" (Theorem 2.3.2(ii)) and use the martingale property of W to write

$$\begin{aligned} \mathbb{E}\big[\Delta(t_\ell)\big(W(t_{\ell+1}) - W(t_\ell)\big)\big|\mathcal{F}(s)\big] &= \Delta(t_\ell)\big(\mathbb{E}\big[W(t_{\ell+1})\big|\mathcal{F}(s)\big] - W(t_\ell)\big) \\ &= \Delta(t_\ell)\big(W(s) - W(t_\ell)\big). \qquad (4.2.5) \end{aligned}$$

Adding (4.2.4) and (4.2.5), we obtain $I(s)$.

It remains to show that the conditional expectations of the third and fourth terms on the right-hand side of (4.2.3) are zero. We will then have $\mathbb{E}\big[I(t)\big|\mathcal{F}(s)\big] = I(s)$.

The summands in the third term are of the form $\Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big]$, where $t_j \geq t_{\ell+1} > s$. This permits us to use the following iterated conditioning trick, which is based on properties (iii) (iterated conditioning) and (ii) (taking out what is known) of Theorem 2.3.2:

$$
\begin{aligned}
&\mathbb{E}\Big\{\Delta(t_j)\big(W(t_{j+1}) - W(t_j)\big)\Big|\mathcal{F}(s)\Big\}\\
&\quad= \mathbb{E}\Big\{\mathbb{E}\big[\Delta(t_j)\big(W(t_{j+1}) - W(t_j)\big)\big|\mathcal{F}(t_j)\big]\Big|\mathcal{F}(s)\Big\}\\
&\quad= \mathbb{E}\Big\{\Delta(t_j)\big(\mathbb{E}\big[W(t_{j+1})\big|\mathcal{F}(t_j)\big] - W(t_j)\big)\Big|\mathcal{F}(s)\Big\}\\
&\quad= \mathbb{E}\Big\{\Delta(t_j)\big(W(t_j) - W(t_j)\big)\Big|\mathcal{F}(s)\Big\} = 0.
\end{aligned}
$$

At the end, we have used the fact that W is a martingale. Because the conditional expectation of each of the summands in the third term on the right-hand side of (4.2.3) is zero, the conditional expectation of the whole term is zero:

$$
\mathbb{E}\left\{\sum_{j=\ell+1}^{k-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big]\,\middle|\, \mathcal{F}(s)\right\} = 0.
$$

The fourth term on the right-hand side of (4.2.3) is treated like the summands in the third term, with the result that

$$
\begin{aligned}
&\mathbb{E}\Big\{\Delta(t_k)\big(W(t) - W(t_k)\big)\Big|\mathcal{F}(s)\Big\}\\
&\quad= \mathbb{E}\Big\{\mathbb{E}\big[\Delta(t_k)\big(W(t) - W(t_k)\big)\big|\mathcal{F}(t_k)\big]\Big|\mathcal{F}(s)\Big\}\\
&\quad= \mathbb{E}\Big\{\Delta(t_k)\big(\mathbb{E}\big[W(t)\big|\mathcal{F}(t_k)\big] - W(t_k)\big)\Big|\mathcal{F}(s)\Big\}\\
&\quad= \mathbb{E}\Big\{\Delta(t_k)\big(W(t_k) - W(t_k)\big)\Big|\mathcal{F}(s)\Big\} = 0.
\end{aligned}
$$

This concludes the proof. □

Because $I(t)$ is a martingale and $I(0) = 0$, we have $\mathbb{E}I(t) = 0$ for all $t \geq 0$. It follows that $\operatorname{Var} I(t) = \mathbb{E}\,I^2(t)$, a quantity that can be evaluated by the formula in the next theorem.

Theorem 4.2.2 (Itô isometry). *The Itô integral defined by (4.2.2) satisfies*

$$
\mathbb{E}\,I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)\,du. \tag{4.2.6}
$$

PROOF: To simplify the notation, we set $D_j = W(t_{j+1}) - W(t_j)$ for $j = 0, \dots, k-1$ and $D_k = W(t) - W(t_k)$ so that (4.2.2) may be written as $I(t) = \sum_{j=0}^{k} \Delta(t_j) D_j$ and

$$I^2(t) = \sum_{j=0}^{k} \Delta^2(t_j) D_j^2 + 2 \sum_{0 \le i < j \le k} \Delta(t_i)\Delta(t_j) D_i D_j.$$

We first show that the expected value of each of the cross terms is zero. For $i < j$, the random variable $\Delta(t_i)\Delta(t_j)D_i$ is $\mathcal{F}(t_j)$-measurable, while the Brownian increment D_j is independent of $\mathcal{F}(t_j)$. Furthermore, $\mathbb{E}D_j = 0$. Therefore,

$$\mathbb{E}\big[\Delta(t_i)\Delta(t_j)D_iD_j\big] = \mathbb{E}\big[\Delta(t_i)\Delta(t_j)D_i\big] \cdot \mathbb{E}D_j = \mathbb{E}\big[\Delta(t_i)\Delta(t_j)D_i\big] \cdot 0 = 0.$$

We next consider the square terms $\Delta^2(t_j)D_j^2$. The random variable $\Delta^2(t_j)$ is $\mathcal{F}(t_j)$-measurable, and the squared Brownian increment D_j^2 is independent of $\mathcal{F}(t_j)$. Furthermore, $\mathbb{E}D_j^2 = t_{j+1} - t_j$ for $j = 0, \dots, k-1$ and $\mathbb{E}D_k^2 = t - t_k$. Therefore,

$$\begin{aligned}
\mathbb{E}I^2(t) = \sum_{j=0}^{k} \mathbb{E}\big[\Delta^2(t_j)D_j^2\big] &= \sum_{j=1}^{k} \mathbb{E}\Delta^2(t_j) \cdot \mathbb{E}D_j^2 \\
&= \sum_{j=0}^{k-1} \mathbb{E}\Delta^2(t_j)(t_{j+1} - t_j) + \mathbb{E}\Delta^2(t_k)(t - t_k).
\end{aligned} \tag{4.2.7}$$

But $\Delta(t_j)$ is constant on the interval $[t_j, t_{j+1})$, and hence $\Delta^2(t_j)(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \Delta^2(u)\,du$. Similarly, $\Delta^2(t_k)(t - t_k) = \int_{t_k}^{t} \Delta^2(u)\,du$. We may thus continue (4.2.7) to obtain

$$\begin{aligned}
\mathbb{E}I^2(t) &= \sum_{j=0}^{k-1} \mathbb{E}\int_{t_j}^{t_{j+1}} \Delta^2(u)\,du + \mathbb{E}\int_{t_k}^{t} \Delta^2(u)\,du \\
&= \mathbb{E}\left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Delta^2(u)\,du + \int_{t_k}^{t} \Delta^2(u)\,du\right] = \mathbb{E}\int_0^t \Delta^2(u)\,du. \qquad \square
\end{aligned}$$

Finally, we turn to the quadratic variation of the Itô integral $I(t)$ thought of as a process in its upper limit of integration t. Brownian motion accumulates quadratic variation at rate one per unit time. However, Brownian motion is scaled in a time- and path-dependent way by the integrand $\Delta(u)$ as it enters the Itô integral $I(t) = \int_0^t \Delta(u)\,dB(u)$. Because increments are squared in the computation of quadratic variation, the quadratic variation of Brownian motion will be scaled by $\Delta^2(u)$ as it enters the Itô integral. The following theorem gives the precise statement.

Theorem 4.2.3. *The quadratic variation accumulated up to time t by the Itô integral (4.2.2) is*

$$[I,I](t) = \int_0^t \Delta^2(u)\,du. \tag{4.2.8}$$

PROOF: We first compute the quadratic variation accumulated by the Itô integral on one of the subintervals $[t_j, t_{j+1}]$ on which $\Delta(u)$ is constant. For this, we choose partition points

$$t_j = s_0 < s_1 < \cdots < s_m = t_{j+1}$$

and consider

$$\begin{aligned}\sum_{i=0}^{m-1} \big[I(s_{i+1}) - I(s_i)\big]^2 &= \sum_{i=0}^{m-1} \big[\Delta(t_j)\big(W(s_{i+1}) - W(s_i)\big)\big]^2 \\ &= \Delta^2(t_j) \sum_{i=0}^{m-1} \big(W(s_{i+1}) - W(s_i)\big)^2. \end{aligned} \tag{4.2.9}$$

As $m \to \infty$ and the step size $\max_{i=0,\ldots,m-1}(s_{i+1} - s_i)$ approaches zero, the term $\sum_{i=0}^{m-1}\big(W(s_{i+1}) - W(s_i)\big)^2$ converges to the quadratic variation accumulated by Brownian motion between times t_j and t_{j+1}, which is $t_{j+1} - t_j$. Therefore, the limit of (4.2.9), which is the quadratic variation accumulated by the Itô integral between times t_j and t_{j+1}, is

$$\Delta^2(t_j)(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \Delta^2(u)\,du,$$

where again we have used the fact that $\Delta(u)$ is constant for $t_j \le u < t_{j+1}$. Analogously, the quadratic variation accumulated by the Itô integral between times t_k and t is $\int_{t_k}^t \Delta^2(u)\,du$. Adding up all these pieces, we obtain (4.2.8). □

In Theorems 4.2.2 and 4.2.3, we finally see how the quadratic variation and the variance of a process can differ. The quadratic variation is computed path-by-path, and the result can depend on the path. If along one path of the Brownian motion we choose large positions $\Delta(u)$, the Itô integral will have a large quadratic variation. Along a different path, we could choose small positions $\Delta(u)$ and the Itô integral would have a small quadratic variation. The quadratic variation can be regarded as a measure of risk, and it depends on the size of the positions we take. The variance of $I(t)$ is an average over all possible paths of the quadratic variation. Because it is the expectation of something, it cannot be random. As an average over all possible paths, realized and unrealized, it is a more theoretical concept than quadratic variation. We emphasize here that what we are calling variance is not the empirical variance. Empirical (or sample) variance is computed from a realized path and is an estimator of the theoretical variance we are discussing. The empirical variance is sometimes carelessly called variance, which creates the possibility of confusion.

Finally, we recall the equation (3.4.10), $dW(t)\,dW(t) = dt$, of Remark 3.4.4. We interpret this equation as the statement that Brownian motion accumulates quadratic variation at rate one per unit time. It is another way of writing $[W,W](t) = t$, $t \geq 0$. The Itô integral formula $I(t) = \int_0^t \Delta(u)\,dW(u)$ can be written in differential form as $dI(t) = \Delta(t)\,dW(t)$, and we can then use (3.4.10) to square $dI(t)$:

$$dI(t)\,dI(t) = \Delta^2(t)\,dW(t)\,dW(t) = \Delta^2(t)\,dt. \tag{4.2.10}$$

This equation says that the Itô integral $I(t)$ accumulates quadratic variation at rate $\Delta^2(t)$ per unit time. The rate of accumulation is typically both time- and path-dependent. Equation (4.2.10) is another way of reporting the result of Theorem 4.2.3.

Remark 4.2.4 (on notation). The notations

$$I(t) = \int_0^t \Delta(u)\,dW(u) \tag{4.2.11}$$

and

$$dI(t) = \Delta(t)\,dW(t) \tag{4.2.12}$$

mean almost the same thing, although the second is probably more intuitive. Equation (4.2.11) has the precise meaning given by (4.2.2). Equation (4.2.12) has the imprecise meaning that when we move forward a little bit in time from time t, the change in the Itô integral I is $\Delta(t)$ times the change in the Brownian motion W. It also has a precise meaning, which one obtains by integrating both sides, remembering to put in a constant of integration $I(0)$:

$$I(t) = I(0) + \int_0^t \Delta(u)\,dW(u). \tag{4.2.13}$$

We say that (4.2.12) is the *differential form* of (4.2.13) and that (4.2.13) is the *integral form* of (4.2.12). These two equations mean exactly the same thing.

The only difference between (4.2.11) and (4.2.13), and hence the only difference between (4.2.11) and (4.2.12), is that (4.2.11) specifies the initial condition $I(0) = 0$, whereas (4.2.12) and (4.2.13) permit $I(0)$ to be any arbitrary constant. □

4.3 Itô's Integral for General Integrands

In this section, we define the Itô integral $\int_0^T \Delta(t)\,dW(t)$ for integrands $\Delta(t)$ that are allowed to vary continuously with time and also to jump. In particular, we no longer assume that $\Delta(t)$ is a simple process as shown in Figure

4.2.1. We do assume that $\Delta(t)$, $t \geq 0$, is adapted to the filtration $\mathcal{F}(t)$, $t \geq 0$. We also assume the square-integrability condition

$$\mathbb{E}\int_0^T \Delta^2(t)\,dt < \infty. \tag{4.3.1}$$

In order to define $\int_0^T \Delta(t)\,dW(t)$, we approximate $\Delta(t)$ by simple processes. Figure 4.3.1 suggests how this can be done. In that figure, the continuously varying $\Delta(t)$ is shown as a solid line and the approximating simple integrand is dashed. Notice that $\Delta(t)$ is allowed to jump. The approximating simple integrand is constructed by choosing a partition $0 = t_0 < t_1 < t_2 < t_3 < t_4$, setting the approximating simple process equal to $\Delta(t_j)$ at each t_j, and then holding the simple process constant over the subinterval $[t_j, t_{j+1})$. As the maximal step size of the partition approaches zero, the approximating integrand will become a better and better approximation of the continuously varying one.

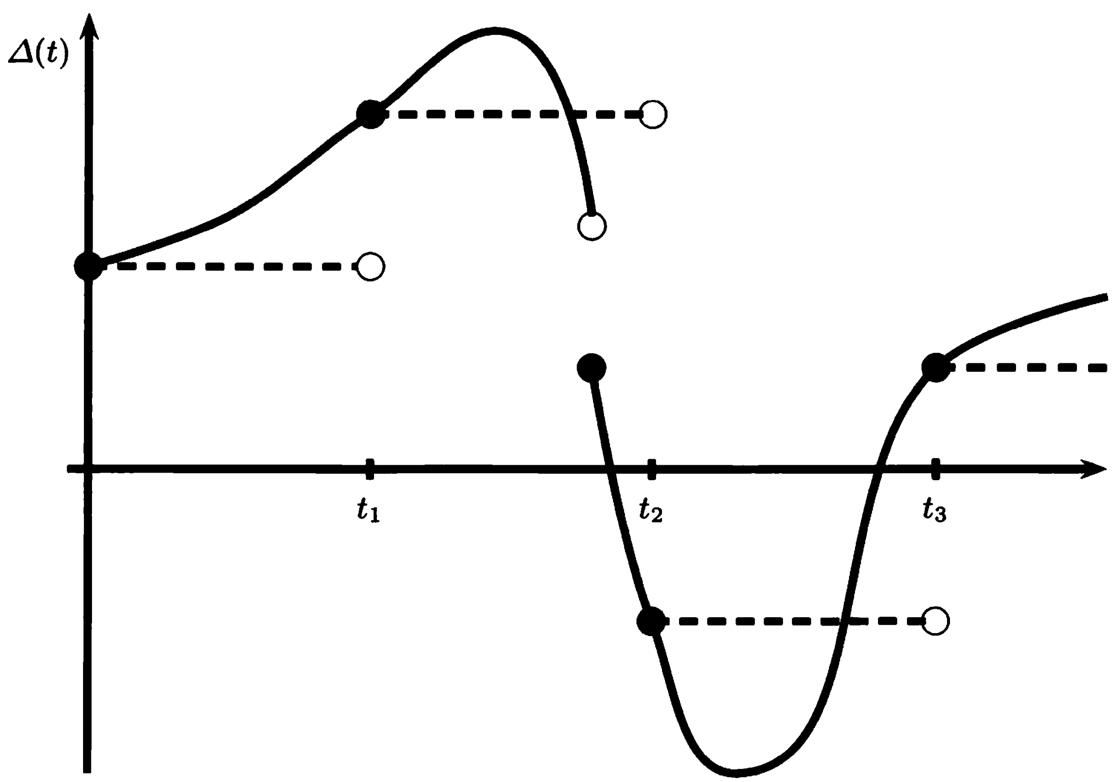

Fig. 4.3.1. Approximating a continuously varying integrand.

In general, then, it is possible to choose a sequence $\Delta_n(t)$ of simple processes such that as $n \to \infty$ these processes converge to the continuously varying $\Delta(t)$. By "converge," we mean that

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |\Delta_n(t) - \Delta(t)|^2\,dt = 0. \tag{4.3.2}$$

For each $\Delta_n(t)$, the Itô integral $\int_0^t \Delta_n(u)\,dW(u)$ has already been defined for $0 \le t \le T$. We define the Itô integral for the continuously varying integrand $\Delta(t)$ by the formula[1]

$$\int_0^t \Delta(u)\,dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(u)\,dW(u), \quad 0 \le t \le T. \tag{4.3.3}$$

This integral inherits the properties of Itô integrals of simple processes. We summarize these in the next theorem.

Theorem 4.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \le t \le T$, be an adapted stochastic process that satisfies (4.3.1). Then $I(t) = \int_0^t \Delta(u)\,dW(u)$ defined by (4.3.3) has the following properties.*

(i) **(Continuity)** *As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*
(ii) **(Adaptivity)** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*
(iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u)\,dW(u)$ and $J(t) = \int_0^t \Gamma(u)\,dW(u)$, then $I(t) \pm J(t) = \int_0^t \big(\Delta(u) \pm \Gamma(u)\big)\,dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)\,dW(u)$.*
(iv) **(Martingale)** *$I(t)$ is a martingale.*
(v) **(Itô isometry)** *$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)\,du$.*
(vi) **(Quadratic variation)** *$[I,I](t) = \int_0^t \Delta^2(u)\,du$.*

Example 4.3.2. We compute $\int_0^T W(t)\,dW(t)$. To do that, we choose a large integer n and approximate the integrand $\Delta(t) = W(t)$ by the simple process

$$\Delta_n(t) = \begin{cases} W(0) = 0 & \text{if } 0 \le t < \frac{T}{n}, \\ W\left(\frac{T}{n}\right) & \text{if } \frac{T}{n} \le t < \frac{2T}{n}, \\ \quad\vdots & \\ W\left(\frac{(n-1)T}{n}\right) & \text{if } \frac{(n-1)T}{n} \le t < T, \end{cases}$$

as shown in Figure 4.3.2. Then $\lim_{n\to\infty} \mathbb{E}\int_0^T |\Delta_n(t) - W(t)|^2\,dt = 0$. By definition,

$$\begin{aligned} \int_0^T W(t)\,dW(t) &= \lim_{n\to\infty} \int_0^T \Delta_n(t)\,dW(t) \\ &= \lim_{n\to\infty} \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]. \end{aligned} \tag{4.3.4}$$

[1] For each t, the limit in (4.3.3) exists because $I_n(t) = \int_0^t \Delta_n(u)\,dW(u)$ is a Cauchy sequence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. This is because of Itô's isometry (Theorem 4.2.2), which yields $\mathbb{E}\big(I_n(t) - I_m(t)\big)^2 = \mathbb{E}\int_0^t \left|\Delta_n(u) - \Delta_m(u)\right|^2 du$. As a consequence of (4.3.2), the right-hand side has limit zero as n and m approach infinity.

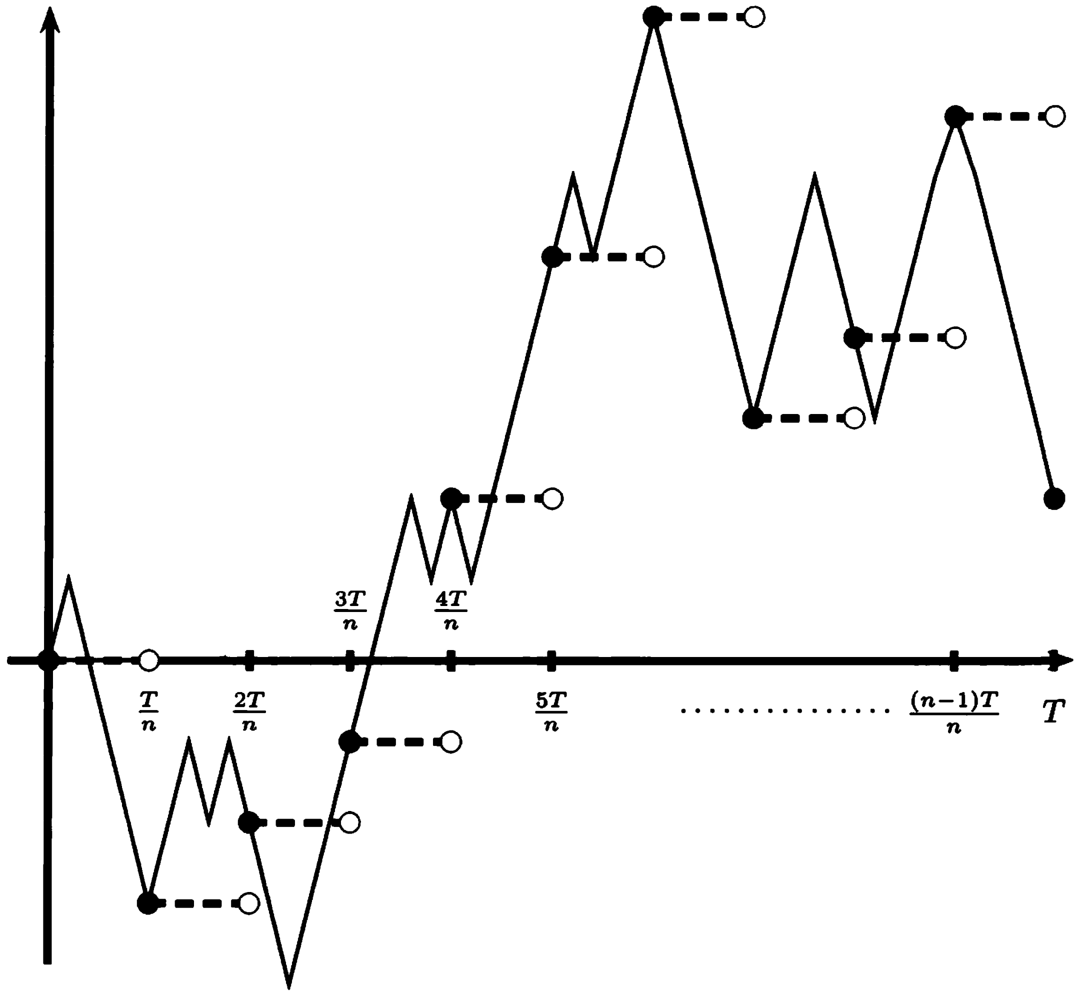

Fig. 4.3.2. Simple process approximating Brownian motion.

To simplify notation, we denote $W_j = W\left(\frac{jT}{n}\right)$. As a precursor to evaluating the limit in (4.3.4), we work out equation (4.3.5) below. The second equality in (4.3.5) is obtained by making the change of index $k = j + 1$ in the first sum. The third equality uses the fact that $W_0 = W(0) = 0$. We have

$$\begin{aligned}
\frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2 &= \frac{1}{2}\sum_{j=0}^{n-1} W_{j+1}^2 - \sum_{j=0}^{n-1} W_j W_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\
&= \frac{1}{2}\sum_{k=1}^{n} W_k^2 - \sum_{j=0}^{n-1} W_j W_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\
&= \frac{1}{2}W_n^2 + \frac{1}{2}\sum_{k=0}^{n-1} W_k^2 - \sum_{j=0}^{n-1} W_j W_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\
&= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_j W_{j+1} \\
&= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1} W_j(W_j - W_{j+1}).
\end{aligned} \tag{4.3.5}$$

From (4.3.5), we conclude that

$$\sum_{j=0}^{n-1} W_j(W_{j+1} - W_j) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2.$$

In the original notation, this is

$$\sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]$$
$$= \frac{1}{2}W^2(T) - \frac{1}{2}\sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2.$$

Letting $n \to \infty$ in (4.3.4) and using this equation, we get

$$\int_0^T W(t)\,dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}[W,W](T) = \frac{1}{2}W^2(T) - \frac{1}{2}T. \tag{4.3.6}$$

We contrast (4.3.6) with ordinary calculus. If g is a differentiable function with $g(0) = 0$, then

$$\int_0^T g(t)\,dg(t) = \int_0^T g(t)g'(t)\,dt = \frac{1}{2}g^2(t)\Big|_0^T = \frac{1}{2}g^2(T).$$

The extra term $-\frac{1}{2}T$ in (4.3.6) comes from the nonzero quadratic variation of Brownian motion and the way we constructed the Itô integral, always evaluating the integrand at the left-hand endpoint of the subinterval (see the right-hand side of (4.3.4)). If we were instead to evaluate at the midpoint, replacing the right-hand side of (4.3.4) by

$$\lim_{n\to\infty}\sum_{j=0}^{n-1} W\left(\frac{(j+\frac{1}{2})T}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right], \tag{4.3.7}$$

then we would not have gotten this term (see Exercise 4.4). The integral obtained by making this replacement is called the *Stratonovich integral*, and the ordinary rules of calculus apply to it. However, it is inappropriate for finance. In finance, the integrand represents a position in an asset and the integrator represents the price of that asset. We cannot decide at 1:00 p.m. which position we took at 9:00 a.m. We must decide the position at the beginning of each time interval, and the Itô integral is the limit of the gain achieved by that kind of trading as the time between trades approaches zero.

For functions $g(t)$ that have a derivative, integrals such as $\int_0^t g(t)\,dg(t)$ are not sensitive to this distinction (i.e., the Itô integral and Stratonovich integral approximations have the same limit, which is $\frac{1}{2}g^2(T)$). For functions that have a nonzero quadratic variation, integrals are sensitive to where in the subintervals the approximating integrands are evaluated.

The upper limit of integration T in (4.3.6) is arbitrary and can be replaced by any $t \geq 0$. In other words,

$$\int_0^t W(u)\,dW(u) = \frac{1}{2}W^2(t) - \frac{1}{2}t, \quad t \geq 0. \tag{4.3.8}$$

Theorem 4.3.1(iv) guarantees that $\int_0^t W(u)\,dW(u)$ is a martingale and hence has constant expectation. At $t = 0$, this martingale is 0, and hence its expectation must always be zero. This is indeed the case because $\mathbb{E}W^2(t) = t$. If the term $-\frac{1}{2}t$ were not present, we would not have a martingale. □

4.4 Itô-Doeblin Formula

The addition of Doeblin's name to what has traditionally been called the Itô formula is explained in the Notes, Section 4.9.

4.4.1 Formula for Brownian Motion

We want a rule to "differentiate" expressions of the form $f(W(t))$, where $f(x)$ is a differentiable function and $W(t)$ is a Brownian motion. If $W(t)$ were also differentiable, then the *chain rule* from ordinary calculus would give

$$\frac{d}{dt}f(W(t)) = f'(W(t))W'(t),$$

which could be written in differential notation as

$$df(W(t)) = f'(W(t))\,W'(t)\,dt = f'(W(t))\,dW(t).$$

Because W has nonzero quadratic variation, the correct formula has an extra term, namely,

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2}f''(W(t))\,dt. \tag{4.4.1}$$

This is the *Itô-Doeblin formula in differential form.* Integrating this, we obtain the *Itô-Doeblin formula in integral form:*

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(u))\,dW(u) + \frac{1}{2}\int_0^t f''(W(u))\,du. \tag{4.4.2}$$

The mathematically meaningful form of the Itô-Doeblin formula is the integral form (4.4.2). This is because we have precise definitions for both terms appearing on the right-hand side. The first, $\int_0^t f'(W(u))\,dW(u)$, is an Itô integral, defined in the previous section. The second, $\int_0^t f''(W(u))\,du$, is an ordinary (Lebesgue) integral with respect to the time variable.

For pencil and paper computations, the more convenient form of the Itô-Doeblin formula is the differential form (4.4.1). There is an intuitive meaning but no precise definition for the terms $df\big(W(t)\big)$, $dW(t)$, and dt appearing in this formula. The intuitive meaning is that $df\big(W(t)\big)$ is the change in $f\big(W(t)\big)$ when t changes a "little bit" dt, $dW(t)$ is the change in the Brownian motion when t changes a "little bit" dt, and the whole formula is exact only when the "little bit" is "infinitesimally small." Because there is no precise definition for "little bit" and "infinitesimally small," we rely on (4.4.2) to give precise meaning to (4.4.1).

The relationship between (4.4.1) and (4.4.2) is similar to that developed in ordinary calculus to assist in changing variables in an integral. If asked to compute the indefinite integral $\int f(u)f'(u)\,du$, we might make the change of variable $v = f(u)$ and write $dv = f'(u)\,du$, so that the indefinite integral becomes $\int v\,dv$, which is $\frac{1}{2}v^2 + C = \frac{1}{2}f^2(u) + C$, where C is a constant of integration. The final formula

$$\int f(u)f'(u)\,du = \frac{1}{2}f^2(u) + C$$

is correct, as can be verified by differentiating $\frac{1}{2}f^2(u)+C$ to get $f(u)f'(u)$. We do not attempt to give precise definitions to the terms dv and du appearing in the equation $dv = f'(u)\,du$ used in deriving it.

We formalize the preceding discussion with a theorem that provides a formula slightly more general than (4.4.2) in that it allows f to be a function of both t and x.

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$\begin{aligned} f\big(T,W(T)\big) &= f\big(0,W(0)\big) + \int_0^T f_t\big(t,W(t)\big)\,dt \\ &\quad + \int_0^T f_x\big(t,W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}\big(t,W(t)\big)\,dt. \end{aligned} \tag{4.4.3}$$

SKETCH OF PROOF: We first show why (4.4.3) holds when $f(x) = \frac{1}{2}x^2$. In this case, $f'(x) = x$ and $f''(x) = 1$. Let x_{j+1} and x_j be numbers. Taylor's formula implies

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2}f''(x_j)(x_{j+1} - x_j)^2. \tag{4.4.4}$$

In this case, Taylor's formula to second order is exact (there is no remainder term) because f''' and all higher derivatives of f are zero. We return to this matter later.

Fix $T > 0$, and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = T$). We are interested in the difference between $f\big(W(0)\big)$ and $f\big(W(T)\big)$. This change in $f\big(W(t)\big)$ between times $t = 0$ and $t = T$ can be written as the sum of the changes in $f\big(W(t)\big)$ over each of the subintervals $[t_j, t_{j+1}]$. We do this and then use Taylor's formula (4.4.4) with $x_j = W(t_j)$ and $x_{j+1} = W(t_{j+1})$ to obtain

$$\begin{aligned} f(W(T)) - f(W(0)) &= \sum_{j=0}^{n-1} \big[f\big(W(t_{j+1})\big) - f\big(W(t_j)\big)\big] \\ &= \sum_{j=0}^{n-1} f'(W(t_j))\big[W(t_{j+1}) - W(t_j)\big] \\ &\quad + \frac{1}{2}\sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2. \end{aligned} \tag{4.4.5}$$

For the function $f(x) = \frac{1}{2}x^2$, the right-hand side of (4.4.5) is

$$\sum_{j=0}^{n-1} W(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \frac{1}{2}\sum_{j=0}^{n-1}\big[W(t_{j+1}) - W(t_j)\big]^2. \tag{4.4.6}$$

If we let $\|\Pi\| \to 0$, the left-hand side of (4.4.5) is unaffected and the terms on the right-hand side converge to an Itô integral and one-half of the quadratic variation of Brownian motion, respectively:

$$\begin{aligned} &f\big(W(T)\big) - f\big(W(0)\big) \\ &= \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} W(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \lim_{\|\Pi\|\to 0}\frac{1}{2}\sum_{j=0}^{n-1}\big[W(t_{j+1}) - W(t_j)\big]^2 \\ &= \int_0^T W(t)\,dW(t) + \frac{1}{2}T \\ &= \int_0^T f'\big(W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f''\big(W(t)\big)\,dt. \end{aligned} \tag{4.4.7}$$

This is the Itô-Doeblin formula in integral form for the function $f(x) = \frac{1}{2}x^2$.

If instead of the quadratic function $f(x) = \frac{1}{2}x^2$ we had a general function $f(x)$, then in (4.4.5) we would have also gotten a sum of terms containing $\big[W(t_{j+1}) - W(t_j)\big]^3$. But according to Exercise 3.4 of Chapter 3, $\sum_{j=0}^{n-1}\big|W(t_{j+1}) - W(t_j)\big|^3$ has limit zero as $\|\Pi\| \to 0$. Therefore, this term would make no contribution to the final answer.

If we take a function $f(t, x)$ of both the time variable t and the variable x, then Taylor's Theorem says that

$$
\begin{aligned}
&f(t_{j+1},x_{j+1}) - f(t_j,x_j)\\
&= f_t(t_j,x_j)(t_{j+1}-t_j) + f_x(t_j,x_j)(x_{j+1}-x_j)\\
&\quad + \frac{1}{2} f_{xx}(t_j,x_j)(x_{j+1}-x_j)^2 + f_{tx}(t_j,x_j)(t_{j+1}-t_j)(x_{j+1}-x_j)\\
&\quad + \frac{1}{2} f_{tt}(t_j,x_j)(t_{j+1}-t_j)^2 + \text{ higher-order terms.}
\end{aligned}
\tag{4.4.8}
$$

We replace x_j by $W(t_j)$, replace x_{j+1} by $W(t_{j+1})$, and sum:

$$
\begin{aligned}
&f\big(T,W(T)\big) - f\big(0,W(0)\big)\\
&= \sum_{j=0}^{n-1} \left[f\big(t_{j+1},W(t_{j+1})\big) - f\big(t_j,W(t_j)\big) \right]\\
&= \sum_{j=0}^{n-1} f_t\big(t_j,W(t_j)\big)(t_{j+1}-t_j) + \sum_{j=0}^{n-1} f_x\big(t_j,W(t_j)\big)\big(W(t_{j+1})-W(t_j)\big)\\
&\quad + \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}\big(t_j,W(t_j)\big)\big(W(t_{j+1})-W(t_j)\big)^2\\
&\quad + \sum_{j=0}^{n-1} f_{tx}\big(t_j,W(t_j)\big)(t_{j+1}-t_j)\big(W(t_{j+1})-W(t_j)\big)\\
&\quad + \frac{1}{2}\sum_{j=0}^{n-1} f_{tt}\big(t_j,W(t_j)\big)(t_{j+1}-t_j)^2 + \text{ higher-order terms.}
\end{aligned}
\tag{4.4.9}
$$

When we take the limit as $\|\Pi\| \to 0$, the left-hand side of (4.4.9) is unaffected. The first term on the right-hand side of (4.4.9) contributes the ordinary (Lebesgue) integral

$$
\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} f_t\big(t_j,W(t_j)\big)(t_{j+1}-t_j) = \int_0^T f_t\big(t,W(t)\big)\,dt
$$

to the final answer. As $\|\Pi\| \to 0$, the second term contributes the Itô integral $\int_0^T f_x\big(t,W(t)\big)\,dW(t)$. The third term contributes another ordinary (Lebesgue) integral, $\frac{1}{2}\int_0^T f_{xx}\big(t,W(t)\big)\,dt$, similar to the way we obtained this integral in (4.4.7). In other words, in the third term we can replace $\big(W(t_{j+1})-W(t_j)\big)^2$ by $t_{j+1}-t_j$. This is not an exact substitution, but when we sum the terms this substitution gives the correct limit as $\|\Pi\| \to 0$. See Remark 3.4.4 for more discussion of this point. With this substitution, the third term on the right-hand side of (4.4.9) contributes $\frac{1}{2}\int_0^T f_{xx}\big(t,W(t)\big)\,dt$. These limits of the first three terms appear on the right-hand side of (4.4.3). The fourth and fifth terms contribute zero. Indeed, for the fourth term, we observe that

$$\begin{aligned}
&\lim_{\|\Pi\|\to 0}\left|\sum_{j=0}^{n-1} f_{tx}(t_j,W(t_j))(t_{j+1}-t_j)\big(W(t_{j+1})-W(t_j)\big)\right| \\
&\le \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tx}(t_j,W(t_j))\right|\cdot(t_{j+1}-t_j)\cdot\left|W(t_{j+1})-W(t_j)\right| \\
&\le \lim_{\|\Pi\|\to 0}\max_{0\le k\le n-1}\left|W(t_{k+1})-W(t_k)\right|\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tx}(t_j,W(t_j))\right|(t_{j+1}-t_j) \\
&= 0\cdot\int_0^T \left|f_{tx}(t,W(t)\right) dt \;=\; 0. \qquad (4.4.10)
\end{aligned}$$

The fifth term is treated similarly:

$$\begin{aligned}
&\lim_{\|\Pi\|\to 0}\left|\frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j,W(t_j))(t_{j+1}-t_j)^2\right| \\
&\qquad\le \lim_{\|\Pi\|\to 0}\frac{1}{2}\sum_{j=0}^{n-1}\left|f_{tt}(t_j,W(t_j))\right|\cdot(t_{j+1}-t_j)^2 \\
&\qquad\le \frac{1}{2}\lim_{\|\Pi\|\to 0}\max_{0\le k\le n-1}(t_{k+1}-t_k)\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tt}(t_j,W(t_j))\right|(t_{j+1}-t_j) \\
&\qquad= \frac{1}{2}\cdot 0\cdot\int_0^T f_{tt}(t,W(t))dt \;=\; 0. \qquad (4.4.11)
\end{aligned}$$

The higher-order terms likewise contribute zero to the final answer. □

Remark 4.4.2. The fact that the sum (4.4.10) of terms containing the product $(t_{j+1}-t_j)\big(W(t_{j+1})-W(t_j)\big)$ has limit zero can be informally recorded by the formula $dt\,dW(t) = 0$. Similarly, the sum (4.4.11) of terms containing $(t_{j+1}-t_j)^2$ also has limit zero, and this can be recorded by the formula $dt\,dt = 0$. We can write these terms if we like in the Itô-Doeblin formula, so that in differential form it becomes

$$\begin{aligned}
&df(t,W(t)) \\
&\quad= f_t(t,W(t))\,dt + f_x(t,W(t))\,dW(t) + \frac{1}{2}f_{xx}(t,W(t))\,dW(t)\,dW(t) \\
&\qquad\qquad + f_{tx}(t,W(t))\,dt\,dW(t) + \frac{1}{2}f_{xx}(t,W(t))\,dt\,dt,
\end{aligned}$$

but

$$dW(t)\,dW(t) = dt, \quad dt\,dW(t) = dW(t)\,dt = 0, \quad dt\,dt = 0, \qquad (4.4.12)$$

and the Itô-Doeblin formula in differential form simplifies to

$$df(t,W(t)) = f_t(t,W(t))\,dt + f_x(t,W(t))\,dW(t) + \frac{1}{2}f_{xx}(t,W(t))\,dt. \quad (4.4.13)$$

In Figure 4.4.1, we illustrate the Taylor series approximation of the difference $f\big(W(t_{j+1})\big) - f\big(W(t_j)\big)$ for a function $f(x)$ that does not depend on t. The first-order approximation, which is $f'\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big)$, has an error due to the convexity of the function $f(x)$. Most of this error is removed by adding in the second-order term $\frac{1}{2}f''\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big)^2$, which captures the curvature of the function $f(x)$ at $x = W(t_j)$.

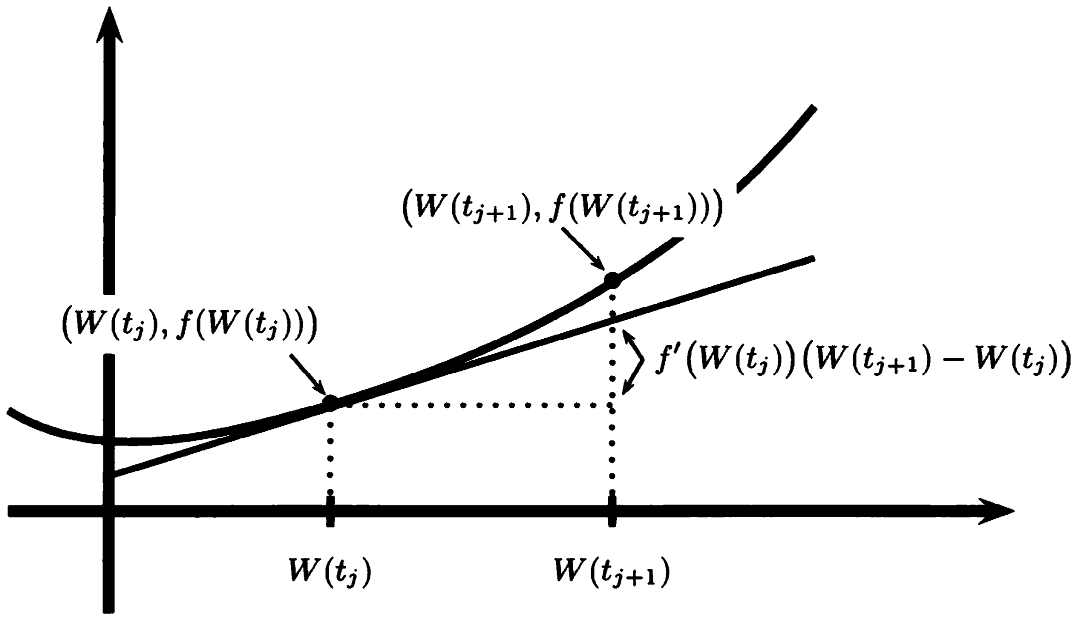

Fig. 4.4.1. Taylor approximation to $f\big(W(t_{j+1})\big) - f\big(W(t_j)\big)$.

In other words,

$$f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) = f'\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big) + \text{ small error,} \tag{4.4.14}$$

and

$$\begin{aligned} f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) = {} & f'\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big) \\ & + \frac{1}{2}f''\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big)^2 \\ & + \text{smaller error.} \end{aligned} \tag{4.4.15}$$

In both (4.4.14) and (4.4.15), as $\|\Pi\| \to 0$, the errors approach zero. However, before we let $\|\Pi\| \to 0$, we must first sum these equations over j, and the smaller we make $\|\Pi\|$, the more terms there are in the sum. When we sum both sides of (4.4.14), the errors accumulate, and although the error in each summand approaches zero as $\|\Pi\| \to 0$, the sum of the errors does not. When we use the more accurate approximation (4.4.15), this does not happen; the limit of the sum of the smaller errors is zero. We need the extra accuracy of (4.4.15) because the paths of Brownian motion are so volatile (i.e., they have nonzero quadratic variation). This extra term makes stochastic calculus different from ordinary calculus.

The Itô-Doeblin formula often simplifies the computation of Itô integrals. For example, with $f(x) = \frac{1}{2}x^2$, this formula says that

$$\begin{aligned}\frac{1}{2}W^2(T) &= f(W(T)) - f(W(0)) \\ &= \int_0^T f'(W(t))\,dW(t) + \frac{1}{2}\int_0^t f''(W(t))\,dt \\ &= \int_0^T W(t)\,dW(t) + \frac{1}{2}T.\end{aligned}$$

Rearranging terms, we have formula (4.3.6) and have obtained it without going through the approximation of the integrand by simple processes as we did in Example 4.3.2.

4.4.2 Formula for Itô Processes

We extend the Itô-Doeblin formula to stochastic processes more general than Brownian motion. The processes for which we develop stochastic calculus are the *Itô processes* defined below. Almost all stochastic processes, except those that have jumps, are Itô processes.

Definition 4.4.3. *Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration. An* Itô process *is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u)\,dW(u) + \int_0^t \Theta(u)\,du, \tag{4.4.16}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes.[2]

In order to understand the volatility associated with Itô processes, we must determine the rate at which they accumulate quadratic variation.

Lemma 4.4.4. *The quadratic variation of the Itô process (4.4.16) is*

$$[X, X](t) = \int_0^t \Delta^2(u)\,du. \tag{4.4.17}$$

PROOF: We introduce the notation $I(t) = \int_0^t \Delta(u)\,dW(u)$, $R(t) = \int_0^t \Theta(u)\,du$. Both these processes are continuous in their upper limit of integration t. To

[2] We assume that $\mathbb{E}\int_0^t \Delta^2(u)\,du$ and $\int_0^t |\Theta(u)|\,du$ are finite for every $t > 0$ so that the integrals on the right-hand side of (4.4.16) are defined and the Itô integral is a martingale. We shall always make such integrability assumptions, but we do not always explicitly state them.

determine the quadratic variation of X on $[0,t]$, we choose a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0,t]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = t$) and we write the sampled quadratic variation

$$\sum_{j=0}^{n-1} \left[X(t_{j+1}) - X(t_j)\right]^2 = \sum_{j=0}^{n-1} \left[I(t_{j+1}) - I(t_j)\right]^2 + \sum_{j=0}^{n-1} \left[R(t_{j+1}) - R(t_j)\right]^2$$
$$+2\sum_{j=0}^{n-1} \left[I(t_{j+1}) - I(t_j)\right]\left[R(t_{j+1}) - R(t_j)\right].$$

As $\|\Pi\| \to 0$, the first term on the right-hand side, $\sum_{j=0}^{n-1} \left[I(t_{j+1}) - I(t_j)\right]^2$, converges to the quadratic variation of I on $[0,t]$, which according to Theorem 4.3.1(vi) is $[I,I](t) = \int_0^t \Delta^2(u)\,du$. The absolute value of the second term is bounded above by

$$\begin{aligned}
&\max_{0 \le k \le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|R(t_{j+1}) - R(t_j)\right| \\
&\quad = \max_{0 \le k \le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|\int_{t_j}^{t_{j+1}} \Theta(u)\,du\right| \\
&\quad \le \max_{0 \le k \le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\Theta(u)|\,du \\
&\quad = \max_{0 \le k \le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \int_0^t |\Theta(u)|\,du,
\end{aligned}$$

and as $\|\Pi\| \to 0$, this has limit $0 \cdot \int_0^t |\Theta(u)|\,du = 0$ because $R(t)$ is continuous. The absolute value of the third term is bounded above by

$$\begin{aligned}
&2 \max_{0 \le k \le n-1} \left|I(t_{k+1}) - I(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|R(t_{j+1}) - R(t_j)\right| \\
&\quad \le 2 \max_{0 \le k \le n-1} \left|I(t_{k+1}) - I(t_k)\right| \cdot \int_0^t |\Theta(u)|\,du,
\end{aligned}$$

and this has limit $0 \cdot \int_0^t |\Theta(u)|^2\,du = 0$ as $\|\Pi\| \to 0$ because $I(t)$ is continuous. We conclude that $[X,X](t) = [I,I](t) = \int_0^t \Delta^2(u)\,du$. □

The conclusion of Lemma 4.4.4 is most easily remembered by first writing (4.4.16) in the differential notation

$$dX(t) = \Delta(t)\,dW(t) + \Theta(t)\,dt \tag{4.4.18}$$

and then using the differential multiplication table (4.4.12) to compute

$$\begin{aligned} dX(t)\,dX(t) &= \Delta^2(t)\,dW(t)\,dW(t) + 2\Delta(t)\Theta(t)\,dW(t)\,dt + \Theta^2(t)\,dt\,dt \\ &= \Delta^2(t)\,dt. \end{aligned} \tag{4.4.19}$$

This says that, at each time t, the process X is accumulating quadratic variation at rate $\Delta^2(t)$ per unit time, and hence the total quadratic variation accumulated on the time interval $[0,t]$ is $[X,X](t) = \int_0^t \Delta^2(u)\,du$. This quadratic variation is solely due to the quadratic variation of the Itô integral $I(t) = \int_0^t \Delta(u)\,dW(u)$. The ordinary integral $R(t) = \int_0^t \Theta(u)\,du$ has zero quadratic variation and thus contributes nothing to the quadratic variation of X.

Notice in this connection that having zero quadratic variation does not necessarily mean that $R(t)$ is nonrandom. Because $\Theta(u)$ can be random, $R(t)$ can also be random. However, $R(t)$ is not as volatile as $I(t)$. At each time t, we have a good estimate of the next increment of $R(t)$. For small time steps $h > 0$,

$$R(t+h) \approx R(t) + \Theta(t)h,$$

and we know both $R(t)$ and $\Theta(t)$ at time t. This is like investing in a money market account at a variable interest rate. At each time, we have a good estimate of the return over the near future because we know today's interest rate. Nonetheless, the return is random because the interest rate (Θ in this analogy) can change. In contrast, I is more volatile. At time t, one estimate of $I(t+h)$ is

$$I(t+h) \approx I(t) + \Delta(t)\big(W(t+h) - W(t)\big),$$

but we do not know $W(t+h) - W(t)$ at time t. In fact, $W(t+h) - W(t)$ is independent of the information available at time t. This is like investing in a stock.

So far we have discussed integrals with respect to time, such as $R(t) = \int_0^t \Theta(u)\,du$ appearing in (4.4.16) and Itô integrals (integrals with respect to Brownian motion) such as $I(t) = \int_0^t \Delta(u)\,dW(u)$, also appearing in (4.4.16). In addition, we shall need integrals with respect to Itô processes (i.e., integrals of the form $\int_0^t \Gamma(u)\,dX(u)$, where Γ is some adapted process). We define such an integral by separating $dX(t)$ into a $dW(t)$ term and a dt term as in (4.4.18).

Definition 4.4.5. *Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $\Gamma(t)$, $t \geq 0$, be an adapted process. We define the* integral with respect to an Itô process[3]

$$\int_0^t \Gamma(u)\,dX(u) = \int_0^t \Gamma(u)\Delta(u)\,dW(u) + \int_0^t \Gamma(u)\Theta(u)\,du. \tag{4.4.20}$$

We again work through the sketch of the proof of Theorem 4.4.1, but with the Itô process $X(t)$ replacing the Brownian motion $W(t)$. In place of (4.4.9), we now have

[3] We assume that $E\int_0^t \Gamma^2(u)\Delta^2(u)\,du$ and $\int_0^t |\Gamma(u)\Theta(u)|\,du$ are finite for every $t > 0$ so that the integrals on the right-hand side of (4.4.20) are defined.

$$
\begin{aligned}
&f(T,X(T)) - f(0,X(0))\\
&= \sum_{j=0}^{n-1} f_t(t_j,X(t_j))(t_{j+1}-t_j) + \sum_{j=0}^{n-1} f_x(t_j,X(t_j))\big(X(t_{j+1})-X(t_j)\big)\\
&\qquad + \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j,X(t_j))\big(X(t_{j+1})-X(t_j)\big)^2\\
&\qquad + \sum_{j=0}^{n-1} f_{tx}(t_j,X(t_j))(t_{j+1}-t_j)\big(X(t_{j+1})-X(t_j)\big)\\
&\qquad + \frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j,X(t_j))(t_{j+1}-t_j)^2 + \text{ higher-order terms.} \qquad (4.4.21)
\end{aligned}
$$

The last two sums on the right-hand side have zero limits as $\|\Pi\| \to 0$ for the same reasons the analogous terms have zero limits in the sketch of the proof of Theorem 4.4.1 (see (4.4.10) and (4.4.11)). The higher-order terms likewise have limit zero. The limit of the first term on the right-hand side of (4.4.21) is $\int_0^T f_t(t,X(t))dt$. The limit of the second term is

$$
\int_0^T f_x(t,X(t))\,dX(t) = \int_0^T f_x(t,X(t))\Delta(t)\,dW(t) + \int_0^T f_x(t,X(t))\Theta(t)\,dt.
$$

Finally, the limit of the third term on the right-hand side of (4.4.19) is

$$
\frac{1}{2}\int_0^T f_{xx}(t,X(t))\,d[X,X](t) = \frac{1}{2}\int_0^T f_{xx}(t,X(t))\Delta^2(t)\,dt
$$

because the Itô process $X(t)$ accumulates quadratic variation at rate $\Delta^2(t)$ per unit time (Lemma 4.4.4). These considerations lead to the following generalization of Theorem 4.4.1.

Theorem 4.4.6 (Itô-Doeblin formula for an Itô process). *Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,*

$$
\begin{aligned}
&f(T,X(T))\\
&= f(0,X(0)) + \int_0^T f_t(t,X(t))\,dt + \int_0^T f_x(t,X(t))\,dX(t)\\
&\qquad + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\,d[X,X](t)\\
&= f(0,X(0)) + \int_0^T f_t(t,X(t))\,dt + \int_0^T f_x(t,X(t))\Delta(t)\,dW(t)\\
&\qquad + \int_0^T f_x(t,X(t))\Theta(t)\,dt + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\Delta^2(t)\,dt. \qquad (4.4.22)
\end{aligned}
$$

Remark 4.4.7 (Summary of stochastic calculus). Theorem 4.4.6 is stated in mathematically precise language. Every term on the right-hand side has a solid definition, and in the end the right-hand side reduces to a sum of a nonrandom quantity $f\big(0, X(0)\big)$, three ordinary (Lebesgue) integrals with respect to time, and an Itô integral.

However, it is easier to remember and use the result of this theorem if we recast it in differential notation. We may rewrite (4.4.22) as

$$df\big(t, X(t)\big) = f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t, X(t)\big)\,dX(t)\,dX(t). \tag{4.4.23}$$

The guiding principle here is that we write out the Taylor series expansion of $f\big(t, X(t)\big)$ with respect to all its arguments, which in this case are t and $X(t)$. We take this Taylor series expansion out to first order for every argument that has zero quadratic variation, which in this case is t, and we take the expansion out to second order for every argument that has nonzero quadratic variation, which in this case is $X(t)$.

We may reduce (4.4.23) to an expression that involves only dt and $dW(t)$ by using the differential form (4.4.18) of the Itô process (i.e., $dX(t) = \Delta(t)\,dW(t) + \Theta(t)\,dt$) and the formula (4.4.19) for the rate at which $X(t)$ accumulates quadratic variation (i.e., $dX(t)\,dX(t) = \Delta^2(t)\,dt$). This is obtained by squaring the formula for $dX(t)$ and using the multiplication table (4.4.12). Making these substitutions in (4.4.23), we obtain

$$\begin{aligned} df\big(t, X(t)\big) &= f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\Delta(t)\,dW(t) \\ &\quad + f_x\big(t, X(t)\big)\Theta(t)\,dt + \frac{1}{2}f_{xx}\big(t, X(t)\big)\Delta^2(t)\,dt. \end{aligned} \tag{4.4.24}$$

Itô calculus is little more than repeated use of this formula in a variety of situations. □

4.4.3 Examples

We conclude this section with three examples illustrating Remark 4.4.7. Many more examples are developed in subsequent sections and in the exercises.

Example 4.4.8 (Generalized geometric Brownian motion). Let $W(t)$, $t \geq 0$, be a Brownian motion, let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Itô process

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \Big(\alpha(s) - \frac{1}{2}\sigma^2(s)\Big)ds. \tag{4.4.25}$$

Then

$$dX(t) = \sigma(t)\,dW(t) + \Big(\alpha(t) - \frac{1}{2}\sigma^2(t)\Big)\,dt,$$

and

$$dX(t)\,dX(t) = \sigma^2(t)\,dW(t)\,dW(t) = \sigma^2(t)\,dt.$$

Consider an asset price process given by

$$S(t) = S(0)e^{X(t)} = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}, \tag{4.4.26}$$

where $S(0)$ is nonrandom and positive. We may write $S(t) = f\big(X(t)\big)$, where $f(x) = S(0)e^x$, $f'(x) = S(0)e^x$, and $f''(x) = S(0)e^x$. According to the Itô-Doeblin formula

$$\begin{aligned}
dS(t) &= df\big(X(t)\big)\\
&= f'\big(X(t)\big)\,dX(t) + \frac{1}{2}f''\big(X(t)\big)\,dX(t)\,dX(t)\\
&= S(0)e^{X(t)}\,dX(t) + \frac{1}{2}S(0)e^{X(t)}\,dX(t)\,dX(t)\\
&= S(t)\,dX(t) + \frac{1}{2}S(t)\,dX(t)\,dX(t)\\
&= \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t).
\end{aligned} \tag{4.4.27}$$

The asset price $S(t)$ has instantaneous mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. Both the instantaneous mean rate of return and the volatility are allowed to be time-varying and random.

This example includes all possible models of an asset price process that is always positive, has no jumps, and is driven by a single Brownian motion. Although the model is driven by a Brownian motion, the distribution of $S(t)$ does not need to be log-normal because $\alpha(t)$ and $\sigma(t)$ are allowed to be time-varying and random. If α and σ are constant, we have the usual geometric Brownian motion model, and the distribution of $S(t)$ is log-normal.

In the case of constant α and σ, (4.4.26) becomes

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}. \tag{4.4.28}$$

One can incorrectly argue from this formula that since Brownian motion is a martingale (i.e., it has no overall tendency to rise or fall), the mean rate of return for $S(t)$ must be $\alpha - \frac{1}{2}\sigma^2$. The error in this argument is that although $W(t)$ is a martingale, $S(0)e^{\sigma W(t)}$ is not. The convexity of the function $e^{\sigma x}$ imparts an upward drift to $S(0)e^{\sigma W(t)}$. In order to correct for this, one must subtract $\frac{1}{2}\sigma^2 t$ in the exponential; the process $S(0)\exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\}$ is a martingale (see Theorem 3.6.1). If we now add αt in the exponential, we get $S(t)$, a process with mean rate of return α.

The Itô-Doeblin formula automatically keeps track of these effects, even when α and σ are time-varying and random. If $\alpha = 0$, then (4.4.27) yields

$$dS(t) = \sigma(t)S(t)\,dW(t).$$

Integration of both sides yields

$$S(t) = S(0) + \int_0^t \sigma(s)S(s)\,dW(s).$$

The right-hand side is the nonrandom constant $S(0)$ plus an Itô integral, which is a martingale, and hence (in the case $\alpha = 0$)

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) - \frac{1}{2}\int_0^t \sigma^2(s)\,ds\right\} \tag{4.4.29}$$

is a martingale. In other words, the term $\sigma(t)S(t)\,dW(t)$ on the right-hand side of (4.4.27) contributes no drift, just pure volatility, to the asset price.

When $\alpha(t)$ is a nonzero random process, (4.4.27) shows that it plays the role of the mean rate of return. In the case of time-varying and random $\alpha(t)$, we will call this the *instantaneous* mean rate of return since it depends on the time (and the sample path) where it is evaluated. □

The preceding example supplies the heart of the proof of the following theorem.

Theorem 4.4.9 (Itô integral of a deterministic integrand). *Let $W(s)$, $s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a* nonrandom *function of time. Define $I(t) = \int_0^t \Delta(s)\,dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)\,ds$.*

PROOF: The mean and variance of $I(t)$ are easy to determine. Since $I(t)$ is a martingale and $I(0) = 0$, we must have $\mathbb{E}I(t) = I(0) = 0$. Itô's isometry (Theorem 4.3.1(v)) implies that

$$\mathrm{Var} I(t) = \mathbb{E}I^2(t) = \int_0^t \Delta^2(s)\,ds.$$

We do not need to take the expected value of $\int_0^t \Delta^2(s)\,ds$ on the right-hand side of this formula because $\Delta(s)$ is not random.

The challenge is to show that $I(t)$ is normally distributed. We shall do this by establishing that $I(t)$ has the moment-generating function of a normal random variable with mean zero and variance $\int_0^t \Delta^2(s)\,ds$, which is (see (3.2.13))

$$\mathbb{E}e^{uI(t)} = \exp\left\{\frac{1}{2}u^2\int_0^t \Delta^2(s)\,ds\right\} \text{ for all } u \in \mathbb{R}. \tag{4.4.30}$$

Because $\Delta(s)$ is not random, (4.4.30) is equivalent to

$$\mathbb{E}\exp\left\{uI(t) - \frac{1}{2}u^2\int_0^t \Delta^2(s)\,ds\right\} = 1,$$

which may be rewritten as

$$\mathbb{E}\exp\left\{\int_0^t u\Delta(s)\,dW(s) - \frac{1}{2}\int_0^t \big(u\Delta(s)\big)^2\,ds\right\} = 1. \tag{4.4.31}$$

But the process

$$\exp\left\{\int_0^t u\Delta(s)\,dW(s) - \frac{1}{2}\int_0^t \big(u\Delta(s)\big)^2\,ds\right\}$$

is a martingale. Indeed, it is a generalized geometric Brownian motion with mean rate of return $\alpha = 0$; see (4.4.29) with $\sigma(s) = u\Delta(s)$. Furthermore, this process takes the value 1 at $t = 0$, and hence its expectation is always 1. This gives us (4.4.31). □

Note that (4.4.31) always holds, regardless of whether $\Delta(s)$ is random. However, we need to assume that $\Delta(s)$ is nonrandom in order to obtain the moment-generating function formula (4.4.30) from (4.4.31). When $\Delta(s)$ is random, there is no reason that the distribution of $\int_0^t \Delta(s)\,dW(s)$ should be normal.

Example 4.4.10 (Vasicek interest rate model). Let $W(t), t \geq 0$, be a Brownian motion. The Vasicek model for the interest rate process $R(t)$ is

$$dR(t) = \big(\alpha - \beta R(t)\big)\,dt + \sigma\,dW(t), \tag{4.4.32}$$

where α, β, and σ are positive constants. Equation (4.4.32) is an example of a *stochastic differential equation.* It defines a random process, $R(t)$ in this case, by giving a formula for its differential, and the formula involves the random process itself and the differential of a Brownian motion.

The solution to the stochastic differential equation (4.4.32) can be determined in closed form and is

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\big(1 - e^{-\beta t}\big) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW(s), \tag{4.4.33}$$

a claim that we now verify. In particular, we compute the differential of the right-hand side of (4.4.33). To do this, we use the Itô-Doeblin formula with

$$f(t,x) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\big(1 - e^{-\beta t}\big) + \sigma e^{-\beta t}x$$

and $X(t) = \int_0^t e^{\beta s}\,dW(s)$. Then the right-hand side of (4.4.33) is $f\big(t, X(t)\big)$. The technique we are using is to separate the right-hand side into two parts: an ordinary function of two variables t and x, which has no randomness in it, and an Itô process $X(t)$, which contains all the randomness. For the Itô-Doeblin formula, we shall need the following partial derivatives of $f(t,x)$:

$$
\begin{aligned}
f_t(t,x) &= -\beta e^{-\beta t}R(0) + \alpha e^{-\beta t} - \sigma\beta e^{-\beta t}x \;=\; \alpha - \beta f(t,x),\\
f_x(t,x) &= \sigma e^{-\beta t},\\
f_{xx}(t,x) &= 0.
\end{aligned}
$$

We shall also need the differential of $X(t)$, which is $dX(t) = e^{\beta t}\,dW(t)$. We shall not need $dX(t)\,dX(t) = e^{2\beta t}\,dt$ because $f_{xx}(t,x) = 0$. The Itô-Doeblin formula states that

$$
\begin{aligned}
&df\big(t,X(t)\big)\\
&= f_t\big(t,X(t)\big)\,dt + f_x\big(t,X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t,X(t)\big)\,dX(t)\,dX(t)\\
&= \Big(\alpha - \beta f\big(t,X(t)\big)\Big)\,dt + \sigma\,dW(t).
\end{aligned}
$$

This shows that $f\big(t,X(t)\big)$ satisfies the stochastic differential equation (4.4.32) that defines $R(t)$. Moreover, $f\big(0,X(0)\big) = R(0)$. Because $f\big(t,X(t)\big)$ satisfies the equation defining $R(t)$ and has the same initial condition as $R(t)$, it must be the case that $f(t,X(t)) = R(t)$ for all $t \geq 0$.

Theorem 4.4.9 implies that the random variable $\int_0^t e^{\beta s}\,dW(s)$ appearing on the right-hand side of (4.4.33) is normally distributed with mean zero and variance

$$\int_0^t e^{2\beta s}\,ds = \frac{1}{2\beta}\big(e^{2\beta t} - 1\big).$$

Therefore, $R(t)$ is normally distributed with mean $e^{-\beta t}R(0) + \frac{\alpha}{\beta}\big(1 - e^{-\beta t}\big)$ and variance $\frac{\sigma^2}{2\beta}\big(1 - e^{-2\beta t}\big)$. In particular, no matter how the parameters $\alpha > 0$, $\beta > 0$, and $\sigma > 0$ are chosen, there is positive probability that $R(t)$ is negative, an undesirable property for an interest rate model.

The Vasicek model has the desirable property that the interest rate is *mean-reverting*. When $R(t) = \frac{\alpha}{\beta}$, the drift term (the dt term) in (4.4.32) is zero. When $R(t) > \frac{\alpha}{\beta}$, this term is negative, which pushes $R(t)$ back toward $\frac{\alpha}{\beta}$. When $R(t) < \frac{\alpha}{\beta}$, this term is positive, which again pushes $R(t)$ back toward $\frac{\alpha}{\beta}$. If $R(0) = \frac{\alpha}{\beta}$, then $\mathbb{E}R(t) = \frac{\alpha}{\beta}$ for all $t \geq 0$. If $R(0) \neq \frac{\alpha}{\beta}$, then $\lim_{t\to\infty}\mathbb{E}R(t) = \frac{\alpha}{\beta}$. □

Example 4.4.11 (Cox-Ingersoll-Ross (CIR) interest rate model). Let $W(t)$, $t \geq 0$, be a Brownian motion. The Cox-Ingersoll-Ross model for the interest rate process $R(t)$ is

$$dR(t) = \big(\alpha - \beta R(t)\big)\,dt + \sigma\sqrt{R(t)}\,dW(t), \tag{4.4.34}$$

where α, β, and σ are positive constants. Unlike the Vasicek equation (4.4.32), the CIR equation (4.4.34) does not have a closed-form solution. The advantage of (4.4.34) over the Vasicek model is that the interest rate in the CIR model does not become negative. If $R(t)$ reaches zero, the term multiplying $dW(t)$

vanishes and the positive drift term $\alpha\,dt$ in equation (4.4.34) drives the interest rate back into positive territory. Like the Vasicek model, the CIR model is mean-reverting.

Although one cannot derive a closed-form solution for (4.4.34), the distribution of $R(t)$ for each positive t can be determined. That computation would take us too far afield. We instead content ourselves with the derivation of the expected value and variance of $R(t)$. To do this, we use the function $f(t,x) = e^{\beta t}x$ and the Itô-Doeblin formula to compute

$$\begin{aligned}
&d\big(e^{\beta t}R(t)\big)\\
&= df\big(t,R(t)\big)\\
&= f_t\big(t,R(t)\big)\,dt + f_x\big(t,R(t)\big)\,dR(t) + \frac{1}{2}f_{xx}\big(t,R(t)\big)\,dR(t)\,dR(t)\\
&= \beta e^{\beta t}R(t)\,dt + e^{\beta t}\big(\alpha - \beta R(t)\big)\,dt + e^{\beta t}\sigma\sqrt{R(t)}\,dW(t)\\
&= \alpha e^{\beta t}\,dt + \sigma e^{\beta t}\sqrt{R(t)}\,dW(t).
\end{aligned} \tag{4.4.35}$$

Integration of both sides of (4.4.35) yields

$$\begin{aligned}
e^{\beta t}R(t) &= R(0) + \alpha\int_0^t e^{\beta u}\,du + \sigma\int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u)\\
&= R(0) + \frac{\alpha}{\beta}\big(e^{\beta t} - 1\big) + \sigma\int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u).
\end{aligned}$$

Recalling that the expectation of an Itô integral is zero, we obtain

$$e^{\beta t}\mathbb{E}R(t) = R(0) + \frac{\alpha}{\beta}\big(e^{\beta t} - 1\big)$$

or, equivalently,

$$\mathbb{E}R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\big(1 - e^{-\beta t}\big). \tag{4.4.36}$$

This is the same expectation as in the Vasicek model.

To compute the variance of $R(t)$, we set $X(t) = e^{\beta t}R(t)$, for which we have already computed

$$dX(t) = \alpha e^{\beta t}\,dt + \sigma e^{\beta t}\sqrt{R(t)}\,dW(t) = \alpha e^{\beta t}\,dt + \sigma e^{\frac{\beta t}{2}}\sqrt{X(t)}\,dW(t)$$

and $\mathbb{E}X(t) = R(0) + \frac{\alpha}{\beta}\big(e^{\beta t} - 1\big)$. According to the Itô-Doeblin formula (with $f(x) = x^2$, $f'(x) = 2x$, and $f''(x) = 2$),

$$\begin{aligned}
d\big(X^2(t)\big) &= 2X(t)\,dX(t) + dX(t)\,dX(t)\\
&= 2\alpha e^{\beta t}X(t)\,dt + 2\sigma e^{\frac{\beta t}{2}}X^{\frac{3}{2}}(t)\,dW(t) + \sigma^2 e^{\beta t}X(t)\,dt.
\end{aligned} \tag{4.4.37}$$

Integration of (4.4.37) yields

$$X^2(t) = X^2(0) + (2\alpha + \sigma^2) \int_0^t e^{\beta u} X(u)\, du + 2\sigma \int_0^t e^{\frac{\beta u}{2}} X^{\frac{3}{2}}(u)\, dW(u).$$

Taking expectations, using the fact that the expectation of an Itô integral is zero and the formula already derived for $\mathbb{E}X(t)$, we obtain

$$\begin{aligned}
\mathbb{E}X^2(t) &= X^2(0) + (2\alpha + \sigma^2) \int_0^t e^{\beta u} \mathbb{E}X(u)\, du \\
&= R^2(0) + (2\alpha + \sigma^2) \int_0^t e^{\beta u} \Big(R(0) + \frac{\alpha}{\beta}(e^{\beta u} - 1) \Big)\, du \\
&= R^2(0) + \frac{2\alpha + \sigma^2}{\beta} \Big(R(0) - \frac{\alpha}{\beta} \Big)(e^{\beta t} - 1) + \frac{2\alpha + \sigma^2}{2\beta} \cdot \frac{\alpha}{\beta}(e^{2\beta t} - 1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}R^2(t) &= e^{-2\beta t} \mathbb{E}X^2(t) \\
&= e^{-2\beta t} R^2(0) + \frac{2\alpha + \sigma^2}{\beta} \Big(R(0) - \frac{\alpha}{\beta} \Big)(e^{-\beta t} - e^{-2\beta t}) \\
&\quad + \frac{\alpha(2\alpha + \sigma^2)}{2\beta^2}(1 - e^{-2\beta t}).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{Var}\big(R(t)\big) &= \mathbb{E}R^2(t) - \big(\mathbb{E}R(t)\big)^2 \\
&= e^{-2\beta t} R^2(0) + \frac{2\alpha + \sigma^2}{\beta} \Big(R(0) - \frac{\alpha}{\beta} \Big)(e^{-\beta t} - e^{-2\beta t}) \\
&\quad + \frac{\alpha(2\alpha + \sigma^2)}{2\beta^2}(1 - e^{-2\beta t}) - e^{-2\beta t} R^2(0) \\
&\quad - \frac{2\alpha}{\beta} R(0)(e^{-\beta t} - e^{-2\beta t}) - \frac{\alpha^2}{\beta^2}(1 - e^{-\beta t})^2 \\
&= \frac{\sigma^2}{\beta} R(0)\Big(e^{-\beta t} - e^{-2\beta t}\Big) + \frac{\alpha\sigma^2}{2\beta^2}\big(1 - 2e^{-\beta t} + e^{-2\beta t}\big).
\end{aligned} \tag{4.4.38}$$

In particular,

$$\lim_{t\to\infty} \mathrm{Var}\big(R(t)\big) = \frac{\alpha\sigma^2}{2\beta^2}.$$

4.5 Black-Scholes-Merton Equation

The addition of Merton's name to what has traditionally been called the Black-Scholes equation is explained in the Notes, Section 4.9.

In this section, we derive the Black-Scholes-Merton partial differential equation for the price of an option on an asset modeled as a geometric Brownian motion. The idea behind this derivation is the same as in the binomial model of Chapter 1 of Volume I, which is to determine the initial capital required to perfectly hedge a short position in the option.

4.5.1 Evolution of Portfolio Value

Consider an agent who at each time t has a portfolio valued at $X(t)$. This portfolio invests in a money market account paying a constant rate of interest r and in a stock modeled by the geometric Brownian motion

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t). \tag{4.5.1}$$

Suppose at each time t, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$, $t \geq 0$. The remainder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market account.

The differential $dX(t)$ for the investor's portfolio value at each time t is due to two factors, the capital gain $\Delta(t)\,dS(t)$ on the stock position and the interest earnings $r\big(X(t) - \Delta(t)S(t)\big)\,dt$ on the cash position. In other words,

$$\begin{aligned} dX(t) &= \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt \\ &= \Delta(t)\big(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\big) + r\big(X(t) - \Delta(t)S(t)\big)\,dt \\ &= rX(t)\,dt + \Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t). \end{aligned} \tag{4.5.2}$$

The three terms appearing in the last line of (4.5.2) can be understood as follows:

(i) an average underlying rate of return r on the portfolio, which is reflected by the term $rX(t)\,dt$,
(ii) a risk premium $\alpha - r$ for investing in the stock, which is reflected by the term $\Delta(t)(\alpha - r)S(t)\,dt$, and
(iii) a volatility term proportional to the size of the stock investment, which is the term $\Delta(t)\sigma S(t)\,dW(t)$.

The discrete-time analogue of equation (4.5.2) appears in Chapter 1 of Volume I as (1.2.12):

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n).$$

We may rearrange terms in this equation to obtain

$$X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n), \tag{4.5.3}$$

which is analogous to the first line of (4.5.2), except in (4.5.3) time steps forward one unit at a time, whereas in (4.5.2) time moves forward continuously.

See Exercise 4.10 for additional discussion of the rationale for equation (4.5.2) in option pricing.

We shall often consider the discounted stock price $e^{-rt}S(t)$ and the discounted portfolio value of an agent, $e^{-rt}X(t)$. According to the Itô-Doeblin formula with $f(t,x) = e^{-rt}x$, the differential of the discounted stock price is

$$\begin{aligned}
&d\big(e^{-rt}S(t)\big)\\
&= df\big(t,S(t)\big)\\
&= f_t\big(t,S(t)\big)\,dt + f_x\big(t,S(t)\big)\,dS(t) + \frac{1}{2}f_{xx}\big(t,S(t)\big)\,dS(t)\,dS(t)\\
&= -re^{-rt}S(t)\,dt + e^{-rt}\,dS(t)\\
&= (\alpha - r)e^{-rt}S(t)\,dt + \sigma e^{-rt}S(t)\,dW(t),
\end{aligned} \tag{4.5.4}$$

and the differential of the discounted portfolio value is

$$\begin{aligned}
&d\big(e^{-rt}X(t)\big)\\
&= df\big(t,X(t)\big)\\
&= f_t\big(t,X(t)\big)\,dt + f_x\big(t,X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t,X(t)\big)\,dX(t)\,dX(t)\\
&= -re^{-rt}X(t)\,dt + e^{-rt}\,dX(t)\\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)\,dt + \Delta(t)\sigma e^{-rt}S(t)\,dW(t)\\
&= \Delta(t)\,d\big(e^{-rt}S(t)\big).
\end{aligned} \tag{4.5.5}$$

Discounting the stock price reduces the mean rate of return from α, the term multiplying $S(t)\,dt$ in (4.5.1), to $\alpha - r$, the term multiplying $e^{-rt}S(t)\,dt$ in (4.5.4). Discounting the portfolio value removes the underlying rate of return r; compare the last line of (4.5.2) to the next-to-last line of (4.5.5). The last line of (4.5.5) shows that change in the discounted portfolio value is solely due to change in the discounted stock price.

4.5.2 Evolution of Option Value

Consider a European call option that pays $\big(S(T)-K\big)^+$ at time T. The strike price K is some nonnegative constant. Black, Scholes, and Merton argued that the value of this call at any time should depend on the time (more precisely, on the time to expiration) and on the value of the stock price at that time, and of course it should also depend on the model parameters r and σ and the contractual strike price K. Only two of these quantities, time and stock price, are variable. Following this reasoning, we let $c(t,x)$ denote the value of the call at time t if the stock price at that time is $S(t) = x$. There is nothing random about the function $c(t,x)$. However, the value of the option is random; it is the stochastic process $c\big(t,S(t)\big)$ obtained by replacing the dummy variable x by the random stock price $S(t)$ in this function. At the initial time, we do not know the future stock prices $S(t)$ and hence do not know the future option values $c(t, S(t))$. Our goal is to determine the function $c(t,x)$ so we at least have a formula for the future option values in terms of the future stock prices.

We begin by computing the differential of $c(t, S(t))$. According to the Itô-Doeblin formula, it is

$$\begin{aligned}
&dc(t,S(t)) \\
&= c_t(t,S(t))\,dt + c_x(t,S(t))\,dS(t) + \frac{1}{2}c_{xx}(t,S(t))\,dS(t)\,dS(t) \\
&= c_t(t,S(t))\,dt + c_x(t,S(t))\big(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\big) \\
&\qquad + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)\,dt \\
&= \left[c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right] dt \\
&\qquad + \sigma S(t)c_x(t,S(t))\,dW(t).
\end{aligned} \tag{4.5.6}$$

We next compute the differential of the discounted option price $e^{-rt}c(t,S(t))$. Let $f(t,x) = e^{-rt}x$. According to the Itô-Doeblin formula,

$$\begin{aligned}
&d\big(e^{-rt}c(t,S(t))\big) \\
&\quad = df\big(t, c(t,S(t))\big) \\
&\quad = f_t\big(t,c(t,S(t))\big)\,dt + f_x\big(t,c(t,S(t))\big)\,dc(t,S(t)) \\
&\qquad\quad + \frac{1}{2}f_{xx}\big(t,c(t,S(t))\big)\,dc(t,S(t))\,dc(t,S(t)) \\
&\quad = -re^{-rt}c(t,S(t))\,dt + e^{-rt}\,dc(t,S(t)) \\
&\quad = e^{-rt}\Big[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) \\
&\qquad\quad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\Big]\,dt + e^{-rt}\sigma S(t)c_x(t,S(t))dW(t).
\end{aligned} \tag{4.5.7}$$

4.5.3 Equating the Evolutions

A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account so that the portfolio value $X(t)$ at each time $t \in [0,T]$ agrees with $c(t,S(t))$. This happens if and only if $e^{-rt}X(t) = e^{-rt}c(t,S(t))$ for all t. One way to ensure this equality is to make sure that

$$d\big(e^{-rt}X(t)\big) = d\big(e^{-rt}c(t,S(t))\big) \text{ for all } t \in [0,T) \tag{4.5.8}$$

and $X(0) = c(0,S(0))$. Integration of (4.5.8) from 0 to t then yields

$$e^{-rt}X(t) - X(0) = e^{-rt}c(t,S(t)) - c(0,S(0)) \text{ for all } t \in [0,T). \tag{4.5.9}$$

If $X(0) = c(0,S(0))$, then we can cancel this term in (4.5.9) and get the desired equality.

Comparing (4.5.5) and (4.5.7), we see that (4.5.8) holds if and only if

$$\begin{aligned}
&\Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t)\\
&= \left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt\\
&\qquad + \sigma S(t)c_x(t,S(t))\,dW(t).
\end{aligned} \tag{4.5.10}$$

We examine what is required in order for (4.5.10) to hold.

We first equate the $dW(t)$ terms in (4.5.10), which gives

$$\Delta(t) = c_x(t,S(t)) \text{ for all } t \in [0,T). \tag{4.5.11}$$

This is called the *delta-hedging rule.* At each time t prior to expiration, the number of shares held by the hedge of the short option position is the partial derivative with respect to the stock price of the option value at that time. This quantity, $c_x(t,S(t))$, is called the *delta* of the option.

We next equate the dt terms in (4.5.10), using (4.5.11), to obtain

$$\begin{aligned}
&(\alpha - r)S(t)c_x(t,S(t))\\
&= -rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\\
&\qquad\qquad \text{for all } t \in [0,T).
\end{aligned} \tag{4.5.12}$$

The term $\alpha S(t)c_x(t,S(t))$ appears on both sides of (4.5.12), and after canceling it, we obtain

$$\begin{aligned}
rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))&\\
\text{for all } t \in [0,T).&
\end{aligned} \tag{4.5.13}$$

In conclusion, we should seek a continuous function $c(t,x)$ that is a solution to the *Black-Scholes-Merton partial differential equation*

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \text{ for all } t \in [0,T),\ x \geq 0, \tag{4.5.14}$$

and that satisfies the *terminal condition*

$$c(T,x) = (x-K)^+. \tag{4.5.15}$$

Suppose we have found this function. If an investor starts with initial capital $X(0) = c(0,S(0))$ and uses the hedge $\Delta(t) = c_x(t,S(t))$, then (4.5.10) will hold for all $t \in [0,T)$. Indeed, the $dW(t)$ terms on the left and right sides of (4.5.10) agree because $\Delta(t) = c_x(t,S(t))$, and the dt terms agree because (4.5.14) guarantees (4.5.13). Equality in (4.5.10) gives us (4.5.9). Canceling $X(0) = c(0,S(0))$ and e^{-rt} in this equation, we see that $X(t) = c(t,S(t))$ for all $t \in [0,T)$. Taking the limit as $t \uparrow T$ and using the fact that both $X(t)$ and

$c(t, S(t))$ are continuous, we conclude that $X(T) = c(T, S(T)) = \big(S(T)-K\big)^+$. This means that the short position has been successfully hedged. No matter which of its possible paths the stock price follows, when the option expires, the agent hedging the short position has a portfolio whose value agrees with the option payoff.

4.5.4 Solution to the Black-Scholes-Merton Equation

The Black-Scholes-Merton equation (4.5.14) does not involve probability. It is a partial differential equation, and the arguments t and x are dummy variables, not random variables. One can solve it by partial differential equation methods. In this section, however, rather than showing how to solve the equation, we shall simply present the solution and check that it works. In Subsection 5.2.5, we present a derivation of this solution based on probability theory.

We want the Black-Scholes-Merton equation to hold for all $x \geq 0$ and $t \in [0, T)$ so that (4.5.14) will hold regardless of which of its possible paths the stock price follows. If the initial stock price is positive, then the stock price is always positive, and it can take any positive value. If the initial stock price is zero, then subsequent stock prices are all zero. We cover both of these cases by asking (4.5.14) to hold for all $x \geq 0$. We do not need (4.5.14) to hold at $t = T$, although we need the function $c(t, x)$ to be continuous at $t = T$. If the hedge works at all times strictly prior to T, it also works at time T because of continuity.

Equation (4.5.14) is a partial differential equation of the type called *backward parabolic*. For such an equation, in addition to the terminal condition (4.5.15), one needs boundary conditions at $x = 0$ and $x = \infty$ in order to determine the solution. The boundary condition at $x = 0$ is obtained by substituting $x = 0$ into (4.5.14), which then becomes

$$c_t(t, 0) = rc(t, 0). \tag{4.5.16}$$

This is an ordinary differential equation for the function $c(t, 0)$ of t, and the solution is

$$c(t, 0) = e^{rt} c(0, 0).$$

Substituting $t = T$ into this equation and using the fact that $c(T, 0) = (0 - K)^+ = 0$, we see that $c(0, 0) = 0$ and hence

$$c(t, 0) = 0 \text{ for all } t \in [0, T]. \tag{4.5.17}$$

This is the *boundary condition at* $x = 0$.

As $x \to \infty$, the function $c(t, x)$ grows without bound. In such a case, we give the boundary condition at $x = \infty$ by specifying the rate of growth. One way to specify a *boundary condition at* $x = \infty$ for the European call is

$$\lim_{x \to \infty} \big[c(t, x) - \big(x - e^{-r(T-t)} K\big)\big] = 0 \text{ for all } t \in [0, T]. \tag{4.5.18}$$

In particular, $c(t, x)$ grows at the same rate as x as $x \to \infty$. Recall that $c(t, x)$ is the value at time t of a call on a stock whose price at time t is x. For large x, this call is deep in the money and very likely to end in the money. In this case, the price of the call is almost as much as the price of the forward contract discussed in Subsection 4.5.6 below (see (4.5.26)). This is the assertion of (4.5.18).

The solution to the Black-Scholes-Merton equation (4.5.14) with terminal condition (4.5.15) and boundary conditions (4.5.17) and (4.5.18) is

$$c(t, x) = xN\big(d_+(T-t, x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t, x)\big), \ 0 \le t < T, \ x > 0, \tag{4.5.19}$$

where

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right], \tag{4.5.20}$$

and N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz. \tag{4.5.21}$$

We shall sometimes use the notation

$$\mathrm{BSM}(\tau, x; K, r, \sigma) = xN\big(d_+(\tau, x)\big) - Ke^{-r\tau}N\big(d_-(\tau, x)\big), \tag{4.5.22}$$

and call $\mathrm{BSM}(\tau, x; K, r, \sigma)$ the *Black-Scholes-Merton function.* In this formula, τ and x denote the time to expiration and the current stock price, respectively. The parameters K, r, and σ are the strike price, the interest rate, and the stock volatility, respectively.

Formula (4.5.19) does not define $c(t, x)$ when $t = T$ (because then $\tau = T - t = 0$ and this appears in the denominator in (4.5.20)), nor does it define $c(t, x)$ when $x = 0$ (because $\log x$ appears in (4.5.20), and $\log 0$ is not a real number). However, (4.5.19) defines $c(t, x)$ in such a way that $\lim_{t\to T} c(t, x) = (x - K)^+$ and $\lim_{x \downarrow 0} c(t, x) = 0$. Verification of all of these claims is given as Exercise 4.9.

4.5.5 The Greeks

The derivatives of the function $c(t, x)$ of (4.5.19) with respect to various variables are called the *Greeks.* Two of these are derived in Exercise 4.9, namely *delta*, which is

$$c_x(t, x) = N\big(d_+(T-t, x)\big), \tag{4.5.23}$$

and *theta*, which is

$$c_t(t, x) = -rKe^{-r(T-t)}N\big(d_-(T-t, x)\big) - \frac{\sigma x}{2\sqrt{T-t}}N'\big(d_+(T-t, x)\big). \tag{4.5.24}$$

Because both N and N' are always positive, delta is always positive and theta is always negative. Another of the Greeks is *gamma*, which is

$$c_{xx}(t,x) = N'\big(d_+(T-t,x)\big)\frac{\partial}{\partial x}d_+(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'\big(d_+(T-t,x)\big). \tag{4.5.25}$$

Like delta, gamma is always positive.

In order to simplify notation in the following discussion, we sometimes suppress the arguments (t,x) of $c(t,x)$ and $(T-t,x)$ of $d_\pm(T-t,x)$. If at time t the stock price is x, then the short option hedge of (4.5.11) calls for holding $c_x(t,x)$ shares of stock, a position whose value is $xc_x = xN(d_+)$. The hedging portfolio value is $c = xN(d_+) - Ke^{-r(T-t)}N(d_-)$, and since $xc_x(t,x)$ of this value is invested in stock, the amount invested in the money market must be

$$c(t,x) - xc_x(t,x) = -Ke^{-r(T-t)}N(d_-),$$

a negative number. To hedge a short position in a call option, one must borrow money. To hedge a long position in a call option, one does the opposite. In other words, to hedge a long call position one should hold $-c_x$ shares of stock (i.e., have a short position in stock) and invest $Ke^{-r(T-t)}N(d_-)$ in the money market account.

Because delta and gamma are positive, for fixed t, the function $c(t,x)$ is increasing and convex in the variable x, as shown in Figure 4.5.1. Suppose at time t the stock price is x_1 and we wish to take a long position in the option and hedge it. We do this by purchasing the option for $c(t,x_1)$, shorting $c_x(t,x_1)$ shares of stock, which generates income $x_1c_x(t,x_1)$, and investing the difference,

$$M = x_1c_x(t,x_1) - c(t,x_1),$$

in the money market account. We wish to consider the sensitivity to stock price changes of the portfolio that has these three components: long option, short stock, and long money market account. The initial portfolio value

$$c(t,x_1) - x_1c_x(t,x_1) + M$$

is zero at the moment t when we set up these positions.

If the stock price were to instantaneously fall to x_0 as shown in Figure 4.5.1 and we do not change our positions in the stock or money market account, then the value of the option we hold would fall to $c(t,x_0)$ and the liability due to our short position in stock would decrease to $x_0c_x(t,x_1)$. Our total portfolio value, including M in the money market account, would be

$$c(t,x_0) - x_0c_x(t,x_1) + M = c(t,x_0) - c_x(t,x_1)(x_0 - x_1) - c(t,x_1).$$

This is the difference at x_0 between the curve $y = c(t,x)$ and the straight line $y = c_x(t,x_1)(x-x_1) + c(t,x_1)$ in Figure 4.5.1. Because this difference is positive, our portfolio benefits from an instantaneous drop in the stock price.

On the other hand, if the stock price were to instantaneously rise to x_2 and we do not change our positions in the stock or money market account, then the value of the option would rise to $c(t,x_2)$ and the liability due to our short position in stock would increase to $x_2 c_x(t, x_1)$. Our total portfolio value, including M in the money market account, would be

$$c(t, x_2) - x_2 c_x(t, x_1) + M = c(t, x_2) - c_x(t, x_1)(x_2 - x_1) - c(t, x_1).$$

This is the difference at x_2 between the curve $y = c(t, x)$ and the straight line $y = c_x(t, x_1)(x - x_1) + c(t, x_1)$ in Figure 4.5.1. This difference is positive, so our portfolio benefits from an instantaneous rise in the stock price.

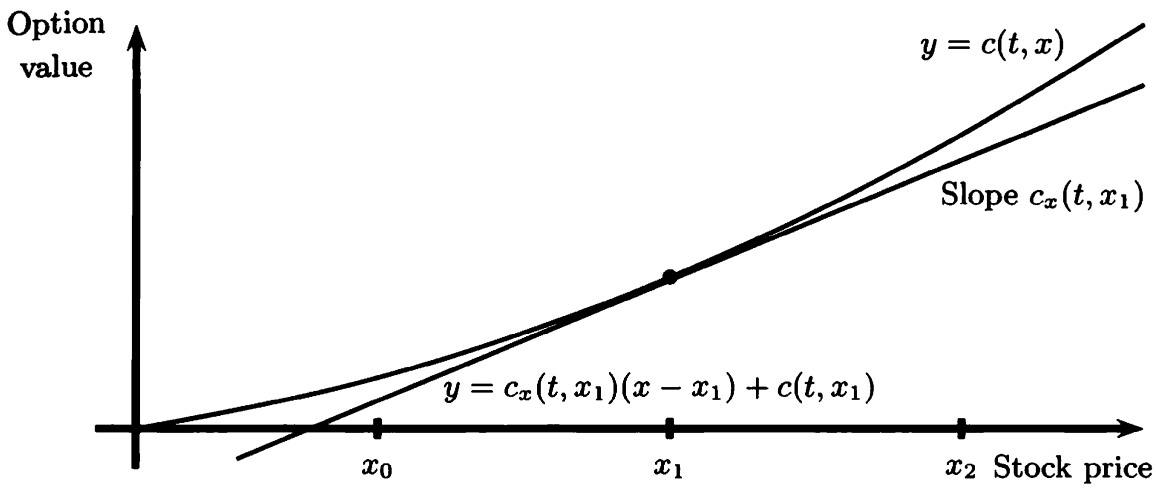

Fig. 4.5.1. Delta-neutral position.

The portfolio we have set up is said to be *delta-neutral* and *long gamma*. The portfolio is long gamma because it benefits from the convexity of $c(t, x)$ as described above. If there is an instantaneous rise or an instantaneous fall in the stock price, the value of the portfolio increases. A long gamma portfolio is profitable in times of high stock volatility.

"Delta-neutral" refers to the fact that the line in Figure 4.5.1 is tangent to the curve $y = c(t, x)$. Therefore, when the stock price makes a small move, the change of portfolio value due to the corresponding change in option price is nearly offset by the change in the value of our short position in the stock. The straight line is a good approximation to the option price *for small stock price moves*. If the straight line were steeper than the option price curve at the starting point x_1, then we would be *short delta*; an upward move in the stock price would hurt the portfolio because the liability from the short position in stock would rise faster than the value of the option. On the other hand, a downward move would increase the portfolio value because the option price would fall more slowly than the rate of decrease in the liability from the short stock position. Unless a trader has a view on the market, he tries to set up portfolios that are delta-neutral. If he expects high volatility, he would at the same time try to choose the portfolio to be long gamma.

The portfolio described above may at first appear to offer an arbitrage opportunity. When we let time move forward, not only does the long gamma position offer an opportunity for profit, but the positive investment in the money market account enhances this opportunity. The drawback is that theta, the derivative of $c(t,x)$ with respect to time, is negative. As we move forward in time, the curve $y = c(t,x)$ is shifting downward. Figure 4.5.1 is misleading because it is drawn with t fixed. In principle, the portfolio can lose money because the curve $c(t,x)$ shifts downward more rapidly than the money market investment and the long gamma position generate income. The essence of the hedging argument in Subsection 4.5.3 is that if the stock really is a geometric Brownian motion and we have determined the right value of the volatility σ, then so long as we continuously rebalance our portfolio, all these effects exactly cancel!

Of course, assets are not really geometric Brownian motions with constant volatility, but the argument above gives a good first approximation to reality. It also highlights volatility as the key parameter. In fact, the mean rate of return α of the stock does not appear in the Black-Scholes-Merton equation (4.5.14). From the point of view of no-arbitrage pricing, it is irrelevant how likely the stock is to go up or down because a delta-neutral position is a hedge against both possibilities. What matters is how much volatility the stock has, for we need to know the amount of profit that can be made from the long gamma position. The more volatile stocks offer more opportunity for profit from the portfolio that hedges a long call position with a short stock position, and hence the call is more expensive. The derivative of the option price with respect to the volatility σ is called *vega*, and it is positive. As volatility increases, so do option prices in the Black-Scholes-Merton model.

4.5.6 Put–Call Parity

A *forward contract* with delivery price K obligates its holder to buy one share of the stock at expiration time T in exchange for payment K. At expiration, the value of the forward contract is $S(T) - K$. Let $f(t,x)$ denote the value of the forward contract at earlier times $t \in [0,T]$ if the stock price at time t is $S(t) = x$.

We argue that the value of a forward contract is given by

$$f(t,x) = x - e^{-r(T-t)}K. \tag{4.5.26}$$

If an agent sells this forward contract at time zero for $f(t,S(0)) = S(0) - e^{-rT}K$, he can set up a *static hedge*, a hedge that does not trade except at the initial time, in order to protect himself. Specifically, the agent should purchase one share of stock. Since he has initial capital $S(0) - e^{-rT}K$ from the sale of the forward contract, this requires that he borrow $e^{-rT}K$ from the money market account. The agent makes no further trades. At expiration of the forward contract, he owns one share of stock and his debt to the money market account has grown to K, so his portfolio value is $S(T) - K$, exactly the value of the forward contract. Because the agent has been able to replicate the payoff of the forward contract with a portfolio whose value at each time t is

$S(t) - e^{-r(T-t)}K$, this must be the value at each time of the forward contract. This is $f(t, S(t))$, where $f(t,x)$ is defined by (4.5.26).

The *forward price* of a stock at time t is defined to be the value of K that causes the forward contract at time t to have value zero (i.e., it is the value of K that satisfies the equation $S(t) - e^{-r(T-t)}K = 0$). Hence, we see that in a model with a constant interest rate, the forward price at time t is

$$\text{For}(t) = e^{r(T-t)}S(t). \tag{4.5.27}$$

Note that the forward price is not the price (or value) of a forward contract. For $0 \le t \le T$, the forward price at time t is the price one can lock in at time t for the purchase of one share of stock at time T, paying the price (*settling*) at time T. No money changes hands at the time the price is locked in.

Let us consider this situation at time $t = 0$. At that time, one can lock in a price $\text{For}(0) = e^{rT}S(0)$ for purchase of the stock at time T. Let us do this, which means we set $K = e^{rT}S(0)$ in (4.5.26). The value of this forward contract is zero at time $t = 0$, but as soon as time begins to move forward, the value of the forward contract changes. Indeed, its value at time t is

$$f(t, S(t)) = S(t) - e^{rt}S(0).$$

Finally, let us consider a *European put*, which pays off $(K - S(T))^+$ at time T. We observe that for any number x, the equation

$$x - K = (x-K)^+ - (K-x)^+ \tag{4.5.28}$$

holds. Indeed, if $x \ge K$, then $(x-K)^+ = x - K$ and $(K-x)^+ = 0$. On the other hand, if $x \le K$, then $(x-K)^+ = 0$ and $-(K-x)^+ = -(K-x) = x - K$. In either case, the right-hand side of (4.5.28) equals the left-hand side. We denote by $p(t,x)$ the value of the European put at time t if the time-t stock price is $S(t) = x$. Similarly, we denote by $c(t,x)$ the value of the European call expiring at time T with strike price K and by $f(t,x)$ the value of the forward contract for the purchase of one share of stock at time T in exchange for payment K. Equation (4.5.28) implies

$$f(T, S(T)) = c(T, S(T)) - p(T, S(T));$$

the payoff of the forward contract agrees with the payoff of a portfolio that is long one call and short one put. Since the value at time T of the forward contract agrees with the value of the portfolio that is long one call and short one put, these values must agree at all previous times:

$$f(t,x) = c(t,x) - p(t,x), \quad x \ge 0,\ 0 \le t \le T. \tag{4.5.29}$$

If this were not the case, one could at some time t either sell or buy the portfolio that is long the forward, short the call, and long the put, realizing an instant profit, and have no liability upon expiration of the contracts. The relationship (4.5.29) is called *put–call parity*.

Note that we have derived the put–call parity formula (4.5.29) without appealing to the Black-Scholes-Merton model of a geometric Brownian motion for the stock price and a constant interest rate. Indeed, without any assumptions on the prices except sufficient liquidity that permits one to form the portfolio that is long one call and short one put, we have put–call parity. If we make the assumption of a constant interest rate r, then $f(t,x)$ is given by (4.5.26). If we make the additional assumption that the stock is a geometric Brownian motion with constant volatility $\sigma > 0$, then we have also the Black-Scholes-Merton call formula (4.5.19). We can then solve (4.5.29) to obtain the Black-Scholes-Merton put formula

$$\begin{aligned} p(t,x) &= x\big(N(d_+(T-t,x)) - 1\big) - Ke^{-r(T-t)}\big(N(d_-(T-t,x)) - 1\big) \\ &= Ke^{-r(T-t)}N(-d_-(T-t,x)) - xN(-d_+(T-t,x)), \qquad (4.5.30) \end{aligned}$$

where $d_\pm(T-t,x)$ is given by (4.5.20).

4.6 Multivariable Stochastic Calculus

4.6.1 Multiple Brownian Motions

Definition 4.6.1. *A* d-dimensional Brownian motion *is a process*

$$W(t) = (W_1(t), \dots, W_d(t))$$

with the following properties.

(i) Each $W_i(t)$ is a one-dimensional Brownian motion.
(ii) If $i \neq j$, then the processes $W_i(t)$ and $W_j(t)$ are independent.

Associated with a d-dimensional Brownian motion, we have a filtration $\mathcal{F}(t)$, $t \geq 0$, such that the following holds.

(iii) **(Information accumulates)** *For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.*
(iv) **(Adaptivity)** *For each $t \geq 0$, the random vector $W(t)$ is $\mathcal{F}(t)$-measurable.*
(v) **(Independence of future increments)** *For $0 \leq t < u$, the vector of increments $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.*

Although we have defined a multidimensional Brownian motion to be a vector of *independent* one-dimensional Brownian motions, we shall see in Example 4.6.6 how to build correlated Brownian motions from this.

Because each component W_i of a d-dimensional Brownian motion is a one-dimensional Brownian motion, we have the quadratic variation formula $[W_i, W_i](t) = t$, which we write informally as

$$dW_i(t)\, dW_i(t) = dt.$$

However, if $i \neq j$, we shall see that independence of W_i and W_j implies $[W_i, W_j](t) = 0$, which we write informally as

$$dW_i(t)\,dW_j(t) = 0, \quad i \neq j.$$

We justify this claim.

Let $\Pi = \{t_0, \ldots, t_n\}$ be a partition of $[0,T]$. For $i \neq j$, define the *sampled cross variation* of W_i and W_j on $[0,T]$ to be

$$C_\Pi = \sum_{k=0}^{n-1} \big[W_i(t_{k+1}) - W_i(t_k)\big]\big[W_j(t_{k+1}) - W_j(t_k)\big].$$

The increments appearing on the right-hand side of the equation above are all independent of one another and all have mean zero. Therefore, $\mathbb{E}C_\Pi = 0$.

We compute $\mathrm{Var}(C_\Pi)$. Note first that

$$\begin{aligned} C_\Pi^2 = &\sum_{k=0}^{n-1} \big[W_i(t_{k+1}) - W_i(t_k)\big]^2\big[W_j(t_{k+1}) - W_j(t_k)\big]^2 \\ &+2\sum_{\ell<k}^{n-1} \big[W_i(t_{\ell+1}) - W_i(t_\ell)\big]\big[W_j(t_{\ell+1}) - W_j(t_\ell)\big] \\ &\qquad\cdot\big[W_i(t_{k+1}) - W_i(t_k)\big]\big[W_j(t_{k+1}) - W_j(t_k)\big]. \end{aligned}$$

All the increments appearing in the sum of cross-terms are independent of one another and all have mean zero. Therefore,

$$\mathrm{Var}(C_\Pi) = \mathbb{E}C_\Pi^2 = \mathbb{E}\sum_{k=0}^{n-1} \big[W_i(t_{k+1}) - W_i(t_k)\big]^2\big[W_j(t_{k+1}) - W_j(t_k)\big]^2.$$

But $\big[W_i(t_{k+1}) - W_i(t_k)\big]^2$ and $\big[W_j(t_{k+1}) - W_j(t_k)\big]^2$ are independent of one another, and each has expectation $(t_{k+1} - t_k)$. It follows that

$$\mathrm{Var}(C_\Pi) = \sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \leq \|\Pi\| \cdot \sum_{k=0}^{n-1}(t_{k+1} - t_k) = \|\Pi\| \cdot T.$$

As $\|\Pi\| \to 0$, we have $\mathrm{Var}(C_\Pi) \to 0$, so C_Π converges to the constant $\mathbb{E}C_\Pi = 0$.

4.6.2 Itô-Doeblin Formula for Multiple Processes

To keep the notation as simple as possible, we write the Itô formula for two processes driven by a two-dimensional Brownian motion. In the obvious way, the formula generalizes to any number of processes driven by a Brownian motion of any number (not necessarily the same number) of dimensions.

Let $X(t)$ and $Y(t)$ be Itô processes, which means they are processes of the form

$$X(t) = X(0) + \int_0^t \Theta_1(u)\,du + \int_0^t \sigma_{11}(u)\,dW_1(u) + \int_0^t \sigma_{12}(u)\,dW_2(u),$$
$$Y(t) = Y(0) + \int_0^t \Theta_2(u)\,du + \int_0^t \sigma_{21}(u)\,dW_1(u) + \int_0^t \sigma_{22}(u)\,dW_2(u).$$

The integrands $\Theta_i(u)$ and $\sigma_{ij}(u)$ are assumed to be adapted processes. In differential notation, we write

$$dX(t) = \Theta_1(t)\,dt + \sigma_{11}(t)\,dW_1(t) + \sigma_{12}(t)\,dW_2(t), \tag{4.6.1}$$
$$dY(t) = \Theta_2(t)\,dt + \sigma_{21}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t). \tag{4.6.2}$$

The Itô integral $\int_0^t \sigma_{11}(u)\,dW_1(u)$ accumulates quadratic variation at rate $\sigma_{11}^2(t)$ per unit time, and the Itô integral $\int_0^t \sigma_{12}(u)\,dW_2(u)$ accumulates quadratic variation at rate $\sigma_{12}^2(t)$ per unit time. Because both of these integrals appear in $X(t)$, the process $X(t)$ accumulates quadratic variation at rate $\sigma_{11}^2(t) + \sigma_{12}^2(t)$ per unit time:

$$[X,X](t) = \int_0^t \left(\sigma_{11}^2(u) + \sigma_{12}^2(u)\right) du.$$

We may write this equation in differential form as

$$dX(t)\,dX(t) = \left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right) dt. \tag{4.6.3}$$

One can informally derive (4.6.3) by squaring (4.6.1) and using the multiplication rules

$$dt\,dt = 0,\ \ dt\,dW_i(t) = 0,\ \ dW_i(t)\,dW_i(t) = dt,\ \ dW_i(t)\,dW_j(t) = 0 \text{ for } i \neq j.$$

In a similar way, we may derive the differential formulas

$$dY(t)\,dY(t) = \left(\sigma_{21}^2(t) + \sigma_{22}^2(t)\right) dt, \tag{4.6.4}$$
$$dX(t)\,dY(t) = \left(\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\right) dt. \tag{4.6.5}$$

Equation (4.6.5) says that, for every $T \geq 0$,

$$[X,Y](T) = \int_0^T \left(\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\right) dt. \tag{4.6.6}$$

The term $[X,Y](T)$ on the left-hand side is defined as follows. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,T]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = T$) and set up the sampled cross variation

$$\sum_{k=0}^{n-1} \left[X(t_{k+1}) - X(t_k)\right]\left[Y(t_{k+1}) - Y(t_k)\right]. \tag{4.6.7}$$

Now let the number of partition points n go to infinity as the length of the longest subinterval $\|\Pi\| = \max_{0\leq k\leq n-1}(t_{k+1} - t_k)$ goes to zero. The limit of the sum in (4.6.7) is $[X,Y](T)$. This limit is given by the right-hand side of (4.6.6). The proof of this assertion is similar to the proof of Lemma 4.4.4, with the additional feature that we must use the fact that $[W_1, W_2](t) = 0$. We omit the details.

The following theorem generalizes the Itô-Doeblin formula of Theorem 4.4.6. The justification, which we omit, is similar to that of Theorem 4.4.6.

Theorem 4.6.2 (Two-dimensional Itô-Doeblin formula). *Let $f(t,x,y)$ be a function whose partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy} are defined and are continuous. Let $X(t)$ and $Y(t)$ be Itô processes as discussed above. The* two-dimensional Itô-Doeblin formula in differential form *is*

$$\begin{aligned} &df\big(t,X(t),Y(t)\big)\\ &=f_t\big(t,X(t),Y(t)\big)\,dt + f_x\big(t,X(t),Y(t)\big)\,dX(t) + f_y\big(t,X(t),Y(t)\big)\,dY(t)\\ &\qquad+\frac{1}{2}f_{xx}\big(t,X(t),Y(t)\big)\,dX(t)\,dX(t) + f_{xy}\big(t,X(t),Y(t)\big)\,dX(t)\,dY(t)\\ &\qquad+\frac{1}{2}f_{yy}\big(t,X(t),Y(t)\big)\,dY(t)\,dY(t). \end{aligned} \tag{4.6.8}$$

Before discussing formula (4.6.8), we rewrite it, leaving out t wherever possible, to obtain the same formula in the more compact notation

$$\begin{aligned} df(t,X,Y) = {}& f_t\,dt + f_x\,dX + f_y\,dY\\ &+\frac{1}{2}f_{xx}\,dX\,dX + f_{xy}\,dX\,dY + \frac{1}{2}f_{yy}\,dY\,dY. \end{aligned} \tag{4.6.9}$$

The right-hand side of (4.6.9) is the Taylor series expansion of f out to second order. The full expansion would have the additional second-order terms $f_{tt}\,dt\,dt$, $\frac{1}{2}f_{tx}\,dt\,dX$, and $\frac{1}{2}f_{ty}\,dt\,dY$, but $dt\,dt$, $dt\,dX$, and $dt\,dY$ are zero. The Taylor series expansion actually has two mixed partial terms, $\frac{1}{2}f_{xy}\,dX\,dY$ and $\frac{1}{2}f_{yx}\,dY\,dX$. For functions f whose second partial derivatives exist and are continuous, $f_{xy} = f_{yx}$, and so we have combined these terms into the single term $f_{xy}\,dX\,dY$ in (4.6.9).

The differentials dX, dY, $dX\,dX$, $dX\,dY$, and $dY\,dY$ appearing in (4.6.9) are given by (4.6.1)–(4.6.5). Making these substitutions and then integrating (4.6.9), we obtain the Itô-Doeblin formula in integral form:

$$\begin{aligned} &f(t,X(t),Y(t)) - f(0,X(0),Y(0))\\ &= \int_0^t \Big[\sigma_{11}(u)f_x(u,X(u),Y(u)) + \sigma_{21}(u)f_y(u,X(u),Y(u))\Big]\,dW_1(u)\\ &\quad+\int_0^t \Big[\sigma_{12}(u)f_x(u,X(u),Y(u)) + \sigma_{22}(u)f_y(u,X(u),Y(u))\Big]\,dW_2(u) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \Big[f_t(u,X(u),Y(u))\\
&\quad+\Theta_1(u)f_x(u,X(u),Y(u))+\Theta_2(u)f_y(u,X(u),Y(u))\\
&\quad+\frac{1}{2}(\sigma_{11}^2(u)+\sigma_{12}^2(u))f_{xx}(u,X(u),Y(u))\\
&\quad+(\sigma_{11}(u)\sigma_{21}(u)+\sigma_{12}(u)\sigma_{22}(u))f_{xy}(u,X(u),Y(u))\\
&\quad+\frac{1}{2}(\sigma_{21}^2(u)+\sigma_{22}^2(u))f_{yy}(u,X(u),Y(u))\Big]\,du. \qquad (4.6.10)
\end{aligned}
$$

The right-hand side of this equation has one ordinary (Lebesgue) integral with respect to du and two Itô integrals, one with respect to $dW_1(u)$ and the other with respect to $dW_2(u)$. All terms have precise mathematical meanings. This equation demonstrates why it is preferable to work with differential notation, such as in (4.6.9).

Corollary 4.6.3 (Itô product rule). *Let $X(t)$ and $Y(t)$ be Itô processes. Then*

$$d\big(X(t)Y(t)\big)=X(t)\,dY(t)+Y(t)\,dX(t)+dX(t)\,dY(t).$$

PROOF: In (4.6.9), take $f(t,x,y)=xy$, so that $f_t=0$, $f_x=y$, $f_y=x$, $f_{xx}=0$, $f_{xy}=1$, and $f_{yy}=0$. □

4.6.3 Recognizing a Brownian Motion

A Brownian motion $W(t)$ is a martingale with continuous paths whose quadratic variation is $[W,W](t)=t$. It turns out that these conditions characterize Brownian motion in the sense of the following theorem.

Theorem 4.6.4 (Lévy, one dimension). *Let $M(t)$, $t\geq 0$, be a martingale relative to a filtration $\mathcal{F}(t)$, $t\geq 0$. Assume that $M(0)=0$, $M(t)$ has continuous paths, and $[M,M](t)=t$ for all $t\geq 0$. Then $M(t)$ is a Brownian motion.*

IDEA OF THE PROOF: A Brownian motion is a martingale whose increments are normally distributed. The surprising feature of Lévy's Theorem is that the assumptions do not say anything about normality, and yet implicit in the conclusion is the assertion that $M(t)$ is normally distributed.

The method used to establish normality is to first check that in the derivation of the Itô-Doeblin formula, Theorem 4.4.1, for Brownian motion, the only properties of Brownian motion that were used are assumed in this theorem: a continuous process with quadratic variation $[M,M](t)=t$. Therefore, the Itô-Doeblin formula may be applied to M with the result that, for any function $f(t,x)$ whose derivatives exist and are continuous,

$$df(t,M(t))=f_t(t,M(t))\,dt+f_x(t,M(t))\,dM(t)+\frac{1}{2}f_{xx}(t,M(t))\,dt. \qquad (4.6.11)$$

The last term uses the fact that $dM(t)\,dM(t) = dt$. In integrated form, (4.6.11) is

$$f(t, M(t)) = f(0, M(0)) + \int_0^t \left[f_t(s, M(s)) + \frac{1}{2} f_{xx}(s, M(s))\right] ds + \int_0^t f_x(s, M(s))\, dM(s). \tag{4.6.12}$$

Because $M(t)$ is a martingale, the stochastic integral $\int_0^t f_x(s, M(s))\, dM(s)$ is also. (See Exercise 4.1 for the case of a simple integrand; the general case follows from this exercise upon passage to the limit.) At $t = 0$, this stochastic integral takes the value zero, and so its expectation is always zero. Taking expectations in (4.6.12), we obtain

$$\mathbb{E}f(t, M(t)) = f(0, M(0)) + \mathbb{E}\int_0^t \left[f_t(s, M(s)) + \frac{1}{2} f_{xx}(s, M(s))\right] ds. \tag{4.6.13}$$

We fix a number u and define

$$f(t, x) = \exp\left\{ux - \frac{1}{2}u^2 t\right\}.$$

Then $f_t(t, x) = -\frac{1}{2}u^2 f(t, x)$, $f_x(t, x) = uf(t, x)$, and $f_{xx}(t, x) = u^2 f(t, x)$. In particular,

$$f_t(t, x) + \frac{1}{2} f_{xx}(t, x) = 0.$$

For this function $f(t, x)$, the second term on the right-hand side of (4.6.13) is zero, and that equation becomes

$$\mathbb{E}\exp\left\{uM(t) - \frac{1}{2}u^2 t\right\} = 1.$$

In other words, we have the moment-generating function formula

$$\mathbb{E}e^{uM(t)} = e^{\frac{1}{2}u^2 t}.$$

This is the moment-generating function for the normal distribution with mean zero and variance t (see (3.2.13)). Hence, that is the distribution that $M(t)$ must have. □

The idea used to justify Theorem 4.6.4 can be combined with the two-dimensional Itô-Doeblin formula used to show independence. In particular, we have the following two-dimensional version of Lévy's Theorem.

Theorem 4.6.5 (Lévy, two dimensions). *Let $M_1(t)$ and $M_2(t)$, $t \geq 0$, be martingales relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. Assume that for $i = 1, 2$, we have $M_i(0) = 0$, $M_i(t)$ has continuous paths, and $[M_i, M_i](t) = t$ for all $t \geq 0$. If, in addition, $[M_1, M_2](t) = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions.*

IDEA OF THE PROOF: The one-dimensional Lévy Theorem, Theorem 4.6.4, implies that M_1 and M_2 are Brownian motions. To show independence, we examine the joint moment-generating function.

Let $f(t,x,y)$ be a function whose derivatives are defined and continuous. The two-dimensional Itô-Doeblin formula implies that

$$\begin{aligned} df(t,M_1,M_2) &= f_t\,dt + f_x\,dM_1 + f_y\,dM_2 \\ &\qquad +\frac{1}{2}f_{xx}\,dM_1\,dM_1 + f_{xy}\,dM_1\,dM_2 + f_{yy}\,dM_2\,dM_2 \\ &= f_t\,dt + f_x\,dM_1 + f_y\,dM_2 + \frac{1}{2}f_{xx}\,dt + \frac{1}{2}f_{yy}\,dt, \end{aligned}$$

where we have used the assumptions $[M_1,M_1](t)=t$, $[M_2,M_2](t)=t$, and $[M_1,M_2](t)=$ in their equivalent form $dM_1(t)\,dM_1(t)=dt$, $dM_2(t)\,dM_2(t)=dt$, and $dM_1(t)\,dM_2(t)=0$. We integrate both sides to obtain

$$\begin{aligned} &f(t,M_1(t),M_2(t)) \\ &= f(0,M_1(0),M_2(0)) + \int_0^t \Big[f_t(s,M_1(s),M_2(s)) + \frac{1}{2}f_{xx}(s,M_1(s),M_2(s)) \\ &\qquad\qquad +\frac{1}{2}f_{yy}(s,M_1(s),M_2(s))\Big]\,ds \\ &\quad + \int_0^t f_x(s,M_1(s),M_2(s))\,dM_1(s) + \int_0^t f_y(x,M_1(s),M_2(s))\,dM_2(s). \end{aligned}$$

The last two terms on the right-hand side are martingales, starting at zero at time zero, and hence having expectation zero. Therefore,

$$\begin{aligned} &\mathbb{E}f(t,M_1(t),M_2(t)) \\ &= f(0,M_1(0),M_2(0)) + \mathbb{E}\int_0^t \Big[f_t(s,M_1(s),M_2(s)) + \frac{1}{2}f_{xx}(s,M_1(s),M_2(s)) \\ &\qquad\qquad +\frac{1}{2}f_{yy}(s,M_1(s),M_2(s))\Big]\,ds. \qquad (4.6.14) \end{aligned}$$

We now fix numbers u_1 and u_2 and define

$$f(t,x,y) = \exp\left\{u_1x + u_2y - \frac{1}{2}(u_1^2+u_2^2)t\right\}.$$

Then $f_t(t,x,y) = -\frac{1}{2}(u_1^2+u_2^2)f(t,x,y)$, $f_x(t,x,y) = u_1f(t,x,y)$, $f_y(t,x,y) = u_2f(t,x,y)$. $f_{xx}(t,x,y) = u_1^2f(t,x,y)$, and $f_{yy}(t,x,y) = u_2^2f(t,x,y)$. For this function $f(t,x,y)$, the second term on the right-hand side of (4.6.14) is zero. We conclude that

$$\mathbb{E}\exp\left\{u_1M_1(t) + u_2M_2(t) - \frac{1}{2}(u_1^2+u_2^2)t\right\} = 1,$$

which gives us the moment-generating function formula

$$\mathbb{E}e^{u_1 M_1(t)+u_2 M_2(t)} = e^{\frac{1}{2}u_1^2 t} \cdot e^{\frac{1}{2}u_2^2 t}.$$

Because the joint moment-generating function factors into the product of moment-generating functions, $M_1(t)$ and $M_2(t)$ must be independent. □

Example 4.6.6 (Correlated stock prices). Suppose

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1\,dt + \sigma_1\,dW_1(t),$$
$$\frac{dS_2(t)}{S_2(t)} = \alpha_2\,dt + \sigma_2\Big[\rho\,dW_1(t) + \sqrt{1-\rho^2}\,dW_2(t)\Big],$$

where $W_1(t)$ and $W_2(t)$ are *independent* Brownian motions and $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 \le \rho \le 1$ are constant. To analyze the second stock price process, we define

$$W_3(t) = \rho W_1(t) + \sqrt{1-\rho^2}\,W_2(t).$$

Then $W_3(t)$ is a continuous martingale with $W_3(0) = 0$, and

$$\begin{aligned} dW_3(t)\,dW_3(t) &= \rho^2\,dW_1(t)\,dW_1(t) + 2\rho\sqrt{1-\rho^2}\,dW_1(t)\,dW_2(t) \\ &\qquad + (1-\rho^2)\,dW_2(t)\,dW_2(t) \\ &= \rho^2\,dt + \big(1-\rho^2\big)\,dt \;=\; dt. \end{aligned}$$

In other words, $[W_3, W_3](t) = t$. According to the one-dimensional Lévy Theorem, Theorem 4.6.4, $W_3(t)$ is a Brownian motion. Because we can write the differential of $S_2(t)$ as

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2\,dt + \sigma_2\,dW_3(t),$$

we see that $S_2(t)$ is a geometric Brownian motion with mean rate of return α_2 and volatility σ_2.

The Brownian motions $W_1(t)$ and $W_3(t)$ are correlated. According to Itô's product rule (Corollary 4.6.3),

$$\begin{aligned} d\big(W_1(t)W_3(t)\big) &= W_1(t)\,dW_3(t) + W_3(t)\,dW_1(t) + dW_1(t)\,dW_3(t) \\ &= W_1(t)\,dW_3(t) + W_3(t)\,dW_1(t) + \rho\,dt. \end{aligned}$$

Integrating, we obtain

$$W_1(t)W_3(t) = \int_0^t W_1(s)\,dW_3(s) + \int_0^t W_3(s)\,dW_1(s) + \rho t.$$

The Itô integrals on the right-hand side have expectation zero, so the covariance of $W_1(t)$ and $W_3(t)$ is

$$\mathbb{E}\big[W_1(t)W_3(t)\big] = \rho t.$$

Because both $W_1(t)$ and $W_3(t)$ have standard deviation $\sqrt{t}$, the number ρ is the correlation between $W_1(t)$ and $W_3(t)$. The case of nonconstant correlation ρ is presented in Exercise 4.17. □

4.7 Brownian Bridge

We conclude this chapter with a the discussion of the Brownian bridge. This is a stochastic process that is like a Brownian motion except that with probability one it reaches a specified point at a specified positive time. We first discuss Gaussian processes in general, the class to which the Brownian bridge belongs, and we then define the Brownian bridge and present its properties. The primary use for the Brownian bridge in finance is as an aid to Monte Carlo simulation. We make no use of it in this text.

4.7.1 Gaussian Processes

Definition 4.7.1. *A* Gaussian process $X(t)$, $t \geq 0$, *is a stochastic process that has the property that, for arbitrary times* $0 < t_1 < t_2 < \cdots < t_n$, *the random variables* $X(t_1), X(t_2), \ldots X(t_n)$ *are jointly normally distributed.*

The joint normal distribution of a set of vectors is determined by their means and covariances. Therefore, for a Gaussian process, the joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ is determined by the means and covariances of these random variables. We denote the mean of $X(t)$ by $m(t)$, and, for $s \geq 0$, $t \geq 0$, we denote the covariance of $X(s)$ and $X(t)$ by $c(s,t)$; i.e.,

$$m(t) = \mathbb{E}X(t), \quad c(s,t) = \mathbb{E}\big[\big(X(s) - m(s)\big)\big(X(t) - m(t)\big)\big].$$

Example 4.7.2 (Brownian motion). Brownian motion $W(t)$ is a Gaussian process. For $0 < t_1 < t_2 < \cdots < t_n$, the increments

$$I_1 = W(t_1),\ I_2 = W(t_2) - W(t_1),\ \ldots,\ I_n = W(t_n) - W(t_{n-1})$$

are independent and normally distributed. Writing

$$W(t_1) = I_1,\ W(t_2) = \sum_{j=1}^{2} I_j,\ \ldots,\ W(t_n) = \sum_{j=1}^{n} I_j,$$

we see that the random variables $W(t_1), W(t_2), \ldots, W(t_n)$ are jointly normally distributed. These random variables are *not* independent. It is the *increments* of Brownian motion that are independent. Of course, the mean function for Brownian motion is

$$m(t) = \mathbb{E}W(t) = 0.$$

We may compute the covariance by letting $0 \leq s \leq t$ be given and noting that

$$\begin{aligned} c(s,t) &= \mathbb{E}\big[W(s)W(t)\big] \\ &= \mathbb{E}\big[W(s)\big(W(t) - W(s) + W(s)\big)\big] \\ &= \mathbb{E}\big[W(s)\big(W(t) - W(s)\big)\big] + \mathbb{E}\big[W^2(s)\big]. \end{aligned}$$

Because $W(s)$ and $W(t) - W(s)$ are independent and both have mean zero, we see that $\mathbb{E}\big[W(s)\big(W(t) - W(s)\big)\big] = 0$. The other term, $\mathbb{E}\big[W^2(s)\big]$, is the variance of $W(s)$, which is s. We conclude that $c(s,t) = s$ when $0 \le s \le t$. Reversing the roles of s and t, we conclude that $c(s,t) = t$ when $0 \le t \le s$. In general, the covariance function for Brownian motion is then

$$c(s,t) = s \wedge t,$$

where $s \wedge t$ denotes the minimum of s and t. □

Example 4.7.3 (Itô integral of a deterministic integrand). Let $\Delta(t)$ be a nonrandom function of time, and define

$$I(t) = \int_0^t \Delta(s)\, dW(s),$$

where $W(t)$ is a Brownian motion. Then $I(t)$ is a Gaussian process, as we now show.

In the proof of Theorem 4.4.9, we showed that, for fixed $u \in \mathbb{R}$, the process

$$M_u(t) = \exp\left\{ uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)\, ds \right\}$$

is a martingale. We used this fact to argue that

$$1 = M_u(0) = \mathbb{E}M_u(t) = e^{-\frac{1}{2}u^2 \int_0^t \Delta^2(s)\, ds} \cdot \mathbb{E}e^{uI(t)},$$

and we thus obtained the moment-generating function formula

$$\mathbb{E}e^{uI(t)} = e^{\frac{1}{2}u^2 \int_0^t \Delta^2(s)\, ds}. \tag{4.7.1}$$

The right-hand side is the moment generating function for a normal random variable with mean zero and variance $\int_0^t \Delta^2(s)\, ds$. Therefore, this is the distribution of $I(t)$.

Although we have shown that $I(t)$ is normally distributed, verification that the process is Gaussian requires more. We must verify that, for $0 < t_1 < t_2 < \cdots < t_n$, the random variables $I(t_1), I(t_2), \ldots, I(t_n)$ are *jointly* normally distributed. It turns out that the increments

$$I(t_1) - I(0) = I(t_1),\ I(t_2) - I(t_1), \ldots, I(t_n) - I(t_{n-1})$$

are normally distributed and independent, and from this the joint normality of $I(t_1), I(t_2), \ldots, I(t_n)$ follows by the same argument as used in Example 4.7.2 for Brownian motion.

We show that, for $0 < t_1 < t_2$, the two random increments $I(t_1) - I(0) = I(t_1)$ and $I(t_2) - I(t_1)$ are normally distributed and independent. The argument we provide can be iterated to prove this result for any number of increments. For fixed $u_2 \in \mathbb{R}$, the martingale property of M_{u_2} implies that

$$M_{u_2}(t_1) = \mathbb{E}[M_{u_2}(t_2)|\mathcal{F}(t_1)].$$

Now let $u_1 \in \mathbb{R}$ be fixed. Because $\frac{M_{u_1}(t_1)}{M_{u_2}(t_1)}$ is $\mathcal{F}(t_1)$-measurable, we may multiply the equation above by this quotient to obtain

$$\begin{aligned}
M_{u_1}(t_1) &= \mathbb{E}\Big[\frac{M_{u_1}(t_1)M_{u_2}(t_2)}{M_{u_2}(t_1)}\Big|\mathcal{F}(t_1)\Big] \\
&= \mathbb{E}\left[\exp\Big\{u_1 I(t_1) + u_2\big(I(t_2) - I(t_1)\big) - \frac{1}{2}u_1^2\int_0^{t_1} \Delta^2(s)\,ds\right. \\
&\qquad\left. -\frac{1}{2}u_2^2\int_{t_1}^{t_2} \Delta^2(s)\,ds\Big\}\Big|\mathcal{F}(t_1)\right].
\end{aligned}$$

We now take expectations

$$\begin{aligned}
1 &= M_{u_1}(0) \\
&= \mathbb{E}M_{u_1}(t_1) \\
&= \mathbb{E}\left[\exp\Big\{u_1 I(t_1) + u_2\big(I(t_2) - I(t_1)\big) - \frac{1}{2}u_1^2\int_0^{t_1} \Delta^2(s)\,ds\right. \\
&\qquad\left. -\frac{1}{2}u_2^2\int_{t_1}^{t_2} \Delta^2(s)\,ds\Big\}\right] \\
&= \mathbb{E}\left[\exp\big\{u_1 I(t_1) + u_2\big(I(t_2) - I(t_1)\big)\big\}\right] \\
&\qquad\cdot\exp\Big\{-\frac{1}{2}u_1^2\int_0^{t_1} \Delta^2(s)\,ds - \frac{1}{2}u_2^2\int_{t_1}^{t_2} \Delta^2(s)\,ds\Big\},
\end{aligned}$$

where we have used the fact that $\Delta^2(s)$ is nonrandom to take the integrals of $\Delta^2(s)$ outside the expectation on the right-hand side. This leads to the moment-generating function formula

$$\begin{aligned}
&\mathbb{E}\left[\exp\big\{u_1 I(t_1) + u_2\big(I(t_2) - I(t_1)\big)\big\}\right] \\
&\qquad= \exp\Big\{\frac{1}{2}u_1^2\int_0^{t_1} \Delta^2(s)\,ds\Big\}\cdot\exp\Big\{\frac{1}{2}u_2^2\int_{t_1}^{t_2} \Delta^2(s)\,ds\Big\}.
\end{aligned}$$

The right-hand side is the product of the moment-generating function for a normal random variable with mean zero and variance $\int_0^{t_1} \Delta^2(s)\,ds$ and the moment-generating function for a normal random variable with mean zero and variance $\int_{t_1}^{t_2} \Delta^2(s)\,ds$. It follows that $I(t_1)$ and $I(t_2) - I(t_1)$ must have these distributions, and because their joint moment-generating function factors into this product of moment-generating functions, they must be independent.

The covariance of $I(t_1)$ and $I(t_2)$ can be computed using the same trick as in Example 4.7.2 for the covariance of Brownian motion. We have

$$
\begin{aligned}
c(t_1, t_2) &= \mathbb{E}\big[I(t_1)I(t_2)\big] \\
&= \mathbb{E}\big[I(t_1)\big(I(t_2) - I(t_1) + I(t_1)\big)\big] \\
&= \mathbb{E}\big[I(t_1)\big(I(t_2) - I(t_1)\big)\big] + \mathbb{E}I^2(t_1) \\
&= \mathbb{E}I(t_1) \cdot \mathbb{E}\big[I(t_2) - I(t_1)\big] + \int_0^{t_1} \Delta^2(s)\,ds \\
&= \int_0^{t_1} \Delta^2(s)\,ds.
\end{aligned}
$$

For the general case where $s \geq 0$ and $t \geq 0$ and we do not know the relationship between s and t, we have the covariance formula

$$c(s,t) = \int_0^{s \wedge t} \Delta^2(u)\,du. \qquad \square$$

4.7.2 Brownian Bridge as a Gaussian Process

Definition 4.7.4. *Let $W(t)$ be a Brownian motion. Fix $T > 0$. We define the* Brownian bridge from 0 to 0 on $[0,T]$ *to be the process*

$$X(t) = W(t) - \frac{t}{T}W(T), \ 0 \leq t \leq T. \tag{4.7.2}$$

Note that $\frac{t}{T}W(T)$ as a function of t is the line from $(0,0)$ to $\big(T, W(T)\big)$. In (4.7.2), we have subtracted this line away from the Brownian motion $W(t)$, so that the resulting process $X(t)$ satisfies

$$X(0) = X(T) = 0.$$

Because $W(T)$ enters the definition of $X(t)$ for $0 \leq t \leq T$, the Brownian bridge $X(t)$ is not adapted to the filtration $\mathcal{F}(t)$ generated by $W(t)$. We shall later obtain a different process that has the same distribution as the process $X(t)$ but is adapted to this filtration.

For $0 < t_1 < t_2 < \cdots < t_n < T$, the random variables

$$X(t_1) = W(t_1) - \frac{t_1}{T}W(T), \ldots, X(t_n) = W(t_n) - \frac{t_n}{T}W(T)$$

are jointly normal because $W(t_1), \ldots, W(t_n), W(T)$ are jointly normal. Hence, the Brownian bridge from 0 to 0 is a Gaussian process. Its mean function is easily seen to be

$$m(t) = \mathbb{E}X(t) = \mathbb{E}\Big[W(t) - \frac{t}{T}W(T)\Big] = 0.$$

For $s, t \in (0,T)$, we compute the covariance function

$$
\begin{aligned}
&c(s,t)\\
&= \mathbb{E}\left[\left(W(s) - \frac{s}{T}W(T)\right)\left(W(t) - \frac{t}{T}W(T)\right)\right]\\
&= \mathbb{E}[W(s)W(t)] - \frac{t}{T}\mathbb{E}[W(s)W(T)] - \frac{s}{T}\mathbb{E}[W(t)W(T)] + \frac{st}{T^2}\mathbb{E}W^2(T)\\
&= s \wedge t - \frac{2st}{T} + \frac{st}{T} \;=\; s \wedge t - \frac{st}{T}.
\end{aligned}
\tag{4.7.3}
$$

Definition 4.7.5. *Let $W(t)$ be a Brownian motion. Fix $T > 0$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$. We define the* Brownian bridge from a to b on $[0,T]$ *to be the process*

$$X^{a\to b}(t) = a + \frac{(b-a)t}{T} + X(t),\ 0 \le t \le T,$$

where $X(t) = X^{0\to 0}$ is the Brownian bridge from 0 to 0 of Definition 4.7.4.

The function $a + \frac{(b-a)t}{T}$, as a function of t, is the line from $(0,a)$ to (T,b). When we add this line to the Brownian bridge from 0 to 0 on $[0,T]$, we obtain a process that begins at a at time 0 and ends at b at time T. Adding a nonrandom function to a Gaussian process gives us another Gaussian process. The mean function is affected:

$$m^{a\to b}(t) = \mathbb{E}X^{a\to b}(t) = a + \frac{(b-a)t}{T}.$$

However, the covariance function is not affected:

$$c^{a\to b}(s,t) = \mathbb{E}\left[\left(X^{a\to b}(s) - m^{a\to b}(s)\right)\left(X^{a\to b}(t) - m^{a\to b}(t)\right)\right] = s \wedge t - \frac{st}{T}.$$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

We cannot write the Brownian bridge as a stochastic integral of a deterministic integrand because the variance of the Brownian bridge,

$$\mathbb{E}X^2(t) = c(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T},$$

increases for $0 \le t \le \frac{T}{2}$ and then decreases for $\frac{T}{2} \le t \le T$. In Example 4.7.3, the variance of $I(t) = \int_0^t \Delta(u)\,dW(u)$ is $\int_0^t \Delta^2(u)\,du$, which is nondecreasing in t. However, we can obtain a process with the same distribution as the Brownian bridge from 0 to 0 as a scaled stochastic integral. In particular, consider

$$Y(t) = (T-t)\int_0^t \frac{1}{T-u}\,dW(u),\ 0 \le t < T. \tag{4.7.4}$$

The integral

$$I(t) = \int_0^t \frac{1}{T-u}\,dW(u)$$

is a Gaussian process of the type discussed in Example 4.7.3, provided $t < T$ so the integrand is defined. For $0 < t_1 < t_2 < \cdots < t_n < T$, the random variables

$$Y(t_1) = (T - t_1)I(t_1),\ Y(t_2) = (T - t_2)I(t_2), \ldots, Y(t_n) = (T - t_n)I(t_n)$$

are jointly normal because $I(t_1), I(t_2), \ldots, I(t_n)$ are jointly normal. In particular, Y is a Gaussian process.

The mean and covariance functions of I are

$$m^I(t) = 0,$$
$$c^I(s,t) = \int_0^{s\wedge t} \frac{1}{(T-u)^2}\,du = \frac{1}{T - s\wedge t} - \frac{1}{T} \text{ for all } s,t \in [0,T).$$

This means that the mean function for Y is $m^Y(t) = 0$. To compute the covariance function for Y, we assume for the moment that $0 \le s \le t < T$ so that

$$c^I(s,t) = \frac{1}{T-s} - \frac{1}{T} = \frac{s}{T(T-s)}.$$

Then

$$\begin{aligned} c^Y(s,t) &= \mathbb{E}\left[(T-s)(T-t)I(s)I(t)\right] \\ &= (T-s)(T-t)\frac{s}{T(T-s)} \\ &= \frac{(T-t)s}{T} \\ &= s - \frac{st}{T}. \end{aligned}$$

If we had taken $0 \le t \le s < T$, the roles of s and t would have been reversed. In general,

$$c^Y(s,t) = s\wedge t - \frac{st}{T} \text{ for all } s,t \in [0,T). \tag{4.7.5}$$

This is the same covariance formula (4.7.3) we obtained for the Brownian bridge. Because the mean and covariance functions for a Gaussian process completely determine the distribution of the process, we conclude that the process Y has the same distribution as the Brownian bridge from 0 to 0 on $[0,T]$.

We now consider the variance

$$\mathbb{E}Y^2(t) = c^Y(t,t) = \frac{t(T-t)}{T},\ 0 < t < T.$$

Note that, as $t \uparrow T$, this variance converges to 0. In other words, as $t \uparrow T$, the random process $Y(t)$, which always has mean zero, has a variance that converges to zero. We did not initially define $Y(T)$, but this observation suggests that it makes sense to define $Y(T) = 0$. If we do that, then $Y(t)$ is continuous at $t = T$. We summarize this discussion with the following theorem.

Theorem 4.7.6. *Define the process*

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u}\,dW(u) & \text{for } 0 \le t < T, \\ 0 & \text{for } t = T. \end{cases}$$

Then $Y(t)$ is a continuous Gaussian process on $[0,T]$ and has mean and covariance functions

$$m^Y(t) = 0, \ t \in [0,T],$$
$$c^Y(s,t) = s \wedge t - \frac{st}{T} \text{ for all } s,t \in [0,T].$$

In particular, the process $Y(t)$ has the same distribution as the Brownian bridge from 0 to 0 on $[0,T]$ (Definition 4.7.5).

We note that the process $Y(t)$ is adapted to the filtration generated by the Brownian motion $W(t)$. It is interesting to compute the stochastic differential of $Y(t)$, which is

$$\begin{aligned} dY(t) &= \int_0^t \frac{1}{T-u}\,dW(u)\cdot d(T-t) + (T-t)\cdot d\int_0^t \frac{1}{T-u}\,dW(u) \\ &= -\int_0^t \frac{1}{T-u}\,dW(u)\cdot dt + dW(t) \\ &= -\frac{Y(t)}{T-t}dt + dW(t). \end{aligned}$$

If $Y(t)$ is positive as t approaches T, the drift term $-\frac{Y(t)}{T-t}dt$ becomes large in absolute value and is negative. This drives $Y(t)$ toward zero. On the other hand, if $Y(t)$ is negative, the drift term becomes large and positive, and this again drives $Y(t)$ toward zero. This strongly suggests, and it is indeed true, that as $t \uparrow T$ the process $Y(t)$ converges to zero almost surely.

4.7.4 Multidimensional Distribution of the Brownian Bridge

We fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and let $X^{a\to b}(t)$ denote the Brownian bridge from a to b on $[0,T]$. We also fix $0 = t_0 < t_1 < t_2 < \cdots < t_n < T$. In this section, we compute the joint density of $X^{a\to b}(t_1), \dots, X^{a\to b}(t_n)$.

We recall that the Brownian bridge from a to b has the mean function

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)a}{T} + \frac{bt}{T}$$

and covariance function

$$c(s,t) = s \wedge t - \frac{st}{T}.$$

When $s \le t$, we may write this as

$$c(s,t) = s - \frac{st}{T} = \frac{s(T-t)}{T}, \; 0 \le s \le t \le T.$$

To simplify notation, we set $\tau_j = T - t_j$ so that $\tau_0 = T$. We define random variables

$$Z_j = \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}.$$

Because $X^{a\to b}(t_1), \ldots, X^{a\to b}(t_n)$ are jointly normal, so are $Z(t_1), \ldots, Z(t_n)$. We compute

$$\begin{aligned}
\mathbb{E}Z_j &= \frac{1}{\tau_j}\mathbb{E}X^{a\to b}(t_j) - \frac{1}{\tau_j}\mathbb{E}X^{a\to b}(t_{j+1}) \\
&= \frac{a}{T} + \frac{bt_j}{T\tau_j} - \frac{a}{T} - \frac{bt_{j-1}}{T\tau_{j-1}} \\
&= \frac{bt_j(T-t_{j-1}) - bt_{j-1}(T-t_j)}{T\tau_j\tau_{j-1}} \\
&= \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j-1}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Var}(Z_j) &= \frac{1}{\tau_j^2}\mathrm{Var}\big(X^{a\to b}(t_j)\big) - \frac{2}{\tau_j\tau_{j-1}}\mathrm{Cov}\big(X^{a\to b}(t_j), X^{a\to b}(t_{j-1})\big) \\
&\quad + \frac{1}{\tau_{j-1}^2}\mathrm{Var}\big(X^{a\to b}(t_{j-1})\big) \\
&= \frac{1}{\tau_j^2}c(t_j,t_j) - \frac{2}{\tau_j\tau_{j-1}}c(t_j,t_{j-1}) + \frac{1}{\tau_{j-1}^2}c(t_{j-1},t_{j-1}) \\
&= \frac{t_j}{T\tau_j} - \frac{2t_{j-1}}{T\tau_{j-1}} + \frac{t_{j-1}}{T\tau_{j-1}} \\
&= \frac{t_j(T-t_{j-1}) - 2t_{j-1}(T-t_j) + t_{j-1}(T-t_j)}{T\tau_j\tau_{j-1}} \\
&= \frac{t_j - t_{j-1}}{\tau_j\tau_{j-1}}.
\end{aligned}$$

Finally, we compute the covariance of Z_i and Z_j when $i < j$. We obtain

$$\begin{aligned}
\mathrm{Cov}(Z_i, Z_j) &= \frac{1}{\tau_i\tau_j}c(t_i,t_j) - \frac{1}{\tau_i\tau_{j-1}}c(t_i,t_{j-1}) - \frac{1}{\tau_{i-1}\tau_j}c(t_{i-1},t_j) \\
&\quad + \frac{1}{\tau_{i-1}\tau_{j-1}}c(t_{i-1},t_{j-1}) \\
&= \frac{t_i(T-t_j)}{T\tau_i\tau_j} - \frac{t_i(T-t_{j-1})}{T\tau_i\tau_{j-1}} - \frac{t_{i-1}(T-t_j)}{T\tau_{i-1}\tau_j} + \frac{t_{i-1}(T-t_{j-1})}{T\tau_{i-1}\tau_{j-1}} \\
&= 0.
\end{aligned}$$

We conclude that the normal random variables $Z_1, \ldots, Z_n$ are independent, and we can write down their joint density, which is

$$\begin{aligned}
&f_{Z(t_1),\ldots,Z(t_n)}(z_1,\ldots,z_n)\\
&\quad= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{t_j - t_{j-1}}{\tau_j\tau_{j-1}}}} \exp\left\{ -\frac{1}{2}\cdot \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j-1}}\right)^2}{\frac{t_j - t_{j-1}}{\tau_j\tau_{j-1}}} \right\}\\
&\quad= \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j-1}}\right)^2}{\frac{t_j - t_{j-1}}{\tau_j\tau_{j-1}}} \right\} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{t_j - t_{j-1}}{\tau_j\tau_{j-1}}}}.
\end{aligned}$$

We make the change of variables

$$z_j = \frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}}, \quad j = 1, \ldots, n,$$

where $x_0 = a$, to find the joint density for $X^{a\to b}(t_1), \ldots, X^{a\to b}(t_n)$. We work first on the sum in the exponent to see the effect of this change of variables. We have

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j-1}}\right)^2}{\frac{t_j - t_{j-1}}{\tau_j\tau_{j-1}}}\\
&= \sum_{j=1}^{n} \frac{\tau_j\tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} - \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j-1}} \right)^2\\
&= \sum_{j=1}^{n} \frac{\tau_j\tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j^2}{\tau_j^2} + \frac{x_{j-1}^2}{\tau_{j-1}^2} + \frac{b^2(t_j - t_{j-1})^2}{\tau_j^2\tau_{j-1}^2} - \frac{2x_jx_{j-1}}{\tau_j\tau_{j-1}} \right.\\
&\qquad\qquad \left. - \frac{2x_jb(t_j - t_{j-1})}{\tau_j^2\tau_{j-1}} + \frac{2x_{j-1}b(t_j - t_{j-1})}{\tau_j\tau_{j-1}^2} \right)\\
&= \sum_{j=1}^{n} \left(\frac{\tau_{j-1}x_j^2}{\tau_j(t_j - t_{j-1})} + \frac{\tau_j x_{j-1}^2}{\tau_{j-1}(t_j - t_{j-1})} + \frac{b^2(t_j - t_{j-1})}{\tau_j\tau_{j-1}} - \frac{2x_jx_{j-1}}{t_j - t_{j-1}} \right.\\
&\qquad\qquad \left. - \frac{2x_jb}{\tau_j} + \frac{2x_{j-1}b}{\tau_{j-1}} \right)\\
&= \sum_{j=1}^{n} \left[\frac{x_j^2}{t_j - t_{j-1}} \left(1 + \frac{\tau_{j-1} - \tau_j}{\tau_j}\right) + \frac{x_{j-1}^2}{t_j - t_{j-1}} \left(1 - \frac{\tau_{j-1} - \tau_j}{\tau_{j-1}}\right) \right.\\
&\qquad\qquad \left. - \frac{2x_jx_{j-1}}{t_j - t_{j-1}} \right] + b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right).
\end{aligned}$$

Now

$$\tau_{j-1} - \tau_j = (T - t_{j-1}) - (T - t_j) = t_j - t_{j-1},$$

and so this last expression is equal to

$$\sum_{j=1}^{n} \left[\frac{x_j^2 - 2x_j x_{j-1} + x_{j-1}^2}{t_j - t_{j-1}} \right] + \sum_{j=1}^{n} \left(\frac{x_j^2}{\tau_j} - \frac{x_{j-1}^2}{\tau_{j-1}} \right)$$
$$+b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right)$$
$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \frac{x_n^2}{T - t_n} - \frac{a^2}{T} + b^2 \left(\frac{1}{T - t_n} - \frac{1}{T} \right)$$
$$-2b \left(\frac{x_n}{T - t_n} - \frac{a}{T} \right)$$
$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \frac{(b - x_n)^2}{T - t_n} - \frac{(b - a)^2}{T}.$$

In conclusion, when we change variables from z_j to x_j, we have the equation

$$\exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_j - 1)}{\tau_j \tau_{j-1}} \right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}} \right\}$$
$$= \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b - a)^2}{2T} \right\}.$$

To change a density, we also need to account for the Jacobian of the change of variables. In this case, we have

$$\frac{\partial z_j}{\partial x_j} = \frac{1}{\tau_j},\ j = 1, \ldots, n,$$
$$\frac{\partial z_j}{\partial x_{j-1}} = -\frac{1}{\tau_{j-1}}, j = 2, \ldots, n,$$

and all other partial derivatives are zero. This leads to the Jacobian matrix

$$J = \begin{bmatrix} \frac{1}{\tau_1} & 0 & \cdots & 0 \\ -\frac{1}{\tau_1} & \frac{1}{\tau_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tau_n} \end{bmatrix},$$

whose determinant is $\prod_{j=1}^{n} \frac{1}{\tau_j}$. Multiplying $f_{Z(t_1),\ldots,Z(t_n)}(z_1, \ldots, z_n)$ by this determinant and using the change of variables worked out above, we obtain the density for $X^{a \to b}(t_1), \ldots, X^{a \to b}(t_n)$,

$$
\begin{aligned}
&f_{X^{a\to b}(t_1),\dots,X^{a\to b}(t_n)}(x_1,\dots,x_n)\\
&=\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\sqrt{\frac{\tau_{j-1}}{\tau_j}}\\
&\qquad\cdot\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}-\frac{(b-x_n)^2}{2(T-t_n)}+\frac{(b-a)^2}{2T}\right\}\\
&=\sqrt{\frac{T}{T-t_n}}\cdot\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\\
&\qquad\cdot\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}-\frac{(b-x_n)^2}{2(T-t_n)}+\frac{(b-a)^2}{2T}\right\}\\
&=\frac{p(T-t_n,x_n,b)}{p(T,a,b)}\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j),
\end{aligned}
\tag{4.7.6}
$$

where

$$p(\tau,x,y)=\frac{1}{\sqrt{2\pi\tau}}\exp\left\{-\frac{(y-x)^2}{2\tau}\right\}$$

is the transition density for Brownian motion.

4.7.5 Brownian Bridge as a Conditioned Brownian Motion

The joint density (4.7.6) for $X^{a\to b}(t_1),\dots,X^{a\to b}(t_n)$ permits us to give one more interpretation for the Brownian bridge from a to b on $[0,T]$. It is a Brownian motion $W(t)$ on this time interval, starting at $W(0)=a$ and conditioned to arrive at b at time T (i.e., conditioned on $W(T)=b$). Let $0=t_0<t_1<t_2<\cdots<t_n<T$ be given. The joint density of $W(t_1),\dots,W(t_n),W(T)$ is

$$f_{W(t_1),\dots,W(t_n),W(T)}(x_1,\dots,x_n,b)=p(T-t_n,x_n,b)\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j),\tag{4.7.7}$$

where $W(0)=x_0=a$. This is because $p(t_1-t_0,x_0,x_1)=p(t_1,a,x_1)$ is the density for the Brownian motion going from $W(0)=a$ to $W(t_1)=x_1$ in the time between $t=0$ and $t=t_1$. Similarly, $p(t_2-t_1,x_1,x_2)$ is the density for going from $W(t_1)=x_1$ to $W(t_2)=x_2$ between time $t=t_1$ and $t=t_2$. The joint density for $W(t_1)$ and $W(t_2)$ is then the product

$$p(t_1,a,x_1)p(t_2-t_1,x_1,x_2).$$

Continuing in this way, we obtain the joint density (4.7.7). The marginal density of $W(T)$ is $p(T,a,b)$. The density of $W(t_1),\dots,W(t_n)$ conditioned on $W(T)=b$ is thus the quotient

$$\frac{p(T-t_n,x_n,b)}{p(T,a,b)}\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j),$$

and this is $f_{X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)}(x_1,\ldots,x_n)$ of (4.7.6).

Finally, let us define

$$M^{a\to b}(T)=\max_{0\le t\le T}X^{a\to b}(t)$$

to be the maximum value obtained by the Brownian bridge from a to b on $[0,T]$. This random variable has the following distribution.

Corollary 4.7.7. *The density of $M^{a\to b}(T)$ is*

$$f_{M^{a\to b}(T)}(y)=\frac{2(2y-b-a)}{T}e^{-\frac{2}{T}(y-a)(y-b)},\quad y>\max\{a,b\}. \tag{4.7.8}$$

PROOF: Because the Brownian bridge from 0 to w on $[0,T]$ is a Brownian motion conditioned on $W(T)=w$, the maximum of $X^{0\to w}$ on $[0,T]$ is the maximum of W on $[0,T]$ conditioned on $W(T)=w$. Therefore, the density of $M^{0\to w}(T)$ was computed in Corollary 3.7.4 and is

$$f_{M^{0\to w}(T)}(m)=\frac{2(2m-w)}{T}e^{-\frac{2m(m-w)}{T}},\quad w<m,m>0. \tag{4.7.9}$$

The density of $f_{M^{a\to b}(T)}(y)$ can be obtained by translating from the initial condition $W(0)=a$ to $W(0)=0$ and using (4.7.9). In particular, in (4.7.9) we replace m by $y-a$ and replace w by $b-a$. This results in (4.7.8). □

4.8 Summary

Let $W(t)$ be a Brownian motion and $\Delta(t)$ a stochastic process adapted to the filtration of the Brownian motion. The Itô integral

$$I(t)=\int_0^t\Delta(u)\,dW(u) \tag{4.8.1}$$

is a martingale. Because it is zero at time $t=0$, its expectation is zero for all t. Its variance is given by *Itô's isometry*

$$\mathbb{E}I^2(t)=\mathbb{E}\int_0^t\Delta^2(u)\,du. \tag{4.8.2}$$

The quadratic variation accumulated by the Itô integral up to time t is

$$[I,I](t)=\int_0^t\Delta^2(u)\,du. \tag{4.8.3}$$

These assertions appear in Theorem 4.3.1. Note that the quadratic variation (4.8.3) is computed path-by-path and the result may depend on the path, whereas the variance (4.8.2) is an average over all paths. In differential notation, we write (4.8.1) as

$$dI(t) = \Delta(t)\, dW(t)$$

and (4.8.3) as

$$dI(t)\, dI(t) = \Delta^2(t)\, dW(t)\, dW(t) = \Delta^2(t)\, dt.$$

An *Itô process* (Definition 4.4.3) is a process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)\, dW(u) + \int_0^t \Theta(u)\, du, \tag{4.8.4}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes. According to Lemma 4.4.4, the quadratic variation accumulated by X up to time t is

$$[X, X](t) = \int_0^t \Delta^2(u)\, du. \tag{4.8.5}$$

In differential notation, we write (4.8.4) as

$$dX(t) = \Delta(t)\, dW(t) + \Theta(t)\, dt$$

and (4.8.5) as

$$\begin{aligned} dX(t)\, dX(t) &= \big(\Delta(t)\, dW(t) + \Theta(t)\, dt\big)^2 \\ &= \Delta^2(t)\, dW(t)\, dW(t) + 2\Delta(t)\, \Theta(t)\, dW(t)\, dt + \Theta^2(t)\, dt\, dt \\ &= \Delta^2(t)\, dt, \end{aligned}$$

where we have used the multiplication table

$$dW(t)\, dW(t) = dt, \quad dW(t)\, dt = dt\, dW(t) = 0, \quad dt\, dt = 0.$$

Suppose X and Y are Itô processes with differentials

$$dX(t) = \Theta_1(t)\, dt + \sigma_{11}(t)\, dW_1(t) + \sigma_{12}(t)\, dW_2(t), \tag{4.8.6}$$
$$dY(t) = \Theta_2(t)\, dt + \sigma_{21}(t)\, dW_1(t) + \sigma_{22}(t)\, dW_2(t), \tag{4.8.7}$$

where W_1 and W_2 are independent Brownian motions. Then

$$dX(t)\, dX(t) = \big(\sigma_{11}^2(t) + \sigma_{12}^2(t)\big)\, dt, \tag{4.8.8}$$
$$dX(t)\, dY(t) = \big(\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\big)\, dt, \tag{4.8.9}$$
$$dY(t)\, dY(t) = \big(\sigma_{21}^2(t) + \sigma_{22}^2(t)\big)\, dt. \tag{4.8.10}$$

Equations (4.8.8)–(4.8.10) can be obtained by multiplying the equations (4.8.6) and (4.8.7) for $dX(t)$ and $dY(t)$ and using the multiplication table

$$dW_i(t)\,dW_i(t) = dt,\ dW_i(t)\,dt = dt\,dW_i(t) = 0,\ dt\,dt = 0,$$

and

$$dW_1(t)\,dW_2(t) = 0. \tag{4.8.11}$$

Equation (4.8.11) holds for *independent* Brownian motions. If instead we had

$$dW_1(t)\,dW_2(t) = \rho\,dt,$$

for a constant $\rho \in [-1, 1]$, then ρ would be the correlation between $W_1(t)$ and $W_2(t)$ (i.e., $\mathbb{E}[W_1(t)W_2(t)] = \rho t$).

Now suppose $f(t, x, y)$ is a function of the time variable t and two dummy variables x and y. The multidimensional Itô-Doeblin formula (Theorem 4.6.2) says

$$\begin{aligned}
&df\big(t, X(t), Y(t)\big)\\
&= f_t\big(t, X(t), Y(t)\big)\,dt + f_x\big(t, X(t), Y(t)\big)\,dX(t) + f_y\big(t, X(t), Y(t)\big)\,dY(t)\\
&\qquad \frac{1}{2}f_{xx}\big(t, X(t), Y(t)\big)\,dX(t)\,dX(t) + f_{xy}\big(t, X(t), Y(t)\big)\,dX(t)\,dY(t)\\
&\qquad +\frac{1}{2}f_{yy}\big(t, X(t), Y(t)\big)\,dY(t)\,dY(t).
\end{aligned} \tag{4.8.12}$$

Replacing all the differentials on the right-hand side of (4.8.12) by their formulas (4.8.6)–(4.8.10) and integrating, one obtains a formula for the stochastic process $f(t, X(t), Y(t))$ as the sum of $f(0, X(0), Y(0))$, an ordinary integral with respect to time, an Itô integral with respect to dW_1, and an Itô integral with respect to dW_2.

There are two important special cases of (4.8.12). If the second process Y is not present, (4.8.12) reduces to the Itô-Doeblin formula for one process (Theorem 4.4.6):

$$df\big(t, X(t)\big) = f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t, X(t)\big)\,dX(t)\,dX(t).$$

If both X and Y are present and $f(t, x, y) = xy$, then (4.8.12) gives us *Itô's product rule* (Corollary 4.6.3):

$$d\big(X(t)Y(t)\big) = X(t)\,dY(t) + Y(t)\,dX(t) + dX(t)\,dY(t).$$

Using the Itô-Doeblin formula, we can derive the Black-Scholes-Merton partial differential equation. This was done in Section 4.5, and that section is summarized here. Let the stock price $S(t)$ be a geometric Brownian motion:

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t).$$

Let $c(t, S(t))$ be the price at time $t \in [0, T]$ of a European call paying $(S(T) - K)^+$ at expiration time T. Suppose we sell this call for $X(0) = c(0, S(0))$ at time zero and, starting with initial capital $X(0)$, invest in a stock and a money market account paying a constant rate of interest r. If $\Delta(t)$ is the number of shares of stock held by the portfolio at time t, then

$$dX(t) = \Delta(t)\, dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\, dt.$$

We compute the differential of the discounted portfolio value $e^{-rt}X(t)$, the differential of the discounted call price $e^{-rt}c(t, S(t))$, and set these two equal. This results in the *delta-hedging rule* (4.5.11),

$$\Delta(t) = c_x(t, S(t)), \tag{4.8.13}$$

and the Black-Scholes-Merton partial differential equation (4.5.14),

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x).$$

In addition to satisfying this partial differential equation, the function $c(t,x)$ must satisfy the boundary conditions

$$c(T,x) = (x-K)^+,\ c(t,0) = 0,\ \lim_{x\to\infty}\left[c(t,x) - \big(x - e^{-r(T-t)}K\big)\right] = 0.$$

The function satisfying these conditions is (see (4.5.19))

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big), \tag{4.8.14}$$

where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right].$$

Using the function given by (4.8.14), if one starts with initial capital $X(0) = c(0, S(0))$ and uses the delta-hedging rule (4.8.13), then at every time t, $X(t) = c(t, S(t))$. In particular, at the final time, the value of the hedging portfolio is $X(T) = c(T, S(T)) = (S(T) - K)^+$ almost surely. The short position in the European call has been hedged.

Lévy's Theorem, Theorem 4.6.4, says that if $M(t)$ is a continuous martingale starting at $M(0) = 0$ and if $[M,M](t) = t$ (i.e., $dM(t)\,dM(t) = dt$), then $M(t)$ is a Brownian motion. If $M_1(t)$ and $M_2(t)$ are two such processes and $[M_1, M_2](t) = 0$ (i.e., $dM_1(t)\,dM_2(t) = 0$), then $M_1(t)$ and $M_2(t)$ are independent Brownian motions (Theorem 4.6.5). One can use this theorem to construct independent Brownian motions from correlated Brownian motions and vice versa (see Exercise 4.13).

A Gaussian process $X(t)$ is one for which $X(t_1), X(t_2), \ldots X(t_n)$ are jointly normally distributed whenever $0 < t_1 < t_2 < \cdots < t_n$ (Definition 4.7.1). Because the joint distribution of jointly normal random variables is determined by means, variances, and covariances, the distribution of a Gaussian process is determined by its mean function $m(t) = \mathbb{E}X(t)$ and covariance function $c(s,t) = \mathrm{Cov}(X(s), X(t))$. Brownian motion is a Gaussian process with $m(t) = 0$ and $c(s,t) = s \wedge t$ (Example 4.7.2). If $\Delta(u)$ is nonrandom, then $I(t) = \int_0^t \Delta(u)\,dW(u)$ is a Gaussian process with $m(t) = 0$ and $c(s,t) = \int_0^{s\wedge t} \Delta^2(u)\,du$ (Example 4.7.3). The Brownian bridge from a to b on $[0,T]$ is a Gaussian process with $m(t) = \frac{(T-t)a+bt}{T}$ for $t \in [0,T]$ and $c(s,t) = s \wedge t - \frac{st}{T}$ for $s,t \in [0,T]$ (see Subsection 10.7.2). The Brownian bridge from a to b on $[0,T]$ is the process one obtains by starting a Brownian motion at a at time $t = 0$ and conditioning on $W(T) = b$ (see Subsection 10.7.5).

4.9 Notes

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remained sealed until May 2000. When it was opened, the document was found to contain a construction of the stochastic integral slightly different from Itô's and a clear statement of the change-of-variable formula. Doeblin's work [52], Yor's [166] analysis of the work, and a detailed history by Bru [24] of the context of the work appeared in the December 2000 issue of *Comptes Rendus de L'Académie des Sciences.* An English translation of this material is [25]. Because of this remarkable development, in this text the change-of-variable formula is called the Itô-Doeblin formula.

We have defined the Itô integral $\int_0^T \Delta^2(t)\,dW(t)$ under the condition

$$\mathbb{E}\int_0^T \Delta^2(t)\,dt < \infty. \tag{4.3.1}$$

The integral can be defined under the weaker condition

$$\int_0^T \Delta^2(t) < \infty \text{ almost surely}$$

but then is not guaranteed to be a martingale. It is still a *local martingale*, a topic discussed in advanced books on stochastic calculus (e.g., [101]). In this text, we do not consider local martingales. We work only under the condition (4.3.1), and every Itô integral we encounter is a martingale.

Brownian motion was introduced to finance by Bachelier [6]. Samuelson [143], [145] presents the argument that geometric Brownian motion is a good model for stock prices. The application of stochastic calculus to finance began with the work of Merton [121]. (The paper [121] and many other papers by Merton that use stochastic calculus in finance are collected in Merton [124].) The Black-Scholes-Merton formula is based on the geometric Brownian motion model for stock prices. However, no-arbitrage pricing theory has now moved far beyond this assumption. As seen in this and subsequent chapters, this theory and the accompanying risk-neutral pricing formula can be applied in the presence of a time-varying random volatility, a time-varying random mean rate of return, and a time-varying random interest rate.

Many finance books, including (in order of increasing mathematical difficulty) Hull [87], Dothan [54], and Duffie [56], include sections on Itô's integral and the Itô-Doeblin formula. Some other books on dynamic models in finance are Cox and Rubinstein [43], Huang and Litzenberger [86], Ingersoll [91], and Jarrow [97]. A comprehensive text is Wilmott [164]. Some good references for practitioners are Baxter and Rennie [8] (reviewed in [134]), Björk [11] (reviewed in [135]), and Musiela and Rutkowski [126] (reviewed in [134]). More mathematical texts on stochastic calculus with applications to finance are Lamberton and Lapeyre [105] (reviewed in [134]) and Steele [150] (reviewed in [136]). Other texts on stochastic calculus are Chung and Williams [36], Karatzas and Shreve [101], Øksendal [129], and Protter [133]. Karatzas and Shreve [102] is a sequel to [101] that focuses on finance. Protter [133] is the easiest place to learn about stochastic calculus for processes with jumps, and this is not at all easy. We introduce this topic in Chapter 11.

No-arbitrage pricing theory and the accompanying risk-neutral pricing formula is predicated on the assumption that there is no arbitrage in the market. An arbitrage is defined to be a trading strategy which begins with zero capital and at a later time has positive capital with positive probability without having any risk of loss. Absence of arbitrage is similar to but different from the *efficient market hypothesis*, which asserts that technical analysis of stock prices is of no value. This hypothesis asserts that patterns in stock prices may be useful to estimate the parameters of the distribution of future returns, but they do not provide clues to whether the next price movement will be up or down. In particular, technical analysis does not permit one to outperform the market. This hypothesis could be violated in a way which permits one to outperform the market with high probability without actually admitting arbitrage because there is still a nonzero probability of underperforming the market. This is sometimes called *statistical arbitrage*. An empirical study supporting the efficient market hypothesis is Fama [64], which also discusses distributions that fit stock prices better than geometric Brownian motion. A criticism of the efficient market hypothesis is provided by LeRoy [106], and a recent paper that finds long-range dependence (but not much) in stock price data is Willinger, Taqqu, and Teverovsky [163]. A provocative article on the source of stock price movements is Black [14].

Geometric fractional Brownian motion has been proposed as an alternative model for stock prices because it has fatter tails than geometric Brownian motion. One can assume such a model and compute the prices of derivative securities as their expected discounted payoffs, but the model is inconsistent with the usual delta-hedging formula. Indeed, geometric fractional Brownian motion violates the efficient market hypothesis so strongly that it admits arbitrage (not just "statistical arbitrage" but actual arbitrage). An example of this is provided by Rogers [138]. Further examples of arbitrage and a market-trading restriction that prevents arbitrage in such markets are provided by Cheridito [33].

The Vasicek model of Example 4.4.10 is taken from [154]. The Cox-Ingersoll-Ross model of Example 4.4.11 is due to [41], where the distribution of the interest rate process in the model is provided.

The derivation of the Black-Scholes-Merton formula in Section 4.5 is similar to that originally given by Black and Scholes [17] but also relies heavily on the no-arbitrage idea appearing in Merton [122]. It is well-documented that the three men cooperated on development of the option-pricing formula, and in recognition of this the 1997 Nobel Prize in Economics was awarded to Scholes and Merton. (Black died in 1995, and the prize is not awarded posthumously). In this text, the role of all three men is acknowledged by the terminology *Black-Scholes-Merton* option-pricing formula. Even though geometric Brownian motion is a less than perfect model for stock prices, the Black-Scholes-Merton pricing formula for vanilla options (i.e., European calls and puts) seems not to be terribly sensitive to deficiencies in the model.

The passage from discrete to continuous time in the model of evolution of the portfolio value, which is touched upon in Subsection 4.5.1, is given a more detailed treatment by Duffie and Protter [60]; see also Exercise 4.10.

4.10 Exercises

Exercise 4.1. Suppose $M(t)$, $0 \le t \le T$, is a martingale with respect to some filtration $\mathcal{F}(t)$, $0 \le t \le T$. Let $\Delta(t)$, $0 \le t \le T$, be a simple process adapted to $\mathcal{F}(t)$ (i.e., there is a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0,T]$ such that, for every j, $\Delta(t_j)$ is $\mathcal{F}(t_j)$-measurable and $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$). For $t \in [t_k, t_{k+1})$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)].$$

We think of $M(t)$ as the price of an asset at time t and $\Delta(t_j)$ as the number of shares of the asset held by an investor between times t_j and t_{j+1}. Then $I(t)$ is the capital gains that accrue to the investor between times 0 and t. Show that $I(t)$, $0 \le t \le T$, is a martingale.

Exercise 4.2. Let $W(t)$, $0 \le t \le T$, be a Brownian motion, and let $\mathcal{F}(t)$, $0 \le t \le T$, be an associated filtration. Let $\Delta(t)$, $0 \le t \le T$, be a *nonrandom* simple process (i.e., there is a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0,T]$ such that for every j, $\Delta(t_j)$ is a nonrandom quantity and $\Delta(t) = \Delta(t_j)$ is constant in t on the subinterval $[t_j, t_{j+1})$). For $t \in [t_k, t_{k+1}]$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)].$$

(i) Show that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is independent of $\mathcal{F}(s)$. (Simplification: If s is between two partition points, we can always insert s as an extra partition point. Then we can relabel the partition points so that they are still called $t_0, t_1, \dots, t_n$, but with a larger value of n and now with $s = t_k$ for some value of k. Of course, we must set $\Delta(s) = \Delta(t_{k-1})$ so that Δ takes the same value on the interval $[s, t_{k+1})$ as on the interval $[t_{k-1}, s)$. Similarly, we can insert t as an extra partition point if it is not already one. Consequently, to show that $I(t) - I(s)$ is independent of $\mathcal{F}(s)$ for all $0 \le s < t \le T$, it suffices to show that $I(t_k) - I(t_\ell)$ is independent of $\mathcal{F}(t_\ell)$ whenever t_k and t_ℓ are two partition points with $t_\ell < t_k$. This is all you need to do.)

(ii) Show that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is a normally distributed random variable with mean zero and variance $\int_s^t \Delta^2(u)\,du$.

(iii) Use (i) and (ii) to show that $I(t)$, $0 \le t \le T$, is a martingale.

(iv) Show that $I^2(t) - \int_0^t \Delta^2(u)\,du$, $0 \le t \le T$, is a martingale.

Exercise 4.3. We now consider a case in which $\Delta(t)$ in Exercise 4.2 is simple but random. In particular, let $t_0 = 0$, $t_1 = s$, and $t_2 = t$, and let $\Delta(0)$ be nonrandom and $\Delta(s) = W(s)$. Which of the following assertions is true? Justify your answers.

(i) $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.

(ii) $I(t) - I(s)$ is normally distributed. (Hint: Check if the fourth moment is three times the square of the variance; see Exercise 3.3 of Chapter 3.)

(iii) $\mathbb{E}[I(t) | \mathcal{F}(s)] = I(s)$.

(iv) $\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\,du \middle| \mathcal{F}(s) \right] = I^2(s) - \int_0^s \Delta^2(u)\,du$.

Exercise 4.4 (Stratonovich integral). Let $W(t)$, $t \ge 0$, be a Brownian motion. Let T be a fixed positive number and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = T$). For each j, define $t_j^* = \frac{t_j + t_{j+1}}{2}$ to be the midpoint of the interval $[t_j, t_{j+1}]$.

(i) Define the *half-sample quadratic variation* corresponding to Π to be

$$Q_{\Pi/2} = \sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j) \right)^2.$$

Show that $Q_{\Pi/2}$ has limit $\frac{1}{2}T$ as $\|\Pi\| \to 0$. (Hint: It suffices to show that $\mathbb{E}Q_{\Pi/2} = \frac{1}{2}T$ and $\lim_{\|\Pi\| \to 0} \mathrm{Var}(Q_{\Pi/2}) = 0$.)

(ii) Define the Stratonovich integral of $W(t)$ with respect to $W(t)$ to be

$$\int_0^T W(t) \circ dW(t) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} W(t_j^*)\big(W(t_{j+1}) - W(t_j)\big). \qquad (4.10.1)$$

In contrast to the Itô integral $\int_0^T W(t)\,dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$ of (4.3.4), which evaluates the integrand at the left endpoint of each subinterval $[t_j, t_{j+1}]$, here we evaluate the integrand at the midpoint t_j^*. Show that

$$\int_0^T W(t) \circ dW(t) = \frac{1}{2}W^2(T).$$

(Hint: Write the approximating sum in (4.10.1) as the sum of an approximating sum for the Itô integral $\int_0^T W(t)\,dW(t)$ and $Q_{\Pi/2}$. The approximating sum for the Itô integral is the one corresponding to the partition $0 = t_0 < t_0^* < t_1 < t_1^* < \cdots < t_{n-1}^* < t_n = T$, not the partition Π.)

Exercise 4.5 (Solving the generalized geometric Brownian motion equation). Let $S(t)$ be a positive stochastic process that satisfies the generalized geometric Brownian motion differential equation (see Example 4.4.8)

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \qquad (4.10.2)$$

where $\alpha(t)$ and $\sigma(t)$ are processes adapted to the filtration $\mathcal{F}(t)$, $t \geq 0$, associated with the Brownian motion $W(t)$, $t \geq 0$. In this exercise, we show that $S(t)$ must be given by formula (4.4.26) (i.e., that formula provides the only solution to the stochastic differential equation (4.10.2)). In the process, we provide a method for solving this equation.

(i) Using (4.10.2) and the Itô-Doeblin formula, compute $d\log S(t)$. Simplify so that you have a formula for $d\log S(t)$ that does not involve $S(t)$.
(ii) Integrate the formula you obtained in (i), and then exponentiate the answer to obtain (4.4.26).

Exercise 4.6. Let $S(t) = S(0)\exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}$ be a geometric Brownian motion. Let p be a positive constant. Compute $d\left(S^p(t)\right)$, the differential of $S(t)$ raised to the power p.

Exercise 4.7. (i) Compute $dW^4(t)$ and then write $W^4(T)$ as the sum of an ordinary (Lebesgue) integral with respect to time and an Itô integral.
(ii) Take expectations on both sides of the formula you obtained in (i), use the fact that $\mathbb{E}W^2(t) = t$, and derive the formula $\mathbb{E}W^4(T) = 3T^2$.
(iii) Use the method of (i) and (ii) to derive a formula for $\mathbb{E}W^6(T)$.

Exercise 4.8 (Solving the Vasicek equation). The Vasicek interest rate stochastic differential equation (4.4.32) is

$$dR(t) = \big(\alpha - \beta R(t)\big)\,dt + \sigma\,dW(t),$$

where α, β, and σ are positive constants. The solution to this equation is given in Example 4.4.10. This exercise shows how to derive this solution.

(i) Use (4.4.32) and the Itô-Doeblin formula to compute $d\big(e^{\beta t}R(t)\big)$. Simplify it so that you have a formula for $d\big(e^{\beta t}R(t)\big)$ that does not involve $R(t)$.
(ii) Integrate the equation you obtained in (i) and solve for $R(t)$ to obtain (4.4.33).

Exercise 4.9. For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x, is

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big),$$

where

$$\begin{aligned} d_+(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right], \\ d_-(\tau,x) &= d_+(\tau,x) - \sigma\sqrt{\tau}, \end{aligned}$$

and $N(y)$ is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz.$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) = rc(t,x),\ \ 0 \le t < T, x > 0, \quad (4.10.3)$$

the *terminal condition*

$$\lim_{t\uparrow T} c(t,x) = (x-K)^+, \quad x > 0, x \neq K, \qquad (4.10.4)$$

and the *boundary conditions*

$$\lim_{x\downarrow 0} c(t,x) = 0, \quad \lim_{x\to\infty}\Big[c(t,x) - \big(x - e^{-r(T-t)}K\big)\Big] = 0, \quad 0 \le t < T. \ (4.10.5)$$

Equation (4.10.4) and the first part of (4.10.5) are usually written more simply but less precisely as

$$c(T,x) = (x-K)^+,\ x \ge 0$$

and

$$c(t,0) = 0,\ 0 \le t \le T.$$

For this exercise, we abbreviate $c(t,x)$ as simply c and $d_\pm(T-t,x)$ as simply $d_\pm$.

(i) Verify first the equation

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+). \tag{4.10.6}$$

(ii) Show that $c_x = N(d_+)$. This is the *delta* of the option. (Be careful! Remember that d_+ is a function of x.)

(iii) Show that

$$c_t = -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+).$$

This is the *theta* of the option.

(iv) Use the formulas above to show that c satisfies (4.10.3).

(v) Show that for $x > K$, $\lim_{t\uparrow T} d_\pm = \infty$, but for $0 < x < K$, $\lim_{t\uparrow T} d_\pm = -\infty$. Use these equalities to derive the terminal condition (4.10.4).

(vi) Show that for $0 \le t < T$, $\lim_{x\downarrow 0} d_\pm = -\infty$. Use this fact to verify the first part of boundary condition (4.10.5) as $x \downarrow 0$.

(vii) Show that for $0 \le t < T$, $\lim_{x\to\infty} d_\pm = \infty$. Use this fact to verify the second part of boundary condition (4.10.5) as $x \to \infty$. In this verification, you will need to show that

$$\lim_{x\to\infty} \frac{N(d_+) - 1}{x^{-1}} = 0.$$

This is an indeterminate form $\frac{0}{0}$, and L'Hôpital's rule implies that this limit is

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\left[N(d_+) - 1\right]}{\frac{d}{dx}x^{-1}}.$$

Work out this expression and use the fact that

$$x = K\exp\left\{\sigma\sqrt{T-t}\,d_+ - (T-t)\left(r + \frac{1}{2}\sigma^2\right)\right\}$$

to write this expression solely in terms of d_+ (i.e., without the appearance of any x except the x in the argument of $d_+(T-t,x)$). Then argue that the limit is zero as $d_+ \to \infty$.

Exercise 4.10 (Self-financing trading). The fundamental idea behind no-arbitrage pricing is to reproduce the payoff of a derivative security by trading in the underlying asset (which we call a stock) and the money market account. In discrete time, we let X_k denote the value of the hedging portfolio at time k and let Δ_k denote the number of shares of stock held between times k and $k+1$. Then, at time k, after rebalancing (i.e., moving from a position of Δ_{k-1} to a position Δ_k in the stock), the amount in the money market account is $X_k - S_k\Delta_k$. The value of the portfolio at time $k+1$ is

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k). \tag{4.10.7}$$

This formula can be rearranged to become

$$X_{k+1} - X_k = \Delta_k\big(S_{k+1} - S_k\big) + r\big(X_k - \Delta_k S_k\big), \tag{4.10.8}$$

which says that the gain between time k and time $k+1$ is the sum of the capital gain on the stock holdings, $\Delta_k(S_{k+1} - S_k)$, and the interest earnings on the money market account, $r(X_k - \Delta_k S_k)$. The continuous-time analogue of (4.10.8) is

$$dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt. \tag{4.10.9}$$

Alternatively, one could define the value of a share of the money market account at time k to be

$$M_k = (1+r)^k$$

and formulate the discrete-time model with two processes, Δ_k as before and Γ_k denoting the number of shares of the money market account held at time k after rebalancing. Then

$$X_k = \Delta_k S_k + \Gamma_k M_k, \tag{4.10.10}$$

so that (4.10.7) becomes

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)\Gamma_k M_k = \Delta_k S_{k+1} + \Gamma_k M_{k+1}. \tag{4.10.11}$$

Subtracting (4.10.10) from (4.10.11), we obtain in place of (4.10.8) the equation

$$X_{k+1} - X_k = \Delta_k\big(S_{k+1} - S_k\big) + \Gamma_k(M_{k+1} - M_k), \tag{4.10.12}$$

which says that the gain between time k and time $k+1$ is the sum of the capital gain on stock holdings, $\Delta_k(S_{k+1} - S_k)$, and the earnings from the money market investment, $\Gamma_k(M_{k+1} - M_k)$.

But Δ_k and Γ_k cannot be chosen arbitrarily. The agent arrives at time $k+1$ with some portfolio of Δ_k shares of stock and Γ_k shares of the money market account and then rebalances. In terms of Δ_k and Γ_k, the value of the portfolio upon arrival at time $k+1$ is given by (4.10.11). After rebalancing, it is

$$X_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1}.$$

Setting these two values equal, we obtain the *discrete-time self-financing condition*

$$S_{k+1}(\Delta_{k+1} - \Delta_k) + M_{k+1}(\Gamma_{k+1} - \Gamma_k) = 0. \tag{4.10.13}$$

The first term is the cost of rebalancing in the stock, and the second is the cost of rebalancing in the money market account. If the sum of these two terms is not zero, then money must either be put into the position or can be taken out as a by-product of rebalancing. The point is that when the two processes Δ_k and Γ_k are used to describe the evolution of the portfolio value

X_k, then two equations, (4.10.12) and (4.10.13), are required rather than the single equation (4.10.8) when only the process Δ_k is used.

Finally, we note that we may rewrite the *discrete-time self-financing condition* (4.10.13) as

$$\begin{aligned} S_k(\Delta_{k+1} - \Delta_k) + (S_{k+1} - S_k)(\Delta_{k+1} - \Delta_k) \\ + M_k(\Gamma_{k+1} - \Gamma_k) + (M_{k+1} - M_k)(\Gamma_{k+1} - \Gamma_k) = 0. \end{aligned} \tag{4.10.14}$$

This is suggestive of the *continuous-time self-financing condition*

$$S(t)\,d\Delta(t) + dS(t)\,d\Delta(t) + M(t)\,d\Gamma(t) + dM(t)\,d\Gamma(t) = 0, \tag{4.10.15}$$

which we derive below.

(i) In continuous time, let $M(t) = e^{rt}$ be the price of a share of the money market account at time t, let $\Delta(t)$ denote the number of shares of stock held at time t, and let $\Gamma(t)$ denote the number of shares of the money market account held at time t, so that the total portfolio value at time t is

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t). \tag{4.10.16}$$

Using (4.10.16) and (4.10.9), derive the continuous-time self-financing condition (4.10.15).

A common argument used to derive the Black-Scholes-Merton partial differential equation and delta-hedging formula goes like this. Let $c(t, x)$ be the price of a call at some time t if the stock price at that time is $S(t) = x$. Form a portfolio that is long the call and short $\Delta(t)$ shares of stock, so that the value of the portfolio at time t is $N(t) = c\big(t, S(t)\big) - \Delta(t)S(t)$. We want to choose $\Delta(t)$ so this is "instantaneously riskless," in which case its value would have to grow at the interest rate. Otherwise, according to this argument, we could arbitrage this portfolio against the money market account. This means we should have $dN(t) = rN(t)\,dt$. We compute the differential of $N(t)$ and get

$$\begin{aligned} dN(t) &= c_t\big(t, S(t)\big)\,dt + c_x\big(t, S(t)\big)\,dS(t) \\ &\qquad + \frac{1}{2}c_{xx}\big(t, S(t)\big)\,dS(t)\,dS(t) - \Delta(t)\,dS(t) \\ &= \big[c_x\big(t, S(t)\big) - \Delta(t)\big]\,dS(t) \\ &\qquad + \left[c_t\big(t, S(t)\big) + \frac{1}{2}\sigma^2 S^2(t) c_{xx}\big(t, S(t)\big)\right] dt. \end{aligned} \tag{4.10.17}$$

In order for this to be instantaneously riskless, we must cancel out the $dS(t)$ term, which contains the risk. This gives us the delta-hedging formula $\Delta(t) = c_x\big(t, S(t)\big)$. Having chosen $\Delta(t)$ this way, we recall that we expect to have $dN(t) = rN(t)\,dt$, and this yields

$$rN(t)\,dt = \left[c_t\big(t, S(t)\big) + \frac{1}{2}\sigma^2 S^2(t) c_{xx}\big(t, S(t)\big)\right] dt. \tag{4.10.18}$$

But

$$N(t) = c\big(t, S(t)\big) - \Delta(t)S(t) = c\big(t, S(t)\big) - S(t)c_x\big(t, S(t)\big), \tag{4.10.19}$$

and substitution of (4.10.19) into (4.10.18) yields the Black-Scholes-Merton partial differential equation

$$c_t\big(t, S(t)\big) + rS(t)c_s\big(t, S(t)\big) + \frac{1}{2}\sigma^2 S^2(t) c_{ss}\big(t, S(t)\big) = rc\big(t, S(t)\big). \tag{4.10.20}$$

One can question the first step of this argument, where we failed to use Itô's product rule (Corollary 4.6.3) on the term $\Delta(t)S(t)$ when we differentiated $N(t)$ in (4.10.17). In discrete time, we hold Δ_k fixed for a period and let S move, computing the capital gain according to the formula $\Delta_k(S_{k+1} - S_k)$, and in (4.10.17) we are attempting something analogous to that in continuous time. However, as soon as we set $\Delta(t) = c_x\big(t, S(t)\big)$, then $\Delta(t)$ moves continuously in time and the differential of $N(t)$ is really

$$\begin{aligned} dN(t) = {} & c_t\big(t, S(t)\big)\,dt + c_x\big(t, S(t)\big)\,dS(t) + \frac{1}{2}c_{xx}\big(t, S(t)\big)\,dS(t)\,dS(t) \\ & -\Delta(t)\,dS(t) - S(t)\,d\Delta(t) - d\Delta(t)\,dS(t) \end{aligned} \tag{4.10.21}$$

rather than the expression in (4.10.17).

This exercise shows that the argument is correct after all. At least, equation (4.10.18) is correct, and from that the Black-Scholes-Merton partial differential equation (4.10.20) follows.

Recall from Subsection 4.5.3 that if we take $X(0) = c\big(0, S(0)\big)$ and at each time t hold $\Delta(t) = c_x\big(t, S(t)\big)$ shares of stock, borrowing or investing in the money market as necessary to finance this, then at each time t we have a portfolio of stock and a money market account valued at $X(t) = c(t, S(t))$. The amount invested in the money market account at each time t is

$$X(t) - \Delta(t)S(t) = c\big(t, S(t)\big) - \Delta(t)S(t) = N(t),$$

and so the number of money market account shares held is

$$\Gamma(t) = \frac{N(t)}{M(t)}.$$

(ii) Now replace (4.10.17) by its corrected version (4.10.21) and use the continuous-time self-financing condition you derived in part (i) to derive (4.10.18).

Exercise 4.11. Let

$$c(t, x) = xN(d_+(T-t, x)) - Ke^{-r(T-t)}N(d_-(T-t, x))$$

denote the price for a European call, expiring at time T with strike price K, where

$$d_\pm(T-t,x) = \frac{1}{\sigma_1\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma_1^2}{2}\right)(T-t)\right].$$

This option price assumes the underlying stock is a geometric Brownian motion with volatility $\sigma_1 > 0$. For this problem, we take this to be the market price of the option.

Suppose, however, that the underlying asset is really a geometric Brownian motion with volatility $\sigma_2 > \sigma_1$, i.e.,

$$dS(t) = \alpha S(t)\,dt + \sigma_2 S(t)\,dW(t).$$

Consequently, the market price of the call is incorrect.

We set up a portfolio whose value at each time t we denote by $X(t)$. We begin with $X(0) = 0$. At each time t, the portfolio is long one European call, is short $c_x(t, S(t))$ shares of stock, and thus has a cash position

$$X(t) - c(t,S(t)) + S(t)c_x(t,S(t)),$$

which is invested at the constant interest rate r. We also remove cash from this portfolio at a rate $\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t,S(t))$. Therefore, the differential of the portfolio value is

$$\begin{aligned} dX(t) = {} & dc(t,S(t)) - c_x(t,S(t))\,dS(t) \\ & + r\big[X(t) - c(t,S(t)) + S(t)c_x(t,S(t))\big]\,dt \\ & - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t,S(t))\,dt,\ 0 \le t \le T. \end{aligned}$$

Show that $X(t) = 0$ for all $t \in [0,T]$. In particular, because $c_{xx}(t,S(t)) > 0$ and $\sigma_2 > \sigma_1$, we have an arbitrage opportunity; we can start with zero initial capital, remove cash at a positive rate between times 0 and T, and at time T have zero liability. (Hint: Compute $d\big(e^{-rt}X(t)\big)$.)

Exercise 4.12. (i) Use formulas (4.5.23)–(4.5.25), (4.5.26), and (4.5.29) to determine the delta $p_x(t,x)$, the gamma $p_{xx}(t,x)$, and the theta $p_t(t,x)$ of a European put.

(ii) Show that an agent hedging a short position in the put should have a short position in the underlying stock and a long position in the money market account.

(iii) Show that $f(t,x)$ of (4.5.26) and $p(t,x)$ satisfy the same Black-Scholes-Merton partial differential equation (4.5.14) satisfied by $c(t,x)$.

Exercise 4.13 (Decomposition of correlated Brownian motions into independent Brownian motions). Suppose $B_1(t)$ and $B_2(t)$ are Brownian motions and

$$dB_1(t)\,dB_2(t) = \rho(t)\,dt,$$

where ρ is a stochastic process taking values strictly between -1 and 1. Define processes $W_1(t)$ and $W_2(t)$ such that

$$\begin{aligned} B_1(t) &= W_1(t), \\ B_2(t) &= \int_0^t \rho(s)\,dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\,dW_2(s), \end{aligned}$$

and show that $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

Exercise 4.14. In the derivation of the Itô-Doeblin formula, Theorem 4.4.1, we considered only the case of the function $f(x) = \frac{1}{2}x^2$, for which $f''(x) = 1$. This made it easy to determine the limit of the last term,

$$\frac{1}{2}\sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2,$$

appearing in (4.4.5). Indeed,

$$\begin{aligned} \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2 &= \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\big[W(t_{j+1}) - W(t_j)\big]^2 \\ &= [W,W](T) \;=\; T \\ &= \int_0^T f''(W(t))\,dt. \end{aligned}$$

If we had been working with an arbitrary function $f(x)$, we could not replace $f''(W(t_j))$ by 1 in the argument above. It is tempting in this case to just argue that $\big[W(t_{j+1}) - W(t_j)\big]^2$ is approximately equal to $(t_{j+1} - t_j)$, so that

$$\sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2$$

is approximately equal to

$$\sum_{j=0}^{n-1} f''(W(t_j))(t_{j+1} - t_j),$$

and this has limit $\int_0^T f''(W(t))\,dt$ as $\|\Pi\| \to 0$. However, as discussed in Remark 3.4.4, it does not make sense to say that $\big[W(t_{j+1}) - W(t_j)\big]^2$ is approximately equal to $(t_{j+1} - t_j)$. In this exercise, we develop a correct explanation for the equation

$$\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2 = \int_0^T f''(W(t))\,dt. \tag{4.10.22}$$

Define

$$Z_j = f''(W(t_j))\big[\big(W(t_{j+1}) - W(t_j)\big)^2 - (t_{j+1} - t_j)\big]$$

so that

$$\sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2 = \sum_{j=0}^{n-1} Z_j + \sum_{j=0}^{n-1} f''(W(t_j))(t_{j+1} - t_j). \tag{4.10.23}$$

(i) Show that Z_j is $\mathcal{F}(t_{j+1})$-measurable and

$$\mathbb{E}[Z_j|\mathcal{F}(t_j)] = 0, \quad \mathbb{E}[Z_j^2|\mathcal{F}(t_j)] = 2\big[f''(W(t_j))\big]^2(t_{j+1} - t_j)^2.$$

It remains to show that

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} Z_j = 0. \tag{4.10.24}$$

This will cause us to obtain (4.10.22) when we take the limit in (4.10.23). Prove (4.10.24) in the following steps.

(ii) Show that $\mathbb{E}\sum_{j=0}^{n-1} Z_j = 0$.

(iii) Under the assumption that $\mathbb{E}\int_0^T [f''(W(t))]^2\,dt$ is finite, show that

$$\lim_{\|\Pi\|\to 0} \mathrm{Var}\left[\sum_{j=0}^{n-1} Z_j\right] = 0.$$

(Warning: The random variables $Z_1, Z_2, \ldots, Z_{n-1}$ are not independent.)

From (iii), we conclude that $\sum_{j=0}^{n-1} Z_j$ converges to its mean, which by (ii) is zero. This establishes (4.10.24).

Exercise 4.15 (Creating correlated Brownian motions from independent ones). Let $(W_1(t), \ldots, W_d(t))$ be a d-dimensional Brownian motion. In particular, these Brownian motions are independent of one another. Let $(\sigma_{ij}(t))_{i=1,\ldots,m;j=1,\ldots,d}$ be an $m \times d$ matrix-valued process adapted to the filtration associated with the d-dimensional Brownian motion. For $i = 1, \ldots, m$, define

$$\sigma_i(t) = \left[\sum_{j=1}^{d} \sigma_{ij}^2(t)\right]^{\frac{1}{2}},$$

and assume this is never zero. Define also

$$B_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\,dW_j(u).$$

(i) Show that, for each i, B_i is a Brownian motion.

(ii) Show that $dB_i(t)\,dB_k(t) = \rho_{ik}(t)$, where

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\sigma_k(t)} \sum_{j=1}^{d} \sigma_{ij}(t)\sigma_{kj}(t).$$

Exercise 4.16 (Creating independent Brownian motions to represent correlated ones). Let $B_1(t), \dots, B_m(t)$ be m one-dimensional Brownian motions with

$$dB_i(t)\,dB_k(t) = \rho_{ik}(t)\,dt \text{ for all } i, k = 1, \dots, m,$$

where $\rho_{ik}(t)$ are adapted processes taking values in $(-1, 1)$ for $i \neq k$ and $\rho_{ik}(t) = 1$ for $i = k$. Assume that the symmetric matrix

$$C(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) & \cdots & \rho_{1m}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \cdots & \rho_{2m}(t) \\ \vdots & \vdots & & \vdots \\ \rho_{m1}(t) & \rho_{m2}(t) & \cdots & \rho_{mm}(t) \end{bmatrix}$$

is positive definite for all t almost surely. Because the matrix $C(t)$ is symmetric and positive definite, it has a *matrix square root.* In other words, there is a matrix

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1m}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2m}(t) \\ \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mm}(t) \end{bmatrix}$$

such that $C(t) = A(t)A^{\mathrm{tr}}(t)$, which when written componentwise is

$$\rho_{ik}(t) = \sum_{j=1}^{m} a_{ij}(t)a_{kj}(t) \text{ for all } i, k = 1, \dots, m. \tag{4.10.25}$$

This matrix can be chosen so that its components $a_{ik}(t)$ are adapted processes. Furthermore, the matrix $A(t)$ has an inverse

$$A^{-1}(t) = \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) & \cdots & \alpha_{1m}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) & \cdots & \alpha_{2m}(t) \\ \vdots & \vdots & & \vdots \\ \alpha_{m1}(t) & \alpha_{m2}(t) & \cdots & \alpha_{mm}(t) \end{bmatrix},$$

which means that

$$\sum_{j=1}^{m} a_{ij}(t)\alpha_{jk}(t) = \sum_{j=1}^{m} \alpha_{ij}(t)a_{jk}(t) = \delta_{ik}, \tag{4.10.26}$$

where we define

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

to be the so-called *Kronecker delta*. Show that there exist m independent Brownian motions $W_1(t), \ldots, W_m(t)$ such that

$$B_i(t) = \sum_{j=1}^{m} \int_0^t a_{ij}(u)\, dW_j(u) \text{ for all } i = 1, \ldots, m. \tag{4.10.27}$$

Exercise 4.17 (Instantaneous correlation). Let

$$\begin{aligned} X_1(t) &= X_1(0) + \int_0^t \Theta_1(u)\, du + \int_0^t \sigma_1(u)\, dB_1(u), \\ X_2(t) &= X_2(0) + \int_0^t \Theta_2(u)\, du + \int_0^t \sigma_2(u)\, dB_2(u), \end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are Brownian motions satisfying $dB_1(t)\, dB_2(t) = \rho(t)$ and $\rho(t)$, $\Theta_1(t)$, $\Theta_2(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are adapted processes. Then

$$dX_1(t)\, dX_2(t) = \sigma_1(t)\sigma_2(t)\, dB_1(t)\, dB_2(t) = \rho(t)\sigma_1(t)\sigma_2(t)\, dt.$$

We call $\rho(t)$ the *instantaneous correlation* between $X_1(t)$ and $X_2(t)$ for the reason explained by this exercise.

We first consider the case when ρ, Θ_1, Θ_2, σ_1, and σ_2 are constant. Then

$$\begin{aligned} X_1(t) &= X_1(0) + \Theta_1 t + \sigma_1 B_1(t), \\ X_2(t) &= X_2(0) + \Theta_2 t + \sigma_2 B_2(t). \end{aligned}$$

Fix $t_0 > 0$, and let $\epsilon > 0$ be given.

(i) Use Itô's product rule to show that

$$\mathbb{E}\left[\big(B_1(t_0+\epsilon) - B_1(t_0)\big)\big(B_2(t_0+\epsilon) - B_2(t_0)\big) \big| \mathcal{F}(t_0)\right] = \rho\epsilon.$$

(ii) Show that, conditioned on $\mathcal{F}(t_0)$, the pair of random variables

$$\big(X_1(t_0+\epsilon) - X_1(t_0), X_2(t_0+\epsilon) - X_2(t_0)\big)$$

has means, variances, and covariance

$$M_i(\epsilon) = \mathbb{E}\left[X_i(t_0+\epsilon) - X_i(t_0) \big| \mathcal{F}(t_0)\right] \;=\; \Theta_i\epsilon \text{ for } i = 1, 2, \tag{4.10.28}$$

$$\begin{aligned} V_i(\epsilon) &= \mathbb{E}\left[\big(X_i(t_0+\epsilon) - X_i(t_0)\big)^2 \Big| \mathcal{F}(t_0)\right] - M_i^2(\epsilon) \\ &= \sigma_i^2\epsilon \text{ for } i = 1, 2, \end{aligned} \tag{4.10.29}$$

$$\begin{aligned} C(\epsilon) &= \mathbb{E}\left[\big(X_1(t_0+\epsilon) - X_1(t_0)\big)\big(X_2(t_0+\epsilon) - X_2(t_0)\big) \big| \mathcal{F}(t_0)\right] \\ &\quad - M_1(\epsilon)M_2(\epsilon) \;=\; \rho\sigma_1\sigma_2\epsilon. \end{aligned} \tag{4.10.30}$$

In particular, conditioned on $\mathcal{F}(t_0)$, the correlation between the increments $X_1(t_0+\epsilon)-X_1(t_0)$ and $X_2(t_0+\epsilon)-X_2(t_0)$ is

$$\frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}}=\rho.$$

We now allow $\rho(t)$, $\Theta_1(t)$, $\Theta_2(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ to be continuous adapted processes, assuming only that there is a constant M such that

$$|\Theta_1(t)|\leq M,\quad |\sigma_1(t)|\leq M,\quad |\Theta_2(t)|\leq M,\quad |\sigma_2(T)|\leq M,\quad |\rho(t)|\leq M \tag{4.10.31}$$

for all $t\geq 0$ almost surely. We again fix $t_0\geq 0$.

(iii) Show that, conditioned on $\mathcal{F}(t_0)$, we have the conditional mean formulas

$$M_i(\epsilon)=E\left[X_i(t_0+\epsilon)-X_i(t_0)\middle|\,\mathcal{F}(t_0)\right]=\Theta_i(t_0)\epsilon+o(\epsilon)\text{ for }i=1,2, \tag{4.10.32}$$

where we denote by $o(\epsilon)$ any quantity that is so small that $\lim_{\epsilon\downarrow 0}\frac{o(\epsilon)}{\epsilon}=0$. In other words, show that

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}M_i(\epsilon)=\Theta_i(t_0)\text{ for }i=1,2. \tag{4.10.33}$$

(Hint: First show that

$$M_i(\epsilon)=\mathbb{E}\left[\int_{t_0}^{t_0+\epsilon}\Theta_i(u)\,du\middle|\,\mathcal{F}(t_0)\right]. \tag{4.10.34}$$

The Dominated Convergence Theorem, Theorem 1.4.9, works for conditional expectations as well as for expectations in the following sense. Let X be a random variable. Suppose for every $\epsilon>0$ we have a random variable $X(\epsilon)$ such that $\lim_{\epsilon\downarrow 0}X(\epsilon)=X$ almost surely. Finally, suppose there is another random variable Y such that $\mathbb{E}Y<\infty$ and $|X(\epsilon)|\leq Y$ almost surely for every $\epsilon>0$. Then

$$\lim_{\epsilon\downarrow 0}\mathbb{E}[X(\epsilon)|\mathcal{F}(t_0)]=\mathbb{E}[X|\mathcal{F}(t_0)].$$

Use this to obtain (4.10.33) from (4.10.34).)

(iv) Show that $D_{ij}(\epsilon)$ defined by

$$\begin{aligned}D_{ij}(\epsilon)=\mathbb{E}\left[\big(X_i(t_0+\epsilon)-X_i(t_0)\big)\big(X_j(t_0+\epsilon)-X_j(t_0)\big)\middle|\,\mathcal{F}(t_0)\right]\\ -M_i(\epsilon)M_j(\epsilon)\end{aligned}$$

for $i=1,2$ and $j=1,2$ satisfies

$$D_{ij}(\epsilon)=\rho_{ij}(t_0)\sigma_i(t_0)\sigma_j(t_0)\epsilon+o(\epsilon), \tag{4.10.35}$$

where we set $\rho_{11}(t)=\rho_{22}(t)=1$ and $\rho_{12}(t)=\rho_{21}(t)=\rho(t)$. (Hint: You should define the martingales

$$Y_i(t) = \int_0^t \sigma_i(u)\,dB_i(u) \text{ for } i = 1, 2,$$

so you can write

$$\begin{aligned} D_{ij}(\epsilon) = \mathbb{E}\Bigg[&\left(Y_i(t_0+\epsilon) - Y_i(t_0) + \int_{t_0}^{t_0+\epsilon} \Theta_i(u)\,du \right) \\ &\cdot \left(Y_j(t_0+\epsilon) - Y_j(t_0) + \int_{t_0}^{t_0+\epsilon} \Theta_j(u)\,du \right) \Bigg| \mathcal{F}(t_0) \Bigg] \\ &-M_i(\epsilon)M_j(\epsilon). \end{aligned} \tag{4.10.36}$$

Then expand the expression on the right-hand side of (4.10.36). You should use Itô's product rule to show that the first term in the expansion is

$$\begin{aligned} \mathbb{E}\left[\big(Y_i(t_0+\epsilon) - Y_i(t_0)\big)\big(Y_j(t_0+\epsilon) - Y_j(t_0)\big) \,\big|\, \mathcal{F}(t_0) \right] \\ = \mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} \rho_{ij}(u)\sigma_i(u)\sigma_j(u)\,du \,\bigg|\, \mathcal{F}(t_0) \right]. \end{aligned}$$

This equation is similar to (4.10.34), and you can use the Dominated Convergence Theorem as stated in the hint for (iii) to conclude that

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}\left[\big(Y_i(t_0+\epsilon) - Y_i(t_0)\big)\big(Y_j(t_0+\epsilon) - Y_j(t_0)\big) \,\big|\, \mathcal{F}(t_0) \right] \\ = \rho_{ij}(t_0)\sigma_i(t_0)\sigma_j(t_0). \end{aligned}$$

To handle the other terms that arise from expanding (4.10.36), you will need (4.10.31) and the fact that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\big[|Y_i(t_0+\epsilon) - Y_1(t_0)| \big| \mathcal{F}(t_0) \big] = 0. \tag{4.10.37}$$

You may use (4.10.37) without proving it.

(v) Show that, conditioned on $\mathcal{F}(t_0)$, the pair of random variables

$$\big(X_1(t_0+\epsilon) - X_1(t_0), X_2(t_0+\epsilon) - X_2(t_0) \big)$$

has variances and covariance

$$\begin{aligned} V_i(\epsilon) &= \mathbb{E}\left[\big(X_i(t_0+\epsilon) - X_i(t_0) \big)^2 \Big| \mathcal{F}(t_0) \right] - M_i^2(\epsilon) \\ &= \sigma_i^2(t_0)\epsilon + o(\epsilon) \text{ for } i = 1, 2, & (4.10.38) \\ C(\epsilon) &= \mathbb{E}\left[\big(X_1(t_0+\epsilon) - X_1(t_0)\big)\big(X_2(t_0+\epsilon) - X_2(t_0)\big) \,\big|\, \mathcal{F}(t_0) \right] \\ &\quad -M_1(\epsilon)M_2(\epsilon) & (4.10.39) \\ &= \rho(t_0)\sigma_1(t_0)\sigma_2(t_0)\epsilon + o(\epsilon). \end{aligned}$$

(vi) Show that

$$\lim_{\epsilon \downarrow 0} \frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}} = \rho(t_0). \tag{4.10.40}$$

In other words, for small values of $\epsilon > 0$, conditioned on $\mathcal{F}(t_0)$, the correlation between the increments $X_1(t_0+\epsilon) - X_1(t_0)$ and $X_2(t_0+\epsilon) - X_2(t_0)$ is approximately equal to $\rho(t_0)$, and this approximation becomes exact as $\epsilon \downarrow 0$.

Exercise 4.18. Let a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t),$$

and let r denote the interest rate. We define the *market price of risk* to be

$$\theta = \frac{\alpha - r}{\sigma}$$

and the *state price density process* to be

$$\zeta(t) = \exp\left\{ - \theta W(t) - \left(r + \frac{1}{2}\theta^2\right)t\right\}.$$

(i) Show that

$$d\zeta(t) = -\theta\zeta(t)\,dW(t) - r\zeta(t)\,dt.$$

(ii) Let X denote the value of an investor's portfolio when he uses a portfolio process $\Delta(t)$. From (4.5.2), we have

$$dX(t) = rX(t)\,dt + \Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t).$$

Show that $\zeta(t)X(t)$ is a martingale. (Hint: Show that the differential $d\big(\zeta(t)X(t)\big)$ has no dt term.)

(iii) Let $T > 0$ be a fixed terminal time. Show that if an investor wants to begin with some initial capital $X(0)$ and invest in order to have portfolio value $V(T)$ at time T, where $V(T)$ is a given $\mathcal{F}(T)$-measurable random variable, then he must begin with initial capital

$$X(0) = \mathbb{E}[\zeta(T)V(T)].$$

In other words, the *present value* at time zero of the random payment $V(T)$ at time T is $\mathbb{E}[\zeta(T)V(T)]$. This justifies calling $\zeta(t)$ the state price density process.

Exercise 4.19. Let $W(t)$ be a Brownian motion, and define

$$B(t) = \int_0^t \operatorname{sign}(W(s))\,dW(s),$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(i) Show that $B(t)$ is a Brownian motion.

(ii) Use Itô's product rule to compute $d[B(t)W(t)]$. Integrate both sides of the resulting equation and take expectations. Show that $\mathbb{E}[B(t)W(t)] = 0$ (i.e., $B(t)$ and $W(t)$ are uncorrelated).

(iii) Verify that

$$dW^2(t) = 2W(t)\,dW(t) + dt.$$

(iv) Use Itô's product rule to compute $d[B(t)W^2(t)]$. Integrate both sides of the resulting equation and take expectations to conclude that

$$\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W^2(t).$$

Explain why this shows that, although they are uncorrelated normal random variables, $B(t)$ and $W(t)$ are not independent.

Exercise 4.20 (Local time). Let $W(t)$ be a Brownian motion. The Itô-Doeblin formula in differential form says that

$$df\big(W(t)\big) = f'\big(W(t)\big)\,dW(t) + \frac{1}{2}f''\big(W(t)\big)\,dt. \tag{4.10.41}$$

In integrated form, this formula is

$$f\big(W(T)\big) = f\big(W(0)\big) + \int_0^T f'\big(W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f''\big(W(t)\big)\,dt. \tag{4.10.42}$$

The usual statement of this formula assumes that the function $f''(x)$ is defined for every $x \in \mathbb{R}$ and is a continuous function of x. In fact, the formula still holds if there are finitely many points x where $f''(x)$ is undefined, provided that $f'(x)$ is defined for every $x \in \mathbb{R}$ and is a continuous function of x (and provided that $|f''(x)|$ is bounded so that the integral $\int_0^T f''\big(W(t)\big)\,dt$ is defined). However, if $f'(x)$ is not defined at some point, naive application of the Itô-Doeblin formula can give wrong answers, as this problem demonstrates.

(i) Let K be a positive constant, and define $f(x) = (x-K)^+$. Compute $f'(x)$ and $f''(x)$. Note that there is a point x where $f'(x)$ is not defined, and note also that $f''(x)$ is zero everywhere except at this point, where $f''(x)$ is also undefined.

(ii) Substitute the function $f(x) = (x-K)^+$ into (4.10.42), replacing the term $\frac{1}{2}\int_0^T f''\big(W(t)\big)\,dt$ by zero since f'' is zero everywhere except at one point, where it is not defined. Show that the two sides of this equation cannot be equal by computing their expected values.

(iii) To get some idea of what is going on here, define a sequence of functions $\{f_n\}_{n=1}^\infty$ by the formula

$$f_n(x) = \begin{cases} 0 & \text{if } x \le K - \frac{1}{2n}, \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & \text{if } K - \frac{1}{2n} \le x \le K + \frac{1}{2n}, \\ x - K & \text{if } x \ge K + \frac{1}{2n}. \end{cases}$$

Show that

$$f_n'(x) = \begin{cases} 0 & \text{if } x \le K - \frac{1}{2n}, \\ n(x-K) + \frac{1}{2} & \text{if } K - \frac{1}{2n} \le x \le K + \frac{1}{2n}, \\ 1 & \text{if } x \ge K + \frac{1}{2n}. \end{cases}$$

In particular, because we get the same value for $f_n'\left(K - \frac{1}{2n}\right)$ regardless of whether we use the formula for $x \le K - \frac{1}{2n}$ or the formula for $K - \frac{1}{2n} \le x \le K + \frac{1}{2n}$, the derivative $f'\left(K - \frac{1}{2n}\right)$ is defined. By the same argument, $f_n'\left(K + \frac{1}{2n}\right)$ is also defined. Verify that

$$f_n''(x) = \begin{cases} 0 \text{ if } x < K - \frac{1}{2n}, \\ n \text{ if } K - \frac{1}{2n} < x < K + \frac{1}{2n}, \\ 0 \text{ if } x < K + \frac{1}{2n}. \end{cases}$$

The second derivative $f''(x)$ is not defined when $x = K \pm \frac{1}{2n}$ because the formulas above disagree at those points.

(iv) Show that

$$\lim_{n\to\infty} f_n(x) = (x-K)^+$$

for every $x \in \mathbb{R}$ and

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 \text{ if } x < K, \\ \frac{1}{2} \text{ if } x = K, \\ 1 \text{ if } x > K. \end{cases}$$

The value of $\lim_{n\to\infty} f_n'(x)$ at a single point will not matter when we integrate in part (v) below, so instead of using the formula just derived, we will replace $\lim_{n\to\infty} f_n'(x)$ by

$$\mathbb{I}_{(K,\infty)}(x) = \begin{cases} 0 \text{ if } x \le K, \\ 1 \text{ if } x > K, \end{cases}$$

in (4.10.44) below. The two functions $\lim_{n\to\infty} f_n'(x)$ and $\mathbb{I}_{(K,\infty)}(x)$ agree except at the single point $x = K$.

For each n, the Itô-Doeblin formula applies to the function f_n because $f_n'(x)$ is defined for every x and is a continuous function of x, $f_n''(x)$ is defined for every x except the two points $x = K \pm \frac{1}{2n}$, and $|f''(x)|$ is bounded above by n. In integrated form, the Itô-Doeblin formula applied to f_n gives

$$f_n(W(T)) = f_n(W(0)) + \int_0^T f_n'(W(t))\, dW(t) + \int_0^T f_n''(W(t))\, dt. \quad (4.10.43)$$

If we now let $n \to \infty$ in this formula, we obtain

$$\begin{aligned}(W(T)-K)^+ &= (W(0)-K)^+ + \int_0^T \mathbb{I}_{(K,\infty)}(W(t))\, dW(t) \\ &\quad + \lim_{n\to\infty} n \int_0^T \mathbb{I}_{\left(K-\frac{1}{2n}, K+\frac{1}{2n}\right)}(W(t))\, dt. \quad (4.10.44)\end{aligned}$$

Let us define the *local time of the Brownian motion at K* to be

$$L_K(T) = \lim_{n\to\infty} n \int_0^T \mathbb{I}_{\left(K-\frac{1}{2n},K+\frac{1}{2n}\right)}\big(W(t)\big)\,dt.$$

(This formula is sometimes written as

$$L_K(T) = \int_0^T \delta_K\big(W(t)\big)\,dt,$$

where δ_K is the so-called "Dirac delta function" at K.) For a fixed n, the expression $\int_0^T \mathbb{I}_{\left(K-\frac{1}{2n},K+\frac{1}{2n}\right)}\big(W(t)\big)\,dt$ measures how much time between time 0 and time T the Brownian motion spends in the band of length $\frac{1}{n}$ centered at K. As $n \to \infty$, this has limit zero because the width of the band is approaching zero. However, before taking the limit, we multiply by n, and now it is not clear whether the limit will be zero, $+\infty$, or something in between. The limit will, of course, be random; it depends on the path of the Brownian motion.

(v) Show that if the path of the Brownian motion stays strictly below K on the time interval $[0,T]$, we have $L_K(T) = 0$.

(vi) We may solve (4.10.44) for $L_K(T)$, using the fact that $W(0) = 0$ and $K > 0$, to obtain

$$L_K(T) = \big(W(T) - K\big)^+ - \int_0^T \mathbb{I}_{(K,\infty)}\big(W(t)\big)\,dW(t). \tag{4.10.45}$$

From this, we see that $L_K(T)$ is never $+\infty$. Show that we cannot have $L_K(T) = 0$ almost surely. In other words, for some paths of the Brownian motion, we must have $L_K(T) > 0$. (It turns out that the paths that reach level K are those for which $L_K(T) > 0$.)

Exercise 4.21 (Stop-loss start-gain paradox). Let $S(t)$ be a geometric Brownian motion with mean rate of return zero. In other words,

$$dS(t) = \sigma S(t)\,dW(t),$$

where the volatility σ is constant. We assume the interest rate is $r = 0$.

Suppose we want to hedge a short position in a European call with strike price K and expiration date T. We assume that the call is initially out of the money (i.e., $S(0) < K$). Starting with zero capital ($X(0) = 0$), we could try the following portfolio strategy: own one share of the stock whenever its price strictly exceeds K, and own zero shares whenever its price is K or less. In other words, we use the hedging portfolio process

$$\Delta(t) = \mathbb{I}_{(K,\infty)}\big(S(t)\big).$$

The value of this hedge follows the stochastic differential equation

$$dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)X(t)\big)\,dt,$$

and since $r = 0$ and $X(0) = 0$, we have

$$X(T) = \sigma \int_0^T \mathbb{I}_{(K,\infty)}(S(t))S(t)\,dW(t). \tag{4.10.46}$$

Executing this hedge requires us to borrow from the money market to buy a share of stock whenever the stock price rises across level K and sell the share, repaying the money market debt, when it falls back across level K. (Recall that we have taken the interest rate to be zero. The situation we are describing can also occur with a nonzero interest rate, but it is more complicated to set up.) At expiration, if the stock price $S(T)$ is below K, there would appear to have been an even number of crossings of the level K, half in the up direction and half in the down direction, so that we would have bought and sold the stock repeatedly, each time at the same price K, and at the final time have no stock and zero debt to the money market. In other words, if $S(T) < K$, then $X(T) = 0$. On the other hand, if at the final time $S(T)$ is above K, we have bought the stock one more time than we sold it, so that we end with one share of stock and a debt of K to the money market. Hence, if $S(T) > K$, we have $X(T) = S(T) - K$. If at the final time $S(T) = K$, then we either own a share of stock valued at K and have a money market debt K or we have sold the stock and have zero money market debt. In either case, $X(T) = 0$. According to this argument, regardless of the final stock price, we have $X(T) = \big(S(T) - K\big)^+$. This kind of hedging is called a *stop-loss start-gain strategy*.

(i) Discuss the practical aspects of implementing the stop-loss start-gain strategy described above. Can it be done?

(ii) Apart from the practical aspects, does the mathematics of continuous-time stochastic calculus suggest that the stop-loss start-gain strategy can be implemented? In other words, with $X(T)$ defined by (4.10.46), is it really true that $X(T) = \big(S(T) - K\big)^+$?

5

Risk-Neutral Pricing

5.1 Introduction

In the binomial asset pricing model of Chapter 1 of Volume I, we showed how to price a derivative security by determining the initial capital required to hedge a short position in the derivative security. In a two-period model, this method led to the six equations (1.2.2), (1.2.3), and (1.2.5)–(1.2.8) in six unknowns in Volume I. Three of these unknowns were the position the hedge should take in the underlying asset at time zero, the position taken by the hedge at time one if the first coin toss results in H, and the position taken by the hedge at time one if the first coin toss results in T. The three other unknowns were the value of the derivative security at time zero, the value of the derivative security at time one if the first coin toss results in H, and the value of the derivative security at time one if the first coin toss results in T. The solution to these six equations provides both the value of the derivative security at all times and the hedge for the short position at all times, regardless of the outcome of the first coin toss.

In Theorem 1.2.2 of Volume I, we discovered a clever way to solve these six equations in six unknowns by first solving for the derivative security values V_n using the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$ in (1.2.16) and then computing the hedge positions from (1.2.17). Equation (1.2.16) says that under the risk-neutral probabilities, the discounted derivative security value is a martingale.

In Section 4.5 of this volume, we repeated the first part of this program. To determine the value of a European call, we determined the initial capital required to set up a portfolio that with probability one hedges a short position in the derivative security. Subsection 4.5.3, in which we equated the evolution of the discounted portfolio value with the evolution of the discounted option value, provides the continuous-time analogue of solving the six equations (1.2.2), (1.2.3), and (1.2.5)–(1.2.8) of Volume I. From that process, we obtained the delta-hedging rule (4.5.11) and we obtained the Black-Scholes-Merton partial differential equation (4.5.14) for the value of the call.

Now we execute the second part of the program. In this chapter, we discover a clever way to solve the partial differential equation (4.5.14) using a risk-neutral probability measure. After solving this equation, we can then compute the short option hedge using (4.5.11).

To accomplish this second part of the program, we show in Section 5.2 how to construct the risk-neutral measure in a model with a single underlying security. This step relies on Girsanov's Theorem, which is presented in Section 5.2. Risk-neutral pricing is a powerful method for computing prices of derivative securities, but it is fully justified only when it is accompanied by a hedge for a short position in the security being priced. In Section 5.3, we provide the conditions under which such a hedge exists in a model with a single underlying security. Section 5.4 generalizes the ideas of Sections 5.2 and 5.3 to models with multiple underlying securities. Furthermore, Section 5.4 provides conditions that guarantee that such a model does not admit arbitrage and that every derivative security in the model can be hedged.

5.2 Risk-Neutral Measure

5.2.1 Girsanov's Theorem for a Single Brownian Motion

In Theorem 1.6.1, we began with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a nonnegative random variable Z satisfying $\mathbb{E}Z = 1$. We then defined a new probability measure $\widetilde{\mathbb{P}}$ by the formula

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, dP(\omega) \text{ for all } A \in \mathcal{F}. \tag{5.2.1}$$

Any random variable X now has two expectations, one under the original probability measure $\mathbb{P}$, which we denote $\mathbb{E}X$, and the other under the new probability measure $\widetilde{\mathbb{P}}$, which we denote $\widetilde{\mathbb{E}}X$. These are related by the formula

$$\widetilde{\mathbb{E}}X = \mathbb{E}[XZ]. \tag{5.2.2}$$

If $\mathbb{P}\{Z > 0\} = 1$, then $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ agree which sets have probability zero and (5.2.2) has the companion formula

$$\mathbb{E}X = \widetilde{\mathbb{E}}\left[\frac{X}{Z}\right]. \tag{5.2.3}$$

We say Z is the *Radon-Nikodým derivative* of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

This is supposed to remind us that Z is like a ratio of these two probability measures. The reader may wish to review Section 3.1 of Volume I, where this concept is discussed in a finite probability model. In the case of a finite probability model, we actually have

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}. \tag{5.2.4}$$

If we multiply both sides of (5.2.4) by $\mathbb{P}(\omega)$ and then sum over ω in a set A, we obtain

$$\widetilde{\mathbb{P}}(A) = \sum_{\omega \in A} Z(\omega) P(\omega) \text{ for all } A \subset \Omega. \tag{5.2.5}$$

In a general probability model, we cannot write (5.2.4) because $\mathbb{P}(\omega)$ is typically zero for each individual ω, but we can write an analogue of (5.2.5). This is (5.2.1).

Example 1.6.6 shows how we can use this change-of-measure idea to move the mean of a normal random variable. In particular, if X is a standard normal random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, θ is a constant, and we define

$$Z = \exp\left\{ -\theta X - \frac{1}{2}\theta^2 \right\},$$

then under the probability measure $\widetilde{\mathbb{P}}$ given by (5.2.1), the random variable $Y = X + \theta$ is standard normal. In particular, $\widetilde{\mathbb{E}}Y = 0$, whereas $\mathbb{E}Y = \mathbb{E}X + \theta = \theta$. By changing the probability measure, we have changed the expectation of Y.

In this section, we perform a similar change of measure in order to change a mean, but this time for a whole process rather than for a single random variable. To set the stage, suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}(t)$, defined for $0 \le t \le T$, where T is a fixed final time. Suppose further that Z is an almost surely positive random variable satisfying $\mathbb{E}Z = 1$, and we define $\widetilde{\mathbb{P}}$ by (5.2.1). We can then define the *Radon-Nikodým derivative process*

$$Z(t) = \mathbb{E}[Z|\mathcal{F}(t)], \quad 0 \le t \le T. \tag{5.2.6}$$

This process in discrete time is discussed in Section 3.2 of Volume I. The Radon-Nikodým derivative process (5.2.6) is a martingale because of iterated conditioning (Theorem 2.3.2(iii)): for $0 \le s \le t \le T$,

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = \mathbb{E}\big[\mathbb{E}[Z|\mathcal{F}(t)]\big|\mathcal{F}(s)\big] = \mathbb{E}[Z|\mathcal{F}(s)] = Z(s). \tag{5.2.7}$$

Furthermore, it has the properties presented in the following two lemmas, which are continuous-time analogues of Lemmas 3.2.5 and 3.2.6 of Volume I.

Lemma 5.2.1. *Let t satisfying $0 \le t \le T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then*

$$\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)]. \tag{5.2.8}$$

PROOF: We use (5.2.2), the unbiasedness of conditional expectations (2.3.25), the property "taking out what is known" (Theorem 2.3.2(ii)), and the definition of $Z(t)$ to write

$$\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ] = \mathbb{E}\big[\mathbb{E}[YZ|\mathcal{F}(t)]\big] = \mathbb{E}\big[Y\mathbb{E}[Z|\mathcal{F}(t)]\big] = \mathbb{E}[YZ(t)]. \qquad \square$$

Lemma 5.2.2. *Let s and t satisfying $0 \le s \le t \le T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then*

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]. \tag{5.2.9}$$

PROOF: It is clear that $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]$ is $\mathcal{F}(s)$-measurable. We must check the partial-averaging property (Definition 2.3.1(ii)), which in this case is

$$\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\,d\widetilde{\mathbb{P}} = \int_A Y\,d\widetilde{\mathbb{P}} \text{ for all } A \in \mathcal{F}(s). \tag{5.2.10}$$

Note that because we are claiming that the right-hand side of (5.2.9) is the conditional expectation of Y under the $\widetilde{\mathbb{P}}$ probability measure, we must integrate with respect to the measure $\widetilde{\mathbb{P}}$ in the statement of the partial-averaging property (5.2.10). We may write the left-hand side of (5.2.10) as

$$\widetilde{\mathbb{E}}\left[\mathbb{I}_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\right]$$

and then use Lemma 5.2.1 for $\mathcal{F}(s)$-measurable random variables, use "taking out what is known," use the unbiasedness of conditional expectations (2.3.25), and finally use Lemma 5.2.1 for $\mathcal{F}(t)$-measurable random variables to write

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[\mathbb{I}_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]\right] &= \mathbb{E}\big[\mathbb{I}_A\mathbb{E}[YZ(t)|\mathcal{F}(s)]\big] \\
&= \mathbb{E}\big[\mathbb{E}[\mathbb{I}_A YZ(t)|\mathcal{F}(s)]\big] \\
&= \mathbb{E}[\mathbb{I}_A YZ(t)] \\
&= \widetilde{\mathbb{E}}[\mathbb{I}_A Y] \\
&= \int_A Y\,d\widetilde{\mathbb{P}}.
\end{aligned}$$

This verifies (5.2.10), which in turn proves (5.2.9). $\square$

Theorem 5.2.3 (Girsanov, one dimension). *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0 \le t \le T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\}, \tag{5.2.11}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du, \tag{5.2.12}$$

and assume that[1]

$$\mathbb{E}\int_0^T \Theta^2(u)Z^2(u)\,du < \infty. \tag{5.2.13}$$

Set $Z = Z(T)$. *Then* $\mathbb{E}Z = 1$ *and under the probability measure* $\widetilde{\mathbb{P}}$ *given by (5.2.1), the process* $\widetilde{W}(t)$, $0 \le t \le T$, *is a Brownian motion.*

PROOF: We use Lévy's Theorem, Theorem 4.6.4, which says that a martingale starting at zero at time zero, with continuous paths and with quadratic variation equal to t at each time t, is a Brownian motion. The process $\widetilde{W}$ starts at zero at time zero and is continuous. Furthermore, $[\widetilde{W},\widetilde{W}](t) = [W,W](t) = t$ because the term $\int_0^t \Theta(u)\,du$ in the definition of $\widetilde{W}(t)$ contributes zero quadratic variation. In other words,

$$d\widetilde{W}(t)\,d\widetilde{W}(t) = \big(dW(t) + \Theta(t)\,dt\big)^2 = dW(t)\,dW(t) = dt.$$

It remains to show that $\widetilde{W}(t)$ is a martingale under $\widetilde{\mathbb{P}}$. We first observe that $Z(t)$ is a martingale under $\mathbb{P}$. With

$$X(t) = -\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du$$

and $f(x) = e^x$ so that $f'(x) = e^x$ and $f''(x) = e^x$, we have

$$\begin{aligned}
dZ(t) &= df\big(X(t)\big)\\
&= f'\big(X(t)\big)\,dX(t) + \frac{1}{2}f''\big(X(t)\big)\,dX(t)\,dX(t)\\
&= e^{X(t)}\Big(-\Theta(t)\,dW(t) - \frac{1}{2}\Theta^2(t)\,dt\Big) + \frac{1}{2}e^{X(t)}\Theta^2(t)\,dt\\
&= -\Theta(t)Z(t)\,dW(t).
\end{aligned}$$

Integrating both sides of the equation above, we see that

$$Z(t) = Z(0) - \int_0^t \Theta(u)Z(u)\,dW(u). \tag{5.2.14}$$

Because Itô integrals are martingales, $Z(t)$ is a martingale. In particular, $\mathbb{E}Z = \mathbb{E}Z(T) = Z(0) = 1$.

Because $Z(t)$ is a martingale and $Z = Z(T)$, we have

$$Z(t) = \mathbb{E}[Z(T)|\mathcal{F}(t)] = \mathbb{E}[Z|\mathcal{F}(t)], \quad 0 \le t \le T.$$

This shows that $Z(t)$, $0 \le t \le T$, is a Radon-Nikodým derivative process as defined in (5.2.6), and Lemmas 5.2.1 and 5.2.2 apply to this situation.

[1] Condition (5.2.13) is imposed to ensure that the Itô integral in (5.2.14) is defined and is a martingale. This is (4.3.1) imposed in the construction of Itô integrals.

We next show that $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$. To see this, we compute the differential using Itô's product rule (Corollary 4.6.3):

$$\begin{aligned} d\big(\widetilde{W}(t)Z(t)\big) &= \widetilde{W}(t)\,dZ(t) + Z(t)\,d\widetilde{W}(t) + d\widetilde{W}(t)\,dZ(t) \\ &= -\widetilde{W}(t)\Theta(t)Z(t)\,dW(t) + Z(t)\,dW(t) + Z(t)\,\Theta(t)\,dt \\ &\qquad + \big(dW(t) + \Theta(t)\,dt\big)\big(-\Theta(t)Z(t)\,dW(t)\big) \\ &= \big(-\widetilde{W}(t)\Theta(t) + 1\big)Z(t)\,dW(t). \end{aligned}$$

Because the final expression has no dt term, the process $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

Now let $0 \le s \le t \le T$ be given. Lemma 5.2.2 and the martingale property for $\widetilde{W}(t)Z(t)$ under $\mathbb{P}$ imply

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\widetilde{W}(s)Z(s) = \widetilde{W}(s).$$

This shows that $\widetilde{W}(t)$ is a martingale under $\widetilde{\mathbb{P}}$. The proof is complete. □

The probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ in Girsanov's Theorem are equivalent (i.e., they agree about which sets have probability zero and hence about which sets have probability one). This is because $\mathbb{P}\{Z > 0\} = 1$; see Definition 1.6.3 and the discussion following it. In the remainder of this section, we set up an asset price model in which $\mathbb{P}$ is the actual probability measure and $\widetilde{\mathbb{P}}$ is the risk-neutral measure. We want these probabilities to agree about what is possible and what is impossible, and they do. In the discrete-time binomial model of Volume I, the actual and risk-neutral probability measures agree about which moves are possible (i.e., they both give positive probability to an up move, positive probability to a down move, and the sizes (but not the probabilities) of the up and down moves are the same whether we are working under the actual probability measure or the risk-neutral probability measure). The set of possible asset price paths is a tree in the binomial model, and both the actual probability measure and the risk-neutral probability measure are based on the same tree. In the continuous-time model, there are infinitely many possible paths, and this agreement about what is possible and what is not possible is the equivalence of Definition 1.6.3.

5.2.2 Stock Under the Risk-Neutral Measure

Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be a filtration for this Brownian motion. Here T is a fixed final time. Consider a stock price process whose differential is

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \quad 0 \le t \le T. \tag{5.2.15}$$

The mean rate of return $\alpha(t)$ and the volatility $\sigma(t)$ are allowed to be adapted processes. We assume that, for all $t \in [0, T]$, $\sigma(t)$ is almost surely not zero.

This stock price is a generalized geometric Brownian motion (see Example 4.4.8, in particular, (4.4.27)), and an equivalent way of writing (5.2.15) is (see (4.4.26))

$$S(t) = S(0) \exp\left\{ \int_0^t \sigma(s)\, dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds \right\}. \tag{5.2.16}$$

In addition, suppose we have an adapted interest rate process $R(t)$. We define the *discount process*

$$D(t) = e^{-\int_0^t R(s)\, ds} \tag{5.2.17}$$

and note that

$$dD(t) = -R(t)D(t)\, dt. \tag{5.2.18}$$

To obtain (5.2.18) from (5.2.17), we can define $I(t) = \int_0^t R(s)\, ds$ so that $dI(t) = R(t)\, dt$ and $dI(t)\, dI(t) = 0$. We introduce the function $f(x) = e^{-x}$, for which $f'(x) = -f(x)$, $f''(x) = f(x)$, and then use the Itô-Doeblin formula to write

$$\begin{aligned} dD(t) &= df\big(I(t)\big) \\ &= f'\big(I(t)\big)\, dI(t) + \frac{1}{2} f''\big(I(t)\big)\, dI(t)\, dI(t) \\ &= -f\big(I(t)\big) R(t)\, dt \\ &= -R(t)D(t)\, dt. \end{aligned}$$

Observe that although $D(t)$ is random, it has zero quadratic variation. This is because it is "smooth." It has a derivative, namely $D'(t) = -R(t)D(t)$, and one does not need stochastic calculus to do this computation. The stock price $S(t)$ is random and has nonzero quadratic variation. It is "more random" than $D(t)$. If we invest in the stock, we have no way of knowing whether the next move of the driving Brownian motion will be up or down, and this move directly affects the stock price. Hence, we face a high degree of uncertainty. On the other hand, consider a money market account with variable interest rate $R(t)$, where money is rolled over at this interest rate. If the price of a share of this money market account at time zero is 1, then the price of a share of this money market account at time t is $e^{\int_0^t R(s)\, ds} = \frac{1}{D(t)}$. If we invest in this account, we know the interest rate at the time of the investment and hence have a high degree of certainty about what the return will be over a short period of time. Over longer periods, we are less certain because the interest rate is variable, and at the time of investment, we do not know the future interest rates that will be applied. However, the randomness in the model affects the money market account only indirectly by affecting the interest rate. Changes in the interest rate do not affect the money market account instantaneously but only when they act over time. (Warning: The money market account is not a bond. For a bond, a change in the interest rate can have an instantaneous effect on price.) Unlike the price of the money market account, the stock price is susceptible to instantaneous unpredictable changes and is, in this sense, "more random" than $D(t)$. Our mathematical model captures this effect because $S(t)$ has nonzero quadratic variation, while $D(t)$ has zero quadratic variation.

The discounted stock price process is

$$D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\}, \tag{5.2.19}$$

and its differential is

$$\begin{aligned} d\big(D(t)S(t)\big) &= \big(\alpha(t) - R(t)\big)D(t)S(t)\,dt + \sigma(t)D(t)S(t)\,dW(t) \\ &= \sigma(t)D(t)S(t)\big[\Theta(t)\,dt + dW(t)\big], \end{aligned} \tag{5.2.20}$$

where we define the *market price of risk* to be

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}. \tag{5.2.21}$$

One can derive (5.2.20) either by applying the Itô-Doeblin formula to the right-hand side of (5.2.19) or by using Itô's product rule and the formulas (5.2.15) and (5.2.18). The first line of (5.2.20), compared with (5.2.15), shows that the mean rate of return of the discounted stock price is $\alpha(t) - R(t)$, which is the mean rate $\alpha(t)$ of the undiscounted stock price, reduced by the interest rate $R(t)$. The volatility of the discounted stock price is the same as the volatility of the undiscounted stock price.

We introduce the probability measure $\widetilde{\mathbb{P}}$ defined in Girsanov's Theorem, Theorem 5.2.3, which uses the market price of risk $\Theta(t)$ given by (5.2.21). In terms of the Brownian motion $\widetilde{W}(t)$ of that theorem, we may rewrite (5.2.20) as

$$d\big(D(t)S(t)\big) = \sigma(t)D(t)S(t)\,d\widetilde{W}(t). \tag{5.2.22}$$

We call $\widetilde{\mathbb{P}}$, the measure defined in Girsanov's Theorem, the *risk-neutral measure* because it is equivalent to the original measure $\mathbb{P}$ and it renders the discounted stock price $D(t)S(t)$ into a martingale. Indeed, according to (5.2.22),

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)\,d\widetilde{W}(u),$$

and under $\widetilde{\mathbb{P}}$ the process $\int_0^t \sigma(u)D(u)S(u)\,d\widetilde{W}(u)$ is an Itô integral and hence a martingale.

The undiscounted stock price $S(t)$ has mean rate of return equal to the interest rate under $\widetilde{\mathbb{P}}$, as one can verify by making the replacement $dW(t) = -\Theta(t)\,dt + d\widetilde{W}(t)$ in (5.2.15). With this substitution, (5.2.15) becomes

$$dS(t) = R(t)S(t)\,dt + \sigma(t)S(t)\,d\widetilde{W}(t). \tag{5.2.23}$$

We can either solve this equation for $S(t)$ or simply replace the Itô integral $\int_0^t \sigma(s)\,dW(s)$ by its equivalent $\int_0^t \sigma(s)\,d\widetilde{W}(s) - \int_0^t \big(\alpha(s) - R(s)\big)\,ds$ in (5.2.16) to obtain the formula

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,d\widetilde{W}(s) + \int_0^t \left(R(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}. \tag{5.2.24}$$

In discrete time, the change of measure does not change the binomial tree, only the probabilities on the branches of the tree. In continuous time, the change from the actual measure $\mathbb{P}$ to the risk-neutral measure $\widetilde{\mathbb{P}}$ changes the mean rate of return of the stock but not the volatility. The volatility tells us which stock price paths are possible—namely those for which the log of the stock price accumulates quadratic variation at rate $\sigma^2(t)$ per unit time. After the change of measure, we are still considering the same set of stock price paths, but we have shifted the probability on them. If $\alpha(t) > R(t)$, as it normally is, then the change of measure puts more probability on the paths with lower return so that the overall mean rate of return is reduced from $\alpha(t)$ to $R(t)$.

5.2.3 Value of Portfolio Process Under the Risk-Neutral Measure

Consider an agent who begins with initial capital $X(0)$ and at each time t, $0 \le t \le T$, holds $\Delta(t)$ shares of stock, investing or borrowing at the interest rate $R(t)$ as necessary to finance this. The differential of this agent's portfolio value is given by the analogue of (4.5.2) for this case of random $\alpha(t)$, $\sigma(t)$, and $R(t)$, and this works out to be

$$\begin{aligned}
dX(t) &= \Delta(t)\,dS(t) + R(t)\big(X(t) - \Delta(t)S(t)\big)\,dt\\
&= \Delta(t)\big(\alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t)\big) + R(t)\big(X(t) - \Delta(t)S(t)\big)dt\\
&= R(t)X(t)\,dt + \Delta(t)\big(\alpha(t) - R(t)\big)S(t)\,dt + \Delta(t)\sigma(t)S(t)\,dW(t)\\
&= R(t)X(t)\,dt + \Delta(t)\sigma(t)S(t)\big[\Theta(t)\,dt + dW(t)\big].
\end{aligned} \tag{5.2.25}$$

Itô's product rule, (5.2.18), and (5.2.20) imply

$$\begin{aligned}
d\big(D(t)X(t)\big) &= \Delta(t)\sigma(t)D(t)S(t)\big[\Theta(t)\,dt + dW(t)\big]\\
&= \Delta(t)\,d\big(D(t)S(t)\big).
\end{aligned} \tag{5.2.26}$$

Changes in the discounted value of an agent's portfolio are entirely due to fluctuations in the discounted stock price. We may use (5.2.22) to rewrite (5.2.26) as

$$d\big(D(t)X(t)\big) = \Delta(t)\sigma(t)D(t)S(t)\,d\widetilde{W}(t). \tag{5.2.27}$$

Our agent has two investment options: (1) a money market account with rate of return $R(t)$, and (2) a stock with mean rate of return $R(t)$ under $\widetilde{\mathbb{P}}$. Regardless of how the agent invests, the mean rate of return for his portfolio will be $R(t)$ under $\widetilde{\mathbb{P}}$, and hence the discounted value of his portfolio, $D(t)X(t)$, will be a martingale. This is the content of (5.2.27).

5.2.4 Pricing Under the Risk-Neutral Measure

In Section 4.5, we derived the Black-Scholes-Merton equation for the value of a European call by asking what initial capital $X(0)$ and portfolio process $\Delta(t)$ an agent would need in order to hedge a short position in the call (i.e., in order to have $X(T) = (S(T) - K)^+$ almost surely). In this section, we generalize the question. Let $V(T)$ be an $\mathcal{F}(T)$-measurable random variable. This represents the payoff at time T of a derivative security. We allow this payoff to be path-dependent (i.e., to depend on anything that occurs between times 0 and T), which is what $\mathcal{F}(T)$-measurability means. We wish to know what initial capital $X(0)$ and portfolio process $\Delta(t)$, $0 \leq t \leq T$, an agent would need in order to hedge a short position in this derivative security, i.e., in order to have

$$X(T) = V(T) \text{ almost surely.} \tag{5.2.28}$$

In Section 4.5, the mean rate of return, volatility, and interest rate were constant. In this section, we do not assume a constant mean rate of return, volatility, and interest rate.

Our agent wishes to choose initial capital $X(0)$ and portfolio strategy $\Delta(t)$, $0 \leq t \leq T$, such that (5.2.28) holds. We shall see in the next section that this can be done. Once it has been done, the fact that $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$ implies

$$D(t)X(t) = \widetilde{\mathbb{E}}\big[D(T)X(T)\big|\mathcal{F}(t)\big] = \widetilde{\mathbb{E}}\big[D(T)V(T)|\mathcal{F}(t)\big]. \tag{5.2.29}$$

The value $X(t)$ of the hedging portfolio in (5.2.29) is the capital needed at time t in order to successfully complete the hedge of the short position in the derivative security with payoff $V(T)$. Hence, we can call this the *price* $V(t)$ of the derivative security at time t, and (5.2.29) becomes

$$D(t)V(t) = \widetilde{\mathbb{E}}\big[D(T)V(T)|\mathcal{F}(t)\big],\ 0 \leq t \leq T. \tag{5.2.30}$$

This is the continuous-time analogue of the risk-neutral pricing formula (2.4.10) in the binomial model of Volume I. Dividing (5.2.30) by $D(t)$, which is $\mathcal{F}(t)$-measurable and hence can be moved inside the conditional expectation on the right-hand side of (5.2.30), and recalling the definition of $D(t)$, we may write (5.2.30) as

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(u)\,du}V(T)\middle|\mathcal{F}(t)\right],\ 0 \leq t \leq T. \tag{5.2.31}$$

This is the continuous-time analogue of (2.4.11) of Volume I. We shall refer to both (5.2.30) and (5.2.31) as the *risk-neutral pricing formula* for the continuous-time model.

5.2.5 Deriving the Black-Scholes-Merton Formula

The addition of Merton's name to what has traditionally been called the Black-Scholes formula is explained in the Notes to Chapter 4, Section 4.9.

To obtain the Black-Scholes-Merton price of a European call, we assume a constant volatility σ, constant interest rate r, and take the derivative security payoff to be $V(T) = \big(S(T) - K\big)^+$. The right-hand side of (5.2.31) becomes

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\big(S(T)-K\big)^+\Big|\mathcal{F}(t)\right].$$

Because geometric Brownian motion is a Markov process, this expression depends on the stock price $S(t)$ and of course on the time t at which the conditional expectation is computed, but not on the stock price prior to time t. In other words, there is a function $c(t,x)$ such that

$$c\big(t,S(t)\big) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\big(S(T)-K\big)^+\Big|\mathcal{F}(t)\right]. \tag{5.2.32}$$

We can compute $c(t,x)$ using the Independence Lemma, Lemma 2.3.4. With constant σ and r, equation (5.2.24) becomes

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\},$$

and we may thus write

$$\begin{aligned} S(T) &= S(t)\exp\left\{\sigma\big(\widetilde{W}(T) - \widetilde{W}(t)\big) + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\} \\ &= S(t)\exp\left\{-\sigma\sqrt{\tau}\,Y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\}, \end{aligned}$$

where Y is the standard normal random variable

$$Y = -\frac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{T-t}},$$

and τ is the "time to expiration" $T-t$. We see that $S(T)$ is the product of the $\mathcal{F}(t)$-measurable random variable $S(t)$ and the random variable

$$\exp\left\{-\sigma\sqrt{\tau}\,Y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\},$$

which is independent of $\mathcal{F}(t)$. Therefore, (5.2.32) holds with

$$\begin{aligned} c(t,x) &= \widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,Y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+ e^{-\frac{1}{2}y^2}\,dy. \end{aligned}$$

The integrand

$$\left(x\exp\left\{-\sigma\sqrt{\tau}\,y+\left(r-\frac{1}{2}\sigma^2\right)\tau\right\}-K\right)^+$$

is positive if and only if

$$y<d_-(\tau,x)=\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau\right]. \tag{5.2.33}$$

Therefore,

$$\begin{aligned}
&c(t,x)\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)}e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,y+\left(r-\frac{1}{2}\sigma^2\right)\tau\right\}-K\right)e^{-\frac{1}{2}y^2}\,dy\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)}x\exp\left\{-\frac{y^2}{2}-\sigma\sqrt{\tau}\,y-\frac{\sigma^2\tau}{2}\right\}dy\\
&\qquad-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)}e^{-r\tau}Ke^{-\frac{1}{2}y^2}\,dy\\
&=\frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)}\exp\left\{-\frac{1}{2}\left(y+\sigma\sqrt{\tau}\right)^2\right\}dy-e^{-r\tau}KN\big(d_-(\tau,x)\big)\\
&=\frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)+\sigma\sqrt{\tau}}\exp\left\{-\frac{z^2}{2}\right\}dz-e^{-r\tau}KN\big(d_-(\tau,x)\big)\\
&=xN\big(d_+(\tau,x)\big)-e^{-r\tau}KN\big(d_-(\tau,x)\big),
\end{aligned}$$

where

$$d_+(\tau,x)=d_-(\tau,x)+\sigma\sqrt{\tau}=\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K}+\left(r+\frac{1}{2}\sigma^2\right)\tau\right]. \tag{5.2.34}$$

For future reference, we introduce the notation

$$\mathrm{BSM}(\tau,x;K,r,\sigma)=\widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,Y+\left(r-\frac{1}{2}\sigma^2\right)\tau\right\}-K\right)^+\right], \tag{5.2.35}$$

where Y is a standard normal random variable under $\widetilde{\mathbb{P}}$. We have just shown that

$$\mathrm{BSM}(\tau,x;K,r,\sigma)=xN\big(d_+(\tau,x)\big)-e^{-r\tau}KN\big(d_-(\tau,x)\big). \tag{5.2.36}$$

In Section 4.5, we derived the Black-Scholes-Merton partial differential equation (4.5.14) and then provided the solution in equation (4.5.19) without explaining how one obtains this solution (although one can verify after the fact that (4.5.19) does indeed solve (4.5.14); see Exercise 4.9 of Chapter 4). Here we have derived the solution by the device of switching to the risk-neutral measure.

5.3 Martingale Representation Theorem

The risk-neutral pricing formula for the price (value) at time t of a derivative security paying $V(T)$ at time T, equation (5.2.31), was derived under the assumption that if an agent begins with the correct initial capital, there is a portfolio process $\Delta(t)$, $0 \le t \le T$, such that the agent's portfolio value at the final time T will be $V(T)$ almost surely. Under this assumption, we determined the "correct initial capital" to be (set $t = 0$ in (5.2.31))

$$V(0) = \widetilde{\mathbb{E}}\big[D(T)V(T)\big],$$

and the value of the hedging portfolio at every time t, $0 \le t \le T$, to be $V(t)$ given by (5.2.31). In this section, in the model with one stock driven by one Brownian motion, we verify the assumption on which the risk-neutral pricing formula (5.2.31) is based. We take up the case of multiple Brownian motions and multiple stocks in Section 5.4.

5.3.1 Martingale Representation with One Brownian Motion

The existence of a hedging portfolio in the model with one stock and one Brownian motion depends on the following theorem, which we state without proof.

Theorem 5.3.1 (Martingale representation, one dimension). *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be the filtration generated by this Brownian motion. Let $M(t)$, $0 \le t \le T$, be a martingale with respect to this filtration (i.e., for every t, $M(t)$ is $\mathcal{F}(t)$-measurable and for $0 \le s \le t \le T$, $\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)$). Then there is an adapted process $\Gamma(u)$, $0 \le u \le T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)\, dW(u),\ 0 \le t \le T. \tag{5.3.1}$$

The Martingale Representation Theorem asserts that when the filtration is the one generated by a Brownian motion (i.e., the only information in $\mathcal{F}(t)$ is that gained from observing the Brownian motion up to time t), then every martingale with respect to this filtration is an initial condition plus an Itô integral with respect to the Brownian motion. The relevance to hedging of this is that the *only* source of uncertainty in the model is the Brownian motion appearing in Theorem 5.3.1, and hence there is only one source of uncertainty to be removed by hedging. This assumption implies that the martingale cannot have jumps because Itô integrals are continuous. If we want to have a martingale with jumps, we will need to build a model that includes sources of uncertainty different from (or in addition to) Brownian motion.

The assumption that the filtration in Theorem 5.3.1 is the one generated by the Brownian motion is more restrictive than the assumption of Girsanov's

Theorem, Theorem 5.2.3, in which the filtration can be larger than the one generated by the Brownian motion. If we include this extra restriction in Girsanov's Theorem, then we obtain the following corollary. The first paragraph of this corollary is just a repeat of Girsanov's Theorem; the second part contains the new assertion.

Corollary 5.3.2. *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be the filtration generated by this Brownian motion. Let $\Theta(t)$, $0 \le t \le T$, be an adapted process, define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\},$$
$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du,$$

and assume that $\widetilde{\mathbb{E}}\int_0^T \Theta^2(u)Z^2(u)\,du < \infty$. Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$, and under the probability measure $\widetilde{\mathbb{P}}$ given by (5.2.1), the process $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

Now let $\widetilde{M}(t)$, $0 \le t \le T$, be a martingale under $\widetilde{\mathbb{P}}$. Then there is an adapted process $\widetilde{\Gamma}(u)$, $0 \le u \le T$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)\,d\widetilde{W}(u),\ 0 \le t \le T. \tag{5.3.2}$$

Corollary 5.3.2 is not a trivial consequence of the Martingale Representation Theorem, Theorem 5.3.1, with $\widetilde{W}(t)$ replacing $W(t)$ because the filtration $\mathcal{F}(t)$ in this corollary is generated by the process $W(t)$, not the $\widetilde{\mathbb{P}}$-Brownian motion $\widetilde{W}(t)$. However, the proof is not difficult and is left to the reader as Exercise 5.5.

5.3.2 Hedging with One Stock

We now return to the hedging problem. We begin with the model of Subsection 5.2.2, which has the stock price process (5.2.15) and an interest rate process $R(t)$ that generates the discount process (5.2.17). Recall the assumption that, for all $t \in [0,T]$, the volatility $\sigma(t)$ is almost surely not zero. We make the additional assumption that the filtration $\mathcal{F}(t)$, $0 \le t \le T$, is generated by the Brownian motion $W(t)$, $0 \le t \le T$.

Let $V(T)$ be an $\mathcal{F}(T)$-measurable random variable and, for $0 \le t \le T$, define $V(t)$ by the risk-neutral pricing formula (5.2.31). Then, according to (5.2.30),

$$D(t)V(t) = \widetilde{\mathbb{E}}\big[D(T)V(T)\big|\mathcal{F}(t)\big].$$

This is a $\widetilde{\mathbb{P}}$-martingale; indeed, iterated conditioning implies that, for $0 \le s \le t \le T$,

$$\begin{aligned}\widetilde{\mathbb{E}}\big[D(t)V(t)\big|\mathcal{F}(s)\big] &= \widetilde{\mathbb{E}}\big[\widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]\big|\mathcal{F}(s)\big]\\ &= \widetilde{\mathbb{E}}\big[D(T)V(T)\big|\mathcal{F}(s)\big]\\ &= D(s)V(s).\end{aligned} \tag{5.3.3}$$

Therefore, $D(t)V(t)$ has a representation as (recall that $D(0)V(0) = V(0)$)

$$D(t)V(t) = V(0) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{W}(u),\ 0 \le t \le T. \tag{5.3.4}$$

On the other hand, for any portfolio process $\Delta(t)$, the differential of the discounted portfolio value is given by (5.2.27), and hence

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)\, d\widetilde{W}(u),\ 0 \le t \le T. \tag{5.3.5}$$

In order to have $X(t) = V(t)$ for all t, we should choose

$$X(0) = V(0) \tag{5.3.6}$$

and choose $\Delta(t)$ to satisfy

$$\Delta(t)\sigma(t)D(t)S(t) = \widetilde{\Gamma}(t),\ 0 \le t \le T, \tag{5.3.7}$$

which is equivalent to

$$\Delta(t) = \frac{\widetilde{\Gamma}(t)}{\sigma(t)D(t)S(t)},\ 0 \le t \le T. \tag{5.3.8}$$

With these choices, we have a hedge for a short position in the derivative security with payoff $V(T)$ at time T.

There are two key assumptions that make the hedge possible. The first is that the volatility $\sigma(t)$ is not zero, so equation (5.3.7) can be solved for $\Delta(t)$. If the volatility vanishes, then the randomness of the Brownian motion does not enter the stock, although it may still enter the payoff $V(T)$ of the derivative security. In this case, the stock is no longer an effective hedging instrument. The other key assumption is that $\mathcal{F}(t)$ is generated by the underlying Brownian motion (i.e., there is no randomness in the derivative security apart from the Brownian motion randomness, which can be hedged by trading the stock). Under these two assumptions, every $\mathcal{F}(T)$-measurable derivative security can be hedged. Such a model is said to be *complete.*

The Martingale Representation Theorem argument of this section justifies the risk-neutral pricing formulas (5.2.30) and (5.2.31), but it does not provide a practical method of finding the hedging portfolio $\Delta(t)$. The final formula (5.3.8) for $\Delta(t)$ involves the integrand $\widetilde{\Gamma}(t)$ in the martingale representation (5.3.4) of the discounted derivative security price. While the Martingale Representation Theorem guarantees that such a process $\widetilde{\Gamma}$ exists and hence a hedge $\Delta(t)$ exists, it does not provide a method for finding $\widetilde{\Gamma}(t)$. We return to this point in Chapter 6.

5.4 Fundamental Theorems of Asset Pricing

In this section, we extend the discussions of Sections 5.2 and 5.3 to the case of multiple stocks driven by multiple Brownian motions. In the process, we develop and illustrate the two fundamental theorems of asset pricing. In addition to providing these theorems, in this section we give precise definitions of some of the basic concepts of derivative security pricing in continuous-time models

5.4.1 Girsanov and Martingale Representation Theorems

The two theorems on which this section is based are the multidimensional Girsanov Theorem and the multidimensional Martingale Representation Theorem. We state them here.

Throughout this section,

$$W(t) = \big(W_1(t), \dots, W_d(t)\big)$$

is a multidimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We interpret $\mathbb{P}$ to be the *actual probability measure*, the one that would be observed from empirical studies of price data. Associated with this Brownian motion, we have a filtration $\mathcal{F}(t)$ (see Definition 3.3.3). We shall have a fixed final time T, and we shall assume that $\mathcal{F} = \mathcal{F}(T)$. We do not always assume that the filtration is the one generated by the Brownian motion. When that is assumed, we say so explicitly.

Theorem 5.4.1 (Girsanov, multiple dimensions). *Let T be a fixed positive time, and let $\Theta(t) = \big(\Theta_1(t), \dots, \Theta_d(t)\big)$ be a d-dimensional adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du\right\}, \tag{5.4.1}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du, \tag{5.4.2}$$

and assume that

$$\mathbb{E}\int_0^T \|\Theta(u)\|^2 Z^2(u)\, du < \infty. \tag{5.4.3}$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$, and under the probability measure $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t)$ is a d-dimensional Brownian motion.

The Itô integral in (5.4.1) is

$$\int_0^t \Theta(u) \cdot dW(u) = \int_0^t \sum_{j=1}^d \Theta_j(u)\, dW_j(u) = \sum_{j=1}^d \int_0^t \Theta_j(u)\, dW_j(u).$$

Also, in (5.4.1), $\|\Theta(u)\|$ denotes the Euclidean norm

$$\|\Theta(u)\| = \left(\sum_{j=1}^d \Theta_j^2(u) \right)^{\frac{1}{2}},$$

and (5.4.2) is shorthand notation for $\widetilde{W}(t) = \big(\widetilde{W}_1(t), \dots, \widetilde{W}_d(t)\big)$ with

$$\widetilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u)\, du, \quad j = 1, \dots, d.$$

The remarkable thing about the conclusion of the multidimensional Girsanov Theorem is that the component processes of $\widetilde{W}(t)$ are independent under $\widetilde{\mathbb{P}}$. This is part of what it means to be a d-dimensional Brownian motion. The component processes of $W(t)$ are independent under $\mathbb{P}$, but each of the $\Theta_j(t)$ processes can depend in a path-dependent, adapted way on all of the Brownian motions $W_1(t), \dots, W_d(t)$. Therefore, under $\mathbb{P}$, the components of $\widetilde{W}(t)$ can be far from independent. Yet, after the change to the measure $\widetilde{\mathbb{P}}$, these components are independent. The proof of Theorem 5.4.1 is like that of the one-dimensional Girsanov Theorem 5.2.3, except it uses a d-dimensional version of Lévy's Theorem. The proof for $d = 2$ based on the two-dimensional Lévy Theorem, Theorem 4.6.5, is left to the reader as Exercise 5.6.

Theorem 5.4.2 (Martingale representation, multiple dimensions). *Let T be a fixed positive time, and assume that $\mathcal{F}(t)$, $0 \le t \le T$, is the filtration generated by the d-dimensional Brownian motion $W(t)$, $0 \le t \le T$. Let $M(t)$, $0 \le t \le T$, be a martingale with respect to this filtration under $\mathbb{P}$. Then there is an adapted, d-dimensional process $\Gamma(u) = \big(\Gamma_1(u), \dots, \Gamma_d(u)\big)$, $0 \le u \le T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u),\ 0 \le t \le T. \tag{5.4.4}$$

If, in addition, we assume the notation and assumptions of Theorem 5.4.1 and if $\widetilde{M}(t)$, $0 \le t \le T$, is a $\widetilde{\mathbb{P}}$-martingale, then there is an adapted, d-dimensional process $\widetilde{\Gamma}(u) = \big(\widetilde{\Gamma}_1(u), \dots, \widetilde{\Gamma}_d(u)\big)$ such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) \cdot d\widetilde{W}(u),\ 0 \le t \le T. \tag{5.4.5}$$

5.4.2 Multidimensional Market Model

We assume there are m stocks, each with stochastic differential

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\,dW_j(t), \quad i = 1,\ldots,m. \tag{5.4.6}$$

We assume that the mean rate of return vector $(\alpha_i(t))_{i=1,\ldots,m}$ and the volatility matrix $(\sigma_{ij}(t))_{i=1,\ldots,m;j=1,\ldots,d}$ are adapted processes. These stocks are typically correlated. To see the nature of this correlation, we set $\sigma_i(t) = \sqrt{\sum_{j=1}^{d}\sigma_{ij}^2(t)}$, which we assume is never zero, and we define processes

$$B_i(t) = \sum_{j=1}^{d}\int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\,dW_j(u), \quad i = 1,\ldots,m. \tag{5.4.7}$$

Being a sum of stochastic integrals, each $B_i(t)$ is a continuous martingale. Furthermore,

$$dB_i(t)\,dB_i(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}\,dt = dt.$$

According to Lévy's Theorem, Theorem 4.6.4, $B_i(t)$ is a Brownian motion. We may rewrite (5.4.6) in terms of the Brownian motion $B_i(t)$ as

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + \sigma_i(t)S_i(t)\,dB_i(t). \tag{5.4.8}$$

From this formula, we see that $\sigma_i(t)$ is the volatility of $S_i(t)$.

For $i \neq k$, the Brownian motions $B_i(t)$ and $B_k(t)$ are typically not independent. To see this, we first note that

$$dB_i(t)\,dB_k(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)}\,dt = \rho_{ik}(t)\,dt, \tag{5.4.9}$$

where

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\sigma_k(t)}\sum_{j=1}^{d}\sigma_{ij}(t)\sigma_{kj}(t). \tag{5.4.10}$$

Itô's product rule implies

$$d\big(B_i(t)B_k(t)\big) = B_i(t)\,dB_k(t) + B_k(t)\,dB_i(t) + dB_i(t)\,dB_k(t),$$

and integration of this equation yields

$$B_i(t)B_k(t) = \int_0^t B_i(u)\,dB_k(u) + \int_0^t B_k(u)\,dB_i(u) + \int_0^t \rho_{ik}(u)\,du. \tag{5.4.11}$$

Taking expectations and using the fact that the expectation of an Itô integral is zero, we obtain the covariance formula

$$\operatorname{Cov}\big[B_i(t), B_k(t)\big] = \mathbb{E}\int_0^t \rho_{ik}(u)\,du. \tag{5.4.12}$$

If the processes $\sigma_{ij}(t)$ and $\sigma_{kj}(t)$ are constant (i.e., independent of t and not random), then so are $\sigma_i(t)$, $\sigma_k(t)$, and $\rho_{ik}(t)$. In this case, (5.4.12) reduces to $\operatorname{Cov}\big[B_i(t), B_k(t)\big] = \rho_{ik}t$. Because both $B_i(t)$ and $B_k(t)$ have standard deviation $\sqrt{t}$, the correlation between $B_i(t)$ and $B_j(t)$ is simply ρ_{ik}. When $\sigma_{ij}(t)$ and $\sigma_{kj}(t)$ are themselves random processes, we call $\rho_{ik}(t)$ the *instantaneous correlation* between $B_i(t)$ and $B_k(t)$. At a fixed time t along a particular path, $\rho_{ik}(t)$ is the conditional correlation between the next increments of B_i and B_k over a "small" time interval following time t (see Exercise 4.17 of Chapter 4 with $\Theta_1 = \Theta_2 = 0$, $\sigma_1 = \sigma_2 = 1$).

Finally, we note from (5.4.8) and (5.4.9) that

$$\begin{aligned} dS_i(t)\,dS_k(t) &= \sigma_i(t)\sigma_k(t)S_i(t)S_k(t)\,dB_i(t)\,dB_k(t) \\ &= \rho_{ik}(t)\sigma_i(t)\sigma_k(t)S_i(t)S_k(t)\,dt. \end{aligned} \tag{5.4.13}$$

Rewriting (5.4.13) in terms of "relative differentials," we obtain

$$\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_k(t)}{S_k(t)} = \rho_{ik}(t)\sigma_i(t)\sigma_k(t)\,dt.$$

The volatility processes $\sigma_i(t)$ and $\sigma_k(t)$ are the respective *instantaneous standard deviations* of the relative changes in S_i and S_k at time t, and the process $\rho_{ik}(t)$ is the *instantaneous correlation* between these relative changes.

Mean rates of return are affected by the change to a risk-neutral measure in the next subsection. Instantaneous standard deviations and correlations are unaffected (Exercise 5.12(ii) and (iii)). If the instantaneous standard deviations and correlations are not random, then (noninstantaneous) standard deviations and correlations are unaffected by the change of measure (see Exercise 5.12(iv) for the case of correlations). However, (noninstantaneous) standard deviations and correlations can be affected by a change of measure when the instantaneous standard deviations and correlations are random (see Exercises 5.12(v) and 5.13 for the case of correlations).

We define a *discount process*

$$D(t) = e^{-\int_0^t R(u)\,du}. \tag{5.4.14}$$

We assume that the interest rate process $R(t)$ is adapted. In addition to stock prices, we shall often work with discounted stock prices. Their differentials are

$$\begin{aligned} d\big(D(t)S_i(t)\big) &= D(t)\big[dS_i(t) - R(t)S_i(t)\,dt\big] \\ &= D(t)S_i(t)\Big[\big(\alpha_i(t) - R(t)\big)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\Big] \\ &= D(t)S_i(t)\big[\big(\alpha_i(t) - R(t)\big)\,dt + \sigma_i(t)\,dB_i(t)\big], \quad i = 1, \dots, m. \end{aligned} \tag{5.4.15}$$

5.4.3 Existence of the Risk-Neutral Measure

Definition 5.4.3. *A probability measure* $\widetilde{\mathbb{P}}$ *is said to be* risk-neutral *if*

(i) $\widetilde{\mathbb{P}}$ *and* $\mathbb{P}$ *are equivalent (i.e., for every* $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ *if and only if* $\widetilde{\mathbb{P}}(A) = 0$*), and*

(ii) under $\widetilde{\mathbb{P}}$*, the discounted stock price* $D(t)S_i(t)$ *is a martingale for every* $i = 1, \dots, m$.

In order to make discounted stock prices be martingales, we would like to rewrite (5.4.15) as

$$d\big(D(t)S_i(t)\big) = D(t)S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)\big[\Theta_j(t)\,dt + dW_j(t)\big]. \tag{5.4.16}$$

If we can find the market price of risk processes $\Theta_j(t)$ that make (5.4.16) hold, with one such process for each source of uncertainty $W_j(t)$, we can then use the multidimensional Girsanov Theorem to construct an equivalent probability measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}(t)$ given by (5.4.2) is a d-dimensional Brownian motion. This permits us to reduce (5.4.16) to

$$d\big(D(t)S_i(t)\big) = D(t)S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)\,d\widetilde{W}_j(t), \tag{5.4.17}$$

and hence $D(t)S_i(t)$ is a martingale under $\widetilde{\mathbb{P}}$. The problem of finding a risk-neutral measure is simply one of finding processes $\Theta_j(t)$ that make (5.4.15) and (5.4.16) agree. Since these equations have the same coefficient multiplying each $dW_j(t)$, they agree if and only if the coefficient multiplying dt is the same in both cases, which means that

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t), \quad i = 1, \dots, m. \tag{5.4.18}$$

We call these the *market price of risk equations*. These are m equations in the d unknown processes $\Theta_1(t), \dots, \Theta_d(t)$.

If one cannot solve the market price of risk equations, then there is an arbitrage lurking in the model; the model is bad and should not be used for pricing. We do not give the detailed proof of this fact. Instead, we give a simple example to illustrate it.

Example 5.4.4. Suppose there are two stocks ($m = 2$) and one Brownian motion ($d = 1$), and suppose further that all coefficient processes are constant. Then, the market price of risk equations are

$$\alpha_1 - r = \sigma_1 \theta, \tag{5.4.19}$$

$$\alpha_2 - r = \sigma_2 \theta. \tag{5.4.20}$$

These equations have a solution θ if and only if

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}.$$

If this equation does not hold, then one can arbitrage one stock against the other. Suppose, for example, that

$$\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}$$

and define

$$\mu = \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} > 0.$$

Suppose that at each time an agent holds $\Delta_1(t) = \frac{1}{S_1(t)\sigma_1}$ shares of stock one and $\Delta_2(t) = -\frac{1}{S_2(t))\sigma_2}$ shares of stock two, borrowing or investing as necessary at the interest rate r to set up and maintain this portfolio. The initial capital required to take the stock positions is $\frac{1}{\sigma_1} - \frac{1}{\sigma_2}$, but if this is positive we borrow from the money market account, and if it is negative we invest in the money market account, so the initial capital required to set up the whole portfolio, including the money market position, is $X(0) = 0$. The differential of the portfolio value $X(t)$ is

$$\begin{aligned}
&dX(t) \\
&= \Delta_1(t)\,dS_1(t) + \Delta_2(t)\,dS_2(t) + r\big(X(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t)\big)\,dt \\
&= \frac{\alpha_1 - r}{\sigma_1}\,dt + dW(t) - \frac{\alpha_2 - r}{\sigma_2}\,dt - dW(t) + rX(t)\,dt \\
&= \mu\,dt + rX(t)\,dt.
\end{aligned}$$

The differential of the discounted portfolio value is

$$d\big(D(t)X(t)\big) = D(t)\big(dX(t) - rX(t)\,dt\big) = \mu D(t)\,dt.$$

The right-hand side $\mu D(t)$ is strictly positive and nonrandom. Therefore, this portfolio will make money for sure and do so faster than the interest rate r because the discounted portfolio value has a nonrandom positive derivative. We have managed to synthesize a second money market account with rate of return higher than r, and now the arbitrage opportunities are limitless. □

When there is no solution to the market price of risk equations, the arbitrage in the model may not be as obvious as in Example 5.4.4, but it does exist. If there is a solution to the market price of risk equations, then there is no arbitrage. To show this, we need to introduce some notation and terminology. In the market with stock prices $S_i(t)$ given by (5.4.6) and interest rate process $R(t)$, an agent can begin with initial capital $X(0)$ and choose adapted portfolio processes $\Delta_i(t)$, one for each stock $S_i(t)$. The differential of the agent's portfolio value will then be

$$\begin{aligned} dX(t) &= \sum_{i=1}^{m} \Delta_i(t)\, dS_i(t) + R(t)\left(X(t) - \sum_{i=1}^{m} \Delta_i(t) S_i(t)\right) dt \\ &= R(t)X(t)\, dt + \sum_{i=1}^{m} \Delta_i(t)\big(dS_i(t) - R(t)S_i(t)\, dt\big) \\ &= R(t)X(t)\, dt + \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)}\, d\big(D(t)S_i(t)\big). \end{aligned} \tag{5.4.21}$$

The differential of the discounted portfolio value is

$$\begin{aligned} d\big(D(t)X(t)\big) &= D(t)\big(dX(t) - R(t)X(t)\, dt\big) \\ &= \sum_{i=1}^{m} \Delta_i(t)\, d\big(D(t)S_i(t)\big). \end{aligned} \tag{5.4.22}$$

If $\widetilde{\mathbb{P}}$ is a risk-neutral measure, then under $\widetilde{\mathbb{P}}$ the processes $D(t)S_i(t)$ are martingales, and hence the process $D(t)X(t)$ must also be a martingale. Put another way, under $\widetilde{\mathbb{P}}$ each of the stocks has mean rate of return $R(t)$, the same as the rate of return of the money market account. Hence, no matter how an agent invests, the mean rate of return of his portfolio value under $\widetilde{\mathbb{P}}$ must also be $R(t)$, and the discounted portfolio value must then be a martingale. We have proved the following result.

Lemma 5.4.5. *Let $\widetilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\widetilde{\mathbb{P}}$, the discounted portfolio value $D(t)X(t)$ is a martingale.*

Definition 5.4.6. *An* arbitrage *is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$*

$$\mathbb{P}\{X(T) \geq 0\} = 1, \quad \mathbb{P}\{X(T) > 0\} > 0. \tag{5.4.23}$$

An arbitrage is a way of trading so that one starts with zero capital and at some later time T is sure not to have lost money and furthermore has a positive probability of having made money. Such an opportunity exists if and only if there is a way to start with positive capital $X(0)$ and to beat the money market account. In other words, there exists an arbitrage if and only if there is a way to start with $X(0)$ and at a later time T have a portfolio value satisfying

$$\mathbb{P}\left\{X(T) \geq \frac{X(0)}{D(T)}\right\} = 1, \quad \mathbb{P}\left\{X(T) > \frac{X(0)}{D(T)}\right\} > 0 \tag{5.4.24}$$

(see Exercise 5.7).

Theorem 5.4.7 (First fundamental theorem of asset pricing). *If a market model has a risk-neutral probability measure, then it does not admit arbitrage.*

PROOF: If a market model has a risk-neutral probability measure $\widetilde{\mathbb{P}}$, then every discounted portfolio value process is a martingale under $\widetilde{\mathbb{P}}$. In particular, every portfolio value process satisfies $\widetilde{\mathbb{E}}\big[D(T)X(T)\big] = X(0)$. Let $X(t)$ be a portfolio value process with $X(0) = 0$. Then we have

$$\widetilde{\mathbb{E}}\big[D(T)X(T)\big] = 0. \tag{5.4.25}$$

Suppose $X(T)$ satisfies the first part of (5.4.23) (i.e., $\mathbb{P}\{X(T) < 0\} = 0$). Since $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, we have also $\widetilde{\mathbb{P}}\{X(T) < 0\} = 0$. This, coupled with (5.4.25), implies $\widetilde{\mathbb{P}}\{X(T) > 0\} = 0$, for otherwise we would have $\widetilde{\mathbb{P}}\{D(T)X(T) > 0\} > 0$, which would imply $\widetilde{\mathbb{E}}\big[D(T)X(T)\big] > 0$. Because $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, we have also $\mathbb{P}\{X(T) > 0\} = 0$. Hence, $X(t)$ is not an arbitrage. In fact, there cannot exist an arbitrage since every portfolio value process $X(t)$ satisfying $X(0) = 0$ cannot be an arbitrage. □

One should never offer prices derived from a model that admits arbitrage, and the First Fundamental Theorem provides a simple condition one can apply to check that the model one is using does not have this fatal flaw. In our model with d Brownian motions and m stocks, this amounts to producing a solution to the market price of risk equations (5.4.18). In models of the term structure of interest rates (i.e., models that provide prices for bonds of every maturity), there are many instruments available for trading, and possible arbitrages in the model prices are a real concern. An application of the First Fundamental Theorem of Asset Pricing in such a model leads directly to the Heath-Jarrow-Morton condition for no arbitrage in term-structure models.

5.4.4 Uniqueness of the Risk-Neutral Measure

Definition 5.4.8. *A market model is* complete *if every derivative security can be hedged.*

Let us suppose we have a market model with a filtration generated by a d-dimensional Brownian motion and with a risk-neutral measure $\widetilde{\mathbb{P}}$ (i.e., we have solved the market price of risk equations (5.4.18), used the resulting market prices of risk $\Theta_1(t),\dots,\Theta_d(t)$ to define the Radon-Nikodým derivative process $Z(t)$, and have changed to the measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}(t)$ defined by (5.4.2) is a d-dimensional Brownian motion). Suppose further that we are given an $\mathcal{F}(T)$-measurable random variable $V(T)$, which is the payoff of some derivative security.

We would like to be sure we can hedge a short position in the derivative security whose payoff at time T is $V(T)$. We can define $V(t)$ by (5.2.31), so that $D(t)V(t)$ satisfies (5.2.30), and just as in (5.3.3), we see that $D(t)V(t)$ is a martingale under $\widetilde{\mathbb{P}}$. According to the Martingale Representation Theorem 5.4.2, there are processes $\widetilde{\Gamma}_1(u),\dots,\widetilde{\Gamma}_d(u)$ such that

$$D(t)V(t) = V(0) + \sum_{j=1}^{d}\int_0^t \widetilde{\Gamma}_j(u)\,d\widetilde{W}_j(u), \quad 0 \le t \le T. \tag{5.4.26}$$

Consider a portfolio value process that begins at $X(0)$. According to (5.4.22) and (5.4.17),

$$\begin{aligned} d\big(D(t)X(t)\big) &= \sum_{i=1}^{m} \Delta_i(t)\,d\big(D(t)S_i(t)\big) \\ &= \sum_{j=1}^{d}\sum_{i=1}^{m} \Delta_i(t)D(t)S_i(t)\sigma_{ij}(t)\,d\widetilde{W}_j(t) \end{aligned} \tag{5.4.27}$$

or, equivalently,

$$D(t)X(t) = X(0) + \sum_{j=1}^{d}\int_0^t \sum_{i=1}^{m} \Delta_i(u)D(u)S_i(u)\sigma_{ij}(u)\,d\widetilde{W}_j(u). \tag{5.4.28}$$

Comparing (5.4.26) and (5.4.28), we see that to hedge the short position, we should take $X(0) = V(0)$ and choose the portfolio processes $\Delta_1(t),\dots,\Delta_m(t)$ so that the *hedging equations*

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t)S_i(t)\sigma_{ij}(t), \quad j = 1,\dots,d, \tag{5.4.29}$$

are satisfied. These are d equations in m unknown processes $\Delta_1(t),\dots,\Delta_m(t)$.

Theorem 5.4.9 (Second fundamental theorem of asset pricing). *Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.*

SKETCH OF PROOF: We first assume that the model is complete. We wish to show that there can be only one risk-neutral measure. Suppose the model has two risk-neutral measures, $\widetilde{\mathbb{P}}_1$ and $\widetilde{\mathbb{P}}_2$. Let A be a set in $\mathcal{F}$, which we assumed at the beginning of this section is the same as $\mathcal{F}(T)$. Consider the derivative security with payoff $V(T) = \mathbb{I}_A \frac{1}{D(T)}$. Because the model is complete, a short position in this derivative security can be hedged (i.e., there is a portfolio value process with some initial condition $X(0)$ that satisfies $X(T) = V(T)$). Since both $\widetilde{\mathbb{P}}_1$ and $\widetilde{\mathbb{P}}_2$ are risk-neutral, the discounted portfolio value process $D(t)X(t)$ is a martingale under both $\widetilde{\mathbb{P}}_1$ and $\widetilde{\mathbb{P}}_2$. It follows that

$$\begin{aligned}\widetilde{\mathbb{P}}_1(A) = \widetilde{\mathbb{E}}_1\big[D(T)V(T)\big] &= \widetilde{\mathbb{E}}_1\big[D(T)X(T)\big] = X(0)\\ = \widetilde{\mathbb{E}}_2\big[D(T)X(T)\big] &= \widetilde{\mathbb{E}}_2\big[D(T)V(T)\big] = \widetilde{\mathbb{P}}_2(A).\end{aligned}$$

Since A is an arbitrary set in $\mathcal{F}$ and $\widetilde{\mathbb{P}}_1(A) = \widetilde{\mathbb{P}}_2(A)$, these two risk-neutral measures are really the same.

For the converse, suppose there is only one risk-neutral measure. This means first of all that the filtration for the model is generated by the d-dimensional Brownian motion driving the assets. If that were not the case (i.e., if there were other sources of uncertainty in the model besides the driving Brownian motions), then we could assign arbitrary probabilities to those sources of uncertainty without changing the distributions of the driving Brownian motions and hence without changing the distributions of the assets. This would permit us to create multiple risk-neutral measures. Because the driving Brownian motions are the only sources of uncertainty, the only way multiple risk-neutral measures can arise is via multiple solutions to the market price of risk equations (5.4.18). Hence, uniqueness of the risk-neutral measure implies that the market price of risk equations (5.4.18) have only one solution $\big(\Theta_1(t), \dots, \Theta_d(t)\big)$. For fixed t and ω, these equations are of the form

$$Ax = b, \tag{5.4.30}$$

where A is the $m \times d$-dimensional matrix

$$A = \begin{bmatrix} \sigma_{11}(t), & \sigma_{12}(t), & \dots, & \sigma_{1d}(t) \\ \sigma_{21}(t), & \sigma_{22}(t), & \dots, & \sigma_{2d}(t) \\ \vdots & \vdots & & \vdots \\ \sigma_{m1}(t), & \sigma_{m2}(t), & \dots, & \sigma_{md}(t) \end{bmatrix}, \tag{5.4.31}$$

x is the d-dimensional column vector

$$x = \begin{bmatrix} \Theta_1(t) \\ \Theta_2(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix},$$

and b is the m-dimensional column vector

$$b = \begin{bmatrix} \alpha_1(t) - R(t) \\ \alpha_2(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{bmatrix}.$$

Our assumption that there is only one risk-neutral measure means that the system of equations (5.4.30) has a unique solution x.

In order to be assured that every derivative security can be hedged, we must be able to solve the hedging equations (5.4.29) for $\Delta_1(t), \dots, \Delta_m(t)$ no matter what values of $\frac{\widetilde{\Gamma}_j(t)}{D(t)}$ appear on the left-hand side. For fixed t and ω, the hedging equations are of the form

$$A^{\mathrm{tr}} y = c, \tag{5.4.32}$$

where A^{tr} is the transpose of the matrix in (5.4.31), y is the m-dimensional vector

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \Delta_1(t)S_1(t) \\ \Delta_2(t)S_2(t) \\ \vdots \\ \Delta_m(t)S_m(t) \end{bmatrix},$$

and c is the d-dimensional vector

$$c = \begin{bmatrix} \frac{\widetilde{\Gamma}_1(t)}{D(t)} \\ \frac{\widetilde{\Gamma}_2(t)}{D(t)} \\ \vdots \\ \frac{\widetilde{\Gamma}_d(t)}{D(t)} \end{bmatrix}.$$

In order to be assured that the market is complete, there must be a solution y to the system of equations (5.4.32), no matter what vector c appears on the right-hand side. If there is always a solution $y_1, \dots, y_m$, then there are portfolio processes $\Delta_i(t) = \frac{y_i}{S_i(t)}$ satisfying the hedging equations (5.4.29), no matter what processes appear on the left-hand side of those equations. We could then conclude that a short position in an arbitrary derivative security can be hedged.

The uniqueness of the solution x to (5.4.30) implies the existence of a solution y to (5.4.32). We give a proof of this fact in Appendix C. Consequently, uniqueness of the risk-neutral measure implies that the market model is complete. □

5.5 Dividend-Paying Stocks

According to Definition 5.4.3, discounted stock prices are martingales under the risk-neutral measure. This is the case provided the stock pays no dividend. The key feature of a risk-neutral measure is that it causes discounted portfolio values to be martingales (see Lemma 5.4.5), and that ensures the absence of arbitrage (First Fundamental Theorem of Asset Pricing, Theorem 5.4.7). In order for the discounted value of a portfolio that invests in a dividend-paying stock to be a martingale, the discounted value of the stock *with the dividends reinvested* must be a martingale, but the discounted stock price itself is not a martingale. This section works out the details of this situation. We consider a single stock price driven by a single Brownian motion, although the results we obtain here also apply when there are multiple stocks and multiple Brownian motions.

5.5.1 Continuously Paying Dividend

Consider a stock, modeled as a generalized geometric Brownian motion, that pays dividends continuously over time at a rate $A(t)$ per unit time. Here $A(t)$, $0 \le t \le T$, is a nonnegative adapted process. A continuously paid dividend is not a bad model for a mutual fund, which collects lump sum dividends at a variety of times on a variety of stocks. In the case of a single stock, it is more reasonable to assume there are periodic lump sum dividend payments. We consider that case in Subsections 5.5.3 and 5.5.4.

Dividends paid by a stock reduce its value, and so we shall take as our model of the stock price

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)\,S(t)\,dW(t) - A(t)S(t)\,dt. \tag{5.5.1}$$

If the stock were to withhold dividends, its mean rate of return would be $\alpha(t)$. Equivalently, if an agent holding the stock were to reinvest the dividends, the mean rate of return on his investment would be $\alpha(t)$. The mean rate of return $\alpha(t)$, the volatility $\sigma(t)$, and the interest rate $R(t)$ appearing in (5.5.2) below are all assumed to be adapted processes.

An agent who holds the stock receives both the capital gain or loss due to stock price movements and the continuously paying dividend. Thus, if $\Delta(t)$ is the number of shares held at time t, then the portfolio value $X(t)$ satisfies

$$\begin{aligned} dX(t) &= \Delta(t)\,dS(t) + \Delta(t)A(t)S(t)\,dt + R(t)\big[X(t) - \Delta(t)S(t)\big]\,dt \\ &= R(t)X(t)\,dt + \big(\alpha(t) - R(t)\big)\Delta(t)S(t)\,dt + \sigma(t)\Delta(t)S(t)\,dW(t) \\ &= R(t)X(t)\,dt + \Delta(t)S(t)\sigma(t)\big[\Theta(t)\,dt + dW(t)\big], \end{aligned} \tag{5.5.2}$$

where

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)} \tag{5.5.3}$$

is the usual market price of risk.

We define

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du \tag{5.5.4}$$

and use Girsanov's Theorem to change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion, so we may rewrite (5.5.2) as

$$dX(t) = R(t)X(t)\,dt + \Delta(t)S(t)\sigma(t)\,d\widetilde{W}(t).$$

The discounted portfolio value satisfies

$$d\big[D(t)X(t)\big] = \Delta(t)D(t)S(t)\sigma(t)\,d\widetilde{W}(t).$$

In particular, under the risk-neutral measure $\widetilde{\mathbb{P}}$, the discounted portfolio process is a martingale. Here we denote by $D(t) = e^{-\int_0^t R(u)du}$ the usual discount process.

If we now wish to hedge a short position in a derivative security paying $V(T)$ at time T, where $V(T)$ is an $\mathcal{F}(T)$-measurable random variable, we will need to choose the initial capital $X(0)$ and the portfolio process $\Delta(t)$, $0 \le t \le T$, so that $X(T) = V(T)$. Because $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we must have

$$D(t)X(t) = \widetilde{\mathbb{E}}\big[D(T)V(T)\big|\mathcal{F}(t)\big],\ 0 \le t \le T.$$

The value $X(t)$ of this portfolio at each time t is the value (price) of the derivative security at that time, which we denote by $V(t)$. Making this replacement in the formula above, we obtain the risk-neutral pricing formula

$$D(t)V(t) = \widetilde{\mathbb{E}}\big[D(T)V(T)\big|\mathcal{F}(t)\big],\ 0 \le t \le T. \tag{5.5.5}$$

We have obtained the same risk-neutral pricing formula (5.2.30) as in the case of no dividends. Furthermore, conditions that guarantee that a short position can be hedged, and hence risk-neutral pricing is fully justified, are the same as in the no-dividend case; see Section 5.3.

The difference between the dividend and no-dividend cases is in the evolution of the underlying stock under the risk-neutral measure. From (5.5.1) and the definition of $\widetilde{W}(t)$, we see that

$$dS(t) = \big[R(t) - A(t)\big]S(t)\,dt + \sigma(t)S(t)\,d\widetilde{W}(t). \tag{5.5.6}$$

Under the risk-neutral measure, the stock does not have mean rate of return $R(t)$, and consequently the discounted stock price is not a martingale. Indeed,

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(u)\,d\widetilde{W}(u) + \int_0^t \Big[R(u) - A(u) - \frac{1}{2}\sigma^2(u)\Big]\,du\right\}. \tag{5.5.7}$$

The process

$$e^{\int_0^t A(u)}D(t)S(t) = \exp\left\{\int_0^t \sigma(u)\,d\widetilde{W}(u) - \frac{1}{2}\int_0^t \sigma^2(u)\,du\right\}$$

is a martingale. This is the interest-rate-discounted value at time t of an account that initially purchases one share of the stock and continuously reinvests the dividends in the stock.

5.5.2 Continuously Paying Dividend with Constant Coefficients

In the event that the volatility σ, the interest rate r, and the dividend rate a are constant, the stock price at time t, given by (5.5.7), is

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - a - \frac{1}{2}\sigma^2\right)t\right\}. \tag{5.5.8}$$

For $0 \le t \le T$, we have

$$S(T) = S(t)\exp\left\{\sigma\big(\widetilde{W}(T) - \widetilde{W}(t)\big) + \left(r - a - \frac{1}{2}\sigma^2\right)(T-t)\right\}.$$

According to the risk-neutral pricing formula, the price at time t of a European call expiring at time T with strike K is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T) - K)^+ \big| \mathcal{F}(t)\right]. \tag{5.5.9}$$

To evaluate this, we first compute

$$\begin{aligned} c(t,x) &\\ = \widetilde{\mathbb{E}}&\left[e^{-r(T-t)}\left(x\exp\left\{\sigma\big(\widetilde{W}(T) - \widetilde{W}(t)\big) + \left(r - a - \frac{1}{2}\sigma^2\right)(T-t)\right\} - K\right)^+\right] \\ = \widetilde{\mathbb{E}}&\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,Y + \left(r - a - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right], \end{aligned} \tag{5.5.10}$$

where $\tau = T - t$ and

$$Y = -\frac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{T-t}}$$

is a standard normal random variable under $\widetilde{\mathbb{P}}$. We define

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r - a \pm \frac{1}{2}\sigma^2\right)\tau\right]. \tag{5.5.11}$$

We note that the random variable whose expectation we are computing in (5.5.10) is nonzero (the call expires in the money) if and only if $Y < d_-(\tau,x)$. Therefore,

$$\begin{aligned} c(t,x) &\\ = \frac{1}{\sqrt{2\pi}}&\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,y + \left(r - a - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)e^{-\frac{1}{2}y^2}\,dy \\ = \frac{1}{\sqrt{2\pi}}&\int_{-\infty}^{d_-(\tau,x)} x\exp\left\{-\sigma\sqrt{\tau}\,y - \left(a + \frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}y^2\right\}dy \\ &-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}Ke^{-\frac{1}{2}y^2}\,dy \\ = \frac{1}{\sqrt{2\pi}}&\int_{-\infty}^{d_-(\tau,x)} xe^{-a\tau}\exp\left\{-\frac{1}{2}(y + \sigma\sqrt{\tau})^2\right\}dy - e^{-r\tau}KN(d_-(\tau,x)). \end{aligned}$$

We make the change of variable $z = y + \sigma\sqrt{\tau}$ in the integral, which leads us to the formula

$$\begin{aligned} c(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+(\tau,x)} x e^{-a\tau} e^{-\frac{z^2}{2}}\, dz - e^{-r\tau} K N(d_-(\tau,x)) \\ &= x e^{-a\tau} N(d_+(\tau,x)) - e^{-r\tau} K N(d_-(\tau,x)). \end{aligned} \tag{5.5.12}$$

According to the Independence Lemma, Lemma 2.3.4, the option price $V(t)$ in (5.5.9) is $c(t, S(t))$. The only differences between this formula and the one for a non-dividend-paying stock is in the definition (5.5.11) of $d_\pm(\tau, x)$ (see (5.2.33) and (5.2.34)) and in the presence of $e^{-a\tau}$ in the first term on the right-hand side of (5.5.12).

5.5.3 Lump Payments of Dividends

Finally, let us consider the case when the dividend is paid in lumps. That is to say there are times $0 < t_1 < t_2 < t_n < T$ and, at each time t_j, the dividend paid is $a_j S(t_j-)$, where $S(t_j-)$ denotes the stock price just prior to the dividend payment. The stock price after the dividend payment is the stock price before the dividend payment less the dividend payment:

$$S(t_j) = S(t_j-) - a_j S(t_j-) = (1 - a_j) S(t_j-). \tag{5.5.13}$$

We assume that each a_j is an $\mathcal{F}(t_j)$-measurable random variable taking values in $[0, 1]$. If $a_j = 0$, no dividend is paid at time t_j. If $a_j = 1$, the full value of the stock is paid as a dividend at time t_j, and the stock value is zero thereafter. To simplify the notation, we set $t_0 = 0$ and $t_{n+1} = T$. However, neither $t_0 = 0$ nor $t_{n+1} = T$ is a dividend payment date (i.e., $a_0 = 0$ and $a_{n+1} = 0$). We assume that, between dividend payment dates, the stock price follows a generalized geometric Brownian motion:

$$dS(t) = \alpha(t) S(t)\, dt + \sigma(t) S(t)\, dW(t), \;\; t_j \le t < t_{j+1}, \;\; j = 0, 1, \ldots, n. \tag{5.5.14}$$

Equations (5.5.13) and (5.5.14) fully determine the evolution of the stock price.

Between dividend payment dates, the differential of the portfolio value corresponding to a portfolio process $\Delta(t)$, $0 \le t \le T$, is

$$\begin{aligned} dX(t) &= \Delta(t)\, dS(t) + R(t)\big[X(t) - \Delta(t)S(t)\big]\, dt \\ &= R(t)X(t)\, dt + \big(\alpha(t) - R(t)\big)\Delta(t)S(t)\, dt + \sigma(t)\Delta(t)S(t)\, dW(t) \\ &= R(t)X(t)\, dt + \Delta(t)\sigma(t)S(t)\big[\Theta(t)\, dt + dW(t)\big], \end{aligned}$$

where the market price of risk $\Theta(t)$ is again defined by (5.5.3). At the dividend payment dates, the value of the portfolio stock holdings drops by $a_j \Delta(t_j) S(t_j-)$, but the portfolio collects the dividend $a_j \Delta(t_j) S(t_j-)$, and so the portfolio value does not jump. It follows that

$$dX(t) = R(t)X(t)\,dt + \Delta(t)\sigma(t)S(t)\big[\Theta(t)\,dt + dW(t)\big] \tag{5.5.15}$$

is the correct formula for the evolution of the portfolio value at all times t. We again define $\widetilde{W}$ by (5.5.4), change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion, and obtain the risk-neutral pricing formula (5.5.5).

5.5.4 Lump Payments of Dividends with Constant Coefficients

We price a European call under the assumption that σ, r, and each a_j are constant. From (5.5.14) and the definition of $\widetilde{W}$, we have

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),\ t_j \le t < t_{j+1},\ j = 0, 1, \ldots, n.$$

Therefore,

$$S(t_{j+1}-) = S(t_j)\exp\left\{\sigma\big(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\big) + \Big(r - \frac{1}{2}\sigma^2\Big)(t_{j+1} - t_j)\right\}. \tag{5.5.16}$$

From (5.5.13), we see that

$$\begin{aligned} &S(t_{j+1})\\ &= (1 - a_{j+1})S(t_j)\exp\left\{\sigma\big(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\big) + \Big(r - \frac{1}{2}\sigma^2\Big)(t_{j+1} - t_j)\right\} \end{aligned}$$

or, equivalently, for $j = 0, 1, \ldots, n$,

$$\frac{S(t_{j+1})}{S(t_j)} = (1 - a_{j+1})\exp\left\{\sigma\big(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\big) + \Big(r - \frac{1}{2}\sigma^2\Big)(t_{j+1} - t_j)\right\}.$$

It follows that

$$\begin{aligned} \frac{S(T)}{S(0)} &= \frac{S(t_{n+1})}{S(t_0)}\\ &= \prod_{j=0}^{n}\frac{S(t_{j+1})}{S(t_j)}\\ &= \prod_{j=0}^{n-1}(1 - a_{j+1})\cdot\exp\left\{\sigma\widetilde{W}(T) + \Big(r - \frac{1}{2}\sigma^2\Big)T\right\}. \end{aligned}$$

In other words,

$$S(T) = S(0)\prod_{j=0}^{n-1}(1 - a_{j+1})\cdot\exp\left\{\sigma\widetilde{W}(T) + \Big(r - \frac{1}{2}\sigma^2\Big)T\right\}. \tag{5.5.17}$$

This is the same formula we would have for the price at time T of a geometric Brownian motion not paying dividends if the initial stock price were

$S(0)\prod_{j=0}^{n-1}(1-a_{j+1})$ rather than $S(0)$. Therefore, the price at time zero of a European call on this dividend-paying asset, a call that expires at time T with strike price K, is obtained by replacing the initial stock price by $S(0)\prod_{j=0}^{n-1}(1-a_{j+1})$ in the classical Black-Scholes-Merton formula. This results in the call price

$$S(0)\prod_{j=0}^{n-1}(1-a_{j+1})N(d_+)-e^{-rT}KN(d_-),$$

where

$$d_\pm=\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K}+\sum_{j=0}^{n-1}\log(1-a_{j+1})+\left(r\pm\frac{1}{2}\sigma^2\right)T\right].$$

A similar formula holds for the call price at times t between 0 and T. In those cases, one includes only the terms $(1-a_{j+1})$ corresponding to the dividend dates between times t and T.

5.6 Forwards and Futures

In this section, we assume there is a unique risk-neutral measure $\widetilde{\mathbb{P}}$, and all assets satisfy the risk-neutral pricing formula. Under this assumption, we study forward and futures prices and the relationship between them. The formulas we develop apply to any tradable, non-dividend-paying asset, not just to a stock. In a binomial model, these topics were addressed in Sections 6.3 and 6.5 of Volume I.

5.6.1 Forward Contracts

Let $S(t)$, $0\le t\le\overline{T}$, be an asset price process, and let $R(t)$, $0\le t\le\overline{T}$, be an interest rate process. We choose here some large time $\overline{T}$, and all bonds and derivative securities we consider will mature or expire at or before time $\overline{T}$. As usual, we define the discount process $D(t)=e^{-\int_0^t R(u)du}$. According to the risk-neutral pricing formula (5.2.30), the price at time t of a zero-coupon bond paying 1 at time T is

$$B(t,T)=\frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)],\quad 0\le t\le T\le\overline{T}. \tag{5.6.1}$$

This pricing formula guarantees that no arbitrage can be found by trading in these bonds because any such portfolio, when discounted, will be a martingale under the risk-neutral measure. The details of this argument in the binomial model are presented in Theorem 6.2.6 and Remark 6.2.7 of Volume I.

Definition 5.6.1. *A* forward contract *is an agreement to pay a specified* delivery price K *at a* delivery date T, *where* $0 \leq T \leq \overline{T}$, *for the asset whose price at time* t *is* $S(t)$. *The* T-forward price $\mathrm{For}_S(t,T)$ *of this asset at time* t, *where* $0 \leq t \leq T \leq \overline{T}$, *is the value of* K *that makes the forward contract have no-arbitrage price zero at time* t.

Theorem 5.6.2. *Assume that zero-coupon bonds of all maturities can be traded. Then*

$$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}, \quad 0 \leq t \leq T \leq \overline{T}. \tag{5.6.2}$$

PROOF: Suppose that at time t an agent sells the forward contract with delivery date T and delivery price K. Suppose further that the value K is chosen so that the forward contract has price zero at time t. Then selling the forward contract generates no income. Having sold the forward contract at time t, suppose the agent immediately shorts $\frac{S(t)}{B(t,T)}$ zero-coupon bonds and uses the income $S(t)$ generated to buy one share of the asset. The agent then does no further trading until time T, at which time she owns one share of the asset, which she delivers according to the forward contract. In exchange, she receives K. After covering the short bond position, she is left with $K - \frac{S(t)}{B(t,T)}$. If this is positive, the agent has found an arbitrage. If it is negative, the agent could instead have taken the opposite position, going long the forward, long the T-maturity bond, and short the asset, to again achieve an arbitrage. In order to preclude arbitrage, K must be given by (5.6.2). □

Remark 5.6.3. The proof of Theorem 5.6.2 does not use the notion of risk-neutral pricing. It shows that the forward price must be given by (5.6.2) in order to preclude arbitrage. Because we have assumed the existence of a risk-neutral measure and are pricing all assets by the risk-neutral pricing formula, we must be able to obtain (5.6.2) from the risk-neutral pricing formula as well. Indeed, using (5.2.30), (5.6.1), and the fact that the discounted asset price is a martingale under $\widetilde{\mathbb{P}}$, we compute the price at time t of the forward contract to be

$$\begin{aligned}
&\frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)\big(S(T)-K\big)\big|\mathcal{F}(t)\big]\\
&= \frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)S(T)\big|\mathcal{F}(t)\big] - \frac{K}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]\\
&= S(t) - KB(t,T).
\end{aligned}$$

In order for this to be zero, K must be given by (5.6.2).

5.6.2 Futures Contracts

Consider a time interval $[0,T]$, which we divide into subintervals using the partition points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. We shall refer to each subinterval $[t_k, t_{k+1})$ as a "day."

Suppose the interest rate is constant within each day. Then the discount process is given by $D(0) = 1$ and, for $k = 0, 1, \dots, n-1$,

$$D(t_{k+1}) = \exp\left\{ - \int_0^{t_{k+1}} R(u)\,du \right\} = \exp\left\{ - \sum_{j=0}^{k} R(t_j)(t_{j+1} - t_j) \right\},$$

which is $\mathcal{F}(t_k)$-measurable. According to the risk-neutral pricing formula (5.6.1), the zero-coupon bond paying 1 at maturity T has time-t_k price

$$B(t_k, T) = \frac{1}{D(t_k)} \widetilde{\mathbb{E}}[D(T) | \mathcal{F}(t_k)].$$

An asset whose price at time t is $S(t)$ has time-t_k forward price

$$\mathrm{For}_S(t_k, T) = \frac{S(t_k)}{B(t_k, T)}, \tag{5.6.3}$$

an $\mathcal{F}(t_k)$-measurable quantity. Suppose we take a long position in the forward contract at time t_k (i.e., agree to receive $S(T)$ and pay $\mathrm{For}_S(t_k, T)$ at time T). The value of this position at time $t_j \geq t_k$ is

$$\begin{aligned}
V_{k,j} &= \frac{1}{D(t_j)} \widetilde{\mathbb{E}}\left[D(T)\left(S(T) - \frac{S(t_k)}{B(t_k, T)} \right) \middle| \mathcal{F}(t_j) \right] \\
&= \frac{1}{D(t_j)} \widetilde{\mathbb{E}}[D(T)S(T) | \mathcal{F}(t_j)] - \frac{S(t_k)}{B(t_k, T)} \cdot \frac{1}{D(t_j)} \widetilde{\mathbb{E}}[D(T) | \mathcal{F}(t_j)] \\
&= S(t_j) - S(t_k) \cdot \frac{B(t_j, T)}{B(t_k, T)}.
\end{aligned}$$

If $t_j = t_k$, this is zero, as it should be. However, for $t_j > t_k$, it is generally different from zero. For example, if the interest rate is a constant r so that $B(t, T) = e^{-r(T-t)}$, then

$$V_{k,j} = S(t_j) - e^{r(t_j - t_k)} S(t_k).$$

If the asset grows faster than the interest rate, the forward contract takes on a positive value. Otherwise, it takes on a negative value. In either case, one of the parties to the forward contract could become concerned about default by the other party.

To alleviate the problem of default risk, parties to a forward contract could agree to settle one day after the contract is entered. The original forward contract purchaser could then seek to purchase a new forward contract one day later than the initial purchase. By repeating this process, the forward contract purchaser could generate the cash flow

$$\begin{aligned}
V_{0,1} &= S(t_1) - S(t_0) \cdot \frac{B(t_1, T)}{B(t_0, T)} \;=\; S(t_1) - S(0) \cdot \frac{B(t_1, T)}{B(0, T)}, \\
V_{1,2} &= S(t_2) - S(t_1) \cdot \frac{B(t_2, T)}{B(t_1, T)}, \\
&\;\;\vdots
\end{aligned}$$

$$V_{n-1,n} = S(t_n) - S(t_{n-1}) \cdot \frac{B(t_n,T)}{B(t_{n-1},T)} = S(T) - \frac{S(t_{n-1})}{B(t_{n-1},T)}.$$

There are two problems with this. First of all, the purchaser of the forward contract was presumably motivated by a desire to hedge against a price increase in the underlying asset. It is not clear the extent to which receiving this cash flow provides such a hedge. Second, this daily buying and selling of forward contracts requires that there be a liquid market each day for forward contracts initiated that day and forward contracts initiated one day before. This is too much to expect.

A better idea than daily repurchase of forward contracts is to create a *futures price* $\mathrm{Fut}_S(t,T)$, and use it as described below. If an agent holds a long futures position between times t_k and t_{k+1}, then at time t_{k+1} he receives a payment

$$\mathrm{Fut}_S(t_{k+1},T) - \mathrm{Fut}_S(t_k,T).$$

This is called *marking to margin.* The stochastic process $\mathrm{Fut}_S(t,T)$ is constructed so that $\mathrm{Fut}_S(t,T)$ is $\mathcal{F}(t)$-measurable for every t and

$$\mathrm{Fut}_S(T,T) = S(T).$$

Therefore, the sum of payments received by an agent who purchases a futures contract at time zero and holds it until delivery date T is

$$\begin{aligned}\big(\mathrm{Fut}_S(t_1,T) - \mathrm{Fut}_S(t_0,T)\big) + \big(\mathrm{Fut}_S(t_2,T) - \mathrm{Fut}_S(t_1,T)\big) + \ldots\\ \cdots + \big(\mathrm{Fut}_S(t_n,T) - \mathrm{Fut}_S(t_{n-1},T) &= \mathrm{Fut}_S(T,T) - \mathrm{Fut}_S(0,T)\\ &= S(T) - \mathrm{Fut}_S(0,T).\end{aligned}$$

If the agent takes delivery of the asset at time T, paying market price $S(T)$ for it, his total income from the futures contract and the delivery payment is $-\mathrm{Fut}_S(0,T)$. Ignoring the time value of money, he has effectively paid the price $\mathrm{Fut}_S(0,T)$ for the asset, a price that was locked in at time zero.

In contrast to the case of a forward contract, the payment from holding a futures contract is distributed over the life of the contract rather than coming solely at the end. The mechanism for these payments is the margin account, which the owner of the futures contract must open at the time of purchase of the contract and to which he must contribute or from which he may withdraw money, depending on the trajectory of the futures price. Whereas the owner of a forward contract is exposed to counterparty default risk, the owner of a futures contract is exposed to the risk that some of the intermediate payments (margin calls) will force him to close out his position prematurely.

In addition to satisfying $\mathrm{Fut}_S(T,T) = S(T)$, the futures price process is chosen so that at each time t_k the value of the payment to be received at time t_{k+1}, and indeed at all future times $t_j > t_k$, is zero. This means that at any time one may enter or close out a position in the contract without incurring any cost other than payments already made. The condition that the value at time t_k of the payment to be received at time t_{k+1} be zero may be written as

$$\begin{aligned} 0 &= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\big[D(t_{k+1})\big(\mathrm{Fut}_S(t_{k+1},T) - \mathrm{Fut}_S(t_k,T)\big)\big|\mathcal{F}(t_k)\big] \\ &= \frac{D(t_{k+1})}{D(t_k)}\big\{\widetilde{\mathbb{E}}[\mathrm{Fut}_S(t_{k+1},T)|\mathcal{F}(t_k)] - \mathrm{Fut}_S(t_k,T)\big\}, \end{aligned}$$

where we have used the fact that $D(t_{k+1})$ is $\mathcal{F}(t_k)$-measurable to take $D(t_{k+1})$ out of the conditional expectation. From the equation above, we see that

$$\widetilde{\mathbb{E}}[\mathrm{Fut}_S(t_{k+1},T)|\mathcal{F}(t_k)] = \mathrm{Fut}_S(t_k,T), \quad k = 0,1,\dots,n-1. \tag{5.6.4}$$

This shows that $\mathrm{Fut}_S(t_k,T)$ must be a discrete-time martingale under $\widetilde{\mathbb{P}}$. But we also require that $\mathrm{Fut}_S(T,T) = S(T)$, from which we conclude that the futures prices must be given by the formula

$$\mathrm{Fut}_S(t_k,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t_k)], \; k = 0,1,\dots,n. \tag{5.6.5}$$

Indeed, under the condition that $\mathrm{Fut}_S(T,T) = S(T)$, equations (5.6.4) and (5.6.5) are equivalent.

We note finally that with $\mathrm{Fut}_S(t,T)$ given by (5.6.5), the value at time t_k of the payment to be received at time t_j is zero for every $j \geq k+1$. Indeed, using the $\mathcal{F}(t_{j-1})$-measurability of $D(t_j)$ and the martingale property for $\mathrm{Fut}_S(t,T)$, we have

$$\begin{aligned} &\frac{1}{D(t_k)}\widetilde{\mathbb{E}}\big[D(t_j)\big(\mathrm{Fut}_S(t_j,T) - \mathrm{Fut}_S(t_{j-1},T)\big)\big|\mathcal{F}(t_k)\big] \\ &= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{E}}\big[D(t_j)\big(\mathrm{Fut}_S(t_j,T) - \mathrm{Fut}_S(t_{j-1},T)\big)\big|\mathcal{F}(t_{j-1})\big]\Big|\mathcal{F}(t_k)\Big] \\ &= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[D(t_j)\widetilde{\mathbb{E}}\big[\mathrm{Fut}_S(t_j,T)\big|\mathcal{F}(t_{j-1})\big] - D(t_j)\mathrm{Fut}_S(t_{j-1},T)\Big|\mathcal{F}(t_k)\Big] \\ &= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[D(t_j)\mathrm{Fut}_S(t_{j-1},T) - D(t_j)\mathrm{Fut}_S(t_{j-1},T)\Big|\mathcal{F}(t_k)\Big] \;=\; 0. \end{aligned}$$

These considerations lead us to make the following definition for the fully continuous case (i.e., the case when $R(t)$ is assumed only to be an adapted stochastic process, not necessarily constant on time intervals of the form $[t_k, t_{k+1})$).

Definition 5.6.4. *The* futures price *of an asset whose value at time T is $S(T)$ is given by the formula*

$$\mathrm{Fut}_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)], \quad 0 \leq t \leq T. \tag{5.6.6}$$

A long position in the futures contract *is an agreement to receive as a cash flow the changes in the futures price (which may be negative as well as positive) during the time the position is held. A short position in the futures contract receives the opposite cash flow.*

Theorem 5.6.5. *The futures price is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$, it satisfies* $\mathrm{Fut}_S(T,T) = S(T)$, *and the value of a long (or a short) futures position to be held over an interval of time is always zero.*

OUTLINE OF PROOF: The usual iterated conditioning argument shows that $\mathrm{Fut}_S(t,T)$ given by (5.6.6) is a $\widetilde{\mathbb{P}}$-martingale satisfying the terminal condition $\mathrm{Fut}_S(T,T) = S(T)$. In fact, this is the only $\widetilde{\mathbb{P}}$-martingale satisfying this terminal condition.

If the filtration $\mathcal{F}(t)$, $0 \le t \le T$, is generated by a Brownian motion $W(t)$, $0 \le t \le T$, then Corollary 5.3.2 of the Martingale Representation Theorem implies that

$$\mathrm{Fut}_S(t,T) = \mathrm{Fut}_S(0,T) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{W}(u), \quad 0 \le t \le T,$$

for some adapted integrand process $\widetilde{\Gamma}$ (i.e., $d\mathrm{Fut}_S(t,T) = \widetilde{\Gamma}(t)\, d\widetilde{W}(t)$). Let $0 \le t_0 < t_1 \le T$ be given and consider an agent who at times t between times t_0 and t_1 holds $\Delta(t)$ futures contracts. It costs nothing to change the position in futures contracts, but because the futures contracts generate cash flow, the agent may have cash to invest or need to borrow in order to execute this strategy. He does this investing and/or borrowing at the interest rate $R(t)$ prevailing at the time of the investing or borrowing. The agent's profit $X(t)$ from this trading satisfies

$$dX(t) = \Delta(t)\, d\mathrm{Fut}_S(t,T) + R(t)X(t)\,dt = \Delta(t)\widetilde{\Gamma}(t)\, d\widetilde{W}(t) + R(t)X(t)\,dt,$$

and thus

$$d\big(D(t)X(t)\big) = D(t)\Delta(t)\widetilde{\Gamma}(t)\, d\widetilde{W}(t).$$

Assume that at time t_0 the agent's profit is $X(t_0) = 0$. At time t_1, the agent's profit $X(t_1)$ will satisfy

$$D(t_1)X(t_1) = \int_{t_0}^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\, d\widetilde{W}(u). \tag{5.6.7}$$

Because Itô integrals are martingales, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}[D(t_1)X(t_1)|\mathcal{F}(t_0)]\\
&= \widetilde{\mathbb{E}}\left[\int_0^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\, d\widetilde{W}(u) - \int_0^{t_0} D(u)\Delta(u)\widetilde{\Gamma}(u)\, d\widetilde{W}(u)\middle|\, \mathcal{F}(t_0)\right]\\
&= \widetilde{\mathbb{E}}\left[\int_0^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\, d\widetilde{W}(u)\middle|\, \mathcal{F}(t_0)\right] - \int_0^{t_0} D(u)\Delta(u)\widetilde{\Gamma}(u)\, d\widetilde{W}(u)\\
&= 0.
\end{aligned} \tag{5.6.8}$$

According to the risk-neutral pricing formula, the value at time t_0 of a payment of $X(t_1)$ at time t_1 is $\frac{1}{D(t_0)}\widetilde{\mathbb{E}}\big[D(t_1)X(t_1)\big|\mathcal{F}(t_0)\big]$, and we have just shown that this is zero. The value of owning a long futures position over the interval t_0 to t_1 is obtained by setting $\Delta(u) = 1$ for all u; the value of holding a short position is obtained by setting $\Delta(u) = -1$ for all u. In both cases, we see that this value is zero.

If the filtration $\mathcal{F}(t)$, $0 \le t \le T$, is not generated by a Brownian motion, so that we cannot use Corollary 5.3.2, then we must write (5.6.7) as

$$D(t_1)X(t_1) = \int_{t_0}^{t_1} D(u)\Delta(u)\, d\mathrm{Fut}_S(u,T). \tag{5.6.9}$$

This integral can be defined and it will be a martingale. We will again have

$$\widetilde{\mathbb{E}}[D(t_1)X(t_1)|\mathcal{F}(t_0)] = 0. \qquad \square$$

Remark 5.6.6 (Risk-neutral valuation of a cash flow). Suppose an asset generates a cash flow so that between times 0 and u a total of $C(u)$ is paid, where $C(u)$ is $\mathcal{F}(u)$-measurable. Then a portfolio that begins with one share of this asset at time t and holds this asset between times t and T, investing or borrowing at the interest rate r as necessary, satisfies

$$dX(u) = dC(u) + R(u)X(u)\,du,$$

or equivalently

$$d\big(D(u)X(u)\big) = D(u)\,dC(u).$$

Suppose $X(t) = 0$. Then integration shows that

$$D(T)X(T) = \int_t^T D(u)\,dC(u).$$

The risk-neutral value at time t of $X(T)$, which is the risk-neutral value at time t of the cash flow received between times t and T, is thus

$$\frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)X(T)] = \frac{1}{D(t)}\widetilde{\mathbb{E}}\left[\int_t^T D(u)\,dC(u)\middle|\mathcal{F}(t)\right], \quad 0 \le t \le T. \tag{5.6.10}$$

Formula (5.6.10) generalizes the risk-neutral pricing formula (5.2.30) to allow for a cash flow rather than payment at the single time T. In (5.6.10), the process $C(u)$ can represent a succession of lump sum payments $A_1, A_2, \ldots, A_n$ at times $t_1 < t_2 < \cdots < t_n$, where each A_i is an $\mathcal{F}(t_i)$-measurable random variable. The formula for this is

$$C(u) = \sum_{i=1}^{n} A_i \mathbb{I}_{[0,u]}(t_i).$$

In this case,

$$\int_t^T D(u)\,dC(u) = \sum_{i=1}^n D(t_i)A_i \mathbb{I}_{(t,T]}(t_i).$$

Only payments made strictly later than time t appear in this sum. Equation (5.6.10) says that the value at time t of the string of payments to be made strictly later than time t is

$$\frac{1}{D(t)}\widetilde{\mathbb{E}}\left[\sum_{i=1}^n D(t_i)A_i\mathbb{I}_{(t,T]}(t_i)\middle|\mathcal{F}(t)\right] = \sum_{i=1}^n \mathbb{I}_{(t,T]}(t_i)\frac{1}{D(t)}\widetilde{\mathbb{E}}[D(t_i)A_i|\mathcal{F}(t)],$$

which is the sum of the time-t values of the payments made strictly later than time t.

The process $C(u)$ can also be continuous, as in (5.6.9). The process $C(u)$ may decrease as well as increase (i.e., the cash flow may be negative as well as positive). □

5.6.3 Forward–Futures Spread

We conclude with a comparison of forward and futures prices. We have defined these prices to be

$$\begin{aligned}\mathrm{For}_S(t,T) &= \frac{S(t)}{B(t,T)},\\ \mathrm{Fut}_S(t,T) &= \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)].\end{aligned}$$

If the interest rate is a constant r, then $B(t,T) = e^{-r(T-t)}$ and

$$\begin{aligned}\mathrm{For}_S(t,T) &= e^{r(T-t)}S(t),\\ \mathrm{Fut}_S(t,T) &= e^{rT}\widetilde{\mathbb{E}}\big[e^{-rT}S(T)\big|\mathcal{F}(t)\big] \;=\; e^{rT}e^{-rt}S(t) \;=\; e^{r(T-t)}S(t).\end{aligned}$$

In this case, the forward and futures prices agree.

We compare $\mathrm{For}_S(0,T)$ and $\mathrm{Fut}_S(0,T)$ in the case of a random interest rate. In this case, $B(0,T) = \widetilde{\mathbb{E}}D(T)$, and the so-called *forward–futures spread* is

$$\begin{aligned}\mathrm{For}_S(0,T) - \mathrm{Fut}_S(0,T) &= \frac{S(0)}{\widetilde{\mathbb{E}}D(T)} - \widetilde{\mathbb{E}}S(T)\\ &= \frac{1}{\widetilde{\mathbb{E}}D(T)}\left\{\widetilde{\mathbb{E}}[D(T)S(T)] - \widetilde{\mathbb{E}}D(T)\cdot\widetilde{\mathbb{E}}S(T)\right\}\\ &= \frac{1}{B(0,T)}\widetilde{\mathrm{Cov}}\big(D(T),S(T)\big), \qquad (5.6.11)\end{aligned}$$

where $\widetilde{\mathrm{Cov}}\big(D(T),S(T)\big)$ denotes the covariance of $D(T)$ and $S(T)$ under the risk-neutral measure. If the interest rate is nonrandom, this covariance is zero and the futures price agrees with the forward price.

One can explain this last formula as follows. If $D(T)$ and $S(T)$ are positively correlated, then higher asset prices tend to correspond to higher discount levels, which tend to correspond to lower interest rates. But when the asset goes up, the long position in the futures contract receives a payment (because the futures price is positively correlated with the underlying asset price). The long position in the futures contract thus receives money when the interest rate for investing is unfavorable (low) and conversely must pay money when the interest rate at which money can be borrowed is also unfavorable (high). The owner of the futures contract would have rather owned the forward contract, in which all payments are postponed until the end. Therefore, to make the futures contract attractive, the futures price must be lower than the forward price. (Recall that this price is what the investor ultimately pays for the asset.) This creates a positive forward–futures spread when the discount factor $D(T)$ and the asset price $S(T)$ are positively correlated. Note that all correlations in this argument are computed under the risk-neutral measure, not the actual probability measure. In a Brownian-motion-driven model, in which the multidimensional Girsanov Theorem, Theorem 5.4.1, is used to change to the risk-neutral measure, instantaneous asset correlations are the same under both measures (see Exercise 5.12). However, correlations between random variables (as opposed to instantaneous correlations between stochastic processes) can be affected by changes of measure (see Exercise 5.13).

5.7 Summary

This chapter treats the application to finance of two major theorems, Girsanov (Theorem 5.4.1) and Martingale Representation (Theorem 5.4.2). These lead to the two *Fundamental Theorems of Asset Pricing*, Theorem 5.4.7 and Theorem 5.4.9. Both of these are stated for models with multiple assets whose prices are driven by multiple Brownian motions.

According to the Fundamental Theorems of Asset Pricing, there are three possible situations when we build a mathematical model of a multiasset market.

Case 1. There is no risk-neutral measure (i.e., the market price of risk equations (5.4.18) cannot be solved for $\Theta_1(t), \dots, \Theta_d(t)$). This is a bad model. There must be some way to form an arbitrage by trading at the prices given by this model. Do not use this model.

Case 2. There are multiple risk-neutral measures (i.e., the market price of risk equations (5.4.18) have more than one solution). The different risk-neutral measures lead to different prices for derivative securities in the model. Any derivative security that has more than one price cannot be synthesized by trading in the model (i.e., a position in this derivative security cannot be hedged). (If the derivative security could be hedged, this would determine a unique price; see the proof of Theorem 5.4.9.) It may still be possible to calibrate the model (i.e., determine its parameters by getting it to match market prices, and the model might then give reasonable prices for nontraded instruments). However, it cannot be used to fully hedge the exposure associated with derivative positions.

At the present time, credit derivative models fall into Case 2. They are used for pricing, but are incomplete because the derivatives in question pay off contingent upon the default of some party and it is impossible to perfectly hedge default risk by trading in primary assets. These models have multiple risk-neutral measures, all of which can be consistent with market prices of the primary assets but give different prices for derivatives. In practical applications, one of these risk-neutral measures is singled out and used for pricing. Which of the risk-neutral measures is chosen for this purpose depends on the way the model is specified and calibrated.

Case 3. There is one and only one set of processes $\Theta_1(t), \dots, \Theta_d(t)$ that solve the market price of risk equations (5.4.18). There is a unique risk-neutral measure, and risk-neutral pricing is justified. In other words,the price (value) at time t of any security that pays $V(T)$ at time T is

$$V(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}\big[D(T)V(T)\big|\mathcal{F}(t)\big]. \tag{5.7.1}$$

In particular, the price at time zero of the security is its risk-neutral expected discounted payoff. The risk-neutral price of a derivative security is the initial capital that permits an agent to set up a perfect hedge for a short position in that derivative security. These perfect hedges are the solutions $\Delta_1(t), \dots, \Delta_m(t)$ of the hedging equations (5.4.29), and these solutions are guaranteed to exist (by the second part of the proof of Theorem 5.4.9). However, we do not generally attempt to determine the hedging positions $\Delta_1(t), \dots, \Delta_m(t)$ by solving (5.4.29). Instead, we determine hedges by the technique presented in Chapter 6.

When assets pay *dividends*, their discounted prices are no longer martingales under the risk-neutral measure. Instead, the martingale under the risk-neutral measure is the discounted value of any portfolio that trades in the assets and receives dividends in proportion to its position in the assets at the time of dividend payment. For the case of a continuous payment of dividends at a constant rate, the Black-Scholes-Merton formula is given by (5.5.12). If dividend payments are made in lump sums, the necessary modification to the classical Black-Scholes-Merton formula is presented in Subsection 11.5.4.

The *forward price* of an asset is defined to be that price that one can agree today to pay at a future delivery date so that the present value of the forward contract is zero. For assets that pay no dividends (and, unlike most commodities, cost nothing to hold), the forward price is the asset price divided by the price of a zero-coupon bond maturing on the delivery date and having face value 1:

$$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}, \quad 0 \le t \le T.$$

The *futures price* of an asset is an adapted stochastic process $\mathrm{Fut}_S(t,T)$ with two properties.

(i) The futures price agrees with the asset price on the delivery date (i.e., $\mathrm{Fut}_S(T,T) = S(T)$).
(ii) The value of holding the futures contract over a period of time and receiving the cash flows associated with this position is zero:

$$\frac{1}{D(t_0)}\widetilde{\mathbb{E}}\left[\int_{t_0}^{t_1} D(u)d\mathrm{Fut}_S(u,T)\,\middle|\,\mathcal{F}(t)\right] = 0, \quad 0 \le t_0 < t_1 \le T.$$

The unique process having these two properties is

$$\mathrm{Fut}_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)], \quad 0 \le t \le T.$$

When the interest rate process is nonrandom, forward and futures prices agree. When interest rates are random, the difference between forward and futures prices is proportional to the covariance under the risk-neutral measure between the discount factor $D(T)$ and the underlying asset price $S(T)$ (see (5.6.11)).

5.8 Notes

The idea of risk-neutral pricing is implicit in the classical papers by Black and Scholes [17] and Merton [122] but was not fully developed and appreciated until the work of Ross [140], Harrison and Kreps [77], and Harrison and Pliska [78], [79]. Ross [140] treats a one-period model, Harrison and Kreps [77] treat a continuous-time model with trading at discrete dates, and Harrison and Pliska [78], [79] treat a continuous-time model with continuous trading. The closely related concept of state price density (see Exercise 5.2) is due to Arrow and Debreu [5].

Girsanov's Theorem, Theorem 5.2.3, in the generality stated here is due to Girsanov [72], although the result for constant θ was established much earlier by Cameron and Martin [26]. The theorem requires a technical condition to ensure that $\mathbb{E}Z(T) = 1$ so that $\widetilde{\mathbb{P}}$ is a probability measure. For this purpose, we imposed (5.2.13). An easier condition to verify, due to Novikov [128], is

$$\mathbb{E}\left\{\frac{1}{2}\int_0^T \Theta^2(u)\,du\right\} < \infty;$$

see Karatzas and Shreve [101], page 198. The multidimensional version of both Girsanov's Theorem and the Martingale Representation Theorem (Theorems 5.4.1 and 5.4.2) can be found in Karatzas and Shreve [101] as Theorems 5.1 and 4.15 of Chapter 3. A mathematically rigorous application of these theorems to Brownian-motion-driven models in finance is provided by Karatzas and Shreve [102].

The application of the Girsanov Theorem to risk-neutral pricing is due to Harrison and Pliska [78]. This methodology frees the Brownian-motion-driven model from the assumption of a constant interest rate and volatility. When both of these are stochastic, the Brownian-motion-driven model is mathematically the most general possible for continuous stock prices that do not admit arbitrage. In particular, *the log-normal model for asset prices is just one special case of the Brownian-motion-driven model.*

The Fundamental Theorems of Asset Pricing, Theorems 5.4.7 and 5.4.9, can be found in Harrison and Pliska [78], [79]. It is tempting to believe the converse of Theorem 5.4.7 (i.e., that the absence of arbitrage implies the existence of a risk-neutral measure). This is true in discrete-time models (see Dalang, Morton, and Willinger [45]), but in continuous-time models a slightly stronger condition is needed to guarantee existence of a risk-neutral measure. See Delbaen and Schachermayer [49] for a summary of relevant results.

The distinction between forward contracts and futures was pointed out by Margrabe [118] and Black [13]. No-arbitrage pricing of futures in a discrete-time model was developed by Cox, Ingersoll, and Ross [40] and Jarrow and Oldfield [98].

5.9 Exercises

Exercise 5.1. Consider the discounted stock price $D(t)S(t)$ of (5.2.19). In this problem, we derive the formula (5.2.20) for $d\big(D(t)S(t)\big)$ by two methods.

(i) Define $f(x) = S(0)e^x$ and set

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \Big(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\Big)\,ds$$

so that $D(t)S(t) = f(X(t))$. Use the Itô-Doeblin formula to compute $df(X(t))$.

(ii) According to Itô's product rule,

$$d\big(D(t)S(t)\big) = S(t)\,dD(t) + D(t)\,dS(t) + dD(t)\,dS(t).$$

Use (5.2.15) and (5.2.18) to work out the right-hand side of this equation.

Exercise 5.2 (State price density process). Show that the risk-neutral pricing formula (5.2.30) may be rewritten as

$$D(t)Z(t)V(t) = \mathbb{E}\big[D(T)Z(T)V(T)\big|\mathcal{F}(t)\big]. \tag{5.9.1}$$

Here $Z(t)$ is the Radon-Nikodým derivative process (5.2.11) when the market price of risk process $\Theta(t)$ is given by (5.2.21) and the conditional expectation on the right-hand side of (5.9.1) is taken under the actual probability measure

$\mathbb{P}$, not the risk-neutral measure $\widetilde{\mathbb{P}}$. In particular, if for some $A \in \mathcal{F}(T)$ a derivative security pays off $\mathbb{I}_A$ (i.e., pays 1 if A occurs and 0 if A does not occur), then the value of this derivative security at time zero is $\mathbb{E}[D(T)Z(T)\mathbb{I}_A]$. The process $D(t)Z(t)$ appearing in (5.9.1) is called the *state price density process.*

Exercise 5.3. According to the Black-Scholes-Merton formula, the value at time zero of a European call on a stock whose initial price is $S(0) = x$ is given by

$$c(0,x) = xN(d_+(T,x)) - Ke^{-rT}N(d_-(T,x)),$$

where

$$d_+(T,x) = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right],$$
$$d_-(T,x) = d_+(T,x) - \sigma\sqrt{T}.$$

The stock is modeled as a geometric Brownian motion with constant volatility $\sigma > 0$, the interest rate is constant r, the call strike is K, and the call expiration time is T. This formula is obtained by computing the discounted expected payoff of the call under the risk-neutral measure,

$$\begin{aligned} c(0,x) &= \widetilde{\mathbb{E}}\left[e^{-rT}(S(T) - K)^+\right] \\ &= \widetilde{\mathbb{E}}\left[e^{-rT}\left(x\exp\left\{\sigma\widetilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\} - K\right)^+\right], \end{aligned} \tag{5.9.2}$$

where $\widetilde{W}$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. In Exercise 4.9(ii), the *delta* of this option is computed to be $c_x(0,x) = N(d_+(T,x))$. This problem provides an alternate way to compute $c_x(0,x)$.

(i) We begin with the observation that if $h(s) = (s-K)^+$, then

$$h'(s) = \begin{cases} 0 \text{ if } s < K, \\ 1 \text{ if } s > K. \end{cases}$$

If $s = K$, then $h'(s)$ is undefined, but that will not matter in what follows because $S(T)$ has zero probability of taking the value K. Using the formula for $h'(s)$, differentiate inside the expected value in (5.9.2) to obtain a formula for $c_x(0,x)$.

(ii) Show that the formula you obtained in (i) can be rewritten as

$$c_x(0,x) = \widehat{\mathbb{P}}(S(T) > K),$$

where $\widehat{\mathbb{P}}$ is a probability measure equivalent to $\widetilde{\mathbb{P}}$. Show that

$$\widehat{W}(t) = \widetilde{W}(t) - \sigma t$$

is a Brownian motion under $\widehat{\mathbb{P}}$.

(iii) Rewrite $S(T)$ in terms of $\widehat{W}(T)$, and then show that

$$\widehat{\mathbb{P}}\{S(T) > K\} = \widehat{\mathbb{P}}\left\{-\frac{\widehat{W}(T)}{\sqrt{T}} < d_+(T,x)\right\} = N(d_+(T,x)).$$

Exercise 5.4 (Black-Scholes-Merton formula for time-varying, nonrandom interest rate and volatility). Consider a stock whose price differential is

$$dS(t) = r(t)S(t)\,dt + \sigma(t)\,d\widetilde{W}(t),$$

where $r(t)$ and $\sigma(t)$ are nonrandom functions of t and $\widetilde{W}$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Let $T > 0$ be given, and consider a European call, whose value at time zero is

$$c(0,S(0)) = \widetilde{\mathbb{E}}\left[\exp\left\{-\int_0^T r(t)\,dt\right\}(S(T)-K)^+\right].$$

(i) Show that $S(T)$ is of the form $S(0)e^X$, where X is a normal random variable, and determine the mean and variance of X.

(ii) Let

$$\begin{aligned}\mathrm{BSM}(T,x;K,R,\Sigma) = &\, xN\left(\frac{1}{\Sigma\sqrt{T}}\left[\log\frac{x}{K} + (R+\Sigma^2/2)T\right]\right)\\ &-e^{-RT}KN\left(\frac{1}{\Sigma\sqrt{T}}\left[\log\frac{x}{K} + (R-\Sigma^2/2)T\right]\right)\end{aligned}$$

denote the value at time zero of a European call expiring at time T when the underlying stock has constant volatility Σ and the interest rate R is constant. Show that

$$c(0,S(0)) = \mathrm{BSM}\left(S(0), T, \frac{1}{T}\int_0^T r(t)dt, \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}\right).$$

Exercise 5.5. Prove Corollary 5.3.2 by the following steps.

(i) Compute the differential of $\frac{1}{Z(t)}$, where $Z(t)$ is given in Corollary 5.3.2.

(ii) Let $\widetilde{M}(t)$, $0 \le t \le T$, be a martingale under $\widetilde{\mathbb{P}}$. Show that $M(t) = Z(t)\widetilde{M}(t)$ is a martingale under $\mathbb{P}$.

(iii) According to Theorem 5.3.1, there is an adapted process $\Gamma(u)$, $0 \le u \le T$, such that

$$M(t) = M(0) + \int_0^T \Gamma(u)\,dW(u),\ 0 \le t \le T.$$

Write $\widetilde{M}(t) = M(t)\cdot\frac{1}{Z(t)}$ and take its differential using Itô's product rule.

(iv) Show that the differential of $\widetilde{M}(t)$ is the sum of an adapted process, which we call $\widetilde{\Gamma}(t)$, times $d\widetilde{W}(t)$, and zero times dt. Integrate to obtain (5.3.2).

Exercise 5.6. Use the two-dimensional Lévy Theorem, Theorem 4.6.5, to prove the two-dimensional Girsanov Theorem (i.e., Theorem 5.4.1 with $d = 2$).

Exercise 5.7. (i) Suppose a multidimensional market model as described in Section 5.4.2 has an arbitrage. In other words, suppose there is a portfolio value process satisfying $X_1(0) = 0$ and

$$\mathbb{P}\{X_1(T) \geq 0\} = 1, \quad \mathbb{P}\{X_1(T) > 0\} > 0, \tag{5.4.23}$$

for some positive T. Show that if $X_2(0)$ is positive, then there exists a portfolio value process $X_2(t)$ starting at $X_2(0)$ and satisfying

$$\mathbb{P}\left\{X_2(T) \geq \frac{X_2(0)}{D(T)}\right\} = 1, \quad \mathbb{P}\left\{X_2(T) > \frac{X_2(0)}{D(T)}\right\} > 0. \tag{5.4.24}$$

(ii) Show that if a multidimensional market model has a portfolio value process $X_2(t)$ such that $X_2(0)$ is positive and (5.4.24) holds, then the model has a portfolio value process $X_1(0)$ such that $X_1(0) = 0$ and (5.4.23) holds.

Exercise 5.8 (Every strictly positive asset is a generalized geometric Brownian motion). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)$, $0 \leq t \leq T$. Let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Assume there is a unique risk-neutral measure $\widetilde{\mathbb{P}}$, and let $\widetilde{W}(t)$, $0 \leq t \leq T$, be the Brownian motion under $\widetilde{\mathbb{P}}$ obtained by an application of Girsanov's Theorem, Theorem 5.2.3.

Corollary 5.3.2 of the Martingale Representation Theorem asserts that every martingale $\widetilde{M}(t)$, $0 \leq t \leq T$, under $\widetilde{\mathbb{P}}$ can be written as a stochastic integral with respect to $\widetilde{W}(t)$, $0 \leq t \leq T$. In other words, there exists an adapted process $\widetilde{\Gamma}(t)$, $0 \leq t \leq T$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{B}(u), \quad 0 \leq t \leq T.$$

Now let $V(T)$ be an almost surely positive ("almost surely" means with probability one under both $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ since these two measures are equivalent), $\mathcal{F}(T)$-measurable random variable. According to the risk-neutral pricing formula (5.2.31), the price at time t of a security paying $V(T)$ at time T is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du} V(T) \Big| \mathcal{F}(t)\right], \quad 0 \leq t \leq T.$$

(i) Show that there exists an adapted process $\widetilde{\Gamma}(t)$, $0 \leq t \leq T$, such that

$$dV(t) = R(t)V(t)\, dt + \frac{\widetilde{\Gamma}(t)}{D(t)}\, d\widetilde{W}(t), \quad 0 \leq t \leq T.$$

(ii) Show that, for each $t \in [0,T]$, the price of the derivative security $V(t)$ at time t is almost surely positive.

(iii) Conclude from (i) and (ii) that there exists an adapted process $\sigma(t)$, $0 \leq t \leq T$, such that

$$dV(t) = R(t)V(t)\,dt + \sigma(t)V(t)\,d\widetilde{W}(t), \quad 0 \leq t \leq T.$$

In other words, prior to time T, the price of every asset with almost surely positive price at time T follows a generalized (because the volatility may be random) geometric Brownian motion.

Exercise 5.9 (Implying the risk-neutral distribution). Let $S(t)$ be the price of an underlying asset, which is not necessarily a geometric Brownian motion (i.e., does not necessarily have constant volatility). With $S(0) = x$, the risk-neutral pricing formula for the price at time zero of a European call on this asset, paying $\big(S(T) - K\big)^+$ at time T, is

$$c(0,T,x,K) = \widetilde{\mathbb{E}}\left[e^{-rT}\big(S(T) - K\big)^+\right].$$

(Normally we consider this as a function of the current time 0 and the current stock price x, but in this exercise we shall also treat the expiration time T and the strike price K as variables, and for that reason we include them as arguments of c.) We denote by $\tilde{p}(0,T,x,y)$ the risk-neutral density in the y variable of the distribution of $S(T)$ when $S(0) = x$. Then we may rewrite the risk-neutral pricing formula as

$$c(0,T,x,K) = e^{-rT}\int_K^\infty (y - K)\tilde{p}(0,T,x,y)\,dy. \tag{5.9.3}$$

Suppose we know the market prices for calls of all strikes (i.e., we know $c(0,T,x,K)$ for all $K > 0$).[2] We can then compute $c_K(0,T,x,K)$ and $c_{KK}(0,T,x,K)$, the first and second derivatives of the option price with respect to the strike. Differentiate (5.9.3) twice with respect to K to obtain the equations

$$\begin{aligned} c_K(0,T,x,K) &= -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy \;=\; -e^{-rT}\widetilde{\mathbb{P}}\{S(T) > K\},\\ c_{KK}(0,T,x,K) &= e^{-rT}\tilde{p}(0,T,x,K). \end{aligned}$$

The second of these equations provides a formula for the risk-neutral distribution of $S(T)$ in terms of call prices:

$$\tilde{p}(0,T,x,K) = e^{rT}c_{KK}(0,T,x,K) \text{ for all } K > 0.$$

[2] In practice, we do not have this many prices. We have the prices of calls at some strikes, and we can infer the prices of calls at other strikes by knowing the prices of puts and using put–call parity. We must create prices for the calls of other strikes by interpolation of the prices we do have.

Exercise 5.10 (Chooser option). Consider a model with a unique risk-neutral measure $\widetilde{\mathbb{P}}$ and constant interest rate r. According to the risk-neutral pricing formula, for $0 \le t \le T$, the price at time t of a European call expiring at time T is

$$C(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \Big| \mathcal{F}(t)\right],$$

where $S(T)$ is the underlying asset price at time T and K is the strike price of the call. Similarly, the price at time t of a European put expiring at time T is

$$P(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(K-S(T))^+ \Big| \mathcal{F}(t)\right].$$

Finally, because $e^{-rt}S(t)$ is a martingale under $\widetilde{\mathbb{P}}$, the price at time t of a forward contract for delivery of one share of stock at time T in exchange for a payment of K at time T is

$$\begin{aligned}
F(t) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)\Big| \mathcal{F}(t)\right]\\
&= e^{rt}\widetilde{\mathbb{E}}\left[e^{-rT}S(T)\big| \mathcal{F}(t)\right] - e^{-r(T-t)}K\\
&= S(t) - e^{-r(T-t)}K.
\end{aligned}$$

Because

$$(S(T)-K)^+ - (K-S(T))^+ = S(T)-K,$$

we have the *put–call parity* relationship

$$\begin{aligned}
C(t) - P(t) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ - e^{-r(T-t)}(K-S(T))^+\Big| \mathcal{F}(t)\right]\\
&= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)\Big| \mathcal{F}(t)\right] \;=\; F(t).
\end{aligned}$$

Now consider a date t_0 between 0 and T, and consider a *chooser option*, which gives the right at time t_0 to choose to own either the call or the put.

(i) Show that at time t_0 the value of the chooser option is

$$C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+.$$

(ii) Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

Exercise 5.11 (Hedging a cash flow). Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be the filtration generated by this Brownian motion. Let the mean rate of return $\alpha(t)$, the interest rate $R(t)$, and the volatility $\sigma(t)$ be adapted processes, and assume that $\sigma(t)$ is never zero. Consider a stock price process whose differential is given by (5.2.15):

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \quad 0 \le t \le T.$$

Suppose an agent must pay a cash flow at rate $C(t)$ at each time t, where $C(t)$, $0 \le t \le T$, is an adapted process. If the agent holds $\Delta(t)$ shares of stock at each time t, then the differential of her portfolio value will be

$$dX(t) = \Delta(t)\,dS(t) + R(t)\big(X(t) - \Delta(t)S(t)\big)\,dt - C(t)\,dt. \tag{5.9.4}$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t)$, $0 \le t \le T$, such that $X(T) = 0$ almost surely. (Hint: Define the risk-neutral measure and apply Corollary 5.3.2 of the Martingale Representation Theorem to the process

$$\widetilde{M}(t) = \widetilde{\mathbb{E}}\left[\int_0^T D(u)C(u)\,du \,\middle|\, \mathcal{F}(t)\right], \quad 0 \le t \le T, \tag{5.9.5}$$

where $D(t)$ is the discount process (5.2.17).)

Exercise 5.12 (Correlation under change of measure). Consider the multidimensional market model of Subsection 5.4.2, and let $B_i(t)$ be defined by (5.4.7). Assume that the market price of risk equations (5.4.18) have a solution $\Theta_1(t), \dots, \Theta_d(t)$, and let $\widetilde{\mathbb{P}}$ be the corresponding risk-neutral measure under which

$$\widetilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u)\,du, \quad j = 1, \dots, d,$$

are independent Brownian motions.

(i) For $i = 1, \dots, d$, define $\gamma_i(t) = \sum_{j=1}^d \frac{\sigma_{ij}(t)\theta_j(t)}{\sigma_i(t)}$. Show that

$$\widetilde{B}_i(t) = B_i(t) + \int_0^t \gamma_i(u)\,du$$

is a Brownian motion under $\widetilde{\mathbb{P}}$.

(ii) We saw in (5.4.8) that

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + \sigma_i(t)S_i(t)\,dB_i(t), \quad i = 1, \dots, m.$$

Show that

$$dS_i(t) = R(t)S_i(t)\,dt + \sigma_i S_i(t)\,d\widetilde{B}_i(t), \quad i = 1, \dots, m.$$

(iii) We saw in (5.4.9) that $dB_i(t)\,dB_k(t) = \rho_{ik}(t)$. This is the *instantaneous correlation* between $B_i(t)$ and $B_k(t)$. Because (5.4.9) makes no reference to the probability measure, Exercise 4.17 of Chapter 4 implies that under both $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, the correlation between the pair of increments $B_1(t_0 + \epsilon) - B_1(t_0)$ and $B_2(t_0 + \epsilon) - B_2(t_0)$ is approximately $\rho_{ik}(t_0)$. Show that

$$d\widetilde{B}_i(t)\, d\widetilde{B}_k(t) = \rho_{ik}(t).$$

This formula means that, conditioned on $\mathcal{F}(t_0)$, under both $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ the correlation between the pair of increments $\widetilde{B}_1(t_0+\epsilon) - \widetilde{B}_1(t_0)$ and $\widetilde{B}_2(t_0+\epsilon) - \widetilde{B}_2(t_0)$ is approximately $\rho_{ik}(t_0)$.

(iv) Show that if $\rho_{ik}(t)$ is not random (although it may still depend on t), then for every $t \geq 0$,

$$\mathbb{E}\big[B_i(t)B_k(t)\big] = \widetilde{\mathbb{E}}\big[\widetilde{B}_i(t)\widetilde{B}_k(t)\big] = \int_0^t \rho_{ik}(u)\, du.$$

Since $B_i(t)$ and $B_k(t)$ both have variance t under $\mathbb{P}$ and $\widetilde{B}_i(t)$ and $\widetilde{B}_k(t)$ both have variance t under $\widetilde{\mathbb{P}}$, this shows that the correlation between $B_i(t)$ and $B_k(t)$ under $\mathbb{P}$ is the same as the correlation between $\widetilde{B}_i(t)$ and $\widetilde{B}_k(t)$ under $\widetilde{\mathbb{P}}$. In both cases, this correlation is $\frac{1}{t}\int_0^t \rho_{ik}(u)\, du$. If ρ_{ik} is constant, then the correlation is simply ρ_{ik}.

(v) When $\rho_{ik}(t)$ is random, we can have

$$\mathbb{E}\big[B_i(t)B_k(t)\big] \neq \widetilde{\mathbb{E}}\big[\widetilde{B}_i(t)\widetilde{B}_k(t)\big].$$

Even though instantaneous correlations are unaffected by a change of measure, correlations can be. To see this, we take $m = d = 2$ and let $W_1(t)$ and $W_2(t)$ be independent Brownian motions under $\mathbb{P}$. Take $\sigma_{11}(t) = \sigma_{21}(t) = 0$, $\sigma_{12}(t) = 1$, and $\sigma_{22}(t) = \operatorname{sign}(W_1(t))$, where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then $\sigma_1(t) = 1$, $\sigma_2(t) = 1$, $\rho_{11}(t) = 1$, $\rho_{22}(t) = 1$ and $\rho_{12}(t) = \rho_{21}(t) = \operatorname{sign}(W_1(t))$. Take $\Theta_1(t) = 1$ and $\Theta_2(t) = 0$, so that $\widetilde{W}_1(t) = W_1(t) + t$ and $\widetilde{W}_2(t) = W_2(t)$. Then $\gamma_1(t) = \gamma_2(t) = 0$. We have

$$\begin{aligned} B_1(t) &= W_2(t), \quad & B_2(t) &= \int_0^t \operatorname{sign}(W_1(u))\, dW_2(u), \\ \widetilde{B}_1(t) &= B_1(t), \quad & \widetilde{B}_2(t) &= B_2(t). \end{aligned}$$

Show that

$$\mathbb{E}\big[B_1(t)B_2(t)\big] \neq \widetilde{\mathbb{E}}\big[\widetilde{B}_1(t)\widetilde{B}_2(t)\big] \text{ for all } t > 0.$$

Exercise 5.13. In part (v) of Exercise 5.12, we saw that when we change measures and change Brownian motions, correlations can change if the instantaneous correlations are random. This exercise shows that a change of measure without a change of Brownian motions can change correlations if the market prices of risk are random.

Let $W_1(t)$ and $W_2(t)$ be independent Brownian motions under a probability measure $\widetilde{\mathbb{P}}$. Take $\Theta_1(t) = 0$ and $\Theta_2(t) = W_1(t)$ in the multidimensional Girsanov Theorem, Theorem 5.4.1. Then $\widetilde{W}_1(t) = W_1(t)$ and $\widetilde{W}_2(t) = W_2(t) + \int_0^t W_1(u)\,du$.

(i) Because $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are Brownian motions under $\widetilde{\mathbb{P}}$, the equation $\widetilde{\mathbb{E}}\widetilde{W}_1(t) = \widetilde{\mathbb{E}}\widetilde{W}_2(t) = 0$ must hold for all $t \in [0,T]$. Use this equation to conclude that

$$\widetilde{\mathbb{E}}W_1(t) = \widetilde{\mathbb{E}}W_2(t) = 0 \text{ for all } t \in [0,T].$$

(ii) From Itô's product rule, we have

$$d\big(W_1(t)W_2(t)\big) = W_1(t)\,dW_2(t) + W_2(t)\,dW_1(t).$$

Use this equation to show that

$$\widetilde{\mathrm{Cov}}[W_1(T), W_2(T)] = \widetilde{\mathbb{E}}[W_1(T)W_2(T)] = -\frac{1}{2}T^2.$$

This is different from

$$\mathrm{Cov}[W_1(T), W_2(T)] = \mathbb{E}[W_1(T)W_2(T)] = 0.$$

Exercise 5.14 (Cost of carry). Consider a commodity whose unit price at time t is $S(t)$. Ownership of a unit of this commodity requires payment at a rate a per unit time (*cost of carry*) for storage. Note that this payment is per unit of commodity, not a fraction of the price of the commodity. Thus, the value of a portfolio that holds $\Delta(t)$ units of the commodity at time t and also invests in a money market account with constant rate of interest r has differential

$$dX(t) = \Delta(t)\,dS(t) - a\Delta(t)\,dt + r\big(X(t) - \Delta(t)S(t)\big)\,dt. \tag{5.9.6}$$

As with the dividend-paying stock in Section 5.5, we must choose the risk-neutral measure so that the discounted portfolio value $e^{-rt}X(t)$ is a martingale. We shall assume a constant volatility, so in place of (5.5.6) we have

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t) + a\,dt, \tag{5.9.7}$$

where $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$.

(i) Show that when $dS(t)$ is given by (5.9.7), then under $\widetilde{\mathbb{P}}$ the discounted portfolio value process $e^{-rt}X(t)$, where $X(t)$ is given by (5.9.6), is a martingale.

(ii) Define

$$Y(t) = \exp\left\{\sigma\widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}.$$

Verify that, for $0 \le t \le T$,

$$dY(t) = rY(t)\,dt + \sigma Y(t)\,d\widetilde{W}(t),$$

that $e^{-rt}Y(t)$ is a martingale under $\widetilde{\mathbb{P}}$, and that

$$S(t) = S(0)Y(t) + Y(t)\int_0^t \frac{a}{Y(s)}\,ds \tag{5.9.8}$$

satisfies (5.9.7).

(iii) For $0 \le t \le T$, derive a formula for $\widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)]$ in terms of $S(t)$ by writing

$$\widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)] = S(0)\widetilde{\mathbb{E}}[Y(T)|\mathcal{F}(t)] + \widetilde{\mathbb{E}}[Y(T)|\mathcal{F}(t)]\int_0^t \frac{a}{Y(s)}\,ds$$
$$+a\int_t^T \widetilde{\mathbb{E}}\left[\frac{Y(T)}{Y(s)}\middle|\mathcal{F}(t)\right]ds \tag{5.9.9}$$

and then simplifying the right-hand side of this equation.

(iv) The process $\widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)]$ is the futures price process for the commodity (i.e., $\mathrm{Fut}_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)]$). This must be a martingale under $\widetilde{\mathbb{P}}$. To check the formula you obtained in (iii), differentiate it and verify that $\widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)]$ is a martingale under $\widetilde{\mathbb{P}}$.

(v) Let $0 \le t \le T$ be given. Consider a forward contract entered at time t to purchase one unit of the commodity at time T for price K paid at time T. The value of this contract at time t when it is entered is

$$\widetilde{\mathbb{E}}\big[e^{-r(T-t)}\big(S(T)-K\big)\big|\mathcal{F}(t)\big]. \tag{5.9.10}$$

The forward price $\mathrm{For}_S(t,T)$ is the value of K that makes the contract value (5.9.10) equal to zero. Show that $\mathrm{For}_S(t,T) = \mathrm{Fut}_S(t,T)$.

(vi) Consider an agent who takes a short position in a forward contract at time zero. This costs nothing and generates no income at time zero. The agent hedges this position by borrowing $S(0)$ from the money market account and purchasing one unit of the commodity, which she holds until time T. At time T, the agent delivers the commodity under the forward contract and receives the forward price $\mathrm{For}_S(0,T)$ set at time zero. Show that this is exactly what the agent needs to cover her debt to the money market account, which has two parts. First of all, at time zero, the agent borrows $S(0)$ from the money market account in order to purchase the unit of the commodity. Second, between times zero and T, the agent pays the cost of carry a per unit time, borrowing from the money market account to finance this. (Hint: The value of the agent's portfolio of commodity and money market account begins at $X(0) = 0$ (one unit of the commodity and a money market position of $-S(0)$) and is governed by (5.9.6) with $\Delta(t) = 1$. Write this equation, determine $d(e^{-rt}X(t))$, integrate both sides from zero to T, and solve for $X(T)$. You will need the fact that $e^{-rt}\big(dS(t) - rS(t)\,dt\big) = d\big(e^{-rt}S(t)\big)$. You should get $X(T) = S(T) - \mathrm{For}_S(0,T)$.)

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