CONVOLUTION AND APPROXIMATE IDENTITIES

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2. Examples of Topological Groups

Definition 2.1. *Topological Group*

Definition 2.2. *Locally Compact*

Definition 2.3. *Haar Measure*

Example 2.1. $\mathbb{R}^n, \mathbb{Z}^n, \mathbb{T}^n$

Example 2.2. $dx/|x|$

Example 2.3. *Heisenberg Group* $\mathbb{H}^n$

3. Convolution

Definition 3.1. *Let $f, g \in L^1(G)$. Define the* ***convolution*** $f * g$ *by*

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\lambda(y) \tag{1}$$

Remark 3.1. *Note that on $\mathbb{R}^n$ with an additive structure (our preferred environment for later chapters), we will simply have:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

Example 3.1. *Let $G = \mathbb{R}$,*

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & else \end{cases} \tag{2}$$

Then we calculate:

$$\begin{aligned}(f * f)(x) &= \int_{\mathbb{R}} f(y)f(x-y)d\lambda(y) \\ &= \begin{cases} \int_{\mathbb{R}} 0 d\lambda(y) & -1 \leq x \leq 1 \\ \int_{\mathbb{R}} \chi_{[-1,1]\cap[x-1,x+1]}(x)d\lambda(x) & else \end{cases}\end{aligned}$$

Notice that the convolution operator has a natural smoothing effect on f, as it does on every function.

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LEMMA 3.1. *Convolution is defined λ almost everywhere.*

Proof. To see this we take the L_1 norm on the definition to find it finite:

$$\begin{aligned}
\|(f*g)(x)\|_{L^1} &= \int_G \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| d\lambda(x) && \text{(Apply Norm)}\\
&\leq \int_G \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y)d\lambda(x) && \text{(Tri. Ineq.)}\\
&= \int_G \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(x)d\lambda(y) && \text{(Fubini)}\\
&= \int_G |f(y)| \int_G |g(y^{-1}x)|\,d\lambda(x)d\lambda(y) && \text{(Measure-Invariance)}\\
&= \int_G |f(y)| \int_G |g(x)|\,d\lambda(x)d\lambda(y) && \text{(Left Haar)}\\
&= \|f\|_{L^1}\|g\|_{L^1} && \text{(Def.)}\\
&< \infty && \text{(Def.)}
\end{aligned}$$

□

LEMMA 3.2.

$$(f*g)(x) = \int_G f(xz)g(z^{-1})d\lambda(z)$$

Proof. We perform a change of variables $z = x^{-1}y$:

$$\begin{aligned}
(f*g)(x) &= \int_G f(y)g(y^{-1}x)d\lambda(y)\\
&= \int_G f(xx^{-1}y)g((yx^{-1})^{-1})d\lambda(y)\\
&= \int_G f(xz)g(z^{-1})d\lambda(x^{-1}y) && \text{(Left Invariance)}\\
&= \int_G f(xz)g(z^{-1})d\lambda(z)
\end{aligned}$$

□

PROPOSITION 3.1. $\forall f, g, h \in L^1(G)$:

(1) $f*(g*h) = (f*g)*h$
(2) $f*(g+h) = f*g+f*h \wedge (f+g)*h = f*h+f*g$

Thus convolution is associative and distributive.

Proof. Associativity:

$$ZZZZZZZZZZ$$

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Distributivity:

$$\begin{aligned} f * (g+h) &= \int_G f(y)(g+h)(y^{-1}x)d\lambda(y) \\ &= \int_G f(y)(g(y^{-1}x) + h(y^{-1}x))d\lambda(y) \\ &= \int_G f(y)g(y^{-1}x) + f(y)h(y^{-1}x)d\lambda(y) \\ &= \int_G f(y)g(y^{-1}x)d\lambda(y) + \int_G f(y)h(y^{-1}x)d\lambda(y) \\ &= f*g + f*h \end{aligned}$$

The mirror statement follows analogously. □

Remark 3.2. *Proof.* Notice the following trivial equality:

$$\begin{aligned} \|f\|_{L^p}^{p/q} &= ((\int_G |f(x)|^p \, d\lambda(x))^{1/p})^{p/q} \\ &= (\int_G |f(x)|^p \, d\lambda(x))^{1/q} \end{aligned}$$

□

4. Basic Convolution Inequalities

Definition 4.1. *Define* $p' := p/(p-1)$. *To maintain our desired property in infinity we also declare:* $1/\infty = 0$

Remark 4.1. *Notice then:* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{p} + \frac{p-1}{p} = \frac{p}{p} = 1$

Remark 4.2. *In the following proofs we often make use of this useful inequality without further explication, it allows us to manipulate inequalities without worrying about negatives:*

$$\begin{aligned} f*g &= \int_G f(y)g(y^{-1}x)d\lambda(y) \\ &\leq \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| \\ &\leq \int_G |f(y)| \, |g(y^{-1}x)| \, d\lambda(y) \\ &= |f| * |g| \end{aligned}$$

What follows is akin to the triangle inequality for L^p spaces.

Theorem 4.1. *Minkowskis Inequality: Let* $1 \leq p \leq \infty$, $f \in L^p(G)$, $g \in L^1(G)$ *then it follows that:* $g * f$ *exists* λ*-almost-everywhere and* $\|g * f\|_{L^p(G)} \leq \|g\|_{L^1(G)} \|f\|_{L^p(G)}$

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Proof. First we notice that the case of $p = 1$ follows from (Lemma 3.1). Similarly we may rid ourselves of the easy case $p = \infty$ since:

$$ZZZ$$

For $1 < p < \infty$, we must work a little harder. We have:

$$(|g| * |f|)(x) = \int_G \left|f(y^{-1}x)\right| \left|g(y)\right| d\lambda(y)$$

We shall apply Hölders inequality with the to following criteria,

Measure: $\left|g(y)\right| d\lambda(y)$
Functions: 1 and $\left|f(y^{-1}x)\right|$
Exponents: p and p' since $\frac{1}{p} + \frac{1}{p'} = 1$

to discover:

$$\begin{aligned}(|g| * |f|)(x) &\leq (\int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y))^{1/p} (\int_G 1^{p'} \left|g(y)\right| d\lambda(y))^{1/p'} \\ &= (\int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y))^{1/p} (\int_G \left|g(y)\right| d\lambda(y))^{1/p'}\end{aligned}$$

Since we are working with positive functions, by the monotonicity of the exponent and the integral, we may take the L^p norm on both sides while preserving the inequality.

$$\begin{aligned}\left\||g| * |f|\right\|_{L^p} &\leq (\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y) d\lambda(x))^{\frac{1}{p}} (ZZZZZZZZZ) \\ &= (\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(y) d\lambda(x))^{\frac{1}{p}} \\ &= (\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p \left|g(y)\right| d\lambda(x) d\lambda(y))^{\frac{1}{p}} \\ &= (\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(y^{-1}x)\right|^p d\lambda(x) \left|g(y)\right| d\lambda(y))^{\frac{1}{p}} \\ &= (\|g\|_{L^1}^{p-1} \int_G \int_G \left|f(x)\right|^p d\lambda(x) \left|g(y)\right| d\lambda(y))^{\frac{1}{p}} \\ &= (\|g\|_{L^1}^{p-1} \int_G \left|f(x)\right|^p d\lambda(x) \int_G \left|g(y)\right| d\lambda(y))^{\frac{1}{p}} \\ &= (\|g\|_{L^1}^{p-1} \|f\|_{L^p}^p \|g\|_{L^1})^{\frac{1}{p}} \\ &= (\|g\|_{L^1}^p \|f\|_{L^p}^p)^{\frac{1}{p}} \\ &= \|g\|_{L^1} \|f\|_{L^p} \\ &< \infty\end{aligned}$$

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This implies that $|g| * |f|$ is finite λ-a.e. and satisfies the required inequality. We have shown before that $g * f \leq |g| * |f|$ which concludes our proof.

□

We shall find that we can prove this same inequality in a more general case, strengthening our tool set by a lot.

THEOREM 4.2. *Young's Inequality: Let $1 \leq p, q, r \leq \infty$ such that*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$$

and $f \in L^p \wedge g \in L^r$. Then $f * g$ exists λ-a.e and the following inequality holds:

$$\|f * g\|_{L^q} \leq \|g\|_{L^r} \|f\|_{L^p}$$

Proof. First we shall rid ourselves of the case $r = \infty$. We notice that then $p = 1 \wedge q = \infty$ follows by satisfaction of the requirements. This case then simply follows from Minkowski's Inequality.

Therefore let us now assume $r < \infty$. We examine first the relationship between p, q, r:

(1) $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p} \Leftrightarrow 1 - \frac{1}{r} + \frac{1}{q} + 1 - \frac{1}{p} = 1 \Leftrightarrow \frac{r-1}{r} + \frac{1}{q} + \frac{p-1}{p} = 1 \Leftrightarrow \frac{1}{r/(r-1)} + \frac{1}{q} + \frac{1}{p/(p-1)} = 1 \Leftrightarrow \frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1$

(2) other

We work with $|f| * |g|$ as before. □

THEOREM 4.3. *Youngs Inequality for Weak Type Spaces ouch proof*

5. Approximate Identities

Approximation of dirac delta function , identity element of convolutions

DEFINITION 5.1. *An* ***approximate identity*** *(as $\varepsilon \to 0$) is a family of $L^1(G)$ functions k_ε with the following three properties:*

(i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \leqslant c$ for all $\varepsilon > 0$.

(ii) $\int_G k_\varepsilon(x) d\lambda(x) = 1$ for all $\varepsilon > 0$.

(iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)| d\lambda(x) \to 0$ as $\varepsilon \to 0$.

THEOREM 5.1. *Any approximate identity has the following two properties:*

(1) $f \in L^p(G) \wedge 1 \leq p < \infty \implies \|k_\epsilon * f - f\|_{L^p(G)} \to 0$ as $\epsilon \to 0$

(2) $f \in L^\infty(G)$ uniformly continuous on $K \subset G \implies \|k_\epsilon * f - f\|_{L^\infty(K)} \to 0$ as $\epsilon \to 0$ Furthermore, if f is bounded and continuous at $x \in G$ then $(k_\epsilon *)(x) \to f(x)$ as $\epsilon \to 0$

Proof. Let us first prove the statement in finite L^p spaces. We shall make use of the following equality:

$$\forall p \in \mathbb{N} : \left| g(h^{-1}x) - g(x) \right|^p \leq (\left| g(h^{-1}x) \right| + \left| g(x) \right|)^p$$

$$\begin{aligned}
&\leq (\operatorname*{ess.sup}_x(\left|g(h^{-1}x)\right|) + \operatorname*{ess.sup}_x(\left|g(x)\right|))^p \\
&\leq (2\|g\|_{L^\infty})^p
\end{aligned}$$

Applying the Lebesgue Dominated Convergence theorem (*) we find:

$$\int_G \left|g(h^{-1}x) - g(x)\right|^p d\lambda(x) \to 0 \text{ as } h \to e$$

Where e is the neutral element of G. In $\mathbb{R}^n$ this is simply 0. We can approximate any $g \in L^p(G)$ with a continuous function f with compact support. Thus the property still holds:

$$\int_G \left|f(h^{-1}x) - f(x)\right|^p d\lambda(x) \to 0 \text{ as } h \to e$$

However we can now say that since f is continuous,

$$\delta > 0 : \exists V(e) : h \in V(e) \implies \int_G \left|f(h^{-1}x) - f(x)\right|^p d\lambda(x) < (\frac{\delta}{2})^p(\frac{1}{c}) \tag{3}$$

Where $c*$, $V(e)$ is a neighborhood of e. We shall fix this neighborhood for later. We have picked the value on the right side for later convenience. As with most proofs in this area, we shall split the object of our analysis into two parts we can evaluate:

$$(k_\epsilon * f)(x) - f(x) = \int_G (f(y^{-1}x) - f(x))k_\epsilon(y)d\lambda(y)$$

We take L^p norms on both sides. This preserves the equality since *.

$$\begin{aligned}
\left\|(k_\epsilon * f)(x) - f(x)\right\|_{L^p(G)} &= \left\|\int_G (f(y^{-1}x) - f(x))k_\epsilon(y)d\lambda(y)\right\|_{L^p(G)} \\
&= (\int_G \left|\int_G (f(y^{-1}x) - f(x))k_\epsilon(y)d\lambda(y)\right|^p d\lambda(x))^{\frac{1}{p}} \\
&\leq (\int_G \int_G \left|f(y^{-1}x) - f(x)\right|^p \left|k_e(y)\right| d\lambda(y)d\lambda(x))^{\frac{1}{p}} \\
&= (\int_G \int_V \left|f(y^{-1}x) - f(x)\right|^p \left|k_e(y)\right| d\lambda(y)d\lambda(x) + \int_G \int_{V^c} \left|f(y^{-1}x) - f(x)\right|^p \left|k_e(y)\right| d\lambda
\end{aligned}$$

Where the inequality originates from Jensen's inequality *. Notice that we can now inspect on V and V^c, we shall call the respective parts of the function F_V and F_{V^c} such that our last statement can be rewritten as $(F_V + F_{V_c})^{\frac{1}{p}}$ for convenience. We now look at them individually.

$$\begin{aligned}
F_V &= \int_G \int_V \left|f(y^{-1}x) - f(x)\right|^p \left|k_e(y)\right| d\lambda(y)d\lambda(x) \\
&= \int_V \int_G \left|f(y^{-1}x) - f(x)\right|^p \left|k_e(y)\right| d\lambda(x)d\lambda(y)
\end{aligned}$$

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$$
\begin{aligned}
&= \int_V \int_G \left| f(y^{-1}x) - f(x) \right|^p d\lambda(x) \left| k_e(y) \right| d\lambda(y) \\
&\leq \int_V ((\frac{\delta}{2})^p \frac{1}{c}) \left| k_e(y) \right| d\lambda(y) \\
&= ((\frac{\delta}{2})^p \frac{1}{c}) \int_V \left| k_e(y) \right| d\lambda(y) \\
&\leq ((\frac{\delta}{2})^p \frac{1}{c}) \int_G \left| k_e(y) \right| d\lambda(y) \\
&\leq ((\frac{\delta}{2})^p \frac{1}{c}) c \\
&= ((\frac{\delta}{2})^p)
\end{aligned}
$$

Here we have used Fubini *, substituted our previously discovered upper bound, and used property (i) intrinsic of any approximate identity, as defined. We bound the other half now:

$$
\begin{aligned}
F_{V^p} &= \int_G \int_{V^c} \left| f(y^{-1}x) - f(x) \right|^p \left| k_\epsilon(y) \right| d\lambda(y) d\lambda(x) \\
&= \int_{V^c} \int_G \left| f(y^{-1}x) - f(x) \right|^p \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&= \int_{V^c} \left\| f(y^{-1}x) - f(x) \right\|_{L^p}^p \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&\leq \int_{V^c} (\|f\|_{L^p}^p + \|f\|_{L^p}^p) \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&= \int_{V^c} (2\|f\|_{L^p}^p) \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&= (2\|f\|_{L^p}^p) \int_{V^c} \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
&\leq (2\|f\|_{L^p}^p) \int_{V^c} \left| k_\epsilon(y) \right| d\lambda(x) d\lambda(y) \\
ZZZ \\
&\leq ((\frac{\delta}{2})^p)
\end{aligned}
$$

Were we have once again used Fubini *, the definition of our norm, minkowski's *, and have choosen a correct neighborhood V^c... *. Putting our function back together we find:

$$
(F_V + F_{V^c}) \leq (\frac{\delta}{2})^p + (\frac{\delta}{2})^p)^{\frac{1}{p}}
$$

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$$
\begin{aligned}
&\le (\frac{\delta}{2}) + (\frac{\delta}{2})) \\
&= \delta
\end{aligned}
$$

Where the final inequality follows from the fact that $p \geq 1 \implies \frac{1}{p} \leq 1$ and Ex. 1.1.4 from the previous section. We have proven property (1) of our theorem since the norm becomes arbitrarily small and thus converges to 0.

We shall no prove the case of $p = \infty$. First we prepare some ingredients. We have required that f is bounded and uniformly continuous. Recall:

$$
f \text{uniformly continuous} \implies \delta > 0 : \forall V(e) : \left| f(y^{-1}x) - f(x) \right| < \frac{\delta}{2c}
$$

Furthermore from property (iii) of an approximate identity we have:

$$
\exists \epsilon_0 > 0 : \forall \epsilon \in (0, \epsilon_0) : \int_{V^c} \left| k_\epsilon(y) \right| d\lambda(y) \leq \frac{\delta}{2(\|f\|_{L^\infty} + 1)}
$$

Once more we shall split our normed function:

$$
\begin{aligned}
\|k_\epsilon * f - f\|_{L^\infty}(G) &= \sup_{x \in G} \left| (k_\epsilon * f)(x) - f(x) \right| \\
&= \sup_{x \in G} \left| \int_G f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) \right| \\
&= \sup_{x \in G} \left| \int_V f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) + \int_{V^c} f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) \right| \\
&\leq \sup_{x \in G} \left| \int_V f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) \right| + \left| \int_{V^c} f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) \right| \\
&\leq \sup_{x \in G} \left| \int_V f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) \right| + \sup_{x \in G} \left| \int_{V^c} f(y^{-1}x) - f(x) k_\epsilon(y) d\lambda(y) \right| \\
&\leq \sup_{x \in G} \int_V \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right| + \sup_{x \in G} \int_{V^c} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right| \\
&\leq \int_V \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right| \\
&\leq \int_V (\frac{\delta}{2c}) \left| k_\epsilon(y) d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right| \\
&\leq (\frac{\delta}{2c}) \int_V \left| k_\epsilon(y) d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right| \\
&\leq (\frac{\delta}{2c}) \int_G \left| k_\epsilon(y) d\lambda(y) \right| + \int_{V^c} \sup_{x \in G} \left| f(y^{-1}x) - f(x) \right| \left| k_\epsilon(y) d\lambda(y) \right|
\end{aligned}
$$

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$$\leq (\frac{\delta}{2c})c + \int_{V^c} \sup_{x\in G}\Big|f(y^{-1}x) - f(x)\Big|\Big|k_\epsilon(y)d\lambda(y)\Big|$$
$$= (\frac{\delta}{2}) + \int_{V^c} \sup_{x\in G}\Big|f(y^{-1}x) - f(x)\Big|\Big|k_\epsilon(y)d\lambda(y)\Big|$$
$$*$$
$$\leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$$

Here we have normed our required equality, split the integral into neighborhood and complement, used the triangle inequality, and applied previous definitions and preparatory discoveries *. Note at the end we have added one case of 0 *.

This concludes the proof of both parts. □

THEOREM 5.2. *Let k_ϵ be an approximate identity with a slight variation, instead of satisfying property (ii) is satisfies:*

$$(ii)' :\Leftrightarrow a \in \mathbb{C} \wedge \forall \epsilon > 0 : \int_G k_\epsilon(x)d\lambda(x) = a$$

then similarly the following is true:

(1) $f \in L^p(G) \wedge 1 \leq p < \infty \implies \|k_\epsilon * f - af\|_{L^p(G)} \to 0$ as $\epsilon \to 0$
(2) $f \in L^\infty(G)$ uniformly continuous on $K \subset G$such that

$$\forall \delta > 0 : \exists V(e) : sup_{x\in G} sup_{y\in V(e)} \Big|f(y^{-1}x) - f(x)\Big| \leq \delta$$

then $\|k_\epsilon * f - af\|_{L^\infty(K)} \to 0$ as $\epsilon \to 0$

Proof. The proof is analogous to the previous proof, working with a instead of 1. □

EXAMPLE 5.1. *A useful example of an approximate identity is the Poisson kernel on $\mathbb{R}$, defined as:*

$$P(x) := (\pi(x^2 + 1))^{-1}$$
$$P_\epsilon(x) := \epsilon^{-1}P(\epsilon^{-1}x)$$

Notice first a convenience:

$$\|P(x)\|_{L^1} = \int_{\mathbb{R}} \Big|\frac{1}{\pi(x^2+1)}\Big| d\lambda(x)$$
$$= \int_{\mathbb{R}} \frac{1}{\pi(x^2+1)} d\lambda(x)$$
$$= \int_{\mathbb{R}} \frac{1}{\epsilon(\pi(\frac{x}{\epsilon})^2 + 1)} d\lambda(x)$$

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$$= \int_{\mathbb{R}} \left| \frac{1}{\epsilon(\pi(\frac{x}{\epsilon})^2 + 1)} \right| d\lambda(x)$$
$$= \left\| P_\epsilon(x) \right\|_{L^1}$$

By a change of variables x to $\frac{x}{\epsilon}$ and noticing that the function is positive. With this insight we can prove required property (i) and (ii):

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\pi(x^2+1)} d\lambda(x) &= \int_{-\infty}^{+\infty} \frac{1}{\pi(x^2+1)} dx \\
&= \lim_{a\to+\infty} \int_{-a}^{+a} \frac{1}{\pi(x^2+1)} dx \\
&= \lim_{a\to+\infty} \frac{1}{\pi}(\arctan(x)|_{-a}^{+a}) \\
&= \lim_{a\to+\infty} \frac{1}{\pi}(\arctan(a) - \arctan(-a)) \\
&= \frac{1}{\pi}(\frac{\pi}{2} - (-\frac{\pi}{2})) \\
&= 1
\end{aligned}$$

Proving property (ii). Notice that (ii) $\wedge (k_\epsilon \leq 0) \implies$ (i) on $\mathbb{R}$. To prove (iii) we notice that $V^c(e)$ on $\mathbb{R}$ is simply the set $\left\{x \in \mathbb{R} : |x| \geq \delta\right\}$ for some $\delta > 0$. We integrate:

$$\begin{aligned}
\int_{\left\{x\in\mathbb{R}:|x|\geq\delta\right\}} |P_\epsilon| \, d\lambda(x) &= \int_{\left\{x\in\mathbb{R}:|x|\geq\delta\right\}} \left| \frac{1}{\epsilon} \frac{1}{\pi((\frac{x}{\epsilon})^2+1)} \right| d\lambda(x) \\
&= \frac{1}{\pi} \int_{\left\{x\in\mathbb{R}:|x|\geq\delta\right\}} \left| \frac{1}{\epsilon} \frac{1}{((\frac{x}{\epsilon})^2+1)} \right| d\lambda(x) \\
zzz &= 1 - \frac{2}{\pi} \arctan(\frac{\delta}{\epsilon})
\end{aligned}$$

Which goes to 0 as $\epsilon \to 0$ since $\arctan(\frac{\delta}{\epsilon}) \to \frac{\pi}{2}$. Thus we have shown that the Poisson kernel is an approximate identity.

REMARK 5.1. *It is possible to state a more general form on $\mathbb{R}^n$ for an approximate identity which contains the Poisson kernel. Let: **

EXAMPLE 5.2. *Another example is the fejer kernel depicted below:*

zz

THEOREM 5.3. *approx. id. on locally compact group G with left Haar measure*

THEOREM 5.4. *ke familz of funcs on loc compact group G with properties...*

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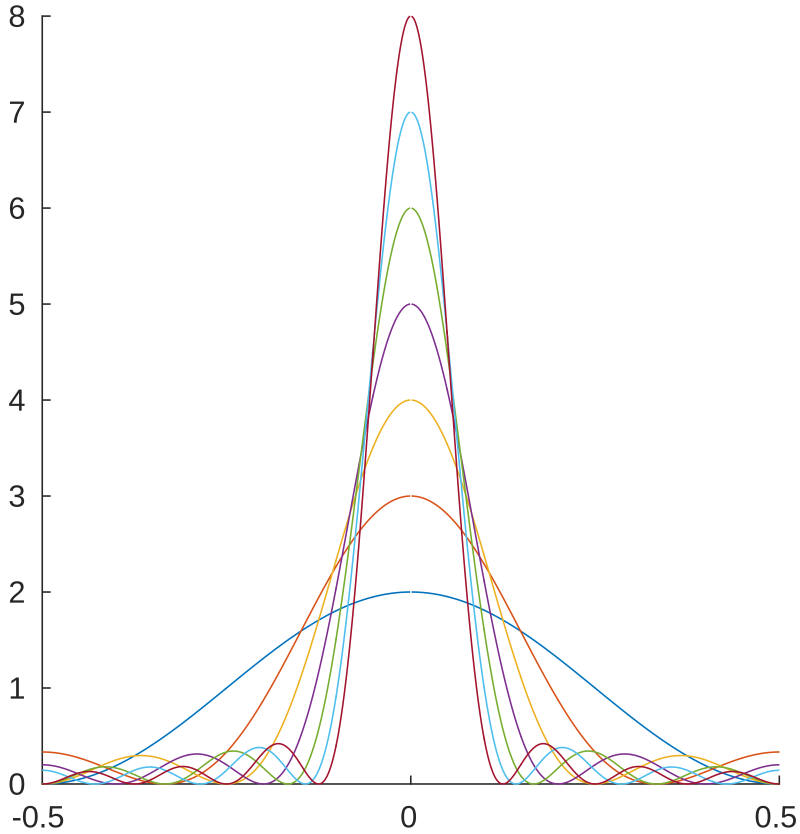

FIGURE 1. Fejer Kernel

6. REQUIRED STUFF

(1) hausdorf topological space
(2) counting measure
(3) area of intersecting circles
(4) banach algebra
(5) hoelders inequality

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(6) fubini
(7) chebyschevs inequality
(8) lebesgue dominated conv. thm.
(9) measure theoretic support

chapter 1 stuff:

(1) Lp norms and other defs etc.
(2) distr. functions

Appendix A. Poisson Kernel

Here's the code*.