CONVOLUTION AND APPROXIMATE IDENTITIES

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1. SETTING THE SCENE

DEFINITION 1.1. *G is a Topological Group if and only if the following are true:*

(1) G is a topological space
(2) G is Hausdorff
(3) G satisfies the group axioms
(4) The maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous

DEFINITION 1.2. *A Topological Group G is Locally Compact if*

$$\exists U \subset G : e \in U \wedge U \text{ is compact}$$

where e is the identity element of G.

In the remaining text G shall be a Locally Compact Topological Group. It is not necessary for G to be abelian in most cases, however we shall assume so. Where it is relevant, we have included footnotes. We shall also make use of a specific measure on G:

LEMMA 1.1. *There exists a measure λ on any topological group G defined on the Borel sets such that:*

(1) $U \subset G \wedge U \neq \emptyset \wedge U$ open $\implies \lambda(U) > 0$
(2) $U \subset G \wedge U$ compact $\implies \lambda(U) < \infty$
(3) λ is unique up to multiplication
(4) $\forall t \in G : A \subset G$ measurable $\implies \lambda(tA) = \lambda(A)$

This measure is called the Left Haar Measure, highlighted by the final property of left invariance.

Proof. For conclusive proof see ZZZZ □

In the remaining text λ shall denote the Left Haar Measure on G. We shall make one final assumption, that G is a countable union of compact subsets. This makes (G, λ) a σ-finite measure space. We now have that which is necessary to begin.

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2. Convolution

DEFINITION 2.1. *Let $f, g \in L^1(G)$. Define the **convolution** $f * g$ by*

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\lambda(y) \tag{1}$$

REMARK 2.1. *Note that on $\mathbb{R}^n$ with an additive structure (our preferred environment for later chapters), we will simply have:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)dy$$

EXAMPLE 2.1. *Let $G = \mathbb{R}$ with the usual additive structure and*

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & else \end{cases} \tag{2}$$

Then we calculate:

$$(f * f)(x) = \int_{\mathbb{R}} f(y)f(x - y)d\lambda(y)$$

$$= \begin{cases} \int_{\mathbb{R}} \chi_{[-1,1] \cap [x-1,x+1]}(x)d\lambda(x) & -2 \leq x \leq 2 \\ \int_{\mathbb{R}} 0 d\lambda(y) & else \end{cases}$$

$$= \begin{cases} 2 - |x| & -2 \leq x \leq 2 \\ 0 & else \end{cases}$$

Notice that the convolution operator has a natural smoothing effect on f, as it does on every function.

LEMMA 2.1. *Convolution is defined λ almost everywhere.*

Proof. To see this we take the L_1 norm on the definition to find it finite:

$$\left\| (f * g)(x) \right\|_{L^1} = \int_G \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| d\lambda(x)$$

$$\leq \int_G \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(y)d\lambda(x)$$

$$= \int_G \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(x)d\lambda(y)$$

$$= \int_G |f(y)| \int_G |g(y^{-1}x)| \, d\lambda(x)d\lambda(y)$$

$$= \int_G |f(y)| \int_G |g(x)| \, d\lambda(x)d\lambda(y)$$

$$= \|f\|_{L^1} \|g\|_{L^1}$$

$$< \infty$$

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This proof followed simply from the triangle inequality for integrals, the definition of our norm, Fubini, and the properties of our measure.   □

LEMMA 2.2.

$$(f * g)(x) = \int_G f(xz)g(z^{-1})d\lambda(z)$$

Proof. We perform a change of variables $z = x^{-1}y$:

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\lambda(y)$$
$$= \int_G f(xx^{-1}y)g((yx^{-1})^{-1})d\lambda(y)$$
$$= \int_G f(xz)g(z^{-1})d\lambda(x^{-1}y)$$
$$= \int_G f(xz)g(z^{-1})d\lambda(z)$$

Where we have made use of the inherent property of a Left Haar Measure.   □

PROPOSITION 2.1. $\forall f, g, h \in L^1(G)$:

    (1) $f * (g * h) = (f * g) * h$
    (2) $f * (g + h) = f * g + f * h \wedge (f + g) * h = f * h + f * g$

Thus convolution is associative and distributive.

Proof. Associativity, as so many things, follows from an application of Fubini:

$$((f * g) * h)(x) = \int_G (f * g)(y)h(y^{-1}x)d\lambda(y)$$
$$= \int_G (\int_G f(z)g(z^{-1}y)d\lambda(z))h(y^{-1}x)d\lambda(y)$$
$$= \int_G \int_G f(z)g(z^{-1}y)h(y^{-1}x)d\lambda(z)d\lambda(y)$$
$$= \int_G f(z) \int_G g(z^{-1}y)h(y^{-1}x)d\lambda(y)d\lambda(z)$$
$$= \int_G f(z) \int_G g(z^{-1}y)h(y^{-1}x)d\lambda(y)d\lambda(z)ZZZZ$$
$$= \int_G f(z)(g * h)(xz^{-1})d\lambda(z)$$
$$= (f * (g * h))(x)$$

Distributivity follow from the additive property of the integral:

$$f * (g + h) = \int_G f(y)(g + h)(y^{-1}x)d\lambda(y)$$

$$= \int_G f(y)(g(y^{-1}x) + h(y^{-1}x))d\lambda(y)$$

$$= \int_G f(y)g(y^{-1}x) + f(y)h(y^{-1}x)d\lambda(y)$$

$$= \int_G f(y)g(y^{-1}x)d\lambda(y) + \int_G f(y)h(y^{-1}x)d\lambda(y)$$

$$= f * g + f * h$$

The mirror statement follows analogously. $\qquad\square$

REMARK 2.2. *ZZZZ (why does this even exist?) ZZZZ Notice also the following trivial equality:*

$$\|f\|_{L^p}^{p/q} = ((\int_G |f(x)|^p \, d\lambda(x))^{1/p})^{p/q}$$

$$= (\int_G |f(x)|^p \, d\lambda(x))^{1/q}$$

3. Basic Convolution Inequalities

DEFINITION 3.1. *Define $p' := p/(p-1)$. To maintain our desired property in infinity we also declare: $1/\infty = 0$*

REMARK 3.1. *Notice then $\frac{1}{p} + \frac{1}{p'} = \frac{1}{p} + \frac{p-1}{p} = \frac{p}{p} = 1$ for any $p \in (0, \infty]$*

REMARK 3.2. *In the following proofs we often make use of this useful inequality without further explication, it allows us to manipulate inequalities without worrying about negatives:*

$$f * g = \int_G f(y)g(y^{-1}x)d\lambda(y)$$

$$\leq \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right|$$

$$\leq \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(y)$$

$$= |f| * |g|$$

What follows is akin to the triangle inequality for L^p spaces.

THEOREM 3.1. *Minkowskis Inequality: Let $1 \leq p \leq \infty$, $f \in L^p(G)$, $g \in L^1(G)$ then it follows that: $g * f$ exists λ-almost-everywhere and $\|g * f\|_{L^p(G)} \leq \|g\|_{L^1(G)}\|f\|_{L^p(G)}$*

Proof. First we notice that the case of $p = 1$ follows from (Lemma 3.1). Similarly[1] we may rid ourselves of the case $p = \infty$ by again applying the triangle inequality and recalling

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[1]This only works with the Abelian property, however it is possible to assume some small additional necessities to make it work in non-commutative groups

that $|f| \leq \|f\|_{L^\infty}$:

$$
\begin{aligned}
\left|(f * g)\right| &= \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| \\
&\leq \int_G \left|f(y)\right|\left|g(y^{-1}x)\right| d\lambda(y) \\
&\leq \|f\|_{L^\infty} \int_G \left|g(y^{-1}x)\right| d\lambda(y) \\
&= \|f\|_{L^\infty} \int_G \left|g(y)\right| d\lambda(y) \\
&= \|f\|_{L^\infty} \|g\|_{L^1}
\end{aligned}
$$

For $1 < p < \infty$, we must work a little harder. We have:

$$
(|g| * |f|)(x) = \int_G \left|f(y^{-1}x)\right|\left|g(y)\right| d\lambda(y)
$$

We shall apply Hölders inequality with the to following criteria,

    Measure: $\left|g(y)\right| d\lambda(y)$

    Functions: 1 and $\left|f(y^{-1}x)\right|$

    Exponents: p and p' since $\frac{1}{p} + \frac{1}{p'} = 1$

to discover:

$$
\begin{aligned}
(|g| * |f|)(x) &\leq \left(\int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y)\right)^{1/p}\left(\int_G 1^{p'}\left|g(y)\right| d\lambda(y)\right)^{1/p'} \\
&= \left(\int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y)\right)^{1/p}\left(\int_G \left|g(y)\right| d\lambda(y)\right)^{1/p'}
\end{aligned}
$$

Since we are working with positive functions, by the monotonicity of the exponent and the integral, we may take the L^p norm on both sides while preserving the inequality.

$$
\begin{aligned}
\left\|\,|g| * |f|\,\right\|_{L^p} &\leq \left\|\left(\int_G \left|g(y)\right| d\lambda(y)\right)^{1/p'}\left(\int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y)\right)^{1/p}\right\|_{L^p} \\
&= \left(\left(\int_G \left|g(y)\right| d\lambda(y)\right)^{\frac{p}{p'}} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y)d\lambda(x)\right)^{\frac{1}{p}} \\
&= \left(\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y)d\lambda(x)\right)^{\frac{1}{p}} \\
&= \left(\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y)d\lambda(x)\right)^{\frac{1}{p}} \\
&= \left(\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(x)d\lambda(y)\right)^{\frac{1}{p}} \\
&= \left(\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p d\lambda(x)\left|g(y)\right| d\lambda(y)\right)^{\frac{1}{p}}
\end{aligned}
$$

$$= (\|g\|_{L^1}^{p-1} \int_G \int_G |f(x)|^p \, d\lambda(x) |g(y)| \, d\lambda(y))^{\frac{1}{p}}$$

$$= (\|g\|_{L^1}^{p-1} \int_G |f(x)|^p \, d\lambda(x) \int_G |g(y)| \, d\lambda(y))^{\frac{1}{p}}$$

$$= (\|g\|_{L^1}^{p-1} \|f\|_{L^p}^p \|g\|_{L^1})^{\frac{1}{p}}$$

$$= (\|g\|_{L^1}^{p} \|f\|_{L^p}^p)^{\frac{1}{p}}$$

$$= \|g\|_{L^1} \|f\|_{L^p}$$

$$< \infty$$

This implies that $|g| * |f|$ is finite λ-a.e. and satisfies the required inequality. We have shown before that $g * f \leq |g| * |f|$ which concludes our proof.

$\square$

We shall find that we can prove this same inequality in a more general case, strengthening our tool set by a lot.

THEOREM 3.2. *Young's Inequality: Let $1 \leq p, q, r \leq \infty$ such that*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$$

and $f \in L^p \wedge g \in L^r$. Then $f * g$ exists λ-a.e and the following inequality holds:

$$\|f * g\|_{L^q} \leq \|g\|_{L^r} \|f\|_{L^p}$$

Proof. First we shall rid ourselves of the case $r = \infty$. We notice that then $p = 1 \wedge q = \infty$ follows by satisfaction of the requirements. This case then simply follows from Minkowski's Inequality.

Therefore let us now assume $r < \infty$. We examine first the beautiful relationship between p, q, r:

(1) $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p} \Leftrightarrow 1 - \frac{1}{r} + \frac{1}{q} + 1 - \frac{1}{p} = 1 \Leftrightarrow \frac{r-1}{r} + \frac{1}{q} + \frac{p-1}{p} = 1 \Leftrightarrow \frac{1}{r/(r-1)} + \frac{1}{q} + \frac{1}{p/(p-1)} = 1 \Leftrightarrow \frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1$

(2) $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p} \Leftrightarrow \frac{p}{q} + p = \frac{p}{r} + 1 \Leftrightarrow \frac{p}{q} + p - \frac{p}{r} = 1 \Leftrightarrow \frac{p}{q} + \frac{p(r-1)}{r} = 1 \Leftrightarrow \frac{p}{q} + \frac{p}{r/(r-1)} = 1 \Leftrightarrow \frac{p}{q} + \frac{p}{r'} = 1$

(3) $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r} \Leftrightarrow \frac{1}{q} + 1 - \frac{1}{p} = \frac{1}{r} \Leftrightarrow \frac{r}{q} + r - \frac{r}{p} = 1 \Leftrightarrow \frac{r}{q} + \frac{r(p-1)}{p} = 1 \Leftrightarrow \frac{r}{q} + \frac{r}{p/(p-1)} = 1 \Leftrightarrow \frac{r}{q} + \frac{r}{p'} = 1$

We shall make use of these lovely final 2 statements in what follows to split our terms into more suitable pieces. We work with $|f| * |g|$ as before:

$$\left|(|f| * |g|)(x)\right| \leq \int_G |f(y)| \left|g(y^{-1}x)\right| \, d\lambda(y)$$

$$= \int_G |f(y)|^{\frac{p}{r'}} |f(y)|^{\frac{p}{q}} \left|g(y^{-1}x)\right|^{\frac{r}{q}} \left|g(y^{-1}x)\right|^{\frac{r}{p'}} \, d\lambda(y)$$

$$= \left\| f(y)^{\frac{p}{r'}} f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} g(y^{-1}x)^{\frac{r}{p'}} \right\|_{L^1}$$

To proceed we shall now make use of Hölder's inducted inequality (Exercise 1.1.2(a)) for multiple exponents with:

    Measure: Simply $d\lambda(y)$

    Functions: $\left| f(y) \right|^{\frac{p}{r'}}, \left| g(y^{-1}x) \right|^{\frac{r}{p'}}$, and by combining the rest, $\left| f(y) \right|^{\frac{p}{q}} \left| g(y^{-1}x) \right|^{\frac{r}{q}}$

    Exponents: r', q, and p'. See our first discovery above.

To receive:

$$\left\| f(y)^{\frac{p}{r'}} f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} g(y^{-1}x)^{\frac{r}{p'}} \right\|_{L^1} \leq \left\| f(y)^{\frac{p}{r'}} \right\|_{L^{r'}} \left\| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right\|_{L^q} \left\| g(y^{-1}x)^{\frac{r}{p'}} \right\|_{L^{p'}}$$

$$= \left(\int_G \left| f(y)^{\frac{p}{r'}} \right|^{r'} d\lambda(y) \right)^{\frac{1}{r'}} \left\| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right\|_{L^q} \left\| g(y^{-1}x)^{\frac{r}{p'}} \right\|_{L^{p'}}$$

$$= \left(\int_G \left| f(y)^p \right| d\lambda(y) \right)^{\frac{1}{r'}} \left\| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right\|_{L^q} \left\| g(y^{-1}x)^{\frac{r}{p'}} \right\|_{L^{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left\| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right\|_{L^q} \left\| g(y^{-1}x)^{\frac{r}{p'}} \right\|_{L^{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left\| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right\|_{L^q} \left(\int_G \left(\left| g(y^{-1}x) \right|^{\frac{r}{p'}} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left\| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right\|_{L^q} \left(\int_G \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left(\int_G \left(\left| f(y)^{\frac{p}{q}} g(y^{-1}x)^{\frac{r}{q}} \right| \right)^q \right)^{\frac{1}{q}} \left(\int_G \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left(\int_G \left| f(y) \right|^p \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{q}} \left(\int_G \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left(\int_G \left| f(y) \right|^p \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{q}} \left(\int_G \left| g(y^{-1}) \right|^r d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \| f \|_{L^p}^{\frac{p}{r'}} \left(\int_G \left| f(y) \right|^p \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{q}} \| g \|_{L^r}^{\frac{r}{p'}}$$

$$= \left(\int_G \left| f(y) \right|^p \left| g(y^{-1}x) \right|^r d\lambda(y) \right)^{\frac{1}{q}} \| g \|_{L^r}^{\frac{r}{p'}} \| f \|_{L^p}^{\frac{p}{r'}}$$

Where, in the final calculating step, we have made use of the fact that we are integrating over everything and the abelian property of our topological group [2]. We now once more take the L^q norm of the inequality since everything is absolute and thus the inequality will be preserved. Notice we shall take the norm in x, respecting the parameter of our convolution whilst preserving the two other norms:

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[2]Note however that this can also be proven without the abelian property, albeit with a few minor prerequisites.

$$\big\||f|*|g|\big\|_{L^q} \le \left(\int_G \left|\left(\int_G |f(y)|^p \big|g(y^{-1}x)\big|^r \, d\lambda(y)\right)^{\frac{1}{q}}\right|^q \, d\lambda(x)\right)^{\frac{1}{q}} \|g\|_{L^r}^{\frac{r}{p'}} \|f\|_{L^p}^{\frac{p}{r'}}$$

$$= \left(\int_G \int_G |f(y)|^p \big|g(y^{-1}x)\big|^r \, d\lambda(x)d\lambda(y)\right)^{\frac{1}{q}} \|g\|_{L^r}^{\frac{r}{p'}} \|f\|_{L^p}^{\frac{p}{r'}}$$

$$= \left(\int_G \int_G \big|g(y^{-1}x)\big|^r \, d\lambda(x) |f(y)|^p \, d\lambda(y)\right)^{\frac{1}{q}} \|g\|_{L^r}^{\frac{r}{p'}} \|f\|_{L^p}^{\frac{p}{r'}}$$

$$= \left(\int_G |g(x)|^r \, d\lambda(x) \int_G |f(y)|^p \, d\lambda(y)\right)^{\frac{1}{q}} \|g\|_{L^r}^{\frac{r}{p'}} \|f\|_{L^p}^{\frac{p}{r'}}$$

$$= \|g\|_{L^r}^{\frac{r}{q}} \|f\|_{L^p}^{\frac{p}{q}} \|g\|_{L^r}^{\frac{r}{p'}} \|f\|_{L^p}^{\frac{p}{r'}}$$

$$= \|g\|_{L^r} \|f\|_{L^p}$$

$$< \infty$$

As in Minkowski's Proof previously, we conclude $f * g$ exists λ-a.e. and satisfies the theorem. $\qquad\square$

We are left wondering if there are similar results for weak L_p spaces, fortunately Young has not left us hanging.

THEOREM 3.3. *Young's Inequality for Weak Type Spaces:*
ZZZZZZZZZZZz

4. APPROXIMATE IDENTITIES

We are left wondering if there is an identity element under convolution, a function g such that

$$\forall f \in L^1(G) : g * f = f * g = f$$

In the next chapter the Dirac delta distribution, which full fills this quality, is introduced. However since this is not a function, we shall make do with the concept of approximating such a g with a series of functions with appropriate characteristics.

DEFINITION 4.1. *An **approximate identity** (as $\varepsilon \to 0$) is a family of $L^1(G)$ functions k_ε with the following three properties:*

    (i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \le c$ for all $\varepsilon > 0$.
    (ii) $\int_G k_\varepsilon(x)d\lambda(x) = 1$ for all $\varepsilon > 0$.
    (iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)|d\lambda(x) \to 0$ as $\varepsilon \to 0$.

THEOREM 4.1. *Any approximate identity has the following two properties:*

    (1) $f \in L^p(G) \wedge 1 \le p < \infty \implies \|k_\epsilon * f - f\|_{L^p(G)} \to 0$ as $\epsilon \to 0$
    (2) $f \in L^\infty(G)$ uniformly continuous on $K \subset G \implies \|k_\epsilon * f - f\|_{L^\infty(K)} \to 0$ as $\epsilon \to 0$
        Furthermore, if f is bounded and continuous at $x \in G$ then $(k_\epsilon*)(x) \to f(x)$ as $\epsilon \to 0$

Proof. Let us first prove the statement in finite L^p spaces. We shall make use of the following equality:

$$\forall p \in \mathbb{N} : \left| g(h^{-1}x) - g(x) \right|^p \leq \left(\left| g(h^{-1}x) \right| + \left| g(x) \right| \right)^p$$

$$\leq \left(\operatorname*{ess.\,sup}_x \left(\left| g(h^{-1}x) \right| \right) + \operatorname*{ess.\,sup}_x \left(\left| g(x) \right| \right) \right)^p$$

$$\leq \left(2 \| g \|_{L^\infty} \right)^p$$

Applying the Lebesgue Dominated Convergence theorem (*) we find:

$$\int_G \left| g(h^{-1}x) - g(x) \right|^p d\lambda(x) \to 0 \text{ as } h \to e$$

Where e is the neutral element of G. In $\mathbb{R}^n$ this is simply 0. We can approximate any $g \in L^p(G)$ with a continuous function f with compact support. Thus the property still holds:

$$\int_G \left| f(h^{-1}x) - f(x) \right|^p d\lambda(x) \to 0 \text{ as } h \to e$$

However we can now say that since f is continuous,

$$\delta > 0 : \exists V(e) : h \in V(e) \implies \int_G \left| f(h^{-1}x) - f(x) \right|^p d\lambda(x) < \left(\frac{\delta}{2} \right)^p \left(\frac{1}{c} \right) \tag{3}$$

Where $c*$, $V(e)$ is a neighborhood of e. We shall fix this neighborhood for later. We have picked the value on the right side for later convenience. As with most proofs in this area, we shall split the object of our analysis into two parts we can evaluate:

$$(k_\epsilon * f)(x) - f(x) = \int_G (f(y^{-1}x) - f(x)) k_\epsilon(y) d\lambda(y)$$

We take L^p norms on both sides. This preserves the equality since *.

$$\left\| (k_\epsilon * f)(x) - f(x) \right\|_{L^p(G)} = \left\| \int_G (f(y^{-1}x) - f(x)) k_\epsilon(y) d\lambda(y) \right\|_{L^p(G)}$$

$$= \left(\int_G \left| \int_G (f(y^{-1}x) - f(x)) k_\epsilon(y) d\lambda(y) \right|^p d\lambda(x) \right)^{\frac{1}{p}}$$

$$\leq \left(\int_G \int_G \left| f(y^{-1}x) - f(x) \right|^p \left| k_e(y) \right| d\lambda(y) d\lambda(x) \right)^{\frac{1}{p}}$$

$$= \left(\int_G \int_V \left| f(y^{-1}x) - f(x) \right|^p \left| k_e(y) \right| d\lambda(y) d\lambda(x) + \int_G \int_{V^c} \left| f(y^{-1}x) - f(x) \right|^p \left| k_e(y) \right| d\lambda \right.$$

Where the inequality originates from Jensen's inequality *. Notice that we can now inspect on V and V^c, we shall call the respective parts of the function F_V and F_{V^c} such that our last statement can be rewritten as $(F_V + F_{V_c})^{\frac{1}{p}}$ for convenience. We now look at them individually.

$$
\begin{aligned}
F_V &= \int_G \int_V \left| f(y^{-1}x) - f(x) \right|^p \left| k_e(y) \right| d\lambda(y) d\lambda(x) \\
&= \int_V \int_G \left| f(y^{-1}x) - f(x) \right|^p \left| k_e(y) \right| d\lambda(x) d\lambda(y) \\
&= \int_V \int_G \left| f(y^{-1}x) - f(x) \right|^p d\lambda(x) \left| k_e(y) \right| d\lambda(y) \\
&\leq \int_V \left((\frac{\delta}{2})^p \frac{1}{c} \right) \left| k_e(y) \right| d\lambda(y) \\
&= \left((\frac{\delta}{2})^p \frac{1}{c} \right) \int_V \left| k_e(y) \right| d\lambda(y) \\
&\leq \left((\frac{\delta}{2})^p \frac{1}{c} \right) \int_G \left| k_e(y) \right| d\lambda(y) \\
&\leq \left((\frac{\delta}{2})^p \frac{1}{c} \right) c \\
&= \left((\frac{\delta}{2})^p \right)
\end{aligned}
$$

Here we have used Fubini *, substituted our previously discovered upper bound, and used property (i) intrinsic of any approximate identity, as defined. We bound the other half now:

$$
\begin{aligned}
F_{V^p} &= \int_G \int_{V^c} \left| f(y^{-1}x) - f(x) \right|^p \left| k_\epsilon(y) \right| d\lambda(y) d\lambda(x) \\
&= \int_{V^c} \int_G \left| f(y^{-1}x) - f(x) \right|^p \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&= \int_{V^c} \left\| f(y^{-1}x) - f(x) \right\|_{L^p}^p \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&\leq \int_{V^c} \left(\| f \|_{L^p}^p + \| f \|_{L^p}^p \right) \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&= \int_{V^c} \left(2\| f \|_{L^p}^p \right) \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&= \left(2\| f \|_{L^p}^p \right) \int_{V^c} \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&\leq \left(2\| f \|_{L^p}^p \right) \int_{V^c} \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
ZZZ& \\
&\leq \left((\frac{\delta}{2})^p \right)
\end{aligned}
$$

Were we have once again used Fubini *, the definition of our norm, minkowski's *, and have choosen a correct neighborhood V^c... *. Putting our function back together we find:

$$
\begin{aligned}
(F_V + F_{V^c}) &\leq (\frac{\delta}{2})^p + (\frac{\delta}{2})^p)^{\frac{1}{p}} \\
&\leq (\frac{\delta}{2}) + (\frac{\delta}{2})) \\
&= \delta
\end{aligned}
$$

Where the final inequality follows from the fact that $p \geq 1 \implies \frac{1}{p} \leq 1$ and Ex. 1.1.4 from the previous section. We have proven property (1) of our theorem since the norm becomes arbitrarily small and thus converges to 0.

We shall no prove the case of $p = \infty$. First we prepare some ingredients. We have required that f is bounded and uniformly continuous. Recall:

$$
f \text{uniformly continuous} \implies \delta > 0 : \forall V(e) : \left| f(y^{-1}x) - f(x) \right| < \frac{\delta}{2c}
$$

Furthermore from property (iii) of an approximate identity we have:

$$
\exists \epsilon_0 > 0 : \forall \epsilon \in (0, \epsilon_0) : \int_{V^c} \left| k_\epsilon(y) \right| d\lambda(y) \leq \frac{\delta}{2(\|f\|_{L^\infty} + 1)}
$$

Once more we shall split our normed function:

$$
\begin{aligned}
\|k_\epsilon * f - f\|_{L^\infty}(G) &= \sup_{x \in G} \left| (k_\epsilon * f)(x) - f(x) \right| \\
&= \sup_{x \in G} \left| \int_G f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) \right| \\
&= \sup_{x \in G} \left| \int_V f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) + \int_{V^c} f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) \right| \\
&\leq \sup_{x \in G} \left| \int_V f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) \right| + \left| \int_{V^c} f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) \right| \\
&\leq \sup_{x \in G} \left| \int_V f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) \right| + \sup_{x \in G} \left| \int_{V^c} f(y^{-1}x) - f(x)k_\epsilon(y)d\lambda(y) \right| \\
&\leq \sup_{x \in G} \int_V \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y)d\lambda(y) \right| + \sup_{x \in G} \int_{V^c} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y)d\lambda(y) \right| \\
&\leq \int_V \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y)d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y)d\lambda(y) \right| \\
&\leq \int_V (\frac{\delta}{2c}) \left| k_\epsilon(y)d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y)d\lambda(y) \right|
\end{aligned}
$$

$$\leq (\frac{\delta}{2c}) \int_V \left| k_\epsilon(y) d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right|$$

$$\leq (\frac{\delta}{2c}) \int_G \left| k_\epsilon(y) d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right|$$

$$\leq (\frac{\delta}{2c})c + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right|$$

$$= (\frac{\delta}{2}) + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right|$$

$$*$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$$

Here we have normed our required equality, split the integral into neighborhood and complement, used the triangle inequality, and applied previous definitions and preparatory discoveries *. Note at the end we have added one case of 0 *.

This concludes the proof of both parts. □

THEOREM 4.2. *Let k_ϵ be an approximate identity with a slight variation, instead of satisfying property (ii) is satisfies:*

$$(ii)' :\Leftrightarrow a \in \mathbb{C} \wedge \forall \epsilon > 0 : \int_G k_\epsilon(x) d\lambda(x) = a$$

then similarly the following is true:

(1) $f \in L^p(G) \wedge 1 \leq p < \infty \implies \|k_\epsilon * f - af\|_{L^p(G)} \to 0$ as $\epsilon \to 0$

(2) $f \in L^\infty(G)$ uniformly continuous on $K \subset G$ such that

$$\forall \delta > 0 : \exists V(e) : \sup_{x \in G} \sup_{y \in V(e)} \left| f(y^{-1}x) - f(x) \right| \leq \delta$$

then $\|k_\epsilon * f - af\|_{L^\infty(K)} \to 0$ as $\epsilon \to 0$

Proof. The proof is analogous to the previous proof, working with a instead of 1. □

Let us inspect a typical case in order to help us visualize the behaviour of an approximate identity:

EXAMPLE 4.1. *A useful example of an approximate identity is the Poisson kernel on $\mathbb{R}$, defined as:*

$$P(x) := (\pi(x^2 + 1))^{-1}$$
$$P_\epsilon(x) := \epsilon^{-1} P(\epsilon^{-1} x)$$

Notice first a convenience:

$$\|P(x)\|_{L^1} = \int_\mathbb{R} \left| \frac{1}{\pi(x^2 + 1)} \right| d\lambda(x)$$

$$= \int_{\mathbb{R}} \frac{1}{\pi(x^2+1)} d\lambda(x)$$

$$= \int_{\mathbb{R}} \frac{1}{\epsilon(\pi(\frac{x}{\epsilon})^2+1)} d\lambda(x)$$

$$= \int_{\mathbb{R}} \left| \frac{1}{\epsilon(\pi(\frac{x}{\epsilon})^2+1)} \right| d\lambda(x)$$

$$= \left\| P_\epsilon(x) \right\|_{L^1}$$

by a change of variables x to $\frac{x}{\epsilon}$ and noticing that the function is positive. With this insight we can prove required property (i) and (ii):

$$\int_{\mathbb{R}} \frac{1}{\pi(x^2+1)} d\lambda(x) = \int_{-\infty}^{+\infty} \frac{1}{\pi(x^2+1)} dx$$

$$= \lim_{a\to+\infty} \int_{-a}^{+a} \frac{1}{\pi(x^2+1)} dx$$

$$= \lim_{a\to+\infty} \frac{1}{\pi} (\arctan(x)|_{-a}^{+a})$$

$$= \lim_{a\to+\infty} \frac{1}{\pi} (\arctan(a) - \arctan(-a))$$

$$= \frac{1}{\pi} (\frac{\pi}{2} - (-\frac{\pi}{2}))$$

$$= 1$$

Notice that $(ii) \wedge (k_\epsilon \leq 0) \implies (i)$ on $\mathbb{R}$. To prove (iii) we notice that $V^c(e)$ on $\mathbb{R}$ is simply the set $\{x \in \mathbb{R} : |x| \geq \delta\}$ for some $\delta > 0$. We integrate:

$$\int_{\{x\in\mathbb{R} : |x|\geq\delta\}} |P_\epsilon| \, d\lambda(x) = \int_{\{x\in\mathbb{R} : |x|\geq\delta\}} \left| \frac{1}{\epsilon} \frac{1}{\pi((\frac{x}{\epsilon})^2+1)} \right| d\lambda(x)$$

$$= \frac{1}{\pi} \int_{\{x\in\mathbb{R} : |x|\geq\delta\}} \left| \frac{1}{\epsilon} \frac{1}{((\frac{x}{\epsilon})^2+1)} \right| d\lambda(x)$$

$$= 1 - \frac{2}{\pi} \arctan(\frac{\delta}{\epsilon})$$

Which goes to 0 as $\epsilon \to 0$ since $\arctan(\frac{\delta}{\epsilon}) \to \frac{\pi}{2}$. Thus we have shown that the Poisson kernel is an approximate identity. Figure 1 depicts this kernel.

REMARK 4.1. *It is possible to state a more general form on $\mathbb{R}^n$ for an approximate identity which contains the Poisson kernel. Let $\int_{\mathbb{R}^n} k(x) = 1$ and define $k_\epsilon(x) = \epsilon^{-n} k(\epsilon^{-1}x)$. We see k_ϵ is an approximate identity by a similar notion to the proof of the Poisson kernel.*

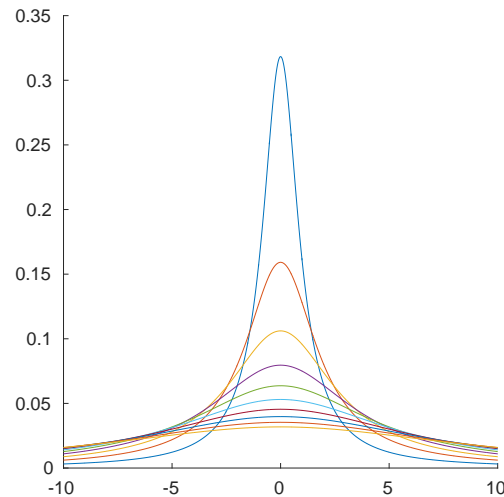FIGURE 1. Poisson Kernel

EXAMPLE 4.2. *Another example of an approximate identity is the Fejer Kernel defined as follows:*

$$F_N(t) := \frac{1}{N+1}\Big(\frac{sin(\pi(N+1)t)}{sin(\pi t)}\Big)^2$$

on the circle group T^1. It is graphed beautifully in Figure 2.

APPENDIX A. POISSON KERNEL

The matlab code for graphing the Poisson kernel:

```matlab
% Arbitrary Approximate Identity on R
% First Example is the Poisson Kernel
% Author: Simon Gruening

figure('position',[0,0,400,400])
hold on;

% ---
n = 1;
k = @(x) (pi .* (x.^2 + 1)).^-1; % poisson
%k = @(x) sqrt(1-x.^2)./pi; % half circle
```
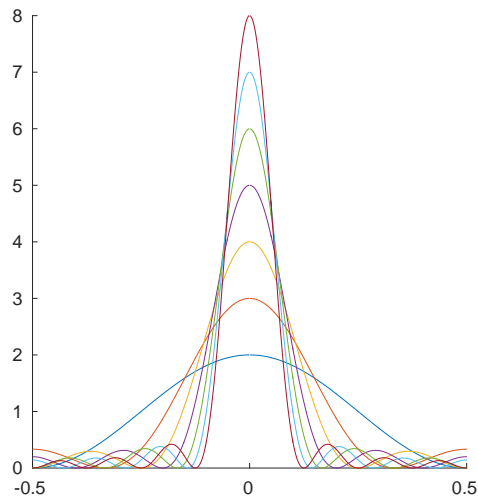
FIGURE 2. Fejer Kernel

```
ke = @(x,e) e.^-n .* k(e.^-1 .* x);

I = -10:0.01:10;

hold on;
for ep = 1:1:10;
    plot(I, ke(I,ep));
end
% ---

set(gcf,'PaperPositionMode','auto','PaperSize',[5,5])
saveas(gcf,'poissonkernel.pdf')
```

APPENDIX B. FEJER KERNEL

The matlab code for graphing the Fejer kernel:

```
% The Fejer Kernel
% Author: Simon Gruening

figure('position',[0,0,400,400])
hold on;
```

```matlab
% ---
F = @(t,N) (1./(1+N)).*((sin(pi.*(N+1).*t)./sin(pi.*t))).^2;
I = -0.5:0.001:0.5;

for n=1:7
    plot(I,F(I,n));
end
% ---

set(gcf,'PaperPositionMode','auto','PaperSize',[5,5])
saveas(gcf,'fejerkernel.pdf')
```