CONVOLUTION AND APPROXIMATE IDENTITIES

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2. Examples of Topological Groups

Definition 2.1. *Topological Group*

Definition 2.2. *Locally Compact*

Definition 2.3. *Haar Measure*

Example 2.1. $\mathbb{R}^n, \mathbb{Z}^n, \mathbb{T}^n$

Example 2.2. $dx/|x|$

Example 2.3. *Heisenberg Group* $\mathbb{H}^n$

3. Convolution

Definition 3.1. *Let* $f, g \in L^1(G)$*. Define the* ***convolution*** $f * g$ *by*

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\lambda(y) \tag{1}$$

Remark 3.1. *Note that on* $\mathbb{R}^n$ *with an additive structure (our preferred environment for later chapters), we will simply have:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

Example 3.1. *Let* $G = \mathbb{R}$*,*

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & else \end{cases} \tag{2}$$

Then we calculate:

$$\begin{aligned} (f * f)(x) &= \int_{\mathbb{R}} f(y)f(x-y)d\lambda(y) \\ &= \begin{cases} \int_{\mathbb{R}} 0 d\lambda(y) & -1 \leq x \leq 1 \\ \int_{\mathbb{R}} \chi_{[-1,1]\cap[x-1,x+1]}(x)d\lambda(x) & else \end{cases} \end{aligned}$$

Notice that the convolution operator has a natural smoothing effect on f*, as it does on every function.*

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Lemma 3.1. *Convolution is defined λ almost everywhere.*

Proof. To see this we take the L_1 norm on the definition to find it finite:

$$\begin{aligned}
\left\|(f*g)(x)\right\|_{L^1} &= \int_G \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| d\lambda(x) && \text{(Apply Norm)} \\
&\leq \int_G \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y)d\lambda(x) && \text{(Tri. Ineq.)} \\
&= \int_G \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(x)d\lambda(y) && \text{(Fubini)} \\
&= \int_G |f(y)| \int_G |g(y^{-1}x)|\,d\lambda(x)d\lambda(y) && \text{(Measure-Invariance)} \\
&= \int_G |f(y)| \int_G |g(x)|\,d\lambda(x)d\lambda(y) && \text{(Left Haar)} \\
&= \|f\|_{L^1}\|g\|_{L^1} && \text{(Def.)} \\
&< \infty && \text{(Def.)}
\end{aligned}$$

□

Lemma 3.2.

$$(f*g)(x) = \int_G f(xz)g(z^{-1})d\lambda(z)$$

Proof. We perform a change of variables $z = x^{-1}y$:

$$\begin{aligned}
(f*g)(x) &= \int_G f(y)g(y^{-1}x)d\lambda(y) \\
&= \int_G f(xx^{-1}y)g((yx^{-1})^{-1})d\lambda(y) \\
&= \int_G f(xz)g(z^{-1})d\lambda(x^{-1}y) && \text{(Left Invariance)} \\
&= \int_G f(xz)g(z^{-1})d\lambda(z)
\end{aligned}$$

□

Proposition 3.1. $\forall f, g, h \in L^1(G)$:

(1) $f*(g*h) = (f*g)*h$
(2) $f*(g+h) = f*g + f*h \wedge (f+g)*h = f*h + f*g$

Thus convolution is associative and distributive.

Proof. Associativity:

$$ZZZZZZZZZ$$

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Distributivity:

$$\begin{aligned}
f * (g + h) &= \int_G f(y)(g + h)(y^{-1}x)d\lambda(y) \\
&= \int_G f(y)(g(y^{-1}x) + h(y^{-1}x))d\lambda(y) \\
&= \int_G f(y)g(y^{-1}x) + f(y)h(y^{-1}x)d\lambda(y) \\
&= \int_G f(y)g(y^{-1}x)d\lambda(y) + \int_G f(y)h(y^{-1}x)d\lambda(y) \\
&= f * g + f * h
\end{aligned}$$

The mirror statement follows analogously. □

4. Basic Convolution Inequalities

Definition 4.1. $p' := p/(p - 1)$

Remark 4.1. *ZZZZZZZZZZ*

Theorem 4.1. *Minkowskis Inequality, triangle inequality for Lp spaces*

Remark 4.2. *We can work with absolute value functions...ZZZ due to the triangle inequality:*

$$\begin{aligned}
f * g &= \int_G f(y)g(y^{-1}x)d\lambda(y) \\
&\leq \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| \\
&\leq \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(y) \\
&= |f| * |g|
\end{aligned}$$

Proof. Case $p = 1$: ZZZZ Case $p = \infty$: ZZZZ

For $1 < p < \infty$, we

$$ZZZ$$

□

Theorem 4.2. *Youngs Inequality*

Theorem 4.3. *Youngs Inequality for Weak Type Spaces ouch proof*

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5. Approximate Identities

Approximation of dirac delta function , identity element of convolutions

Definition 5.1. *An* ***approximate identity*** *(as $\varepsilon \to 0$) is a family of $L^1(G)$ functions k_ε with the following three properties:*

(i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \leqslant c$ for all $\varepsilon > 0$.
(ii) $\int_G k_\varepsilon(x)d\lambda(x) = 1$ for all $\varepsilon > 0$.
(iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)|d\lambda(x) \to 0$ as $\varepsilon \to 0$.

Example 5.1.

Theorem 5.1. *approx. id. on locallz compact group G with left Haar measure*

Theorem 5.2. *ke familz of funcs on loc compact group G with properties...*

6. Required Stuff

(1) hausdorf topological space
(2) counting measure
(3) area of intersecting circles
(4) banach algebra
(5) hoelders inequality
(6) fubini
(7) chebyschevs inequality
(8) lebesgue dominated conv. thm.
(9) measure theoretic support

chapter 1 stuff:

(1) Lp norms and other defs etc.
(2) distr. functions

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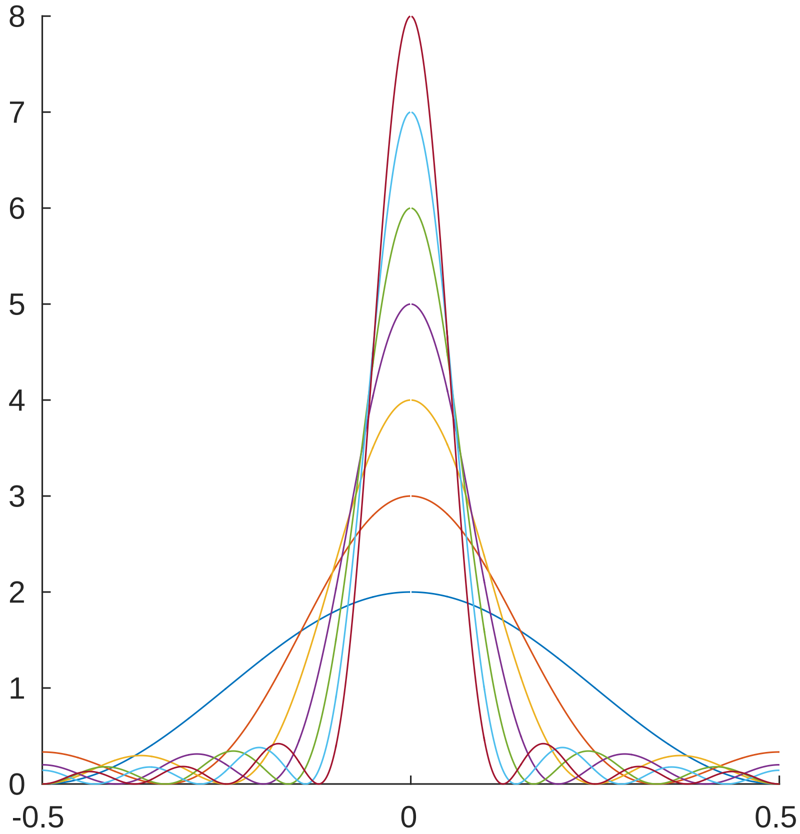

Figure 1. Fejer Kernel