CONVOLUTION AND APPROXIMATE IDENTITIES

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2. Examples of Topological Groups

Definition 2.1. *Topological Group*

Definition 2.2. *Locally Compact*

Definition 2.3. *Haar Measure*

Example 2.1. $\mathbb{R}^n, \mathbb{Z}^n, \mathbb{T}^n$

Example 2.2. $dx/|x|$

Example 2.3. *Heisenberg Group* $\mathbb{H}^n$

3. Convolution

Definition 3.1. *Let $f, g \in L^1(G)$. Define the* ***convolution*** *$f * g$ by*

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\lambda(y) \tag{1}$$

Remark 3.1. *Note that on $\mathbb{R}^n$ with an additive structure (our preferred environment for later chapters), we will simply have:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

Example 3.1. *Let $G = \mathbb{R}$,*

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & else \end{cases} \tag{2}$$

Then we calculate:

$$\begin{aligned} (f * f)(x) &= \int_{\mathbb{R}} f(y)f(x-y)d\lambda(y) \\ &= \begin{cases} \int_{\mathbb{R}} 0 d\lambda(y) & -1 \leq x \leq 1 \\ \int_{\mathbb{R}} \chi_{[-1,1]\cap[x-1,x+1]}(x)d\lambda(x) & else \end{cases} \end{aligned}$$

Notice that the convolution operator has a natural smoothing effect on f, as it does on every function.

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LEMMA 3.1. *Convolution is defined λ almost everywhere.*

Proof. To see this we take the L_1 norm on the definition to find it finite:

$$\begin{aligned}
\left\|(f*g)(x)\right\|_{L^1} &= \int_G \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| d\lambda(x) && \text{(Apply Norm)}\\
&\leq \int_G \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y)d\lambda(x) && \text{(Tri. Ineq.)}\\
&= \int_G \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(x)d\lambda(y) && \text{(Fubini)}\\
&= \int_G |f(y)| \int_G |g(y^{-1}x)|\,d\lambda(x)d\lambda(y) && \text{(Measure-Invariance)}\\
&= \int_G |f(y)| \int_G |g(x)|\,d\lambda(x)d\lambda(y) && \text{(Left Haar)}\\
&= \|f\|_{L^1}\|g\|_{L^1} && \text{(Def.)}\\
&< \infty && \text{(Def.)}
\end{aligned}$$

□

LEMMA 3.2.

$$(f*g)(x) = \int_G f(xz)g(z^{-1})d\lambda(z)$$

Proof. We perform a change of variables $z = x^{-1}y$:

$$\begin{aligned}
(f*g)(x) &= \int_G f(y)g(y^{-1}x)d\lambda(y)\\
&= \int_G f(xx^{-1}y)g((yx^{-1})^{-1})d\lambda(y)\\
&= \int_G f(xz)g(z^{-1})d\lambda(x^{-1}y) && \text{(Left Invariance)}\\
&= \int_G f(xz)g(z^{-1})d\lambda(z)
\end{aligned}$$

□

PROPOSITION 3.1. $\forall f, g, h \in L^1(G)$:

(1) $f*(g*h) = (f*g)*h$
(2) $f*(g+h) = f*g + f*h \wedge (f+g)*h = f*h + f*g$

Thus convolution is associative and distributive.

Proof. Associativity:

$$ZZZZZZZZZ$$

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Distributivity:

$$\begin{aligned} f * (g+h) &= \int_G f(y)(g+h)(y^{-1}x)d\lambda(y) \\ &= \int_G f(y)(g(y^{-1}x) + h(y^{-1}x))d\lambda(y) \\ &= \int_G f(y)g(y^{-1}x) + f(y)h(y^{-1}x)d\lambda(y) \\ &= \int_G f(y)g(y^{-1}x)d\lambda(y) + \int_G f(y)h(y^{-1}x)d\lambda(y) \\ &= f * g + f * h \end{aligned}$$

The mirror statement follows analogously. □

4. Basic Convolution Inequalities

Definition 4.1. $p' := p/(p-1)$

Remark 4.1. *ZZZZZZZZZZ*

Theorem 4.1. *Minkowskis Inequality, triangle inequality for Lp spaces*

Remark 4.2. *We can work with absolute value functions...ZZZ due to the triangle inequality:*

$$\begin{aligned} f * g &= \int_G f(y)g(y^{-1}x)d\lambda(y) \\ &\leq \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| \\ &\leq \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y) \\ &= |f| * |g| \end{aligned}$$

Proof. Case $p = 1$: ZZZZ Case $p = \infty$: ZZZZ

For $1 < p < \infty$, we

$$ZZZ$$

□

Theorem 4.2. *Youngs Inequality*

Theorem 4.3. *Youngs Inequality for Weak Type Spaces ouch proof*

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5. Approximate Identities

Approximation of dirac delta function , identity element of convolutions

Definition 5.1. *An **approximate identity** (as $\varepsilon \to 0$) is a family of $L^1(G)$ functions k_ε with the following three properties:*

(i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \leqslant c$ for all $\varepsilon > 0$.
(ii) $\int_G k_\varepsilon(x)d\lambda(x) = 1$ for all $\varepsilon > 0$.
(iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)|d\lambda(x) \to 0$ as $\varepsilon \to 0$.

Theorem 5.1. *Any approximate identity has the following two properties:*

(1) $f \in L^p(G) \wedge 1 \leq p < \infty \implies \|k_\epsilon * f - f\|_{L^p(G)} \to 0$ as $\epsilon \to 0$
(2) $f \in L^\infty(G)$ uniformly continuous on $K \subset G \implies \|k_\epsilon * f - f\|_{L^\infty(K)} \to 0$ as $\epsilon \to 0$ Furthermore, if f is bounded and continuous at $x \in G$ then $(k_\epsilon *)(x) \to f(x)$ as $\epsilon \to 0$

Proof. Let us first prove the statement in finite L^p spaces. We shall make use of the following equality:

$$\begin{aligned}\forall p \in \mathbb{N} : \left|g(h^{-1}x) - g(x)\right|^p &\leq (\left|g(h^{-1}x)\right| + \left|g(x)\right|)^p \\ &\leq (ess.sup_x(\left|g(h^{-1}x)\right|) + ess.sup_x(\left|g(x)\right|))^p \\ &\leq (2\|g\|_{L^\infty})^p\end{aligned}$$

□

Example 5.1. *A useful example of an approximate identity is the poison kernel on $\mathbb{R}$, defined as:*

$$P(x) := (\pi(x^2+1))^{-1}$$
$$P_\epsilon(x) := \epsilon^{-1}P(\epsilon^{-1}x)$$

Notice first a convenience:

$$\left\|P(x)\right\|_{L^1} = \int_{\mathbb{R}} \frac{1}{\epsilon\pi(\frac{x^2}{\epsilon^2}+1)}\epsilon d\lambda(y)$$

ZZZ

From this it follows that

Example 5.2. *Another example is the fejer kernel depicted below:*

zz

Theorem 5.2. *approx. id. on locallz compact group G with left Haar measure*

Theorem 5.3. *ke familz of funcs on loc compact group G with properties...*

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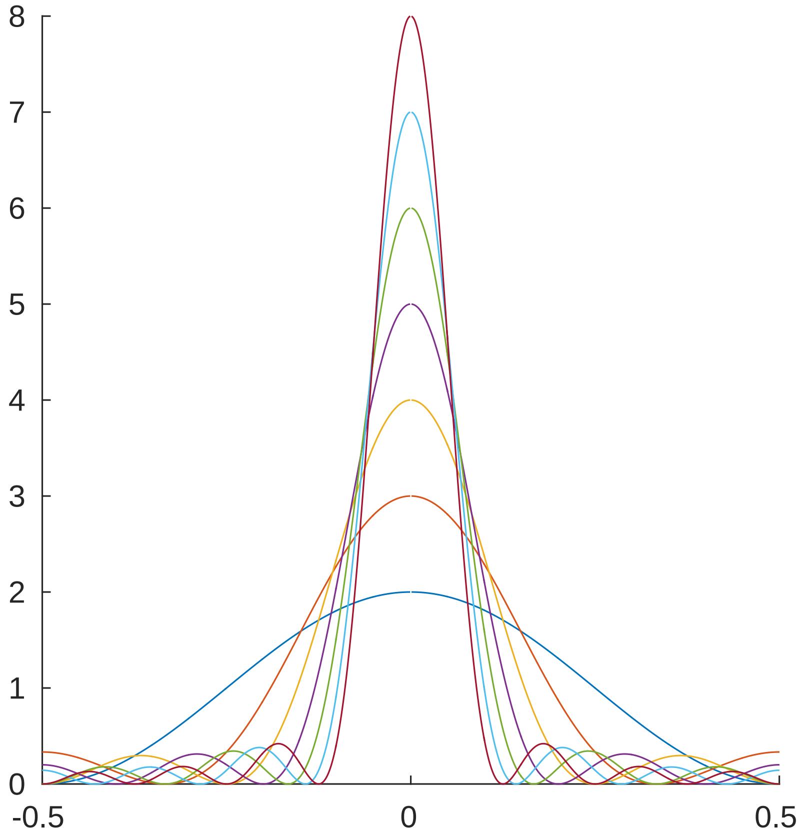

FIGURE 1. Fejer Kernel

6. REQUIRED STUFF

(1) hausdorf topological space
(2) counting measure
(3) area of intersecting circles
(4) banach algebra
(5) hoelders inequality

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(6) fubini
(7) chebyschevs inequality
(8) lebesgue dominated conv. thm.
(9) measure theoretic support

chapter 1 stuff:

(1) Lp norms and other defs etc.
(2) distr. functions