CONVOLUTION AND APPROXIMATE IDENTITIES

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2. Examples of Topological Groups

Definition 2.1. *Topological Group*

Definition 2.2. *Locally Compact*

Definition 2.3. *Haar Measure*

Example 2.1. $\mathbb{R}^n, \mathbb{Z}^n, \mathbb{T}^n$

Example 2.2. $dx/|x|$

Example 2.3. *Heisenberg Group* $\mathbb{H}^n$

3. Convolution

Definition 3.1. *Let $f, g \in L^1(G)$. Define the **convolution** $f * g$ by*

$$(f * g)(x) := \int_G f(y) g(y^{-1}x) d\lambda(y) \tag{1}$$

Remark 3.1. *Note that on $\mathbb{R}^n$ with an additive structure (our preferred environment for later chapters), we will simply have:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) dy$$

Example 3.1. *Let $G = \mathbb{R}$,*

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & else \end{cases} \tag{2}$$

Then we calculate:

$$\begin{aligned} (f * f)(x) &= \int_{\mathbb{R}} f(y) f(x - y) d\lambda(y) \\ &= \begin{cases} \int_{\mathbb{R}} 0 d\lambda(y) & -1 \leq x \leq 1 \\ \int_{\mathbb{R}} \chi_{[-1,1] \cap [x-1, x+1]}(x) d\lambda(x) & else \end{cases} \end{aligned}$$

Notice that the convolution operator has a natural smoothing effect on f, as it does on every function.

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LEMMA 3.1. *Convolution is defined λ almost everywhere.*

Proof. To see this we take the L_1 norm on the definition to find it finite:

$$\begin{aligned}
\left\|(f*g)(x)\right\|_{L^1} &= \int_G \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| d\lambda(x) && \text{(Apply Norm)} \\
&\leq \int_G \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(y)d\lambda(x) && \text{(Tri. Ineq.)} \\
&= \int_G \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(x)d\lambda(y) && \text{(Fubini)} \\
&= \int_G |f(y)| \int_G |g(y^{-1}x)| \, d\lambda(x)d\lambda(y) && \text{(Measure-Invariance)} \\
&= \int_G |f(y)| \int_G |g(x)| \, d\lambda(x)d\lambda(y) && \text{(Left Haar)} \\
&= \|f\|_{L^1} \|g\|_{L^1} && \text{(Def.)} \\
&< \infty && \text{(Def.)}
\end{aligned}$$

□

LEMMA 3.2.

$$(f*g)(x) = \int_G f(xz)g(z^{-1})d\lambda(z)$$

Proof. We perform a change of variables $z = x^{-1}y$:

$$\begin{aligned}
(f*g)(x) &= \int_G f(y)g(y^{-1}x)d\lambda(y) \\
&= \int_G f(xx^{-1}y)g((yx^{-1})^{-1})d\lambda(y) \\
&= \int_G f(xz)g(z^{-1})d\lambda(x^{-1}y) && \text{(Left Invariance)} \\
&= \int_G f(xz)g(z^{-1})d\lambda(z)
\end{aligned}$$

□

PROPOSITION 3.1. $\forall f, g, h \in L^1(G):$

(1) $f*(g*h) = (f*g)*h$
(2) $f*(g+h) = f*g + f*h \wedge (f+g)*h = f*h + f*g$

Thus convolution is associative and distributive.

Proof. Associativity:

$$ZZZZZZZZZZ$$

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Distributivity:

$$\begin{aligned}
f*(g+h) &= \int_G f(y)(g+h)(y^{-1}x)d\lambda(y)\\
&= \int_G f(y)(g(y^{-1}x)+h(y^{-1}x))d\lambda(y)\\
&= \int_G f(y)g(y^{-1}x)+f(y)h(y^{-1}x)d\lambda(y)\\
&= \int_G f(y)g(y^{-1}x)d\lambda(y)+\int_G f(y)h(y^{-1}x)d\lambda(y)\\
&= f*g+f*h
\end{aligned}$$

The mirror statement follows analogously. □

Remark 3.2. *Proof.* Notice the following trivial equality:

$$\begin{aligned}
\|f\|_{L^p}^{p/q} &= ((\int_G |f(x)|^p \, d\lambda(x))^{1/p})^{p/q}\\
&= (\int_G |f(x)|^p \, d\lambda(x))^{1/q}
\end{aligned}$$

□

4. Basic Convolution Inequalities

Definition 4.1. *Define $p' := p/(p-1)$. To maintain our desired property in infinity we also declare: $1/\infty = 0$*

Remark 4.1. *Notice then: $\frac{1}{p}+\frac{1}{p'}=\frac{1}{p}+\frac{p-1}{p}=\frac{p}{p}=1$*

Remark 4.2. *In the following proofs we often make use of this useful inequality without further explication, it allows us to manipulate inequalities without worrying about negatives:*

$$\begin{aligned}
f*g &= \int_G f(y)g(y^{-1}x)d\lambda(y)\\
&\leq \left|\int_G f(y)g(y^{-1}x)d\lambda(y)\right|\\
&\leq \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y)\\
&= |f|*|g|
\end{aligned}$$

What follows is akin to the triangle inequality for L^p spaces.

Theorem 4.1. *Minkowskis Inequality: Let $1 \leq p \leq \infty$, $f \in L^p(G)$, $g \in L^1(G)$ then it follows that: $g*f$ exists λ-almost-everywhere and $\|g*f\|_{L^p(G)} \leq \|g\|_{L^1(G)}\|f\|_{L^p(G)}$*

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Proof. First we shall inspect the easier case of $p = 1$: ZZZZZZprooofZZZZ

Similarly we may rid ourselves of the other easy case $p = \infty$: ZZZZproofZZZZ

For $1 < p < \infty$, we must work a little harder. We have:

$$(|g| * |f|)(x) = \int_G \left|f(y^{-1}x)\right| \left|g(y)\right| d\lambda(y)$$

We shall apply Hölders inequality as follows ZZZ to recieve:

$$(|g| * |f|)(x) \leq (\int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y))^{1/p} (\int_G \left|g(y)\right| d\lambda(y))^{1/p'}$$

Since we are insane ZZZZ we may take the L^p norm on both sides while preserving the inequality.

$$\begin{aligned} \left\| |g| * |f| \right\|_{L^p} &= () \\ &= blah \end{aligned}$$

□

THEOREM 4.2. *Youngs Inequality*

THEOREM 4.3. *Youngs Inequality for Weak Type Spaces ouch proof*

5. Approximate Identities

Approximation of dirac delta function , identity element of convolutions

DEFINITION 5.1. *An **approximate identity** (as $\varepsilon \to 0$) is a family of $L^1(G)$ functions k_ε with the following three properties:*

- *(i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \leqslant c$ for all $\varepsilon > 0$.*
- *(ii) $\int_G k_\varepsilon(x) d\lambda(x) = 1$ for all $\varepsilon > 0$.*
- *(iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)| d\lambda(x) \to 0$ as $\varepsilon \to 0$.*

THEOREM 5.1. *Any approximate identity has the following two properties:*

- *(1) $f \in L^p(G) \wedge 1 \leq p < \infty \implies \|k_\epsilon * f - f\|_{L^p(G)} \to 0$ as $\epsilon \to 0$*
- *(2) $f \in L^\infty(G)$ uniformly continuous on $K \subset G \implies \|k_\epsilon * f - f\|_{L^\infty(K)} \to 0$ as $\epsilon \to 0$ Furthermore, if f is bounded and continuous at $x \in G$ then $(k_\epsilon *)(x) \to f(x)$ as $\epsilon \to 0$*

Proof. Let us first prove the statement in finite L^p spaces. We shall make use of the following equality:

$$\begin{aligned} \forall p \in \mathbb{N} : \left|g(h^{-1}x) - g(x)\right|^p &\leq (\left|g(h^{-1}x)\right| + \left|g(x)\right|)^p \\ &\leq (\operatorname*{ess.\,sup}_x(\left|g(h^{-1}x)\right|) + \operatorname*{ess.\,sup}_x(\left|g(x)\right|))^p \\ &\leq (2\|g\|_{L^\infty})^p \end{aligned}$$

□

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Example 5.1. *A useful example of an approximate identity is the poison kernel on $\mathbb{R}$, defined as:*

$$P(x) := (\pi(x^2+1))^{-1}$$
$$P_\epsilon(x) := \epsilon^{-1}P(\epsilon^{-1}x)$$

Notice first a convenience:

$$\|P(x)\|_{L^1} = \int_{\mathbb{R}} \frac{1}{\epsilon\pi(\frac{x^2}{\epsilon^2}+1)}\epsilon d\lambda(y)$$
$$ZZZ$$

From this it follows that

Example 5.2. *Another example is the fejer kernel depicted below:*

zz

Theorem 5.2. *approx. id. on locallz compact group G with left Haar measure*

Theorem 5.3. *ke familz of funcs on loc compact group G with properties...*

6. Required Stuff

(1) hausdorf topological space
(2) counting measure
(3) area of intersecting circles
(4) banach algebra
(5) hoelders inequality
(6) fubini
(7) chebyschevs inequality
(8) lebesgue dominated conv. thm.
(9) measure theoretic support

chapter 1 stuff:

(1) Lp norms and other defs etc.
(2) distr. functions

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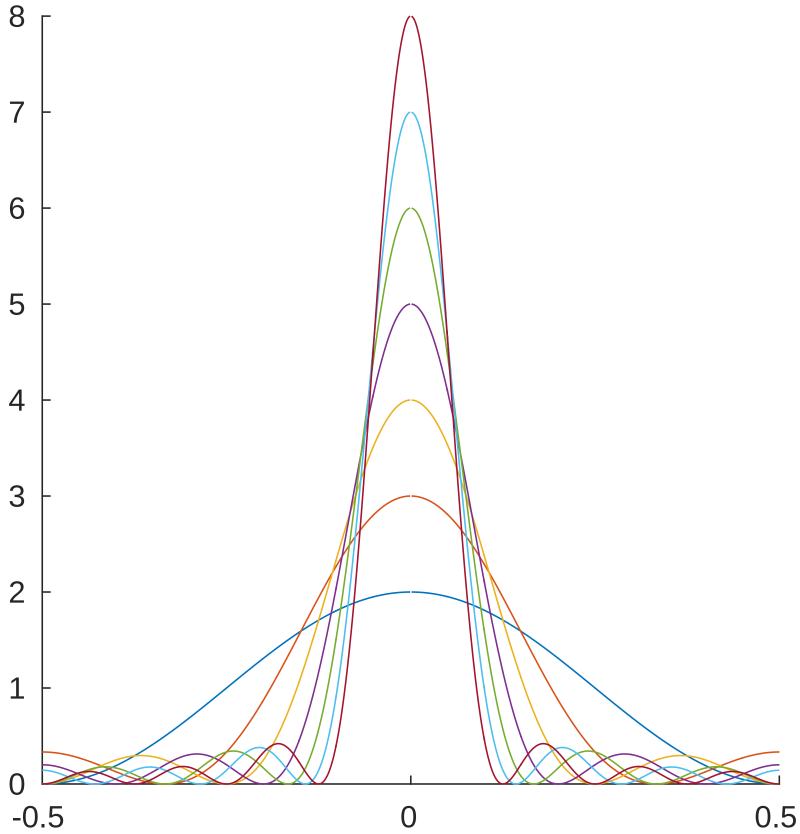

FIGURE 1. Fejer Kernel